Curl-Conforming Vector Bases for Hybrid Meshes: A New Paradigm for Pyramid Elements

Curl-Conforming Vector Bases for Hybrid Meshes: A New Paradigm for Pyramid Elements

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***Abstract*—A simple procedure for obtaining hierarchical curl-conforming pyramid bases has been obtained by shifting to a new paradigm that requires the mapping of the pyramidal cell into a cube and then directly imposing the conformity of the vector bases with those used on adjacent differently shaped cells (tetrahedra, hexahedra and triangular prisms). This summary discusses and generalizes some features of the new construction method recently published elsewhere.**

I. Introduction

Curl- and divergence-conforming bases for tetrahedra, hexahedra (bricks) and prismatic cells are reported in [1]. Hybrid meshes using these higher order bases provide highly accurate models that balance computational efficiency with geometric flexibility. However, it is difficult to create hybrid meshes without pyramidal elements as they are the natural fillers for discretizations formed mainly by the other differently-shaped cells. Unfortunately, it has been extremely complicated to obtain vector bases for the pyramid while it is quite simple for tetrahedral, brick and prismatic cells. These latter cells are in fact accompanied by polynomial vector bases in a completely natural way because only three edges and three faces branch off from their vertices while a pyramid has four edges and faces converging at one vertex. A simpler procedure for obtaining higher-order hierarchical curl-conforming vector bases for pyramids was recently published in [2]. These bases have a polynomial form in the so-called *grandparent* space, where the pyramid of the *"child"* space (x, y, z) is mapped by a unit-cube (see Fig. 1). The object space is called the *child* space to distinguish it from the *parent* space which is the one where, for all cell types, the shape functions that specify the geometry of the cells and the vector basis functions are defined [1], [2]. In [2], the curl-conforming bases are obtained by imposing the continuity of the tangential components of the basis functions across adjacent elements of equal order but different shape, according to the following new paradigm:

1) The vector basis functions are subdivided from the outset into three different groups of edge, face, and volume-based functions.
2) Each higher order vector function is obtained by multiplying one edge-based vector function of the lowest order by a scalar polynomial. In the case of hierarchical bases, the polynomials are the product of normalized orthogonal polynomials.
3) The polynomials are defined in a cell whose vertices are points of intersection of only three edges and faces (i.e., for the pyramid, the grandparent cube of Fig. 1).

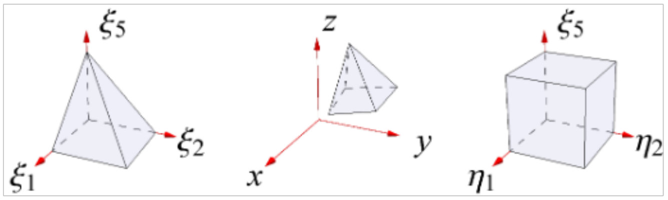

Fig. 1. The parent-space pyramid shown on the left maps the *child* pyramid of the *object* space shown at center. In turn, the unitary cube of the grandparent space shown on the right maps the parent pyramid.

4) On the cell border, the polynomials that generate the edge and face based functions equal those of the adjacent elements, no matter what shape they have.
5) The vector components of the basis functions and their curl are polynomials of the parent or grandparent (for the pyramid) variables. Unisolvency and base completeness is proved in this space.

The reader is referred to [1], [2] for the construction of the bubble (volume-based) functions which are obtained by following a different construction path. Several numerical results show that higher-order functions provide faster convergence with avoidance of spurious modes and more accurate results than those achievable with low-order elements [1], [2].

For the sake of brevity, in [2] we prove the tangential continuity at the boundary of each cell by expressing the pyramidal basis functions in terms of parent-coordinates despite the fact that the bases in [2] have been obtained precisely by imposing this continuity. The demonstration is simple and immediate because in the parent space the quadrilateral and triangular faces of differently shaped cells are mapped by the same unit-square or triangular simplex, respectively.

In the following we list the main characteristics of the multiplicative polynomials and display the edge- and face-based vectors functions behavior on the cell faces, up to the first order.

II. Orthogonal Polynomials and Vector Bases

To build the vector functions as stated in point 2 of our paradigm it is necessary to distinguish between three types of edges:

a) *Quadrilateral* edges in common with two rectangular faces, such as the twelve edges of the brick or three edges (out of 9) of the triangular prism;

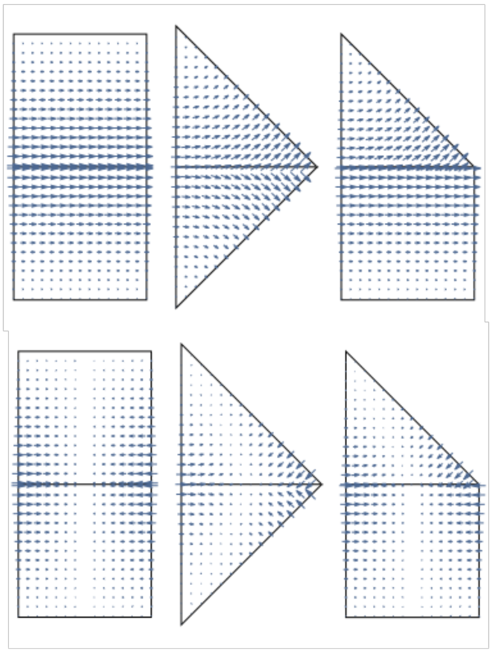

Fig. 2. An edge-based function has a zero tangential component on all faces but the two sharing that edge. The edge can be a quadrilateral edge (left), a triangle edge (center) or a mixed edge (right) depending on the shape of the two faces it has in common. The figure shows, in the parent domain, the component tangent to the two faces of the edge-based function of order zero (above) and of the first order (below).

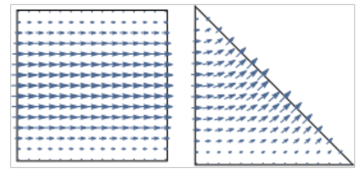

Fig. 3. A face-based function has a zero tangential component on all faces except the one where it is based. On the left we show the tangential component of the 1-st order function based on a quadrilateral face, The tangential component of the 1-st order function based on a triangular face is shown at right. The zeroth order edge-based function used to construct these vector functions is associated to the edge at bottom. The tangent component of the face-based basis functions vanishes on all the bounding edges.

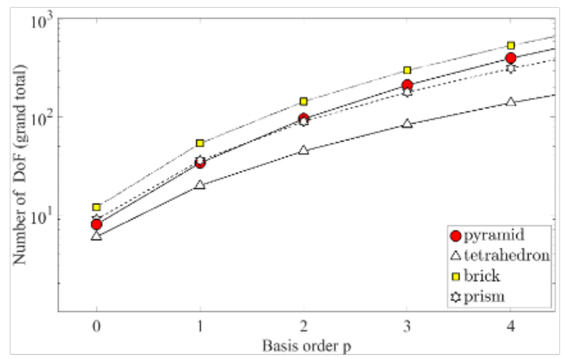

Fig. 4. Total number of degrees of freedom (DoF) for curl-conforming vector bases of order p on single, differently shaped canonical cells.

b) *Triangle* edges shared by two triangular faces of the cell, for example the six edges of a tetrahedron or the four edges of the pyramid connected to the vertex;
c) *Mixed* edges shared by a triangular and a rectangular face. that is to say the edges of the quadrilateral base of the pyramid or the edges bounding the two triangular bases of the prism.

The edge-based polynomials of a quadrilateral edge are obtained by using Legendre polynomials [3] while shifted scaled Legendre polynomials build the edge-based functions associated with triangle and mixed edges [2], [4], [5]. Obviously, for a given polynomial order, the trend of the multiplicative polynomial along the edge does not depend on the type of edge (whether quadrilateral, triangular or mixed) and therefore all the edge-based polynomials simplify with the same functional behavior along the edge. This is because an edge can be shared by several cells and therefore be, for example, of mixed type in one cell and quadrilateral in another.

These polynomials are in turn multiplied by orthogonal Jacobi polynomials to build the face-based multiplicative functions. Finally, as illustrated in [1], it is possible to obtain better conditioned vector bases by linearly combining the polynomials thus obtained with the volume-based polynomials (which, for the sake of brevity, are not discussed here) in order to obtain multiplicative polynomials which are mutually orthogonal for integrals on the parent cell or, in case of the pyramid, on the grandparent cube.

These scalar polynomials of order p are then multiplied by the lowest order vector function (of order zero) associated with the edge at issue to get a vector function of order p. In this regard, bear in mind that each zero-order vector function has a non vanishing tangential component only on the two faces sharing the aforesaid edge.

Not all the face-based vector functions are independent of each other. To form a p-th order base, we must discard the dependent functions and count the total number of Degrees of Freedom (DoF), as done in [1], [2]. Fig. 4 shows the total number of DoF for curl-conforming vector bases of order p on single, differently shaped canonical cells.

Acknowledgement

This work was supported in part by the Italian Ministry of University and Research (MUR) under PRIN Grant 2017NT5W7Z.

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