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***Abstract*—We present here an innovative treatment of nonlinear oscillators subject to colored noise sources, namely Lorentzian fluctuations. Exploiting averaging techniques, the full system is transformed into an equivalent oscillator with white noise sources in the limit of small noise correlation time, thus avoiding the Itô-Stratonovich ambiguity. The equivalent noisy oscillator is then transformed into a phase-amplitude model, thus simplifying phase noise analysis. The soundness of the approach is demonstrated for a 2 dimensional example.**

***Index Terms*—Oscillator noise, phase noise, colored noise, stochastic averaging, phase models**

I. Introduction

Phase noise, in the analog world, and its time-domain counterpart time-jitter (more commonly used for digital applications) are essential figures of merit in assessing the quality of oscillators. In biological and neural systems, phase noise plays a fundamental role explaining synchronization/desynchronization processes which, in turn, influence neural information processing. In electronic systems, phase noise is responsible for errors in bit transmission rate, modulation and demodulation of data. As a consequence the study of oscillator phase noise plays a major role.

The physics and, as a consequence, the modeling of oscillators subject to white Gaussian noise is rather well understood. However, real world random fluctuations are characterized by a finite, non null time-correlation, i.e. the power spectral density (PSD) is often characterized by a larger value at low frequency. A more realistic description of such a perturbation is represented by an exponentially correlated process, known as colored (Lorentzian) noise.

In this contribution we present a novel approach for phase noise analysis in nonlinear oscillators subject to colored noise. The model we study can be decomposed into two parts: the oscillator dynamics and the stochastic process modeling noise. We consider oscillators of a generic order N, subject to colored noise that can be either modulated (multiplicative) or un-modulated (additive). We do not impose any *a priori* restriction on the noise intensity. The colored noise is modeled as an Ornstein-Uhlenbeck process (OUP), where the OUP is expressed as the solution of a linear stochastic differential equation (SDE) with an un-modulated (additive) white Gaussian noise. OUP has exponentially decaying expectation value and correlation, and it is characterized by a Lorentz (Cauchy) distribution. The only assumption we use is that the noise correlation time is small, although not vanishingly small (in the zero correlation time limit, OUP reduces to white noise).

Using a method proposed in [1], [2] the oscillator with colored noise is first transformed into an equivalent system subject to white Gaussian noise. In other words, the $N+1$ dimensional, coupled SDEs describing oscillator dynamics and OUP are reduced to an N dimensional SDE describing the evolution of an equivalent nonlinear oscillator subject to white Gaussian noise. The solution of the reduced oscillator model converges weakly to the solution of full system, in the sense that all of the statistical properties of the exact (strong) solution are retained by the reduced system: As a consequence, for most practical applications this information is quite adequate. The advantage in considering the reduced system is twofold. First, not only numerical simulations are greatly simplified, because the OUP SDE doesn't need to be solved, but also expected quantities and other useful information can be found by direct computation on the reduced system applying stochastic calculus. Second, the transformation resolves the Itô-Stratonovich dilemma [3]: even if the original system (oscillator + OUP noise modeling) has un-modulated (additive) noise, any order reduction and/or nonlinear coordinate transformation introduces noise modulation, making the noise source multiplicative [4], [5]. This implies that the resulting SDE yields different results depending on whether it is interpreted as a Stratonovich or an Itô SDE.

Exploiting Floquet theory [6], the reduced system with white noise is then transformed into a phase-amplitude model [7]. The phase variable describes a random walk process along a direction tangent to the limit cycle of the unperturbed oscillator, while the amplitude describes motion transversal to the cycle. The main findings that we provide are twofold: 1) the transformation into an equivalent system with white Gaussian noise highlights the effect of finite noise correlation time; 2) the transformation to amplitude and phase equations greatly simplifies phase noise analysis, and represents the idal starting point to derive further simplified phase models.

II. Influence of colored noise

We consider a nonlinear oscillator subject to colored noise modeled as an Ornstein-Uhlenbeck process. Adopting standard notation for stochastic differential equations, we write the system in the form

$$\mathrm{d}\mathbf{x}_t = [\mathbf{a}(\mathbf{x}_t) + \mathbf{B}(\mathbf{x}_t)\eta_t]\,\mathrm{d}t \tag{1a}$$

$$\tau \mathrm{d}\eta_t = -\eta_t \mathrm{d}t + D\mathrm{d}W_t \tag{1b}$$

where $\mathbf{x}_t : \mathbb{R} \mapsto \mathbb{R}^n$ denotes the state of the system, $\mathbf{a} : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a smooth vector field that defines the system internal dynamics, $\mathbf{B} : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a smooth vector valued function representing noise modulation, $\eta_t : \mathbb{R} \mapsto \mathbb{R}$ is a scalar function describing the source fluctuations, both internal and external, and W_t denotes a Wiener process, i.e. the integral of a white noise. Finally, parameters τ and D represent the noise correlation time and diffusion coefficient, respectively. It is worth noting that equation (1) describes a diffusion process with unmodulated white Gaussian noise $\mathrm{d}W_t$. In the limit of uncorrelated noise $\tau \to 0$, (1) reduces to a SDE describing a nonlinear oscillator subject to white Gaussian noise

$$\mathrm{d}\mathbf{x}_t = \mathbf{a}(\mathbf{x}_t)\mathrm{d}t + D\mathbf{B}(\mathbf{x}_t)\mathrm{d}W_t \tag{2}$$

However the problem arises about the correct interpretation of (2) (Stratonovich, Itô or others), because white Gaussian noise is now multiplied by the modulating function $\mathbf{B}(\mathbf{x})$. The problem can be tackled by applying the procedure described in [1], [2]. Using that method, it can be shown that in the limit of short correlation time τ, the solutions of (1) converge in probability to the solutions of the Stratonovich SDE

$$\mathrm{d}\mathbf{x}_t = \mathbf{a}(\mathbf{x}_t)\mathrm{d}t + D\mathbf{B}(\mathbf{x}_t) \circ \mathrm{d}W_t \tag{3}$$

or to the solutions of the equivalent Itô SDE

$$\mathrm{d}\mathbf{x}_t = \left[\mathbf{a}(\mathbf{x}_t) + \frac{D^2}{2}\frac{\partial \mathbf{B}(\mathbf{x}_t)}{\partial \mathbf{x}}\,\mathbf{B}(\mathbf{x}_t)\right]\mathrm{d}t + D\,\mathbf{B}(\mathbf{x}_t)\mathrm{d}W_t \tag{4}$$

where, adopting standard the notation, we used symbol $\circ$ to denote Stratonovich stochastic integration. A short explanation is required. Convergence in probability, or weak convergence, means that the solutions of two equation for a specification realization of the stochastic process differ in details, but they have the same statistical properties. From the point of view of practical applications, a weak solution is all is needed, because the user is typically interested in expected quantities. Equivalence of (3) and (4) means that the two equations, although being solved using two different definitions of stochastic integration, have exactly the same solution. As a consequence, it is just a matter of personal taste and practical convenience which interpretation should be preferred. Here, we prefer Itô interpretation and thus we shall consider (4). This choice will simplify the calculation of expected quantities and numerical simulations.

Equation (4) casts light on the effect of colored noise with respect to white noise. The last term in the equation, $D\mathbf{B}(\mathbf{x}_t)\mathrm{d}W_t$, represents a diffusion contribution. By contrast the term $(D^2/2)(\partial\mathbf{B}(\mathbf{x}_t)/\partial\mathbf{x})\mathbf{B}(\mathbf{x}_t)\mathrm{d}t$ represents an additional drift, that can be ascribed to the finite, non null noise correlation time.

III. Phase dynamics

The concept of phase is one of the most important in the theory of oscillators. While it seems trivial to define the concept of phase for linear second order oscillators, great attention must be paid to extend the same concept for higher order, nonlinear systems. The most general and fruitful way to define the phase of a nonlinear oscillator is based on the concept of asymptotic phase, or the phase function.

In the absence of perturbations, a nonlinear oscillator exhibits a T-periodic solution, represented by a limit cycle $\mathbf{x}_\mathrm{s}(t)$ in its phase space. In order to define the phase of a point $\mathbf{x}_i$ in the basin of attraction of the limit cycle, let us introduce a phase function $\phi : \mathbb{R}^n \to (0, 2\pi)$. We define a reference initial condition $\mathbf{x}_0 \in \mathbf{x}_\mathrm{s}(t)$, and we assign $\phi(\mathbf{x}_0) = 0$. Let $\varphi(t, \mathbf{x}_0)$ denote the trajectory of the unperturbed system at time t with initial condition at $\mathbf{x}_0$. Obviously, $\varphi(t, \mathbf{x}_0)$ solves $\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t))$ and $\varphi(0, \mathbf{x}_0) = \mathbf{x}_0$. The phase of the trajectory $\varphi(t, \mathbf{x}_0)$ is $\phi(\varphi(t, \mathbf{x}_0)) = 2\pi t/T = \omega_0\, t$, where ω_0 is the oscillator free running angular frequency. In other words, the phase function represents a re-parametrization of the limit cycle. The concept of phase can be extended to the basin of attraction of the limit cycle $B(\mathbf{x}_\mathrm{s})$, introducing isochrons. Isochrons are $(n-1)$-dimensional manifolds (surfaces) transverse to the limit cycle, representing the set of initial conditions $\mathbf{x}_\mathrm{i} \in B(\mathbf{x}_s)$ such that the trajectories leaving from $\mathbf{x}_\mathrm{i}$ meet asymptotically on the limit cycle $\mathbf{x}_\mathrm{s}(t)$. Mathematically, the isochron transversal to the limit cycle at $\mathbf{x}_0$ is

$$I_{\mathbf{x}_0} = \left\{ \mathbf{x}_\mathrm{i} \in B(\mathbf{x}_s) \;\middle|\; \lim_{t\to+\infty} ||\varphi(t, \mathbf{x}_\mathrm{i}) - \varphi(t, \mathbf{x}_0)|| = 0 \right\} \tag{5}$$

where $||\cdot||$ denotes the Euclidean distance. The phase of points in $B(\mathbf{x}_s)$ remains defined if we assign the same phase to all points belonging to the same isochron. Thus, given a point on the limit cycle $\mathbf{x}_j \in \mathbf{x}_s$ and set of initial conditions on the isochron based at $\mathbf{x}_j$, $\{\mathbf{x}_i\} \in I_{\mathbf{x}_j}$, the phase of the trajectories starting at $\mathbf{x}_i$ are $\phi(\varphi(t, \mathbf{x}_\mathrm{i})) = \omega t + \phi(\mathbf{x}_j)$. In analogy with the flow box theorem, the phase function ϕ is the diffeomorphism that realizes a rectification of trajectories in the basin of attraction of the limit cycle, mapping solutions of $\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t))$ onto those of $\dot{\phi} = \omega_0 t + \phi_0$, and isochrons are the level sets of the phase function.

To study the influence of random perturbations we consider a vector basis $\{\mathbf{u}_1(t), \ldots, \mathbf{u}_n(t)\}$, where $\mathbf{u}_1(t)$ is chosen as the unit vector tangent to the limit cycle at any t

$$\mathbf{u}_1(t) = \frac{\mathbf{a}(\mathbf{x}_\mathrm{s}(t))}{||\mathbf{a}(\mathbf{x}_\mathrm{s}(t))||} \tag{6}$$

while the remaining $n-1$ vectors $\mathbf{u}_2(t), \ldots, \mathbf{u}_n(t)$ are chosen as the Floquet vectors (apart from the limit cycle tangent $\mathbf{u}_1(t)$) of the linearized variational equation [5]–[7]

$$\frac{d\widetilde{\mathbf{x}}(t)}{dt} = \boldsymbol{J}(t)\,\widetilde{\mathbf{x}}(t) \tag{7}$$

Define the matrix $\mathbf{U}(t) = [\mathbf{u}_1(t), \ldots, \mathbf{u}_n(t)]$, and the reciprocal vectors $\mathbf{v}_1^T(t), \ldots, \mathbf{v}_n^T(t)$ as the rows of the inverse matrix $\mathbf{V}(t) = \mathbf{U}^{-1}(t)$. Thus $\{\mathbf{v}_1(t), \ldots, \mathbf{v}_n(t)\}$ is also a basis for $\mathbb{R}^n$ and the vectors satisfy the bi-orthogonality condition $\mathbf{v}_i^T \mathbf{u}_j = \mathbf{u}_i^T \mathbf{v}_j = \delta_{ij}$. Finally, introduce matrices $\mathbf{Y}(t) = [\mathbf{u}_2(t), \ldots, \mathbf{u}_n(t)]$, $\mathbf{Z}(t) = [\mathbf{v}_2(t), \ldots, \mathbf{v}_n(t)]$, and the modulus of the vector field evaluated on the limit cycle, $r(t) = ||\mathbf{a}(\mathbf{x}_{\mathrm{s}}(t))||$.

We look for a solution of (4) in the form

$$\mathbf{x}_t = \mathbf{x}_{\mathrm{s}}(\theta_t) + \mathbf{Y}(\theta_t)\mathbf{R}_t \tag{8}$$

where $\mathbf{x}_{\mathrm{s}}(\theta_t)$ represents the projection of the stochastic process $\mathbf{x}_t$ onto the limit cycle, evaluated at an unknown time instant θ_t. The second component $\mathbf{Y}(\theta_t)\mathbf{R}_t$ represents an unknown distance between the solution and the limit cycle, measured along the directions spanned by the vectors $\mathbf{v}_2, \ldots, \mathbf{v}_n$ at the random time θ_t. Following the procedure given in [5, Theorem 3.1] and [7, Theorem 1], the following Itô SDEs for the stochastic processes $\theta : \mathbb{R} \mapsto \mathbb{R}$ and $\mathbf{R} : \mathbb{R} \mapsto \mathbb{R}^{n-1}$ are found

$$\begin{aligned} \mathrm{d}\theta_t =& [1 + a_\theta(\theta_t, \mathbf{R}_t) + \hat{a}_\theta(\theta_t, \mathbf{R}_t) + b_\theta(\theta_t, \mathbf{R}_t)]\, \mathrm{d}t \\ &+ B_\theta(\theta_t, \mathbf{R}_t)\, \mathrm{d}W_t \end{aligned} \tag{9}$$

$$\begin{aligned} \mathrm{d}\mathbf{R}_t =& [\mathbf{L}(\theta_t)\mathbf{R}_t + \mathbf{a}_\mathbf{R}(\theta_t, \mathbf{R}_t) + \\ &+ \hat{\mathbf{a}}_\mathbf{R}(\theta_t, \mathbf{R}_t) + \mathbf{b}_\mathbf{R}(\theta_t, \mathbf{R}_t)]\, \mathrm{d}t + \\ &+ \mathbf{B}_\mathbf{R}(\theta_t, \mathbf{R}_t)\, \mathrm{d}W_t \end{aligned} \tag{10}$$

where (the $'$ sign denotes the derivative with respect to θ)

$$a_\theta(\theta, \mathbf{R}) = \kappa\, \mathbf{v}_1^T \left[\mathbf{a}(\mathbf{x}_{\mathrm{s}} + \mathbf{Y}\mathbf{R}) - \mathbf{a}(\mathbf{x}_{\mathrm{s}}) - \mathbf{Y}'\mathbf{R}\right] \tag{11}$$

$$\begin{aligned} \hat{a}_\theta(\theta, \mathbf{R}) =& -\kappa\, \mathbf{v}_1^T \Big[\mathbf{Y}'\mathbf{B}_\mathbf{R}(\theta, \mathbf{R}) B_\theta(\theta, \mathbf{R}) \\ &+ \frac{1}{2} B_\theta^2(\theta, \mathbf{R})(\mathbf{x}_{\mathrm{s}}'' + \mathbf{Y}''\mathbf{R})\Big] \end{aligned} \tag{12}$$

$$b_\theta(\theta, \mathbf{R}) = \frac{D^2}{2}\, \kappa\, \mathbf{v}_1^T \frac{\partial \mathbf{B}(\mathbf{x}_{\mathrm{s}} + \mathbf{Y}\mathbf{R})}{\partial \mathbf{x}} \mathbf{B}(\mathbf{x}_{\mathrm{s}} + \mathbf{Y}\mathbf{R}) \tag{13}$$

$$B_\theta(\theta, \mathbf{R}) = D\, \kappa\, \mathbf{v}_1^T \mathbf{B}(\mathbf{x}_{\mathrm{s}} + \mathbf{Y}\mathbf{R}) \tag{14}$$

$$\mathbf{L}(\theta) = -\mathbf{Z}^T\, \mathbf{Y}' \tag{15}$$

$$\mathbf{a}_\mathbf{R}(\theta, \mathbf{R}) = \mathbf{Z}^T \left[\mathbf{a}(\mathbf{x}_{\mathrm{s}} + \mathbf{Y}\mathbf{R}) - \mathbf{Y}'\mathbf{R}\, a_\theta(\theta, \mathbf{R})\right] \tag{16}$$

$$\begin{aligned} \hat{\mathbf{a}}_\mathbf{R}(\theta, \mathbf{R}) =& -\mathbf{Z}^T \Big[\mathbf{Y}'\mathbf{R}\, \hat{a}_\theta(\theta, \mathbf{R}) + \mathbf{Y}'\mathbf{B}_\mathbf{R}(\theta, \mathbf{R}) B_\theta(\theta, \mathbf{R}) \\ &+ \frac{1}{2} B_\theta^2(\theta, \mathbf{R})(\mathbf{x}_{\mathrm{s}}'' + \mathbf{Y}''\mathbf{R})\Big] \end{aligned} \tag{17}$$

$$\begin{aligned} \mathbf{b}_\mathbf{R}(\theta, \mathbf{R}) =& -\mathbf{Z}^T\mathbf{Y}'\mathbf{R}\, b_\theta(\theta, \mathbf{R}) \\ &+ \frac{D^2}{2}\, \mathbf{Z}^T \frac{\partial \mathbf{B}(\mathbf{x}_{\mathrm{s}} + \mathbf{Y}\mathbf{R})}{\partial \mathbf{x}} \mathbf{B}(\mathbf{x}_{\mathrm{s}} + \mathbf{Y}\mathbf{R}) \end{aligned} \tag{18}$$

$$\mathbf{B}_\mathbf{R}(\theta, \mathbf{R}) = -\mathbf{Z}^T\mathbf{Y}'\mathbf{R}\, B_\theta(\theta, \mathbf{R}) + D\, \mathbf{Z}^T\mathbf{B}(\mathbf{x}_{\mathrm{s}} + \mathbf{Y}\mathbf{R}) \tag{19}$$

and

$$\kappa = \left(r + \mathbf{v}_1^T\mathbf{Y}'\mathbf{R}\right)^{-1} \tag{20}$$

In the neighborhood of the limit cycle, the phase θ is a local first order approximation of the phase function ϕ defined through isochrons [7], [8].

Analysis of the SDEs (9) and (10) shows that the phase noise problem in a nonlinear oscillator is a drift-diffusion process. Function B_θ is the diffusion coefficient. Apart for the constant D, (14) is not different from the one found for the case of white Gaussian noise [7], [8]. Therefore we conclude that, as far as phase diffusion is concerned, colored noise does not play a different role from a white noise of the same intensity. The situation is significantly different for phase drift. Comparing with the results of [7], [8], SDE (9) shows the additional drift coefficient b_θ. This term represents a further drift effect, not present for the case of white noise, that can be ascribed to the finite noise correlation time.

IV. Example

As an example we study a second order oscillator subject to colored noise, described by the SDE

$$\mathrm{d}x_1 = \left[\mu\left(x_1 - \frac{x_1}{3} - x_2\right) + x_1\, y\right] \mathrm{d}t \tag{21a}$$

$$\mathrm{d}x_2 = \left(\frac{1}{\mu} x_2 + x_2\, y\right) \mathrm{d}t \tag{21b}$$

$$\tau \mathrm{d}\eta_t = -\eta_t\, \mathrm{d}t + D\, \mathrm{d}W_t \tag{21c}$$

where μ is a real valued parameter. After transformation into the equivalent system with white Gaussian noise we obtain

$$\mathrm{d}x_1 = \left[\mu\left(x_1 - \frac{x_1^3}{3} - x_2\right) + \frac{D^2}{2} x_2\right] \mathrm{d}t + D x_1 \mathrm{d}W_t \tag{22a}$$

$$\mathrm{d}x_2 = \left(\frac{1}{\mu} x_1 + \frac{D^2}{2} x_2\right) \mathrm{d}t + D\, x_2\, \mathrm{d}W_t \tag{22b}$$

Figure 1 shows the probability density function (PDF) $p(x_1, x_2)$ for different values of the correlation time τ. The PDF is obtained through integration of (21) and (22) exploiting the Euler-Maruyama numerical integration scheme with a time integration step $\Delta t = 3 \times 10^{-6}$. The probability to find the system in state $x_1{+}\mathrm{d}x_1, x_2{+}\mathrm{d}x_2$ is evaluated as the fraction of time spent in that interval, normalized to the total simulation length. As the correlation time decreases, the PDF approaches that of the equivalent system with white Gaussian noise.

Figure 2 **(a)** shows the detailed phase portrait for the nonlinear oscillator (21) in absence of noise. The thick blue line is the limit cycle, blue and red arrows are the tangent and the transversal Floquet vectors $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$, respectively. Floquet vectors are computed with the help of the analytical formulas given in [9], however for higher order systems specialized numerical algorithms are available [10]–[12]. Thin black lines are some of the isochrons, computed using the algorithm given in [13]. Figure 2 **(b)** shows the Jacobian of the coordinate transformation (8) for the nonlinear oscillator under investigation. Thick black lines identify where the determinant is null, and thus the transformation is not invertible. Clearly the Jacobian is regular for large values of the amplitude deviation

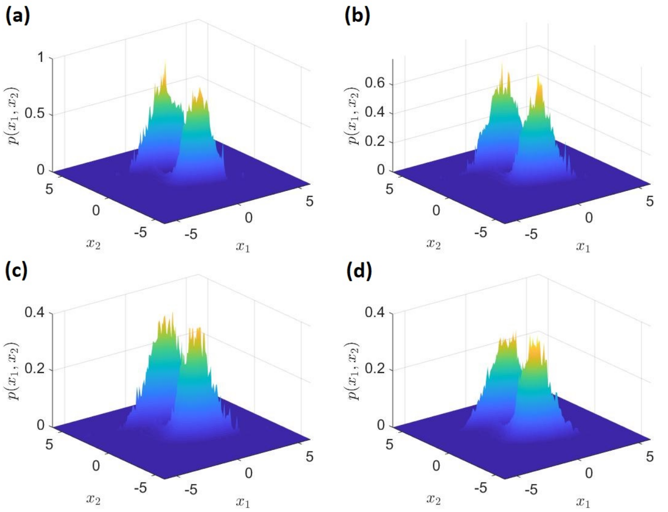

Figure 1. Probability density function for the nonlinear oscillator (21). **(a)** $\tau = 1$. **(b)** $\tau = 0.5$. **(c)** $\tau = 0.05$. **(d)** Equivalent white Gaussian noise. Parameter $\mu = 2.5$.

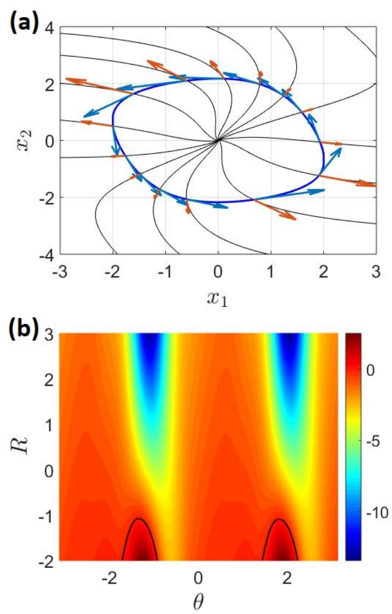

Figure 2. **(a)** Phase portrait for the nonlinear oscillator. **(b)** Jacobian of the coordinate transformation (8) for (21).

R, suggesting that the method can be applied even for quite a large noise intensity.

Finally, figure 3 shows the expected normalized angular frequency $\langle \omega/\omega_0 \rangle$ for the noisy oscillators (21) and (22), where ω_0 is the oscillator free running angular frequency in the absence of noise. The influence of colored noise on the expected angular frequency is clearly visible. Furthermore, the excellent agreement between equivalent system subject to white Gaussian noise, and the system with colored noise for short correlation time is also confirmed. As expected, the agreement deteriorates when noise correlation time is increased.

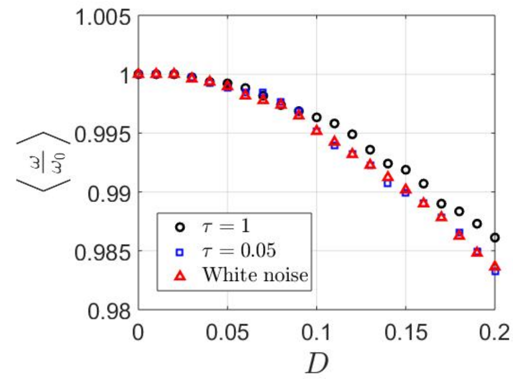

Figure 3. Expected normalized angular frequencies.

V. Conclusions

Exploiting stochastic averaging, we have obtained an Itô-Stratonovich dilemma free formulation for the noisy behavior of nonlinear oscillators subject to colored fluctuation with small noise correlation time. Transformation to amplitude and phase description is also presented, along with an example of application that enables to verify the correctness of the approach.

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