

Uncertainty in identifying contact stiffness in a dovetail attachment for turbine blades

*Original*

Uncertainty in identifying contact stiffness in a dovetail attachment for turbine blades / Botto, D.; Glorioso, M.; Occhipinti, S.; Cuccovillo, F.. - In: MECHANICAL SYSTEMS AND SIGNAL PROCESSING. - ISSN 0888-3270. - ELETTRONICO. - 197:(2023). [10.1016/j.ymssp.2023.110379]

*Availability:*

This version is available at: 11583/2978403 since: 2023-05-09T08:25:56Z

*Publisher:*

ACADEMIC PRESS LTD- ELSEVIER SCIENCE

*Published*

DOI:10.1016/j.ymssp.2023.110379

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(Article begins on next page)

# Uncertainty in identifying contact stiffnesses in a dovetail attachment for turbine blades

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## Abstract

Designing bladed disks in turbo engines requires accurate dynamic models to correctly estimate resonance frequencies and related stresses. These stresses are of fundamental importance to assess the fatigue strength of blades. The contact parameters - stiffness and damping in the blade attachments - are currently among the most significant uncertainties of such models. Dry friction and alternating relative motions between the contact interfaces determine the transition between stick and slip that causes the nonlinear behavior of attachments. As a consequence, the joint contact parameters depend on centrifugal loads and amplitude vibration of blades. Commercial and in-house finite element software make use of specific contact elements to simulate dry friction and the resulting nonlinear behavior. These elements require the friction coefficient as input while the normal and tangential contact stiffness can be directly evaluated by the software or set up by the user. Researchers have been working to evaluate the contact parameters with theoretical models or to measure them with laboratory apparatus or in the field. The main objective of the present research is to discuss the uncertainty associated with identifying with experimental data the normal and tangential stiffness in a dovetail coupling. In addition, the reliability of available theoretical contact models will be addressed. A novel experimen-

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9 tal procedure was developed to evaluate the influence of an attachment on the  
10 blade dynamics. The response of the blade/attachment system was measured  
11 as a function of the axial load (simulating the centrifugal force) and of the  
12 blade vibration amplitude. An identification procedure using a finite element  
13 model was set up to identify the normal and tangential contact stiffness of the  
14 attachment. These stiffnesses were compared with the values predicted by a  
15 theoretical model. The comparison highlights a significant difference between  
16 measured and predicted stiffness. The final discussion focuses on the significance  
17 of theoretical contact stiffness and its use in finite element models.

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23 *Keywords:* Contact stiffness, Contact model, Dovetail, Turbine blade  
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## 25 26 **1. Introduction**

### 27 28 *1.1. Background*

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30 Mechanical design of bladed disks in aero-engines includes the assessment  
31 of structural integrity against fatigue, be it high cycle, low cycle, or fretting  
32 fatigue [1, 2]. To correctly predict stresses due to vibrations, accurate dynamic  
33 models are necessary. A complete resonance decoupling of such components is  
34 5 fatigue [1, 2]. To correctly predict stresses due to vibrations, accurate dynamic  
35 models are necessary. A complete resonance decoupling of such components is  
36 not a viable design choice because of the high modal density of the disk and  
37 the broad spectrum of the external loads. For this reason, damping introduced  
38 by dry friction is one of the tools used to reduce the amplitude of vibration.  
39 Friction damping develops at contact surfaces between shrouds, damper and  
40 10 Friction damping develops at contact surfaces between shrouds, damper and  
41 under-platforms, disk slot and blade attachment, and more generally whenever  
42 interfaces are present.  
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46 Numerical simulations of models with interfaces are challenging because the  
47 tangential force in oscillating contacts is a piecewise function depending on the  
48 state of the contact, stick, slip or even separation of the surfaces. Researchers  
49 15 state of the contact, stick, slip or even separation of the surfaces. Researchers  
50 have been working to develop efficient methods to solve the nonlinear equation  
51 of motion of dynamic systems embedding friction contacts. A new method,  
52 the harmonic balance technique, was proposed by in [3] to avoid the time do-  
53 main solution of the dynamic equations of nonlinear circuits. This technique,  
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known as Harmonic Balance Method (HBM), was widely used to determine the steady-state forced response of mechanical systems with nonlinear behavior [4]. An incremental HBM method, named as Multi-Harmonic Balance Method (MHBM), was developed in [5] for the analysis of damped systems with dry friction. In [6] this method was enhanced using a Fast Fourier Transform algorithm to transfer the equation of motion from time to frequency domain and vice versa. Nowadays, this technique is usually employed to study the dynamic behavior of bladed disks in turbomachines [7]. The method was improved with an analytical formulation of the contact forces [8] or by adding the static component of the contact force [9, 10]. Time integration remains the most used tool to evaluate transients and even more efficient methods were put forward, see for example the one proposed in [11]. The aforementioned techniques make use of contact models based on Jenkins element in which the contact is composed by a linear spring in series with a Coulomb slider. Contact models evolved from the earlier one-dimensional [12] and two-dimensional [13] (2D) models to the more complete three dimensional frictional contacts [14] with variable normal load [15].

These models require three parameters to define the contact properties: the friction coefficient  $\mu$ , the tangential and the normal contact stiffness,  $k_t$  and  $k_n$  respectively. The normal contact stiffness  $k_n$  for non-conforming geometries can be calculated with the Hertz theory as reported in [16]. The normal contact deformation of cylinders, despite their simple geometry, is a challenging task and several solutions are available in the literature [17–19]. The most reliable solution, supported by experimental tests, is the one proposed in [20]. The Hertz theory of contact was extended to account for friction forces and calculate the tangential stiffness between spherical bodies [21, 22]. The solution for the contact of nonspherical bodies subjected to oblique loading was given in [23]. Among conforming geometries the rigid punch with rounded edges has a very important practical application in aero-engines because it is similar to the contact between the lobes in the disk slot and in the blade root. The pressure distribution was given in [24] for a friction-less rigid punch on an elastic half-

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9 space, while the effect of a tangential force on the state of stress was studied  
10 in [25]. The latter determined the boundary between stick and slip regions and  
11 the traction distribution. The theory presented in [25] was developed for 2D  
12 geometries and subsequently extended to three-dimensional (3D) contact bodies  
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15 in [26]. This paper presents a semi-analytical method to evaluate the tangential  
55 force and displacements for a 3D punch with rounded edges. The contact  
16 stiffness and the dissipated energy are determined from the hysteresis loop (the  
17 friction force against the tangential displacement), provided the friction coefficient  
18 is known. The contact stiffness was also evaluated in its complex form by  
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29 using an harmonic balance so that the imaginary part of the complex stiffness,  
60 that is in phase with the velocity [27], represents the damping effect. Calculation  
30 of damping induced by dry friction at blade attachments is still an open  
31 issue as evidenced by recent papers found in the literature [28, 29].

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Specific experimental campaigns were carried out to investigate the component-  
scale dynamic behavior of blades fixed with typical blade-disk joints. These cam-

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9       paigns provided a useful database for the validation of numerical simulations.  
10       The free response of a pulled mock-blade fixed with dovetail and fir-tree joints at  
11       its two ends was obtained in [42] and [43] . Damping and resonance frequencies  
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14       85 were extracted as functions of the amplitude of vibration for different centrifugal  
15       loads. the same test rig was used in [44] to measure the forced-response and  
16       to collect a database of the frequency response with different normal contact  
17       loads and varying the dynamic excitation. A similar test apparatus [45] was  
18       used to measure the damping introduced by the blade-disk attachment. The  
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21       90 joint behavior was simulated with two different contact models, one considering  
22       only macroslip conditions the other based on microslip. Results showed that the  
23       macroslip model was not suitable for accurate simulations. The damping ratio  
24       and the natural frequency on a simple beam with dovetail root joint was also  
25       measured in [46]. The centrifugal load was applied with a loading screw pushing  
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28       95 on the back of the dovetail while a vibration slip table provided the dynamic  
29       excitation of the whole system. The micro-slip characteristic and the tangential  
30       contact stiffness were measured on the same material by using a fretting test  
31       rig. The contact stiffness inferred from the dynamic test performed on the sim-  
32       ple beam with dovetail was compared with the results from the fretting test.  
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35       100 Results showed that the tangential contact stiffness measured on the dovetail is  
36       smaller than the stiffness obtained by the fretting tests. The friction coefficient  
37       was measured on a fir-tree attachment in [47], but the rig used in this work did  
38       not allow measuring the contact stiffness.  
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## 44       1.2. Objective and workflow

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46       105       Although the elastic properties of the contact in blade attachments are of  
47       great significance for engineers few works were devoted to measuring these pa-  
48       rameters under dynamic conditions. The objective of the present work is to  
49       evaluate the uncertainty associated to the experimental identification of the  
50       contact stiffnesses  $k_n$  and  $k_t$ , along the normal and tangential direction, in a  
51       dovetail attachment. Moreover, the contact stiffnesses are also evaluated with  
52       110 the semi-analytical models found in the literature and compared with the ex-  
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perimental results. This comparison was aimed to asses if the elastic contact parameters calculated with these models are feasible as contact elements in a FE model during the design step of bladed disks. The objective is pursued by collecting the dynamic response of a mock-blade with dovetail joints, as depicted in Fig. 1, and analyzing the results with the help of a FE model. The flow chart

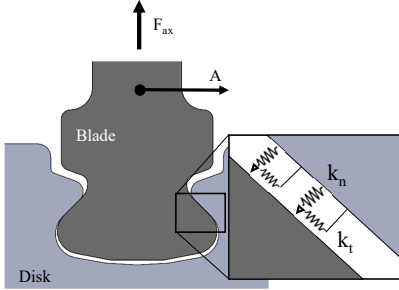


Figure 1: sketch of the dovetail joint investigated in this paper. The contact stiffnesses are highlighted in the panel.

in Fig. 2 shows the analysis procedure described in detail below.

- Section 2 reports the main findings of the method proposed in [48] to identify the instantaneous natural frequency  $\omega_n(A)$  and the instantaneous damping  $\eta(A)$  depending on the vibration amplitude  $A$ . More details on the method are given in [49] while properties and assumptions regarding the Hilbert transform can be found in [50]. The reader familiar with the method can skip this section.
- Section 3 describes the experimental apparatus used for the dynamic tests on the mock-blade. This apparatus utilizes the rig described in [42] but in this work a different excitation system is used. The free decay of the mock-blade is measured for different tensile axial forces  $F_{ax}$  and then processed to extract  $\omega_n$  and  $\eta$  of the nonlinear system.
- Section 4 illustrates the FE model of mock-blade and supports. Contact elements containing the unknown stiffnesses  $k_t$  and  $k_n$  connect the corresponding contact nodes on the dovetail and on the slot. These elements

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implement only the stick state. This section also reports the method used to obtain the reduced models of the mock-blade and supports, comprehensive of the stiffening effect due to the axial force.

- 135 • Section 5 describes how the theoretical contact stiffnesses in the attachment  $k_t^{th}$  and  $k_n^{th}$  can be calculated using the semi-analytical model of the flat punch with rounded edges proposed in [26]. The theoretical and experimental results are compared to discuss the feasibility of the theoretical model for the blade design.
- 140 • Section 6 documents the procedure to identify  $k_n$  and  $k_t$  for varying  $F_{ax}$ . The experimentally determined natural frequencies are compared with the frequencies computed by the FE model modal analysis. Appendix A reports the uncertainty of the identified stiffness.

Throughout this paper boldface letters indicate vectors and matrices. Moreover, the greek letter  $\omega$  indicates frequencies in rad/s whereas  $f$  is the frequency expressed in Hz.

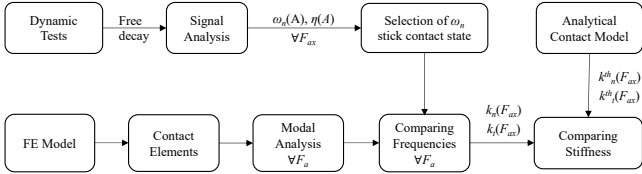


Figure 2: overview of the different phases of the activity.

## 2. Frequency and damping identification

The dependence of the natural frequency and damping on the amplitude of vibration, typical of nonlinear behavior, can be obtained from a free decay test with the method proposed in [48]. This method works in the time domain and is based on the analytic signal  $Y(t)$  corresponding to the real signal  $y(t)$

$$Y(t) = y(t) + i\tilde{y}(t) = A(t) \cdot e^{i\varphi(t)} \tag{1}$$

where  $\tilde{y}(t)$

$$\tilde{y}(t) = H[y(t)] = \frac{1}{\pi t} * y(t) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{y(\tau)}{t - \tau} d\tau \quad (2)$$

is the Hilbert transform of  $y(t)$ . The terms  $A(t)$  and  $\varphi(t)$  in Eq. 1 are the instantaneous amplitude and instantaneous phase of the analytic signal  $Y(t)$

150 respectively.

Dry friction is the dominant source of damping in blade attachments. Friction forces undergoing oscillating motion generate hysteresis loops as sketched in Fig. 4b. Analysis of energy dissipation in hysteresis loops lead to the conclusion that dry friction can be classified as a frequency-dependent damping. This kind of damping is denoted to as hysteretic damping and in dynamic model it is equivalent to an imaginary stiffness. The dynamic equation of motion with hysteretic damping can be written as

$$m \ddot{y} + (k(A) + i h(A)) y = 0 \quad (3)$$

In Eq. 3 the stiffness  $k(A)$  and the hysteretic damping  $h(A)$  are unknown functions and, for non linear systems, depend on the amplitude of vibration  $A$ . Applying the Hilbert transform to Eq. 3 the dynamic equation of motion becomes

$$m \ddot{\tilde{y}} + (k(A) + i h(A)) \tilde{y} = 0 \quad (4)$$

Multiplying each term of Eq. 4 by the imaginary unit  $i$  and adding it to the corresponding term of Eq. 3 we obtain the analytic form of the equation of motion

$$m \ddot{Y} + (k(A) + i h(A)) Y = 0 \quad (5)$$

Equation 5 can be rewritten

$$\ddot{Y} + \omega_n^2(A) (1 + i \eta(A)) Y = 0 \quad (6)$$

with

$$\omega_n^2(A) = \frac{k(A)}{m} \quad (7a)$$

$$\eta(A) = \frac{h(A)}{k(A)} \quad (7b)$$

the instantaneous natural frequency and the loss factor respectively, also referred to as modal parameters. The first and second derivative of the analytic signal in Eq. (1) are

$$\dot{Y} = \dot{y} + i\dot{\tilde{y}} = (y + i\tilde{y}) \left[ \frac{\dot{A}}{A} + i\omega \right] \quad (8a)$$

$$\ddot{Y} = \ddot{y} + i\ddot{\tilde{y}} = (y + i\tilde{y}) \left[ \frac{\ddot{A}}{A} - \omega^2 + i2\omega \frac{\dot{A}}{A} + i\dot{\omega} \right] \quad (8b)$$

where  $\omega = \dot{\varphi}(t)$  is the instantaneous frequency of the signal. By substituting Eq. (8b) into Eq. (5) the dynamic equation of motion becomes

$$\left[ \frac{\ddot{A}}{A} - \omega^2 + \omega_n^2 + i \left( 2\omega \frac{\dot{A}}{A} + \dot{\omega} + \omega_n^2 \eta \right) \right] Y = 0 \quad (9)$$

By solving two equation for the real and imaginary parts of Eq. (9) we obtain the expression for the two unknown parameters

$$\omega_n^2(A) = \omega^2 - \frac{\ddot{A}}{A} \quad (10a)$$

$$\eta(A) = -\frac{1}{\omega_n^2} \left( 2\omega \frac{\dot{A}}{A} + \dot{\omega} \right) \quad (10b)$$

The terms on the right-hand side of Eqs. (10) can be evaluated using the measured signal  $y(t)$  and its Hilbert transform  $\tilde{y}(t)$ . The first derivative (Eq. 8a) gives the free vibration frequency  $\omega$  and the ratio  $\dot{A}/A$

$$\omega(t) = \frac{1}{A^2} (y\dot{\tilde{y}} - \tilde{y}\dot{y}) \quad (11a)$$

$$\frac{\dot{A}}{A}(t) = \frac{1}{A^2} (y\dot{y} + \tilde{y}\dot{\tilde{y}}) \quad (11b)$$

while the second derivative (Eq. 8b) gives the variation of free vibration frequency  $\dot{\omega}$  and the ratio  $\ddot{A}/A$

$$\dot{\omega}(t) = \frac{y\ddot{\tilde{y}} - \ddot{\tilde{y}}y}{A^2} - 2\omega \frac{\dot{A}}{A} \quad (12a)$$

$$\frac{\ddot{A}}{A}(t) = \frac{y\ddot{y} + \ddot{\tilde{y}}\tilde{y}}{A^2} + \omega^2 \quad (12b)$$

The right-hand side in Eqs. 11 and 12 depends on the Amplitude  $A(t)$  that can be evaluated as

$$A(t) = \sqrt{y^2 + \tilde{y}^2} \quad (13)$$

### 3. Experimental measurements

#### 3.1. Test Rig

Figure 3 depicts the working scheme of the test rig used in the present work. The complexity of generating the centrifugal load through a rotating rig led to the development of a static loading technique. The unfeasibility of statically pulling a real blade without introducing additional damping is the reason of the symmetry of the rig, in which a dovetail type attachment is machined at both ends of a beam with constant rectangular cross-section to create the mock-blade. The equivalence of loads and joint kinematics makes the results obtained on this rig comparable with those on a real disk. More detailed considerations on the design principle of the rig can be found in [42].

The attachments are fitted into the slots machined in two supports. These supports are integral with two crossbars: one crossbar is fixed while the other is free to move if loaded with an axial force. The axial force  $F_{ax}$  is exerted through a hydraulic cylinder operated by a manual pump. The axial force is measured with two independent systems. A pressure gauge is connected to the hydraulic cylinder and its value is read directly by the operator. A full-bridge strain gauge is attached with adhesive to the mock-blade and it is read by a signal conditioning card and stored by the Data Acquisition (DAQ) system. The dynamic excitation system is composed by an electrodynamic shaker and its power amplifier. This amplifier is fed with a signal produced by a wave form generator. The drive rod of the shaker is not fixed to the mock-blade but it can be brought into contact or released by displacing the moving coil of the shaker. Beforehand, the drive rod is displaced by a wave form consisting of a ramp followed by a positive constant value. A proper choice of the initial gap between the drive rod and the mock-blade together with the value of the constant signal allows the drive rod to preload the mock-blade. The preload is followed by a sinusoidal wave whose frequency is chosen as close as possible to the resonance of the modal shape under investigation. If the preload has been properly defined the drive rod remains in contact with the mock-blade during

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9 the sinusoidal excitation period. At the end of the excitation period the drive  
10 rod is suddenly detached, with a negative wave form signal, from the mock-  
11 blade that from this point on is free to oscillate so that no additional damping  
12 injected by the shaker. This method will be referred to as the Detached Drive  
13 the  
14 Rod Method (DDRM). The excitation system used in this work differs from the  
15 185 Rod Method (DDRM). The excitation system used in this work differs from the  
16 one used in [42] in which the mock-blade was loaded with a non-contact exciter  
17 based on electromagnets. The main objective of both systems is the same:  
18 to uncouple the mock-blade from the exciter so that no additional damping is  
19 injected during the free decay. With the non-contact exciter this objective is  
20 achieved by stopping the current in the coil whereas with the DDRM the drive  
21 rod is physically detached from the contact point. The main advantage of using  
22 the DDRM is that the excitation force is applied at a specific point and can  
23 be measured with a load cell if the forced-response is needed. The non-contact  
24 exciter needs a complex calibration process to measure the transfer function  
25 between the force measured on the electromagnet and the true force applied to  
26 the mock-blade, as well explained in [44].  
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34 A Laser Doppler Vibrometer together with its controller measures the veloc-  
35 ity of one reference point at time on the mock-blade. The DAQ system collects  
36 and records point velocity and strain gauge signals. The wave form is created  
37 with a in-house code with a sampling rate of 10 kHz and stored on file. This  
38 200 file is then read by the wave form generator that provides the output voltage  
39 signal.  
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### 44 3.2. Testing procedure

45 Experiments were focused on the free decay of oscillations of the first and  
46 second bending mode of the mock-blade. Velocity was measured at the antinode  
47 205 of the modal shape under investigation. For both modes a set of measurements  
48 at different axial loads  $F_{ax}$  was carried out. The testing procedure for each  
49 measurement is described in the following.  
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- 54 1. The mock-blade is pulled with the axial load  $F_{ax}$ , which is evaluated with  
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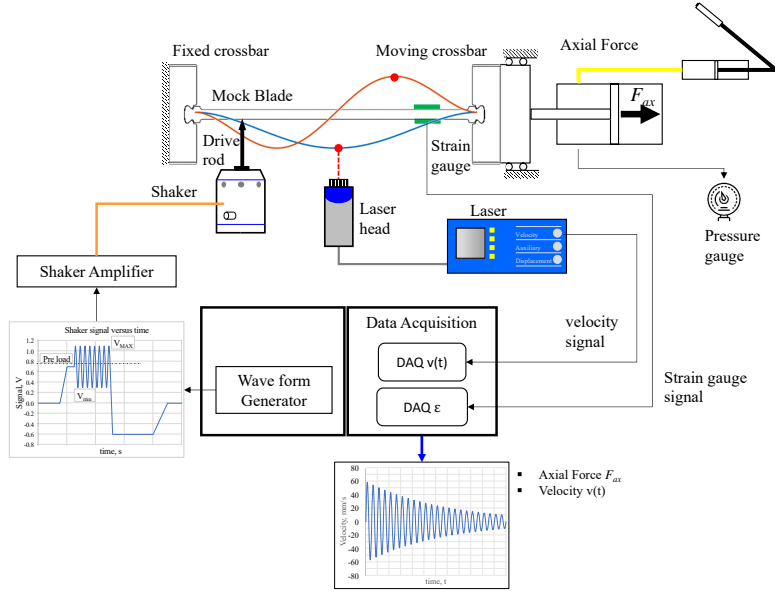


Figure 3: Test Rig scheme

the pressure gauge and strain gauges through the formulae

$$F_{ax} = A_{cyl} p \quad (14a)$$

$$F_{ax} = A_{beam} \sigma = A_{beam} E_{st} \varepsilon \quad (14b)$$

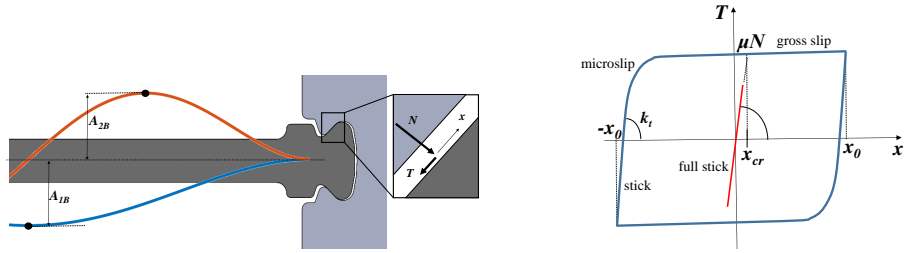
where  $A_{cyl} = 1790 \text{ mm}^2$  is the effective area of the hydraulic cylinder,  $p$  is the measured pressure,  $A_{beam} = 26 \times 10 \text{ mm}^2$  is the rectangular cross-section of the mock-blade,  $E_{st} = 200.3 \text{ GPa}$  the Young Modulus of the blade material (steel) and  $\varepsilon$  the measured strain. The total length of the blade is 325.7mm, comprehensive of the dovetail extremities, see Fig. 9a. As reported in [42], on real engines in cruise condition a typical centrifugal load on blades is about 50 kN. The minimum load, during the flight descent phase when the engine runs at idle speed, is about 2.5 kN. In the present work measurements were performed with tensile loads up to 25 kN, a value that is representative of the real operating condition.

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2. A preliminary hammer test is performed on the loaded mock-blade to obtain a first estimate of the natural frequencies of the first two bending modes.
3. The loaded mock-blade is excited by using the DDRM with an oscillating force whose frequency was estimated as in point 2. The free vibrations is obtained once the drive rod is detached from the excitation point.
4. The useful portion of the velocity  $v(t)$  is limited to the time interval of free decay. This portion of the signal is filtered with a bandpass Parks-McClellan FIR filter, centered on the frequency of the examined mode, to reduce noise and other undesired signals.
5. The measured signal is then processed with the method described in Sect. 2. In the present work, the measured signal is the velocity  $v(t) = \dot{y}(t)$ . Thus, displacement  $y(t)$  and acceleration  $\ddot{y}(t)$  were computed by numerical integration and differentiation of  $v(t)$  respectively.

3.3. Test Results

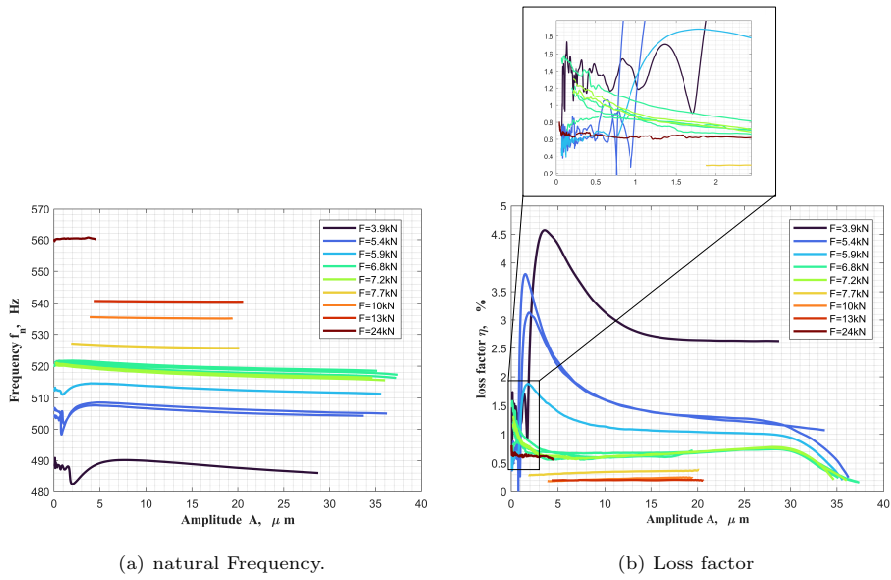
Figures 5 and 6 show the instantaneous frequency and loss factor of the first and second bending mode respectively. These dynamic parameters are evaluated with Eqs. 10. Results are shown as functions of the displacement amplitude  $A$  of the antinode. In both modes frequency and loss factor are constant for small vibration amplitudes, see the detail in Fig. 5b, thus revealing a linear behavior of the system. For small amplitudes slip is negligible whereas slip occurs for higher amplitudes. The loss factor shows a maximum, clearly visible in Fig. 5b, at an optimal amplitude  $A_{opt}$ : before  $A_{opt}$  the loss factor increases whereas after  $A_{opt}$  it decreases. This maximum is not visible for the second mode, Fig. 6b, because the output force available at the shaker was not able to displace the second mode up to the optimal amplitude. Oscillating displacements under slip condition lead to the formation of hysteresis loops as shown in Fig. 4b. Friction at interfaces dissipates energy, the amount of which is related to the loop area that varies with the amplitude  $A$ . The maximum loss factor is found at the optimal amplitude that is not the maximum amplitude.



(a) First (1B) and second (2B) Bending modes of the mock-blade.  $A_{1B}$  and  $A_{2B}$  are the displacement amplitude of their respective antinodes. In the detail the tangential  $T$  and normal  $N$  contact force and the relative displacement  $x$  at interfaces.

(b) Scheme of the Hysteresis loop between tangential force  $T$  and relative displacement  $x$  at contact interfaces.  $N$ : normal load;  $x_0$ : maximum amplitude of vibrating displacement  $x$ ;  $k_t$ : contact tangential stiffness.

Figure 4



(a) natural Frequency.

(b) Loss factor

Figure 5: modal parameters for the first bending mode (1B).

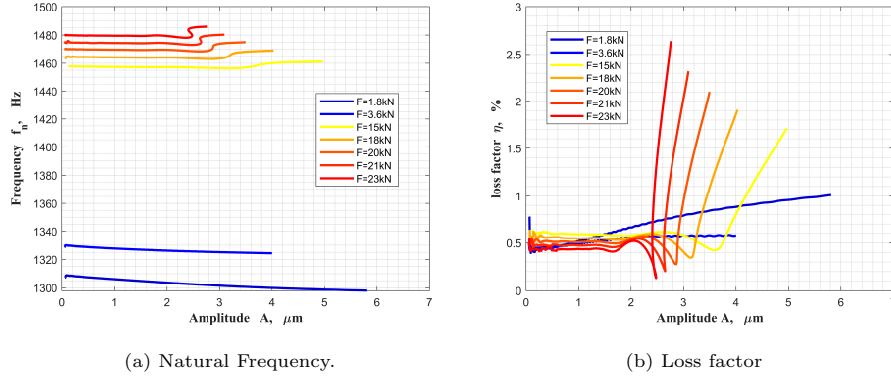
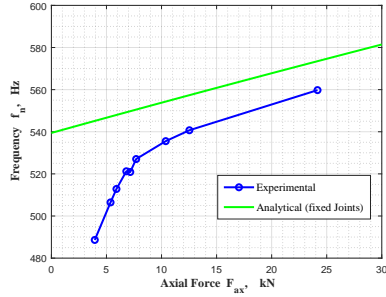


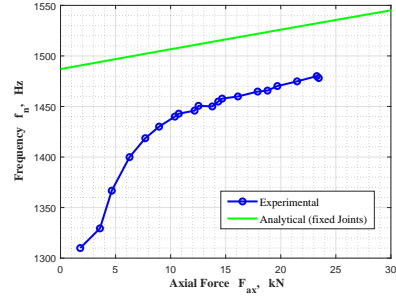
Figure 6: modal parameters for the second bending mode (2B).

The loss factor, see Figs. 5b and 6b, decreases with increasing axial loads  $F_{ax}$ , provided the amplitude is the same. This behavior is due to the increasing contact normal load  $N$  which reduces the gross-slip displacement range in the hysteresis loop, thus decreasing the dissipated energy. The natural frequency increases with axial loads  $F_{ax}$ , as reported in Figs. 5a and 6a. This increment is caused by two factors. First, the axial stress along the beam produces a moment opposing the bending deformation. This effect was considered with an analytical model [51] in which the Euler-Bernoulli beam model was modified to introduce a second order deformation effects. The effect of  $F_{ax}$  on the frequencies, evaluated with this analytical model, is depicted in Fig. 7. The second effect is related to the increasing contact normal load  $N$  that increases the stiffness of the contacts. This phenomenon will be further discussed in Sect. 6. Figure 7 also shows the natural frequencies measured in stick condition, taken as the mean value in the small amplitudes range. The measured frequency is lower than the predicted because the contact stiffness at the interfaces is in series with the mock-blade stiffness and then reduces the global stiffness. For axial loads greater than 12 kN the frequencies increase with a slope slightly higher than that predicted by the analytical model. For axial loads lower than 12 kN the experimental frequencies diverge from the theoretical ones because the interface tends to slip and the contact stiffness decreases, thus reducing the natural frequency of the

entire system.



(a) Natural frequency of the first bending mode.



(b) Natural frequency of the second bending mode.

Figure 7: stiffening effect due to the axial load  $F_{ax}$ . The increase of the frequency predicted by the analytical model is compared with the experimental results.

#### 270 4. Finite Element model

A commercial Finite Element (FE) model was used in the procedure to identify the normal and tangential contact stiffnesses. The mock-blade and the slots, depicted in Fig. 8, were modeled as separate bodies. The material properties of the mock-blade were evaluated with a hammer modal test in free-free condition. The measured modulus of elasticity and density were  $E = 200.3$  GPa and  $\rho = 7592 \text{ kg/m}^3$  respectively. The contact surface on the root blade and the matching surface on the slot were discretized with the same number of nodes. These nodes have coincident coordinates, see the detail in Fig. 8, so that node-to-node contact element is allowed. These contact elements are assembled during the identification procedure.

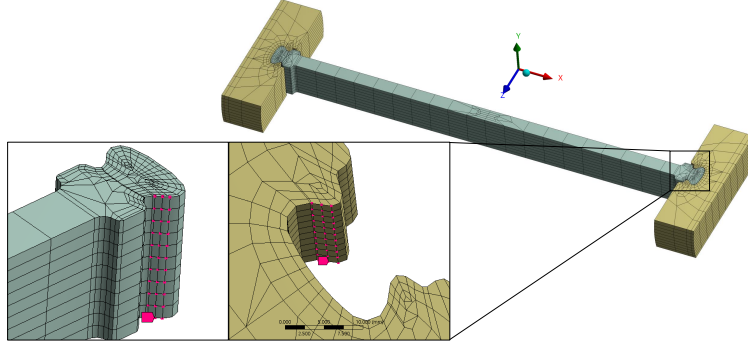


Figure 8: Finite Element model and detail of contact nodes.

#### 4.1. Tensile force stiffening effect

The stiffening effect induced by the axial load  $F_{ax}$  was considered introducing the so-called geometrical stiffness matrix  $\Delta\mathbf{K}_G$

$$\mathbf{K} = \mathbf{K}_0 + \Delta\mathbf{K}_{G,F_{ax}} \quad (15)$$

This matrix depends on the state of stress in the beam, and it was evaluated through a FE preliminary static analysis of the mock-blade for each  $F_{ax}$  applied in the experiments

$$\Delta\mathbf{K}_{G,F_{ax}} = \int_V \mathbf{S}_g^T \boldsymbol{\sigma} \mathbf{S}_g dV \quad (16)$$

In Eq. 16  $\mathbf{S}_g$  is the matrix of the shape functions derivatives while  $\boldsymbol{\sigma}$  is the stress state matrix. The stress stiffening effect is only significant in slender bodies, then it was calculated for the mock-blade but not for the supports.

The preliminary static analysis was performed by applying to the dovetails contact surfaces the normal forces  $N$  balancing the axial load  $F_{ax}$

$$N = \frac{F_{ax}}{2 \sin \beta} \quad (17)$$

with  $\beta = 45^\circ$  as sketched in Fig. 9a. Since results are little affected by contact tangential forces  $T$  these forces were neglected. The mock-blade must be properly constrained during the preliminary analysis to avoid kinematic indeterminacy that would not allow solving the static problem. Since these constraints

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are embedded in the stiffness matrix in Eq. 15 they could invalidate the identification procedure of  $k_n$  and  $k_t$ . To avoid this issue the FE model was constrained with node-to-ground elastic springs with a very small stiffness  $k = 1 \cdot 10^{-4}$  N/mm so that they have no effect on the mock-blade dynamics in the frequency range of interest. Three nodes for each dovetail were constrained along the transverse directions, see Fig. 9b, while the center node of the mock-blade was constrained along the axial direction as in Fig. 9c. The rigid motion of the mock-blade is canceled with a minimum number of constraints and complying with the symmetry of the first two modal shapes.

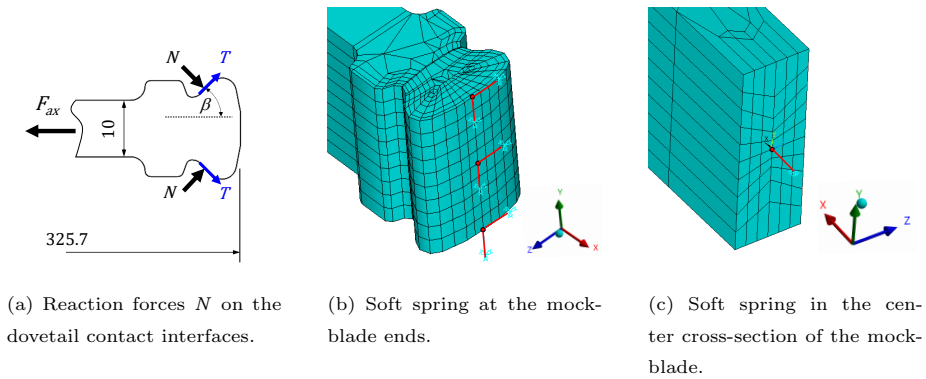


Figure 9: boundary conditions for the preliminary FE static analysis of the mock-blade.

#### 4.2. CMS reduction

The identification of the contact stiffness was performed with a purposely developed in-house code. This code utilizes the stiffness and mass matrix generated by the FE commercial software. To decrease the computational cost and speed up the identification process the Degrees of Freedom (DoF) of the models were reduced using the tool built in the FE software. Then the reduced matrices were exported in a suitable format and read by the in-house code. The reduction was performed with the Component Mode Synthesis (CMS), a technique developed in [52] for sub-structuring, and summarized in the following. The dynamic equations in physical coordinates  $\mathbf{u}$  is

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0} \tag{18}$$

The physical DoFs are divided in active  $\mathbf{u}_a$  and omitted  $\mathbf{u}_o$

$$\mathbf{u} = \begin{Bmatrix} \mathbf{u}_a \\ \mathbf{u}_o \end{Bmatrix} \quad (19)$$

so that Eq. 18 can be rearranged as

$$\begin{bmatrix} \mathbf{M}_{aa} & \mathbf{M}_{ao} \\ \mathbf{M}_{oa} & \mathbf{M}_{oo} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{u}}_a \\ \ddot{\mathbf{u}}_o \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_{aa} & \mathbf{K}_{ao} \\ \mathbf{K}_{oa} & \mathbf{K}_{oo} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_a \\ \mathbf{u}_o \end{Bmatrix} = \mathbf{0} \quad (20)$$

According to the CMS method, only the physical displacement  $\mathbf{u}_a$  of the active nodes are retained in the equation. The omitted DoFs are approximated with a linear combination of the active DoFs and normal modes  $\Phi_{oo}$

$$\mathbf{u} = \begin{Bmatrix} \mathbf{u}_m \\ \mathbf{u}_s \end{Bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{K}_{oo}^{-1}\mathbf{K}_{oa} & \overline{\Phi}_{oo} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_a \\ \boldsymbol{\eta}_o \end{Bmatrix} = \mathbf{T}\mathbf{q} \quad (21)$$

The normal modes  $\Phi_{oo}$  are solution to the eigenproblem of the sub-system composed of the omitted DoFs with proper boundary conditions.  $\overline{\Phi}_{oo}$  is a subset of  $\Phi_{oo}$  while  $\boldsymbol{\eta}_o$  are the modal DoFs. The number of active and modal DoFs defines the degree of reduction. The dynamic equation 18 becomes

$$\overline{\mathbf{M}}\ddot{\mathbf{q}} + \overline{\mathbf{K}}\mathbf{q} = \mathbf{0} \quad (22)$$

with  $\mathbf{q}$  generalized DoFs and

$$\overline{\mathbf{K}} = \mathbf{T}^T \mathbf{K} \mathbf{T} \quad (23a)$$

$$\overline{\mathbf{M}} = \mathbf{T}^T \mathbf{M} \mathbf{T} \quad (23b)$$

are the reduced stiffness and mass matrices. The reduction was performed separately for the mock-blade and the two supports. The mock-blade stiffness matrix is actually a set of matrices, one for each axial load  $F_{ax}$ , according to what reported in Sect. 4.1. The active nodes on the mock-blade comprise 33 nodes on each contact interface and two line of 21 nodes each along the axial direction. These nodes allow visualizing the modal shapes during post processing of the results. One of these latter nodes coincides with the spot on which the velocity is measured, another one with the excitation point. The

subset  $\bar{\Phi}_{oo}$  is composed by the first 100 modal shapes (corresponding to the lowest frequencies) which brings the total number of generalized DoFs to 622. The active nodes on the support are the nodes on the contact interfaces and 117 nodes on the base. The latter nodes allow a static analysis of the reduced model of the support. This analysis was not performed in the present work and these DoFs were constrained. Also for the support 100 modal shapes were chosen so that the number of generalized DoFs amounts to 322.

### 4.3. Contact element

The mock-blade roots and the supports are connected by node-to-node contact elements, namely springs acting along the normal and tangential directions of the contact surface. The stiffness of these nodal springs  $\tilde{k}_n$  and  $\tilde{k}_t$  are obtained by uniformly distributing the contact stiffness of the attachment  $k_n$  and  $k_t$  on the nodes of each interface. The contact element and its stiffness matrix and DoFs in the Local coordinate system are visible in Fig. 10.

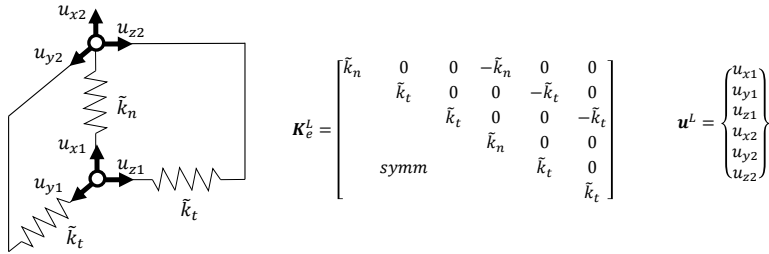


Figure 10: node-to-node contact element.  $\tilde{k}_n$  and  $\tilde{k}_t$  are the nodal stiffnesses.

## 5. Semi-analytical contact model

The unknown contact stiffnesses  $k_n$  and  $k_t$  were identified following the procedure described in Sect. 6. Nevertheless, the contact stiffness can be also estimated by using the semi-analytical solution of the contact between a flat punch on an infinite half-plane, as shown in Fig. 11a. These stiffnesses are referred to as theoretical contact stiffnesses  $k_n^{th}$  and  $k_t^{th}$ . The pressure distribution

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9  $p$  on a punch with rounded edges

$$\frac{bp(\phi)}{P} = \frac{2/\pi}{\pi - 2\phi_0 - \sin 2\phi_0} \left[ (\pi - 2\phi_0) \cos \phi + \sin \phi \ln \left| \frac{\sin(\phi + \phi_0)}{\sin(\phi - \phi_0)} \right| + \sin \phi_0 \ln \left| \tan \frac{\phi + \phi_0}{2} \tan \frac{\phi - \phi_0}{2} \right| \right] \quad (24)$$

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15 was first found in [24] and also recalculated in [25]. Equation 24 is given in  
16 dimensionless form in which  $b$  is the contact half-width and  $P$  the normal load  
17 per unit length. Figure 11a sketches the analogy between the punch geometry  
18 and the contact region in the attachment. The auxiliary angle  $\phi$  is defined as  
19  $\sin \phi = x/b$  and the angle  $\phi_0$  implicitly specifies the half-width  $b = a/\sin \phi_0$ .  
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21 The angle  $\phi_0$  can be found by solving the equation  
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$$\frac{nPR}{a^2 E^*} = \frac{\pi - 2\phi_0}{2 \sin^2 \phi_0} - \cot \phi_0 \quad (25)$$

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27 where  $2a = 2.57$  mm and  $1/E^* = \sum_{i=1,2} (1 - \nu_i^2)/E_i$  with  $E_i = 200.3$  GPa  
28 and  $\nu_i = 0.3$  the Young's modulus and the Poisson's ratio of the  $i$ -body. The  
29 contact length is  $L=26$ mm while the curvature radius of the rounded edges  
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31 is  $R=2.50$ mm (considering a symmetrical punch geometry). Because the real  
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33 profile is not symmetric  $R$  was chosen as the maximum between the two radius  
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35 (1.25 and 2.50 mm) corresponding to the higher values of the contact stiffnesses  
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37  $k_n^{th}$  and  $k_t^{th}$ . However, the variation of the theoretical contact stiffness with the  
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39 radius  $R$  in the above-mentioned range is negligible.

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330 A discrepancy was found about the coefficient  $n$  in Eq. 25: according to  
[24]  $n=2$  whereas [25] reports a coefficient  $n=4$ . Figure 11b shows the pressure  
distribution calculated with the two different coefficients and compares it with  
the results of a FE analysis. The coefficient  $n=4$  overestimate the contact half-  
width and coherently underestimate the pressure peak. For this reason, this  
335 work utilizes the formulation proposed in [24].

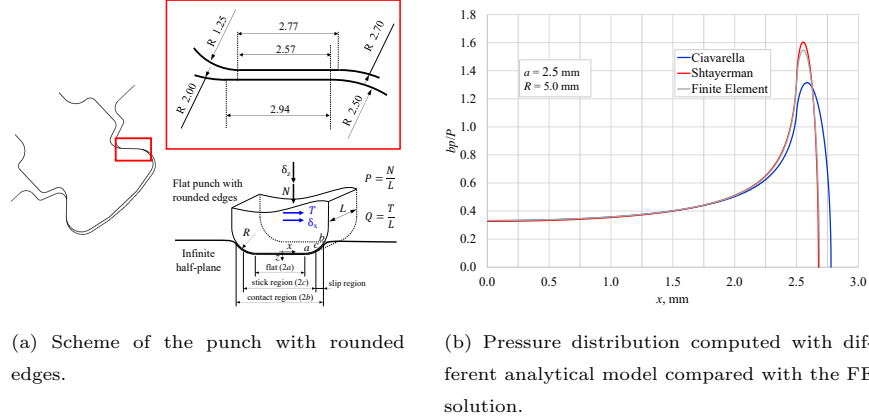


Figure 11: the punch with rounded edges and its pressure distribution.

The results obtained in [24] were extended in [25] in the case of contacts experiencing also a tangential force  $T$ . The shear stress was calculated as the sum of the stress in full slip condition  $\mu p(x)$  ( $\mu$  being the friction coefficient) and a corrective shear stress  $q^*(x)$  in the stick region

$$q(x) = \mu p(x) - q^*(x) \quad (26)$$

The corrective shear stress distribution  $q^*(\theta)$  is

$$\frac{cq^*(\theta)}{\mu P - Q} = -\frac{2/\pi}{\pi - 2\theta_0 - \sin 2\theta_0} \left[ (\pi - 2\theta_0) \cos \theta + \sin \theta \ln \left| \frac{\sin(\theta + \theta_0)}{\sin(\theta - \theta_0)} \right| + \sin \theta_0 \ln \left| \tan \frac{\theta + \theta_0}{2} \tan \frac{\theta - \theta_0}{2} \right| \right] \quad (27)$$

where  $\sin \theta = x/c$ ,  $P=N/L$  and  $Q=T/L$ . The half-width of the stick region  $c = a/\sin \theta_0$ , see Fig. 11a, is computed solving for  $\theta_0$  in the equation

$$\frac{nPR}{a^2 E^*} \left( 1 - \frac{Q}{\mu P} \right) = \frac{\pi - 2\theta_0}{2 \sin^2 \theta_0} - \cot \theta_0 \quad (28)$$

The tangential displacement  $\delta_x$  was found in [26]

$$\delta_x = u_{x1} - u_{x2} = \frac{2}{\pi E^*} \left[ -\int_{-b}^b q_x(r) \ln \left| \frac{r}{b} \right| dr + Q \left( \ln \left| \frac{L}{b} \right| + \frac{\nu}{1-\nu} \right) \right] \quad (29)$$

through the elastic potential theory of Boussinesq-Cerruti as reported in [53]. Hence, the stiffness is calculated as the first derivative of the tangential force at

the onset of the tangential displacement for a given normal force  $N$

$$k_t^{th}(N) = \left( \frac{\partial T}{\partial \delta_x} \right)_{\delta_x=0} = \frac{1}{\left( \frac{\partial \delta_x(N)}{\partial T} \right)_{T=0}} \quad (30)$$

Similarly, the normal displacement of each body is

$$|u_{z_i}| = \frac{1 - \nu_i^2}{\pi E_i} \int_{-b}^b p(r) \left( \int_{-L/2}^{L/2} \frac{1}{\sqrt{r^2 + s^2}} ds \right) dr = \frac{1 - \nu_i^2}{\pi E_i} \int_{-b}^b p(r) 2 \sinh^{-1} \left( \frac{L/2}{|r|} \right) dr \quad (31)$$

thus the relative normal displacement  $\delta_z$  is

$$\delta_z = u_{z1} - u_{z2} = \frac{2}{\pi E^*} \int_{-b}^b p(r) \sinh^{-1} \left( \frac{L/2}{|r|} \right) dr \quad (32)$$

and the normal contact stiffness can be written as

$$k_n^{th}(N) = \frac{\partial N}{\partial \delta_z} = \frac{1}{\frac{\partial \delta_z}{\partial N}} \quad (33)$$

For a given axial force  $F_{ax}$  the corresponding normal load  $N$  ranges from a minimum  $N_{min}$  (corresponding to full slip  $T=\mu N$ ) and a maximum value  $N_{max}$  (corresponding to a friction-less contact  $T=0$ )

$$\frac{F_{ax}}{2(\sin \beta + \mu \cos \beta)} \leq N \leq \frac{F_{ax}}{2 \sin \beta} \quad (34)$$

as visualized in Fig. 9a, Figure 13b shows that the theoretical contact stiffness are weakly affected by the tangential force  $T$  with  $\mu=0.4$ .

## 6. Contact stiffness Identification

The unknown contact stiffnesses  $k_n$  and  $k_t$  were indirectly estimated through their influence on the natural frequencies calculated by the reduced FE model. These frequencies, calculated for both the first (1B) and second (2B) bending mode, were compared with the measured frequencies. This comparison was carried out for each axial load  $F_{ax}$  with which experiments were performed. The analysis was achieved for small vibration amplitudes in which gross slip does not occur. In this condition the frequency remains constant, as shown in Fig. 6a, and the system exhibits a linear behavior. Microslip is always present

but its influence is significant only for axial loads less than 12 kN, as reported in Fig. 7. For this reason, only results obtained with axial loads greater than 12 kN were employed in the comparison. The unknown stiffnesses  $k_n$  and  $k_t$  can be determined by solving at each axial load  $F_{ax}$  the nonlinear system

$$\begin{cases} f_{n,1B}^{FE}(k_n, k_t) - f_{n,1B}^m \\ f_{n,2B}^{FE}(k_n, k_t) - f_{n,2B}^m \end{cases} = \begin{cases} \Delta f_{n,1B} \\ \Delta f_{n,2B} \end{cases} = \begin{cases} 0 \\ 0 \end{cases} \quad (35)$$

where  $f_n^{FE}$  and  $f_n^m$  are the computed and measured natural frequencies respectively. The natural frequencies of the first bending mode as a function of the contact stiffness are shown in Fig. 12a. Equation 35 is solved by finding the locus of the minima of the residual norm  $g := \|\Delta \mathbf{f}_n\|$ . This locus is characterized by a minimum gradient  $\|\nabla g\|$  on the level curves of the residual norm surface

$$\begin{cases} \frac{\partial}{\partial n} \|\nabla g\| = 0 \\ \frac{\partial^2}{\partial n^2} \|\nabla g\| > 0 \end{cases} \quad (36)$$

where direction  $\mathbf{n}$  defines the level curve as  $\nabla g \cdot \mathbf{n} = 0$ . Together with Eq. 36 a further limit is set on the value of the norm that must be less than a selected tolerance  $g < toll_g$ . The locus of the minima can be visualized as the path followed by a sphere free to roll on the residual norm surface. This path is visualized by the solid line marked in Fig. 12b. This locus represents the feasible solutions and is reported in Fig. 12c for different axial loads. Once the loci are defined a single solution must be selected. In this work the following three criteria were considered.

- The most intuitive criterion is to find the global minimum of  $\|\Delta \mathbf{f}_n\|$  on the minima locus. The drawback of this criterion is the low accuracy with which the stiffness is determined. The errors on the stiffness, computed with the procedure described in Appendix A, is about 55% and makes this solution less reliable. The reason for this large error is that the solution  $(k_n, k_t) \simeq (1, 7)$  kN/ $\mu$ m belongs to the region in which the sensitivity of the contact stiffness to the frequency variation is high, and a small

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355 uncertainty on the measured frequencies propagates dramatically on the stiffness value.

- Uncertainty on the identified stiffness also suggests a different criterion: selecting the solution  $(k_n, k_t)$  with the minimum error (according to the procedure described in Appendix A). This solution is denoted to as the “optimal solution pair”.
- The theoretical contact stiffnesses were calculated in Sect. 5 and their values can guide the selection procedure. A third option is to choose the pair  $(k_n, k_t)$  with the minimum distance from theoretical stiffness.

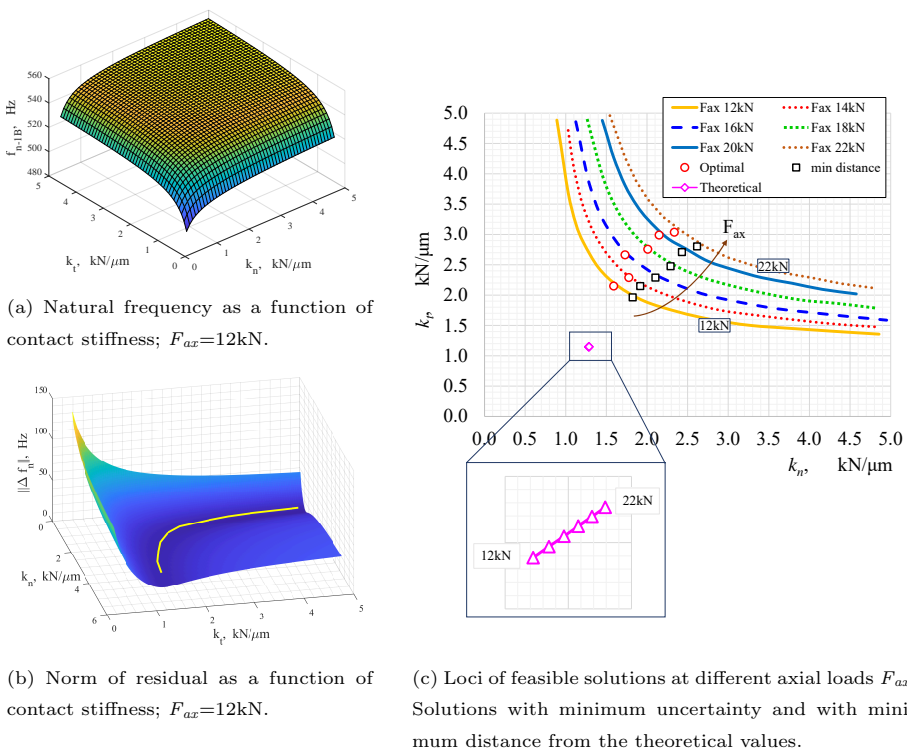


Figure 12: variation of frequency (a) and residual norm (b) with contact stiffness. Loci of the minima with optimal and minimum distance solutions for different axial loads.

Figure 12c reports the solution determined with the last two criteria: these so-

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11 365 solutions are close to each other and almost equivalent. In this work, the optimal  
12 solution criterion was adopted and Table 1 reports the numerical results with  
13 their accuracy. The ratio of tangential to normal stiffness remains almost con-  
14 stant,  $k_t/k_n \simeq 1.37$ . This result differs from what reported in [54] in which this  
15 ratio slowly increases with normal loading.

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19 370 Figure 13a compares the optimal and the theoretical solution and points out  
20 that the difference between the two results is very high. Taking the theoretical  
21 stiffness as reference, the difference in the worst case is about 82% and 164% for  
22  $k_n$  and  $k_t$  respectively. Furthermore, the trend of the two solutions at varying  
23 axial loads is very different. The optimal solution shows a slope of about 73  
24 and  $95 \text{ mm}^{-1}$  for  $k_n$  and  $k_t$  respectively, whereas the theoretical values remains  
25 almost constant at varying axial load as emphasized in Fig. 13b.  
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35 380 Most of this discrepancy is due to the slightly different meaning that the  
36 term “contact stiffness” takes on in the numerical and analytical models. In the  
37 numerical model, the optimal contact stiffness represents the local connection  
38 between the two bodies through the interface that breaks off the continuum. The  
39 flexibility of the bulk, here the lobes on root and slots, is considered in the finite  
40 element model. The global stiffness is given by the series of the optimal and bulk  
41 stiffness. On the contrary, the analytical model calculates the displacements,  
42 from which the theoretical stiffness are deduced, considering the elastic half-  
43 space. The comparison should be made between global and theoretical stiffness  
44 rather than optimal and theoretical stiffness, but this comparison is by no means  
45 an easy task.

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51 390 Furthermore, the assumption of elastic half-space is valid only if the size  $2b$  of  
52 the contact region is much smaller than the curvature radius  $R$  [26], hypothesis  
53 not strictly fulfilled in the real contact geometry of the examined joint.

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65 Moreover, the blade deformation and the lobe flexibility induce rotation of  
the contact surfaces. This rotation generates an asymmetric pressure distribu-  
tion, whereas the theoretical model assumes that the loads and the pressure  
distribution are symmetric. Since the contact problem is nonlinear asymmet-  
ric pressure distributions result in different stiffness even for the same normal

load. An analytical model comprehensive of the moment and asymmetry of the contact patch was put forward in [55]. This model delivers the pressure and tangential distribution, but the displacement field from which the contact stiffness is derived is not ready yet. Furthermore, it is not easy to evaluate the moment at the root when the blade is oscillating also considering that this moment varies during a cycle.

$F_{ax}$	$k_n$	$\sigma_{nn}/k_n$	$k_t$	$\sigma_{tt}/k_t$
kN	kN/ $\mu\text{m}$	%	kN/ $\mu\text{m}$	%
12	1.59	15	2.15	12
14	1.68	17	2.38	15
16	1.82	21	2.57	18
18	2.01	26	2.76	22
20	2.15	27	2.99	25
22	2.34	31	3.04	26

Table 1: normal and tangential contact stiffness with their estimated errors for different axial loads.

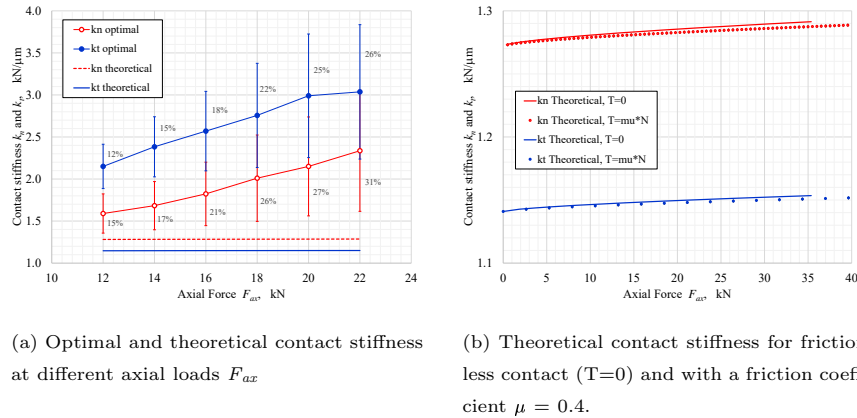


Figure 13: Comparison between the optimal contact stiffness resulting from the identification procedure and the theoretical values obtained with the semi-analytical model.

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9 **7. Conclusions**

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11 This paper has presented a novel procedure to experimentally investigate  
12 normal and tangential contact stiffness in dovetail attachments. Experimental  
13 data were collected on a mock-blade with a dovetail at both ends. With this  
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17 405 symmetric configuration it is possible to apply the load simulating the centrifu-  
18 gal force without adding undesirable damping or stiffness. In our experiment,  
19 a shaker excited the mock-blade whose free decay was measured with a laser  
20 Doppler velocimeter. The Detached Drive Rod Method was developed to ensure  
21 that the free decay was not affected by any coupling with the excitation system.  
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24 410 The measured response was transformed into an analytic signal and processed  
25 to determine the instantaneous modal parameters, namely natural frequency  
26 and loss factor. Results substantiate the dependency of the modal parameters  
27 on the vibration amplitude, a behavior typical of nonlinear systems. Results  
28 also show that the loss factor reaches a maximum at an optimal amplitude, as  
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32 415 predicted by theoretical models.

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34 In order to determine the normal and tangential contact stiffness, part of  
35 the interval of the free decay was utilized. This part of the interval was chosen  
36 because the system behaves linearly and the natural frequencies are constant.  
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38 An FE model of the mock-blade and its supports was built within a commercial  
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40 420 software. The stiffness and mass matrices of this model were reduced using the  
41 component mode synthesis method and imported in an in-house code. This  
42 code identifies the contact stiffness by comparing the natural frequencies of the  
43 first and second bending modes computed by the numerical model with the  
44 corresponding measured frequencies. The presence of two parameters, namely  
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47 425 normal and tangential stiffness, generates a multi-objective optimization prob-  
48 lem. The contact stiffness pair with minimum uncertainty was chosen among  
49 the feasible solutions and named “optimal stiffness pair”. This optimal stiff-  
50 ness pair was also compared with the theoretical stiffness pair calculated with  
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54 a semi-analytical solution. It was found that the optimal pair also minimizes  
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56 430 the distance between optimal and theoretical stiffness. The difference between  
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9 the optimal and theoretical results was very high, reaching 82% and 164% in  
10 the worst case for  $k_t$  and  $k_n$  respectively. The trend of the contact stiffness at  
11 varying axial loads was also very different. The optimal stiffness showed a slope  
12 of about 73 and 95 mm<sup>-1</sup> for  $k_t$  and  $k_n$  respectively, whereas the theoretical  
13 values remained almost constant.  
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16 This disagreement can be explained considering the meaning of the term  
17 "contact stiffness". In the numerical model, contact stiffness represents the  
18 connection between two bodies through the interface. The flexibility of the  
19 bulk, represented by the lobes on the root and the slot, is considered in the fi-  
20 nite element model. Instead, the theoretical model calculates the displacements  
21 and then the stiffness, considering the elastic half-space. When the theoret-  
22 ical stiffnesses are assembled in the FE model it is as if the bulk flexibility is  
23 considered twice.  
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26 Moreover, when contact surfaces are loaded they deform because of the flex-  
27 ibility of the lobes. The pressure distribution becomes asymmetric, whereas the  
28 theoretical model assumes that the loads and the pressure distribution are sym-  
29 metric. Since the contact problem is nonlinear, different pressure distributions  
30 result in different stiffness even if the total loads are the same. Currently, no  
31 theoretical models are available which consider the moment at the interface and  
32 asymmetric pressure distribution.  
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35 From the previous observations, it is clear that measured and theoretical  
36 contact stiffnesses are not interchangeable. This work shows that computing  
37 contact stiffnesses with a theoretical model, such as the one presented in the  
38 paper, and substituting them in a finite element model underestimates the stiff-  
39 ness of the whole system. Designers should be aware that in this case the modal  
40 analysis and the forced response will be inaccurate.  
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## 43 Appendix A. Contact stiffness accuracy

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Uncertainties on contact stiffness were estimated through the procedure shown in [56]. The numerical frequencies as a function of the contact stiffness

can be approximated by a first order Taylor expansion

$$\mathbf{f}(k_n, k_t) = \begin{Bmatrix} f_{n,1B}^{FE}(k_n, k_t) \\ f_{n,2B}^{FE}(k_n, k_t) \end{Bmatrix} \simeq \mathbf{f}_0 + \mathbf{J}(\mathbf{k} - \mathbf{k}_0) \quad (\text{A.1})$$

with

$$\mathbf{k} = \begin{Bmatrix} k_n \\ k_t \end{Bmatrix} \quad (\text{A.2})$$

and

$$\mathbf{J} = \begin{bmatrix} \frac{\partial f_{n,1B}}{\partial k_n} & \frac{\partial f_{n,1B}}{\partial k_t} \\ \frac{\partial f_{n,2B}}{\partial k_n} & \frac{\partial f_{n,2B}}{\partial k_t} \end{bmatrix} \quad (\text{A.3})$$

Vector  $\mathbf{k}_0$  contains a pair of trial solutions and  $\mathbf{f}_0$  its corresponding frequencies. The uncertainties  $\delta f_{n,1B}$  and  $\delta f_{n,2B}$  on the frequencies for the first and second bending mode were obtained from the the uncertainty on the measured axial load  $F_{ax}$  due to pressure gauge accuracy ( $\delta p/p = \delta F_{ax}/F_{ax} = 1\%$ ).

$$\delta f_n = \frac{\partial f_n}{\partial F_{ax}} \delta F_{ax} \quad (\text{A.4})$$

$F_{ax}$	$\delta f_{n,1B}^m$	$\delta f_{n,2B}^m$
kN	Hz	Hz
12	0.23	0.40
14	0.25	0.45
16	0.29	0.50
18	0.33	0.55
20	0.33	0.54
22	0.35	0.55

Table A.2: measured natural frequencies uncertainties due to uncertainties on measured axial load  $F_{ax}$

Sensitivity of the natural frequency  $f_n$  on axial load  $F_{ax}$  was estimated by using the FE model with the identified contact stiffnesses  $k_n$  and  $k_t$ . The

frequency covariance matrix is

$$\boldsymbol{\sigma}^2 := \langle \delta \mathbf{f} \delta \mathbf{f}^T \rangle = \begin{bmatrix} \sigma_{1B}^2 & 0 \\ 0 & \sigma_{2B}^2 \end{bmatrix} = \begin{bmatrix} \delta f_{n,1B}^2 & 0 \\ 0 & \delta f_{n,2B}^2 \end{bmatrix} \quad (\text{A.5})$$

with  $\langle \cdot \rangle$  the expected value. In Eq. A.5 the standard deviations  $\sigma_{1B}$  and  $\sigma_{2B}$  are approximated with the uncertainties  $\delta f_{n,1B}$  and  $\delta f_{n,2B}$ . Thus, the covariance matrix of contact stiffness at  $\mathbf{k}_0$  is

$$\boldsymbol{\sigma}_{\mathbf{k}}^2 = \begin{bmatrix} \sigma_{nn}^2 & \sigma_{nt}^2 \\ \sigma_{tn}^2 & \sigma_{tt}^2 \end{bmatrix} = \langle \delta \mathbf{k} \delta \mathbf{k}^T \rangle = \mathbf{J}^{-1} \langle \delta \mathbf{f} \delta \mathbf{f}^T \rangle (\mathbf{J}^{-1})^T = \mathbf{J}^{-1} \boldsymbol{\sigma}^2 (\mathbf{J}^{-1})^T \quad (\text{A.6})$$

with  $\delta \mathbf{f} = \mathbf{J} \delta \mathbf{k}$ . The solution (named as optimal stiffness pair) was selected as the stiffness pair - on the minima locus - with the minimum 2norm of the stiffness covariance matrix. Standard deviations of  $k_n$  and  $k_t$  were computed from the covariance matrix

$$\mathbf{k}^{opt} = \left\{ \begin{array}{c} k_n^{opt} \\ k_t^{opt} \end{array} \right\} : \quad \min \|\boldsymbol{\sigma}_{\mathbf{k}}^2\|_2 \Rightarrow \left\{ \begin{array}{l} k_n = k_n^{opt} \pm \sigma_{nn} \\ k_t = k_t^{opt} \pm \sigma_{tt} \end{array} \right. \quad (\text{A.7})$$

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