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ON THE GENERATORS OF CLIFFORD SEMIGROUPS: POLYNOMIAL RESOLVENTS AND THEIR INTEGRAL TRANSFORMS

RICCARDO GHILONI AND VINCENZO RECUPERO

ABSTRACT. This paper deals with generators A of strongly continuous right linear semigroups in Banach two-sided spaces whose set of scalars is an arbitrary Clifford algebra $Cl(0, n)$. We study the invertibility of operators of the form $P(A)$, where $P(x) \in \mathbb{R}[x]$ is any real polynomial, and we give an integral representation for $P(A)^{-1}$ by means of a Laplace-type transform of the semigroup $T(t)$ generated by A . In particular, we deduce a new integral representation for the spherical quadratic resolvent of A (also called pseudoresolvent of A). As an immediate consequence, we also obtain a new proof of the well-known integral representation for the spherical resolvent of A .

dedicated to Professor Klaus Gürlebeck

1. INTRODUCTION AND MAIN RESULTS

Quaternionic functional analysis has probably its original motivation in the seminal paper [4], where it is pointed out that quantum mechanics may be formulated, not only on complex Hilbert spaces, but also on Hilbert spaces whose set of scalars is \mathbb{H} , the noncommutative real algebra of quaternions.

Many papers have been devoted to the development of quantum mechanics in the quaternionic framework (see, e.g., [20, 18, 33, 1]), whose natural setting is a Hilbert two-sided \mathbb{H} -module X , with the space of bounded linear operators acting on it replaced by the set $\mathcal{L}^r(X)$ of bounded *right* linear operators. However, a full development of quaternionic quantum mechanics was prevented by the lack of suitable quaternionic spectral notions, indeed, as observed in [9, 23], the classical definitions of spectrum and resolvent operator do not allow to define a noncommutative functional calculus.

A first rigorous formulation of a quaternionic spectral theory has been provided only in [8] where one can find the first definition of the notions of *spherical resolvent set* $\rho_S(A)$, *quadratic resolvent operator* $Q_q(A)$, *spherical resolvent operator* $C_q(A)$ and *spherical spectrum* $\sigma_S(A)$ of a right linear operator A on a quaternionic Banach space X . They are

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given by

$$\begin{aligned}\rho_s(\mathbf{A}) &:= \{q \in \mathbb{H} : \exists(\mathbf{A}^2 - 2\operatorname{Re}(q)\mathbf{A} + |q|^2)^{-1} \in \mathcal{L}^r(X)\}, \\ \mathbf{Q}_q(\mathbf{A}) &:= (\mathbf{A}^2 - 2\operatorname{Re}(q)\mathbf{A} + |q|^2)^{-1}, \quad q \in \rho_s(\mathbf{A}), \\ \mathbf{C}_q(\mathbf{A}) &:= \mathbf{Q}_q(\mathbf{A})\bar{q} - \mathbf{A}\mathbf{Q}_q(\mathbf{A}), \quad q \in \rho_s(\mathbf{A}).\end{aligned}$$

In [8] the name *pseudoresolvent* is used for $\mathbf{Q}_q(\mathbf{A})$ but we choose the term *quadratic resolvent* since in the classical complex case the name “pseudoresolvent” has a different meaning. The above definitions permit to develop a noncommutative functional calculus for right linear operators on a Banach two-sided module over \mathbb{H} (and over a Clifford algebra as well, cf. [14, 11, 12, 9, 10, 15, 23]) and to deduce in [2, 24] the spectral representation theorems for normal operators in the quaternionic Hilbert setting (cf. [2, 24]). A wider bibliography can be found in the recent accounts on the theory [5, 6].

The mentioned noncommutative functional calculus is intimately connected to the theory of slice regular functions, introduced in [22], which extends to quaternions the classical concept of holomorphic function. They form a class of functions admitting a local power series expansion at every point of their domain of definition (cf. [21]), including polynomials with quaternionic coefficients on one side, and they admit a Cauchy-type integral representation formula with a suitable quaternionic version of the kernel proved for the first time in [7] (see also [26]).

The next natural stage in this analysis is the development of a noncommutative theory of right linear operator semigroups which was developed in [13, 27, 28]. In the classical complex theory a fundamental tool is provided by the integral representation of the resolvent operator of a generator \mathbf{A} by means of the Laplace transform of the semigroup $\mathbf{T}(t)$ generated by \mathbf{A} . An analogous integral representation in the quaternionic case for the spherical resolvent operator $\mathbf{C}_q(\mathbf{Q})$ is shown in [13] and a proof is provided in [28] using techniques from slice regular function theory.

The purpose of the present paper is to study the invertibility of operators of the form $P(\mathbf{A})$ where $P(x)$ is an arbitrary polynomial with real coefficients of degree at least 2, including $P(x) = \Delta_q(x) = x^2 - 2\operatorname{Re}(q)x + |q|^2$. Under natural conditions, we prove that $P(\mathbf{A})^{-1}$ exists and belongs to $\mathcal{L}^r(X)$. Furthermore, we provide an integral representation for $P(\mathbf{A})^{-1}$ by means of a Laplace-type transform of the semigroup $\mathbf{T}(t)$ generated by \mathbf{A} . We extend this integral representation to operators of the form $\sum_{j=0}^{d-1} \mathbf{A}^j P(\mathbf{A})^{-1} p_j$, where d is the degree of P and p_0, \dots, p_{d-1} are arbitrarily chosen quaternions. In the case $P(x) = \Delta_q(x)$, we obtain a new integral representation for $\mathbf{Q}_q(\mathbf{A})$ and, setting $p_0 := \bar{q}$ and $p_1 := -1$, we discover again the well-known integral representation for $\mathbf{C}_q(\mathbf{A})$ via the quaternionic Laplace transform. This gives also a new proof of the integral representation for $\mathbf{C}_q(\mathbf{A})$, which avoids the use of slice regular function techniques. Our results are valid not only on quaternions but also on a class of real associative $*$ -algebras including, as the main examples, all Clifford algebras $\mathcal{C}\ell(0, n)$.

Let $n \in \mathbb{N}$, let \mathbb{R}_n be the Clifford algebra $\mathcal{C}\ell(0, n)$ equipped with the Clifford conjugation and the Clifford operator norm $|\cdot|$. Consider a Banach two-sided \mathbb{R}_n -module X with norm $\|\cdot\|$ and the set $\mathcal{L}^r(X)$ of all bounded right linear operators on X (all the precise definitions will be recalled in the next section).

Let $m \in \mathbb{N}$ and let $P(x) = \sum_{k=0}^{m+2} x^k a_k \in \mathbb{R}[x]$ be a polynomial with real coefficients in the indeterminate x . Suppose P has degree $m+2$, that is, $a_{m+2} \neq 0$. Given a right linear

operator $A : D(A) \rightarrow X$, we define the right linear operator $P(A) : D(A^{m+2}) \rightarrow X$ simply by replacing x with A , that is, $P(A) := \sum_{k=0}^{m+2} A^k a_k$. Denote by $C^\infty([0, \infty[; \mathbb{R})$ the set of all infinitely many times differentiable functions $g : [0, \infty[\rightarrow \mathbb{R}$. Consider the following ODE with constant coefficients in the variable $g \in C^\infty([0, \infty[; \mathbb{R})$:

$$\begin{cases} P\left(-\frac{d}{dt}\right)(g) = 0 & \text{on } [0, \infty[, \\ g(0) = g'(0) = \dots = g^{(m)}(0) = 0, \\ g^{(m+1)}(0) = (-1)^m (a_{m+2})^{-1}, \end{cases} \quad (1.1)$$

where $g^{(k)}$ is the k^{th} -derivative of g and $P\left(-\frac{d}{dt}\right)(g) := \sum_{k=0}^{m+2} g^{(k)}(-1)^k a_k$. Denote by $g_P \in C^\infty([0, \infty[; \mathbb{R})$ the unique solution of (1.1). Recall that, if $\lambda_1, \dots, \lambda_h$ are the complex roots of the polynomial $P(x)$ with multiplicity m_1, \dots, m_h , then there exist complex polynomials $Q_1(x), \dots, Q_h(x) \in \mathbb{C}[x]$ such that the degree of each $Q_j(x)$ is $< m_j$ and

$$g_P(t) = \sum_{j=1}^h Q_j(t) e^{-\lambda_j t}. \quad (1.2)$$

Define $r_P \in \mathbb{R}$ by

$$r_P := \min\{\Re(\lambda_1), \dots, \Re(\lambda_h)\}, \quad (1.3)$$

where $\Re(\lambda_j)$ is the real part of the complex number λ_j .

Recall also that if $\mathbb{T} : [0, \infty[\rightarrow \mathcal{L}^r(X)$ is a strongly continuous right linear semigroup then there exists $\omega \in \mathbb{R}$ such that $\sup_{t \in [0, \infty[} \|\mathbb{T}(t)\| e^{-\omega t} < \infty$, see [27, Thm 4.5(b)].

Our main result reads as follows.

Theorem 1.1. *Let $\mathbb{T} : [0, \infty[\rightarrow \mathcal{L}^r(X)$ be a strongly continuous right linear semigroup, let $A : D(A) \rightarrow X$ be its generator and let $\omega \in \mathbb{R}$ be a real constant such that $M := \sup_{t \in [0, \infty[} \|\mathbb{T}(t)\| e^{-\omega t} < \infty$. Then, if $r_P > \omega$, the operator $P(A)$ is bijective, $P(A)^{-1} \in \mathcal{L}^r(X)$ and it holds:*

$$P(A)^{-1}x = \int_0^\infty \mathbb{T}(t) g_P(t) x \, dt \quad \forall x \in X \quad (1.4)$$

and

$$\|P(A)^{-1}\| \leq M \sum_{j=1}^h \sum_{k=1}^{m_j} \frac{|c_{-k}^{(j)}|}{(r_P - \omega)^k}, \quad (1.5)$$

where $c_{-k}^{(j)}$ is the residue at λ_j of the meromorphic function $z \mapsto a_{m+2}(z - \lambda_j)^{k-1}/P(-z)$.

Furthermore, given $(p_0, p_1, \dots, p_{m+1}) \in (\mathbb{R}_n)^{m+2}$, we have

$$\sum_{j=0}^{m+1} A^j P(A)^{-1} p_j x = \int_0^\infty \mathbb{T}(t) \left(\sum_{j=0}^{m+1} g_P^{(j)}(-1)^j p_j x \right) dt \quad \forall x \in X. \quad (1.6)$$

Let $Q_{\mathbb{R}_n}$ be the quadratic cone of \mathbb{R}_n (see (2.5) for the definition) and let $q \in Q_{\mathbb{R}_n}$. Define the function $g_q : \mathbb{R} \rightarrow \mathbb{R}$ by

$$g_q(t) := t e^{-\Re(q)t} \text{sinc}(t|\text{Im}(q)|), \quad t \in \mathbb{R}, \quad (1.7)$$

where $\Re(q)$ and $\text{Im}(q)$ are the real and imaginary parts of q , respectively. We recall that $\text{sinc} : \mathbb{R} \rightarrow \mathbb{R}$ is the *unnormalized sinc function*, that is, the real-valued continuous function ξ on \mathbb{R} defined by $\xi(0) = 1$ and $\xi(r) = \sin(r)/r$ for all $r \neq 0$.

Thanks to the preceding result, we are able to prove the following:

Theorem 1.2. *Let $\mathbb{T} : [0, \infty[\rightarrow \mathcal{L}^r(X)$ be a strongly continuous right linear semigroup, let $\mathbf{A} : D(\mathbf{A}) \rightarrow X$ be its generator, and let $\omega \in \mathbb{R}$ be a real constant such that $M := \sup_{t \in [0, \infty[} \|\mathbb{T}(t)\| e^{-\omega t} < \infty$. Consider any $q \in Q_{\mathbb{R}_n}$ and set $a := \operatorname{Re}(q)$ and $b := |\operatorname{Im}(q)|$. If $\operatorname{Re}(q) > \omega$, then we have that $q \in \rho_s(\mathbf{A})$ and it holds:*

$$\mathbf{Q}_q(\mathbf{A})x = \int_0^\infty \mathbb{T}(t)g_q(t)x \, dt = \int_0^\infty \mathbb{T}(t)t e^{-ta} \operatorname{sinc}(tb)x \, dt, \quad (1.8)$$

$$\mathbf{A}\mathbf{Q}_q(\mathbf{A})x = - \int_0^\infty \mathbb{T}(t)g'_q(t)x \, dt = - \int_0^\infty \mathbb{T}(t)e^{-ta}(\cos(tb) - at \operatorname{sinc}(tb))x \, dt, \quad (1.9)$$

$$\mathbf{C}_q(\mathbf{A})x = \int_0^\infty \mathbb{T}(t)e^{-tq}x \, dt \quad (1.10)$$

for every $x \in X$. Moreover, we have:

$$\|\mathbf{Q}_q(\mathbf{A})\| \leq \frac{M}{(\operatorname{Re}(q) - \omega)^2}, \quad (1.11)$$

$$\|\mathbf{C}_q(\mathbf{A})\| \leq \frac{M}{\operatorname{Re}(q) - \omega}. \quad (1.12)$$

Here is the plan of the paper. In the following section we recall all the needed precise definitions. In Section 3 we present the preceding theorems in the more general case of certain real $*$ -algebras, including all the \mathbb{R}_n 's. Section 4 is devoted to the proofs of these theorems. Finally, in Section 5 we apply the main theorem in order to derive an integral representation of the integer powers of the quadratic resolvent and the estimate of their norms; this extends to $\mathbf{Q}_q(\mathbf{A})$ our Theorem 6.6 in [28] concerning the Laplace-type transform for the integer slice powers of $\mathbf{C}_q(\mathbf{A})$.

2. PRELIMINARIES

Throughout all the paper we will assume that \mathbb{A} is a nontrivial finite dimensional \mathbb{R} -vector space endowed with a bilinear product $\mathbb{A} \times \mathbb{A} \rightarrow \mathbb{A} : (p, q) \mapsto pq$ with unit $1_{\mathbb{A}}$, and with a mapping $\mathbb{A} \rightarrow \mathbb{A} : q \mapsto q^c$ called $*$ -involution, which is an \mathbb{R} -linear mapping such that $(q^c)^c = q$, $(pq)^c = q^c p^c$, and $r^c = r$ for all $p, q \in \mathbb{A}, r \in \mathbb{R} \subseteq \mathbb{A}$, where we are identifying \mathbb{R} with the subalgebra of \mathbb{A} generated by $1_{\mathbb{A}}$ by means of the algebra isomorphism $\mathbb{R} \rightarrow \mathbb{R}1_{\mathbb{A}} : r \mapsto r1_{\mathbb{A}}$. Therefore we can write $1 = 1_{\mathbb{A}}$ and we summarize the previous assumptions by saying that

$$\begin{aligned} \mathbb{A} \text{ is a finite dimensional associative nontrivial real } * \text{-algebra} \\ \text{with } * \text{-involution } q \mapsto q^c \text{ and unit } 1. \end{aligned} \quad (2.1)$$

Under the previous assumptions we define the *imaginary sphere* $\mathbb{S}_{\mathbb{A}}$ in \mathbb{A} by

$$\mathbb{S}_{\mathbb{A}} := \{q \in \mathbb{A} : q^c = -q, q^2 = -1\}. \quad (2.2)$$

In the remainder of the paper we will assume that

$$\mathbb{S}_{\mathbb{A}} \neq \emptyset. \quad (2.3)$$

Condition (2.3) in particular implies that \mathbb{A} cannot be equal to \mathbb{R} . We set

$$\mathbb{C}_{\mathbf{j}} := \{r + s\mathbf{j} \in \mathbb{A} : r, s \in \mathbb{R}\}, \quad \mathbf{j} \in \mathbb{S}_{\mathbb{A}}, \quad (2.4)$$

$$Q_{\mathbb{A}} := \bigcup_{\mathbf{j} \in \mathbb{S}_{\mathbb{A}}} \mathbb{C}_{\mathbf{j}}, \quad (2.5)$$

the set $Q_{\mathbb{A}}$ being called *quadratic cone of \mathbb{A}* . The *real part* $\operatorname{Re}(q)$ and the *imaginary part* $\operatorname{Im}(q)$ of an element $q \in \mathbb{A}$ are defined by

$$\operatorname{Re}(q) := (q + q^c)/2, \quad \operatorname{Im}(q) := (q - q^c)/2, \quad q \in \mathbb{A}. \quad (2.6)$$

Notice that in general $\operatorname{Re}(q)$ and $\operatorname{Im}(q)$ are not real numbers, at variance with the customary complex notations $\Re(z) := (z + \bar{z})/2 \in \mathbb{R}$ and $\Im(z) := (z - \bar{z})/2i \in \mathbb{R}$ for $z \in \mathbb{C}$.

We finally observe that $qq^c \in \mathbb{R}$ for every $q \in Q_{\mathbb{A}}$ and we assume that

$$\begin{aligned} &\mathbb{A} \text{ is endowed with a complete norm } |\cdot| \text{ such that} \\ &|q_1 q_2| \leq |q_1| |q_2| \text{ for every } q_1, q_2 \in \mathbb{A} \text{ and } |q|^2 = qq^c \text{ for every } q \in Q_{\mathbb{A}}. \end{aligned} \quad (2.7)$$

The equivalence of the above definitions with other presentations (e.g. [25] is provided in [28]). We recall here that

$$\mathbb{C}_{\mathbf{j}} \cap \mathbb{C}_{\mathbf{k}} = \mathbb{R} \quad \forall \mathbf{j}, \mathbf{k} \in \mathbb{S}_{\mathbb{A}}, \mathbf{j} \neq \pm \mathbf{k}. \quad (2.8)$$

Example 2.1 (Clifford algebras). For $n \in \mathbb{N} \setminus \{0\}$ let $\mathcal{P}(n)$ be the power set of $\{1, \dots, n\}$. If we identify \mathbb{R} with the vector subspace $\mathbb{R} \times \{0\}$ of $\mathbb{R}^{2^n} = \mathbb{R} \times \mathbb{R}^{2^n-1}$ and we set $e_{\emptyset} := 1$, then we denote by $\{e_K\}_{K \in \mathcal{P}(n)}$ the canonical basis of \mathbb{R}^{2^n} . For convenience, we set $e_k := e_{\{k\}}$ if $k \in \{1, \dots, n\}$ and we define a real bilinear and associative product on \mathbb{R}^{2^n} by imposing that 1 is the neutral element and that

$$\begin{aligned} e_k^2 &= -1 \text{ and } e_k e_h = -e_h e_k \text{ if } k, h \in \{1, \dots, n\} \text{ with } k \neq h, \\ e_K &= e_{k_1} \cdots e_{k_s} \text{ if } K = \{k_1, \dots, k_s\} \in \mathcal{P}(n) \setminus \{\emptyset\} \text{ with } k_1 < \dots < k_s. \end{aligned}$$

The *Clifford conjugation* of \mathbb{R}^{2^n} is the *-involution $q \mapsto q^c := \bar{q}$ defined by

$$\bar{q} := \sum_{K \in \mathcal{P}(n)} (-1)^{|K|(|K|+1)/2} a_K e_K \quad \text{if } q = \sum_{K \in \mathcal{P}(n)} a_K e_K \in \mathbb{R}_n, \quad a_K \in \mathbb{R},$$

where $|K|$ indicates the cardinality of the set K . Endowing \mathbb{R}^{2^n} with the above defined product and with the Clifford conjugation, we obtain a real *-algebra \mathbb{A} satisfying (2.1), called *Clifford algebra $\mathcal{Cl}(0, n)$ of signature $(0, n)$* , which is denoted also by \mathbb{R}_n . Observe that \mathbb{R}_1 and \mathbb{R}_2 are isomorphic to \mathbb{C} and \mathbb{H} , respectively. Moreover \mathbb{R}_n is not commutative if $n \geq 2$. If $n \geq 3$ then \mathbb{R}_n has zero divisors, indeed $(1 - e_{\{1,2,3\}})(1 + e_{\{1,2,3\}}) = 0$. One verifies that a point $q = \sum_{K \in \mathcal{P}(n)} a_K e_K$ of \mathbb{R}_n with $a_K \in \mathbb{R}$ belongs to the quadratic cone $Q_{\mathbb{R}_n}$ of \mathbb{R}_n if and only if it satisfies the following conditions

$$a_K = 0 \quad \text{and} \quad \langle q, q e_K \rangle_{2^n} = 0 \quad \text{for every } K \in \mathcal{P}(n) \setminus \{\emptyset\} \text{ with } e_K^2 = 1,$$

where $\langle \cdot, \cdot \rangle_{2^n}$ denotes the standard scalar product on \mathbb{R}^{2^n} . On \mathbb{R}_n it is defined the following submultiplicative norm, called *Clifford operator norm*: $|q|_{\mathcal{Cl}} := \sup\{|qa|_{2^n} \in \mathbb{R} : |a|_{2^n} = 1\}$, where $|\cdot|_{2^n}$ indicates the Euclidean norm of \mathbb{R}^{2^n} . It turns out that:

- (a) $Q_{\mathbb{R}_n} = \mathbb{R}_n$ if and only if $n \in \{1, 2\}$. In particular, \mathbb{R}_1 and \mathbb{R}_2 are division algebras.
- (b) $|q|_{\mathcal{Cl}} = |x| = \sqrt{x\bar{x}}$ for every $x \in Q_{\mathbb{R}_n}$ and hence $|\cdot|_{\mathcal{Cl}} = |\cdot|$ if $n \in \{1, 2\}$.

Notice that if $n \geq 3$ then the Euclidean norm $|\cdot|_{2^n}$ of \mathbb{R}_n is not submultiplicative (e.g. $|(1 + e_{\{1,2,3\}})^2| = \sqrt{8} > 2 = |1 + e_{\{1,2,3\}}|^2$). Endowing \mathbb{R}_n with Clifford conjugation and Clifford operator norm, we obtain a real $*$ -algebra \mathbb{A} satisfying (2.3) and (2.7). For further details we refer the reader to [29, 31].

Example 2.2 (Complex numbers and quaternions). If $n \in \mathbb{N} \setminus \{0\}$ and \mathbb{R}_n denotes the Clifford algebra of signature $(0, n)$ recalled in the previous Example 2.1, then we have:

- (i) $\mathbb{R}_1 = \mathbb{C}$ with $e_1 = i$, where $z \mapsto z^c = \bar{z}$ is the standard conjugation and $|\cdot|$ is the Euclidean norm;
- (ii) \mathbb{R}_2 is the algebra of quaternions \mathbb{H} with $i := e_1$, $j := e_2$, $k := e_3$, where $q = a + bi + cj + dk \mapsto q^c = \bar{q} = a - bi - cj - dk$ and $|\cdot|$ is the euclidean norm.

Definition 2.3. If \mathbb{A} satisfies (2.1) then a two-sided \mathbb{A} -module is a commutative group $(X, +)$ endowed with a left scalar multiplication $\mathbb{A} \times X \rightarrow X : (q, x) \mapsto qx$ and a right scalar multiplication $X \times \mathbb{A} \rightarrow X : (x, q) \mapsto xq$ such that

$$\begin{aligned}
q(x + y) &= qx + qy, & (x + y)q &= xq + yq & \forall x, y \in X, & \forall q \in \mathbb{A}, \\
(p + q)x &= px + qx, & x(p + q) &= xp + xq & \forall x \in X, & \forall p, q \in \mathbb{A}, \\
1x &= x = x1 & & & \forall x \in X, \\
p(qx) &= (pq)x, & (xp)q &= x(pq) & \forall x \in X, & \forall p, q \in \mathbb{A}, \\
p(xq) &= (px)q & & & \forall x \in X, & \forall p, q \in \mathbb{A}, \\
rx &= xr & & & \forall x \in X, & \forall r \in \mathbb{R}.
\end{aligned}$$

If Y is a commutative subgroup of X then Y is called a left \mathbb{A} -submodule if $qx \in Y$ whenever $x \in Y$ and $q \in \mathbb{A}$. Instead Y is called a right \mathbb{A} -submodule of X if $xq \in Y$ whenever $x \in Y$ and $q \in \mathbb{A}$. Finally Y is called a two-sided \mathbb{A} -submodule of X if it is both a left and a right \mathbb{A} -submodule of X .

Definition 2.4. Assume (2.1) and (2.7) hold. A two-sided \mathbb{A} -module X is called a normed two-sided \mathbb{A} -module if it is endowed with a \mathbb{A} -norm on X , that is, a function $\|\cdot\| : X \rightarrow [0, \infty[$ such that

$$\begin{aligned}
\|x\| &= 0 \iff x = 0, \\
\|x + y\| &\leq \|x\| + \|y\| & \forall x, y \in X, \\
\|qx\| &\leq |q| \|x\|, \quad \|xq\| \leq |q| \|x\| & \forall x \in X, \quad \forall q \in \mathbb{A}.
\end{aligned} \tag{2.9}$$

We say that X is a Banach two-sided \mathbb{A} -module if the metric $d : X \times X \rightarrow [0, \infty[: (x, y) \mapsto \|x - y\|$ is complete.

Let us recall the following result (cf. [28, Lemma 3.3]).

Lemma 2.5. Assume (2.1) and (2.7) hold, and let X be a normed two-sided \mathbb{A} -module. Then

$$\|qx\| = \|xq\| = |q| \|x\| \quad \forall x \in X, \quad \forall q \in Q_{\mathbb{A}}. \tag{2.10}$$

Definition 2.6. Assume (2.1) holds and that X is a two-sided \mathbb{A} -module. Let $D(\mathbb{A})$ be a right \mathbb{A} -submodule of X . We say that $\mathbf{A} : D(\mathbb{A}) \rightarrow X$ is right linear if it is additive and

$$\mathbf{A}(xq) = \mathbf{A}(x)q \quad \forall x \in D(\mathbb{A}), \quad \forall q \in \mathbb{A}.$$

As usual, the notation Ax is often used in place of $A(x)$. We use the symbol $\text{End}^r(X)$ to denote the set of right linear operators A with $D(A) = X$. The identity operator is right linear and is denoted by Id_X or simply by Id . Moreover, if X is a normed two-sided \mathbb{A} -module, then we say that $A : D(A) \rightarrow X$ is closed if its graph is closed in $X \times X$. As in the classical theory, we set $D(A^n) := \{x \in D(A^{n-1}) : A^{n-1}x \in D(A)\}$ for every $n \in \mathbb{N} \setminus \{0\}$, where $A^0 := \text{Id}$.

Let us also recall the following definition (see, e.g., [3, Chapter 1, p. 55-57]).

Definition 2.7. Let $D(A)$ be a right \mathbb{A} -submodule of X and let $q \in \mathbb{A}$. If $A : D(A) \rightarrow X$ is a right linear operator, then we define the mapping $qA : D(A) \rightarrow X$ by setting

$$(qA)(x) := qA(x), \quad x \in D(A). \quad (2.11)$$

If $D(A)$ is also a left \mathbb{A} -submodule of X , then we can define $Aq : D(A) \rightarrow X$ by setting

$$(Aq)(x) := A(qx), \quad x \in D(A). \quad (2.12)$$

The sum of operators is defined in the usual way.

It is easy to see that the operators defined in (2.11) and (2.12) are right linear.

Definition 2.8. Assume X is normed with \mathbb{A} -norm $\|\cdot\|$. For every $B \in \text{End}^r(X)$, we set

$$\|B\| := \sup_{x \neq 0} \frac{\|Bx\|}{\|x\|} \quad (2.13)$$

and we define the set $\mathcal{L}^r(X) := \{B \in \text{End}^r(X) : \|B\| < \infty\}$.

3. MAIN RESULTS IN THEIR GENERAL FORM

In order to state our main result in its general form we recall the noncommutative spectral notions given for the first time in [8] for quaternions and in [14] for arbitrary Clifford algebras \mathbb{R}_n . Here we consider the general case introduced in [27, Definition 2.26]. We will assume that

\mathbb{A} satisfies (2.1), (2.3) and (2.7), and X is a Banach two-sided \mathbb{A} -module.

Definition 3.1. Let $D(A)$ be a right \mathbb{A} -submodule of X and let $A : D(A) \rightarrow X$ be a closed right linear operator.

(i) Given $q \in Q_{\mathbb{A}}$, the right linear operator $\Delta_q(A) : D(A^2) \rightarrow X$ is defined by

$$\Delta_q(A) := A^2 - 2\text{Re}(q)A + |q|^2 \text{Id}, \quad q \in Q_{\mathbb{A}}.$$

(ii) The spherical resolvent set $\rho_S(A)$ of A is defined by

$$\rho_S(A) := \{q \in Q_{\mathbb{A}} : \Delta_q(A) \text{ is bijective, } \Delta_q(A)^{-1} \in \mathcal{L}^r(X)\}$$

and the spherical spectrum $\sigma_S(A)$ of A by $\sigma_S(A) := Q_{\mathbb{A}} \setminus \rho_S(A)$.

(iii) Given $q \in \rho_S(A)$, the quadratic resolvent (or spherical pseudoresolvent) of A at q is the operator $Q_q(A) : X \rightarrow X$ defined by

$$Q_q(A) := \Delta_q(A)^{-1}, \quad q \in \rho_S(A).$$

(iv) Given $q \in \rho_S(A)$, the spherical resolvent of A at q is the operator $C_q(A) : X \rightarrow X$ defined by

$$C_q(A) := Q_q(A)q^c - AQ_q(A), \quad q \in \rho_S(A).$$

Let us observe that

$$\mathbf{Q}_q(\mathbf{A}) \in \mathcal{L}^r(X), \quad \mathbf{C}_q(\mathbf{A}) \in \mathcal{L}^r(X) \quad \forall q \in \rho_s(\mathbf{A}). \quad (3.1)$$

Indeed, by definition, $\mathbf{Q}_q(\mathbf{A})$ is bounded and if we endow $(X, +)$ with the (left) real scalar multiplication $\mathbb{R} \times X \rightarrow X : (r, x) \mapsto rx = xr$, then thanks to (2.10) X can be considered as a real Banach space and \mathbf{A} is a closed \mathbb{R} -linear operator on it, thus the closed graph theorem implies that $\mathbf{A}\mathbf{Q}_q(\mathbf{A})$ is continuous and consequently $\mathbf{C}_q(\mathbf{A})$ is also continuous. Since all these operators are also \mathbb{A} -right linear we infer (3.1).

The introduction of the name ‘‘quadratic resolvent’’ for $\mathbf{Q}_q(\mathbf{A})$, which we slightly prefer, is due to the fact that in the classical complex literature the term ‘‘pseudoresolvent’’ is already used for a different class of operators (cf., e.g., [36, Section 1.9]). We also mention that a definition that has some similarities with the spherical spectrum was given in [34] in the context of real $*$ -algebras.

We now recall the natural definition of right linear operator semigroup (cf. [13] for the quaternionic case and [27] for the general case).

Definition 3.2. *A mapping $\mathbf{S} : [0, \infty[\rightarrow \mathcal{L}^r(X)$ is called strongly continuous if the $t \mapsto \mathbf{S}(t)x$ is continuous from $[0, \infty[$ into X for every $x \in X$.*

Definition 3.3. *A mapping $\mathbf{T} : [0, \infty[\rightarrow \mathcal{L}^r(X)$ is called right linear strongly continuous (operator) semigroup if \mathbf{T} is strongly continuous and if*

$$\begin{aligned} \mathbf{T}(t+s) &= \mathbf{T}(t)\mathbf{T}(s) \quad \forall t, s > 0, \\ \mathbf{T}(0) &= \text{Id}. \end{aligned}$$

The generator of \mathbf{T} is the right linear operator $\mathbf{A} : D(\mathbf{A}) \rightarrow X$ defined by

$$\begin{aligned} D(\mathbf{A}) &:= \left\{ x \in X : \exists \lim_{h \rightarrow 0} \frac{1}{h} (\mathbf{T}(h)x - x) \in X \right\}, \\ \mathbf{A}x &:= \lim_{h \rightarrow 0} \frac{1}{h} (\mathbf{T}(h)x - x), \quad x \in D(\mathbf{A}). \end{aligned}$$

For the classical theory of semigroups in the complex framework we refer, e.g., to [32, 16, 36, 30, 35, 37, 19].

The next result includes Theorem 1.1.

Theorem 3.4. *Let $m \in \mathbb{N}$, let $P(x) = \sum_{k=0}^{m+2} x^k a_k \in \mathbb{R}[x]$ such that $a_{m+2} \neq 0$ and let $g_P \in C^\infty([0, \infty[; \mathbb{R})$ be the unique solution of (1.1). Let $\mathbf{T} : [0, \infty[\rightarrow \mathcal{L}^r(X)$ be a strongly continuous right linear semigroup, let $\mathbf{A} : D(\mathbf{A}) \rightarrow X$ be its generator and let $\omega \in \mathbb{R}$ be a real constant such that $M := \sup_{t \in [0, \infty[} \|\mathbf{T}(t)\| e^{-\omega t} < \infty$. Then, if $r_P > \omega$, the operator $P(\mathbf{A}) = \sum_{k=0}^{m+2} \mathbf{A}^k a_k$ is bijective, $P(\mathbf{A})^{-1} \in \mathcal{L}^r(X)$ and it holds:*

(a) $P(\mathbf{A})^{-1} = \mathbf{L}(g_P)$, that is,

$$P(\mathbf{A})^{-1}x = \int_0^\infty \mathbf{T}(t)g_P(t)x \, dt \quad \forall x \in X. \quad (3.2)$$

(b) Given $(p_0, p_1, \dots, p_{m+1}) \in \mathbb{A}^{m+2}$, we have

$$\sum_{j=0}^{m+1} \mathbf{A}^j P(\mathbf{A})^{-1} p_j x = \mathbf{L} \left(\sum_{j=0}^{m+1} g_P^{(j)}(-1)^j p_j \right) x \quad \forall x \in X. \quad (3.3)$$

Moreover,

$$\|P(\mathbf{A})^{-1}\| \leq M \sum_{j=1}^h \sum_{k=1}^{m_j} \frac{|c_{-k}^{(j)}|}{(r_P - \omega)^k} \quad (3.4)$$

where $c_{-k}^{(j)}$ is the residue at λ_j of the complex rational function $a_{m+2}(z - \lambda_j)^{k-1}/P(-z)$.

Furthermore we have

Theorem 3.5. *The statement of Theorem 1.2 holds true replacing \mathbb{R}_n with \mathbb{A} .*

4. PROOFS

Let us start with a lemma on strongly continuous mapping. The symbol $C([0, \infty[; \mathbb{A})$ denotes the space of continuous functions from $[0, \infty[$ to \mathbb{A} , both endowed with the topology induced by the Euclidean distance.

Lemma 4.1. *If $\mathbb{T} : [0, \infty[\rightarrow \mathcal{L}^r(X)$ is a strongly continuous and $g \in C([0, \infty[; \mathbb{A})$, then the following statements hold true.*

- (a) *The function $t \mapsto \|\mathbb{T}(t)\| |g(t)|$ is Lebesgue measurable on $[0, \infty[$.*
- (b) *For every $x \in X$ the function $t \mapsto \mathbb{T}(t)g(t)x$ is continuous from $[0, \infty[$ into X .*

Proof. Since $\mathbb{T} : [0, \infty[\rightarrow \mathcal{L}^r(X)$ is strongly continuous, by the Banach-Steinhaus theorem it follows that $\|\mathbb{T}(t)\| \leq \liminf_{\tau \rightarrow t} \|\mathbb{T}(\tau)\|$ for every $t \geq 0$, i.e. $t \mapsto \|\mathbb{T}(t)\|$ is lower semicontinuous, and hence it is Lebesgue measurable. Thus (a) is proved. In order to prove (b) fix an arbitrary $t_0 \geq 0$. Since \mathbb{T} is strongly continuous, by the uniform boundedness principle there exists $C > 0$ such that for every $t \geq 0$ with $|t - t_0| < 1$ we have $\|\mathbb{T}(t)\| \leq C$ and

$$\begin{aligned} \|\mathbb{T}(t)g(t)x - \mathbb{T}(t_0)g(t_0)x\| &\leq \|\mathbb{T}(t)g(t)x - \mathbb{T}(t)g(t_0)x\| + \|\mathbb{T}(t)g(t_0)x - \mathbb{T}(t_0)g(t_0)x\| \\ &\leq C|g(t) - g(t_0)|\|x\| + \|\mathbb{T}(t)g(t_0)x - \mathbb{T}(t_0)g(t_0)x\|. \end{aligned}$$

Thus the continuity of $t \mapsto \mathbb{T}(t)g(t)x$ at t_0 follows from the continuity of g and from the strong continuity of \mathbb{T} . \square

If $\mathbb{T} : [0, \infty[\rightarrow \mathcal{L}^r(X)$ is a strongly continuous right linear semigroup, $t \geq 0$, $g \in C([0, \infty[; \mathbb{A})$, and $x \in X$, then Lemma 4.1 and estimate $\|\mathbb{T}(t)g(t)x\| \leq \|\mathbb{T}(t)\| |g(t)| \|x\|$ allow to give the following definition.

Definition 4.2. *Let $\mathbb{T} : [0, \infty[\rightarrow \mathcal{L}^r(X)$ be a strongly continuous right linear semigroup. We denote by $L_{\mathbb{T}}([0, \infty[; \mathbb{A})$ the real vector space of all continuous functions $g : [0, \infty[\rightarrow \mathbb{A}$ such that the function $t \mapsto \|\mathbb{T}(t)\| |g(t)|$ belongs to $L^1([0, \infty[; \mathbb{R})$. For every $g \in L_{\mathbb{T}}([0, \infty[; \mathbb{A})$ we define the operator $\mathbf{L}(g) : X \rightarrow X$ by setting*

$$\mathbf{L}(g)x := \int_0^\infty \mathbb{T}(t)g(t)x dt, \quad x \in X. \quad (4.1)$$

Notice that the assumptions implies that the integral in (4.1) is a convergent Lebesgue integral for functions with values in the Banach space $(X, +)$ endowed with the real scalar multiplication $\mathbb{R} \times X \rightarrow X : (r, x) \mapsto rx = xr$ (thanks to (2.10) $\|\cdot\|$ is a norm on this real vector space). The symbol $L^1(J; X)$ denotes the space of Lebesgue integrable functions from an interval $J \subseteq \mathbb{R}$ into this real Banach space.

In the remainder of the paper, for $g \in C([0, \infty[; \mathbb{A})$, the symbols g' and g'' will denote the first and second derivative of g , respectively.

Lemma 4.3. *If $\mathbb{T} : [0, \infty[\rightarrow \mathcal{L}^r(X)$ is a strongly continuous right linear semigroup, then the following statements hold true.*

- (a) *If $g \in L_{\mathbb{T}}([0, \infty[; \mathbb{A})$ then $t \mapsto \mathbb{T}(t)g(t-h)x$ belongs to $L^1([h, \infty[; X)$ for every $h > 0$ and for every $x \in X$.*
(b) *If $g, g' \in L_{\mathbb{T}}([0, \infty[; \mathbb{A})$ then*

$$\lim_{h \rightarrow 0} \left\| \int_h^\infty \mathbb{T}(t) \frac{g(t-h) - g(t)}{h} x dt + \mathbb{L}(g')x \right\| = 0 \quad \forall x \in X.$$

Proof. The claim (a) follows trivially by the estimate $\|\mathbb{T}(t)g(t-h)x\| \leq \|\mathbb{T}(h)\| \|\mathbb{T}(t-h)\| \|g(t-h)\| \|x\|$ holding for every $x \in X$, $h > 0$, and $t > h$. In order to prove (b) fix $x \in X$ and an arbitrary $\varepsilon > 0$, and let $T > 0$ be such that $\int_T^\infty \|\mathbb{T}(t)\| |g(t)| dt < \varepsilon$. Since \mathbb{T} is a strongly continuous semigroup, there exists $M \geq 1$ such that $\|\mathbb{T}(s)\| \leq M$ whenever $0 \leq s \leq 1$, therefore for every $h \in]0, 1[$ and every $t > h$ we have

$$\begin{aligned} \left\| \mathbb{T}(t) \frac{g(t-h) - g(t)}{h} x \right\| &= \left\| \int_0^1 \mathbb{T}(\xi h) \mathbb{T}(t - \xi h) g'(t - \xi h) x d\xi \right\| \\ &\leq M \|x\| \int_0^1 \|\mathbb{T}(t - \xi h)\| |g'(t - \xi h)| d\xi \end{aligned}$$

hence it follows that

$$\begin{aligned} \left\| \int_T^\infty \mathbb{T}(t) \frac{g(t-h) - g(t)}{h} x dt \right\| &\leq M \|x\| \int_T^\infty \int_0^1 \|\mathbb{T}(t - \xi h)\| |g'(t - \xi h)| d\xi dt \\ &= M \|x\| \int_0^1 \int_T^\infty \|\mathbb{T}(t - \xi h)\| |g'(t - \xi h)| dt d\xi \\ &= M \|x\| \int_0^1 \int_{T-\xi h}^\infty \|\mathbb{T}(\tau)\| |g'(\tau)| d\tau d\xi \\ &\leq M \|x\| \int_0^1 \int_T^\infty \|\mathbb{T}(\tau)\| |g'(\tau)| d\tau d\xi \\ &\leq M \|x\| \int_T^\infty \|\mathbb{T}(\tau)\| |g'(\tau)| d\tau \leq M \|x\| \varepsilon. \end{aligned} \quad (4.2)$$

Moreover it is easily found a $\delta \in]0, 1[$ such that for every $h \in]0, \delta[$ we have

$$\begin{aligned} &\left\| \int_h^T \mathbb{T}(t) \frac{g(t-h) - g(t)}{h} x dt + \int_0^T \mathbb{T}(t) g'(t) x dt \right\| \\ &\leq \left\| \int_h^T \mathbb{T}(t) \left(\frac{g(t-h) - g(t)}{h} + g'(t) \right) x dt \right\| + \left\| \int_0^h \mathbb{T}(t) g'(t) x dt \right\| \leq \varepsilon. \end{aligned} \quad (4.3)$$

Hence assertion (b) follows from (4.2)–(4.3) and from the following estimate

$$\begin{aligned} &\left\| \int_h^\infty \mathbb{T}(t) \frac{g(t-h) - g(t)}{h} x dt + \mathbb{L}(g')x \right\| \\ &\leq \left\| \int_h^T \mathbb{T}(t) \frac{g(t-h) - g(t)}{h} x dt + \int_0^T \mathbb{T}(t) g'(t) x dt \right\| \\ &\quad + \left\| \int_T^\infty \mathbb{T}(t) \frac{g(t-h) - g(t)}{h} x dt \right\| + \left\| \int_T^\infty \mathbb{T}(t) g'(t) x dt \right\|. \end{aligned}$$

□

The next lemma plays a key role in the proof of Theorem 3.5.

Lemma 4.4. *Let $\mathbb{T} : [0, \infty[\rightarrow \mathcal{L}^r(X)$ be a strongly continuous right linear semigroup and for every $g \in L_{\mathbb{T}}([0, \infty[; \mathbb{A})$ let $\mathbf{L}(g) : X \rightarrow X$ be defined by (4.1). Then $\mathbf{L}(g) \in \mathcal{L}^r(X)$ for every $g \in L_{\mathbb{T}}([0, \infty[; \mathbb{A})$ and the resulting mapping $\mathbf{L} : L_{\mathbb{T}}([0, \infty[; \mathbb{A}) \rightarrow \mathcal{L}^r(X)$ is \mathbb{R} -linear. Moreover the following assertions hold.*

(a) *If $g, g' \in L_{\mathbb{T}}([0, \infty[; \mathbb{A})$, then*

$$\mathbf{L}(g)(X) \subseteq D(\mathbf{A}), \quad (4.4)$$

$$\mathbf{A}\mathbf{L}(g)x = -g(0)x - \mathbf{L}(g')x \quad \forall x \in X. \quad (4.5)$$

(b) *If $m \in \mathbb{N}$, $g, g', \dots, g^{(m+2)} \in L_{\mathbb{T}}([0, \infty[; \mathbb{A})$, and $g(0) = g'(0) = \dots = g^{(m)}(0) = 0$, then*

$$\mathbf{L}(g)(X) \subseteq D(\mathbf{A}^{m+2}), \quad (4.6)$$

$$\mathbf{A}^{m+2}\mathbf{L}(g)x = (-1)^{m+2} (g^{(m+1)}(0)x + \mathbf{L}(g^{(m+2)})x) \quad \forall x \in X, \quad (4.7)$$

$$\mathbf{A}^k\mathbf{L}(g)x = (-1)^k \mathbf{L}(g^{(k)})x \quad \forall x \in X, \forall k \in \{0, \dots, m+1\}. \quad (4.8)$$

(c) *If $g, g', g'' \in L_{\mathbb{T}}([0, \infty[; \mathbb{A})$, $g(0) = 0$, and $g'(0) = 1$, then*

$$\Delta_q(\mathbf{A})\mathbf{L}(g)x = x + \mathbf{L}(g'' + 2\operatorname{Re}(q)g' + |q|^2g)x \quad \forall x \in X, \forall q \in \mathbb{Q}_{\mathbb{A}}. \quad (4.9)$$

(d) *If $g, g' \in L_{\mathbb{T}}([0, \infty[; \mathbb{A})$ and g is real-valued, then*

$$\mathbf{A}\mathbf{L}(g)x = \mathbf{L}(g)\mathbf{A}x \quad \forall x \in D(\mathbf{A}). \quad (4.10)$$

(e) *If $m \in \mathbb{N}$, $g, g', \dots, g^{(m+2)} \in L_{\mathbb{T}}([0, \infty[; \mathbb{A})$, $g(0) = g'(0) = \dots = g^{(m)}(0) = 0$, and g is real valued, then*

$$\mathbf{A}^k\mathbf{L}(g)x = \mathbf{L}(g)\mathbf{A}^kx \quad \forall x \in D(\mathbf{A}^k), \forall k \in \{1, \dots, m+2\}. \quad (4.11)$$

Proof. For every $g \in L_{\mathbb{T}}([0, \infty[; \mathbb{A})$ the right linearity of $\mathbf{L}(g)$ follows from the right linearity of $\mathbb{T}(t)$ and from the definition of the X -valued Lebesgue integral. For every $x \in X$ we have

$$\|\mathbf{L}(g)x\| \leq \|x\| \int_0^\infty \|\mathbb{T}(t)\| |g(t)| dt,$$

hence $\mathbf{L}(g)$ is also continuous and $\|\mathbf{L}(g)\| \leq \int_0^\infty \|\mathbb{T}(t)\| |g(t)| dt$. The real linearity of \mathbf{L} is straightforward. Now in the following list of items we prove the assertions from (a) to (e).

(a) For every $h > 0$ and for every $x \in X$ we have

$$\begin{aligned} \frac{\mathbb{T}(h) - \operatorname{Id}}{h} \mathbf{L}(g)x &= \frac{1}{h} \int_0^\infty \mathbb{T}(t+h)g(t)x dt - \frac{1}{h} \int_0^\infty \mathbb{T}(t)g(t)x dt \\ &= \frac{1}{h} \int_h^\infty \mathbb{T}(t)g(t-h)x dt - \frac{1}{h} \int_0^\infty \mathbb{T}(t)g(t)x dt \\ &= -\frac{1}{h} \int_0^h \mathbb{T}(t)g(t)x dt + \int_h^\infty \mathbb{T}(t) \left(\frac{g(t-h) - g(t)}{h} \right) x dt. \end{aligned}$$

Hence, taking the limit as $h \rightarrow 0$, thanks to Lemma 4.1 we find that

$$\mathbf{A}\mathbf{L}(g)x = -\mathbb{T}(0)g(0)x - \int_0^\infty \mathbb{T}(t)g'(t)x dt = -g(0)x - \mathbf{L}(g')x.$$

(b) We proceed by induction on $m \in \{-1\} \cup \mathbb{N}$. The case $m = -1$ follows from (a). Let us assume that the result is true for $m - 1$, and we prove it for m . Therefore if g satisfies the assumptions, in particular we have $\mathbf{L}(g)x \in D(\mathbf{A}^{m+1})$ and $\mathbf{A}^{m+1}\mathbf{L}(g)x = (-1)^{m+1}(g^{(m)}(0)x + \mathbf{L}(g^{(m+1)})x$ for every $x \in X$. But $g^{(m)}(0) = 0$ hence $\mathbf{A}^{m+1}\mathbf{L}(g)x = (-1)^{m+1}\mathbf{L}(g^{(m+1)})x$ and $\mathbf{L}(g^{(m+1)})x \in D(\mathbf{A})$ by virtue of an application of (4.4) with g replaced by $g^{(m+1)}$. Thus $\mathbf{A}^{m+1}\mathbf{L}(g)x \in D(\mathbf{A})$ and (4.6) follows. Using again the validity of the statement for $m - 1$ and the identity $g^{(m)}(0) = 0$ we have

$$\begin{aligned} \mathbf{A}^{m+2}\mathbf{L}(g)x &= \mathbf{A}\mathbf{A}^{m+1}\mathbf{L}(g)x = (-1)^{m+1}\mathbf{A}\mathbf{L}(g^{(m+1)})x \\ &= (-1)^m(g^{(m+1)}(0)x + \mathbf{L}(g^{(m+2)}))x, \end{aligned}$$

where in the last equality we have used (4.5) with g replaced by $g^{(m+1)}$. Therefore (4.7) is proved. Formula (4.8) is trivial for $k = 0$ and follows from (a) for $k = 1$, while for $2 \leq k \leq m + 1$ follows from (4.7) which we have already proved.

(c) Now fix $x \in X$ and $q \in Q_{\mathbb{A}}$. From (b) we obtain $\mathbf{A}^2\mathbf{L}(g)x = x + \mathbf{L}(g'')x$, hence, exploiting again (4.5) and the \mathbb{R} -linearity of \mathbf{L} , we obtain

$$\begin{aligned} \Delta_q(\mathbf{A})\mathbf{L}(g)x &= \mathbf{A}^2\mathbf{L}(g)x - 2\operatorname{Re}(q)\mathbf{A}\mathbf{L}(g)x + |q|^2\mathbf{L}(g)x \\ &= x + \mathbf{L}(g'')x - 2\operatorname{Re}(q)(-\mathbf{L}(g')x) + |q|^2\mathbf{L}(g)x \\ &= x + \mathbf{L}(g'' + 2\operatorname{Re}(q)g' + |q|^2g)x. \end{aligned}$$

(d) If $x \in D(\mathbf{A})$, then for every $h > 0$ and for every $t > 0$ we have $\mathbf{T}(t)g(t)\mathbf{T}(h)x = \mathbf{T}(t)\mathbf{T}(h)g(t)x = \mathbf{T}(t+h)g(t)x = \mathbf{T}(h)\mathbf{T}(t)g(t)x$, because g is real-valued and \mathbf{T} is a semigroup. Therefore

$$\begin{aligned} \mathbf{L}(g)\frac{\mathbf{T}(h) - \operatorname{Id}}{h}x &= \frac{1}{h}\int_0^\infty \mathbf{T}(t)g(t)\mathbf{T}(h)x \, dt - \frac{1}{h}\int_0^\infty \mathbf{T}(t)g(t)x \, dt \\ &= \frac{1}{h}\int_0^\infty \mathbf{T}(h)\mathbf{T}(t)g(t)x \, dt - \frac{1}{h}\int_0^\infty \mathbf{T}(t)g(t)x \, dt \\ &= \frac{\mathbf{T}(h) - \operatorname{Id}}{h}\mathbf{L}(g)x, \end{aligned}$$

and the assertion follows taking the limit as $h \rightarrow 0$ and invoking (4.4).

(e) By induction on $m \in \{-1\} \cup \mathbb{N}$. The case $m = -1$ is true by virtue of (d). Let us assume that the result is true for $m - 1$, and we prove it for m . Therefore if g satisfies the assumptions, in particular we have that $\mathbf{A}^k\mathbf{L}(g) = \mathbf{L}(g)\mathbf{A}^k$ on $D(\mathbf{A}^k)$ for all $k \in \{1, \dots, m + 1\}$. Hence if $x \in D(\mathbf{A}^{m+2}) \subset D(\mathbf{A}^{m+1})$ then $\mathbf{A}^{m+1}x \in D(\mathbf{A})$ and we have that

$$\mathbf{L}(g)\mathbf{A}^{m+2}x = \mathbf{L}(g)\mathbf{A}\mathbf{A}^{m+1}x = \mathbf{A}\mathbf{L}(g)\mathbf{A}^{m+1}x = \mathbf{A}\mathbf{A}^{m+1}\mathbf{L}(g)x = \mathbf{A}^{m+2}\mathbf{L}(g)x,$$

where in the second equality we have used again (d). \square

Proof of Theorem 3.4. Let us first recall that the existence of constant $\omega \in \mathbb{R}$ such that $M := \sup_{t \geq 0} \|\mathbf{T}(t)\|e^{-\omega t} < \infty$ is well known (cf. [27, Thm 4.5(b)]). From (1.2) it follows that $g_P \in C^{m+2}([0, \infty[; \mathbb{A})$ and the Leibniz formula for the higher derivatives of a product yields the existence of a polynomial $p(t, \lambda_1, \dots, \lambda_h)$ such that

$$\|\mathbf{T}(t)\| |g_P^k(t)| \leq M e^{\omega t} |p(t, \lambda_1, \dots, \lambda_h)| e^{-tr_P} = M |p(t, \lambda_1, \dots, \lambda_h)| e^{(\omega - r_P)t}$$

for all $t \geq 0$ and for all $k \in \{0, 1, \dots, m+2\}$. Thus $g_P, g'_P, \dots, g_P^{(m+2)} \in L_{\mathbb{T}}([0, \infty[; \mathbb{A})$ and we can apply part (b) of Lemma 4.4 taking into account of initial conditions for g_P . We infer that

$$\begin{aligned} P(\mathbf{A})\mathbf{L}(g_P)x &= \sum_{k=0}^{m+2} \mathbf{A}^k a_k \mathbf{L}(g_P)x \\ &= \sum_{k=0}^{m+1} (-1)^k \mathbf{L}(g_P^{(k)}) a_k x + (-1)^{m+2} \left(g_P^{(m+1)}(0) + \mathbf{L}(g_P^{(m+2)}) \right) a_{m+2} x \\ &= \mathbf{L} \left(\sum_{k=0}^{m+2} g_P^{(k)} (-1)^k a_k \right) x + (-1)^{m+1} x = x g_P^{(m+1)}(0) (-1)^m a_{m+2} = x. \end{aligned}$$

On the other hand since g_P is real-valued we can also apply parts (d) and (e) of Lemma 4.4 and infer that $\mathbf{L}(g_P)P(\mathbf{A}) = P(\mathbf{A})\mathbf{L}(g_P)$ thus $\mathbf{L}(g_P) = P(\mathbf{A})^{-1}$ and (3.2) is proved. Now take $(p_0, p_1, \dots, p_{m+1}) \in \mathbb{A}^{m+2}$. Using Lemma 4.4(b) we deduce:

$$\sum_{j=0}^{m+1} \mathbf{A}^k P(\mathbf{A})^{-1} p_j x = \sum_{j=0}^{m+1} \mathbf{A}^k \mathbf{L}(g_P) p_j x = \sum_{j=0}^{m+1} (-1)^k \mathbf{L}(g_P^{(k)}) p_j x = \mathbf{L} \left(\sum_{j=0}^{m+1} (-1)^k g_P^{(k)} p_j \right) x$$

and (3.3) is proved. It is well known that

$$Q_j(t) = \sum_{k=1}^{m_j} \frac{c_{-k}^{(j)}}{(k-1)!} t^{k-1},$$

where $c_{-k}^{(j)}$ is the coefficient of $(z - \lambda_j)^{-k}$ within the partial fractions decomposition of $a_{m+2}/P(-z)$, i.e. the residue at λ_j of the function $a_{m+2}(z - \lambda_j)^{k-1}/P(-z)$. Therefore

$$|g_P(t)| \leq \sum_{j=1}^h \sum_{k=1}^{m_j} \frac{|c_{-k}^{(j)}|}{(k-1)!} t^{k-1} e^{-r_P t} \quad \forall t \geq 0$$

and

$$\begin{aligned} \|P(\mathbf{A})^{-1}\| &\leq \int_0^\infty \|\mathbb{T}(t)\| \sum_{j=1}^h \sum_{k=1}^{m_j} \frac{|c_{-k}^{(j)}|}{(k-1)!} t^{k-1} e^{-r_P t} dt \\ &\leq M \sum_{j=1}^h \sum_{k=1}^{m_j} \frac{|c_{-k}^{(j)}|}{(k-1)!} \int_0^\infty t^{k-1} e^{-(r_P - \omega)t} dt \\ &= M \sum_{j=1}^h \sum_{k=1}^{m_j} \frac{|c_{-k}^{(j)}|}{(r_P - \omega)^k} \frac{1}{(k-1)!} \int_0^\infty t^{k-1} e^{-t} dt \\ &= M \sum_{j=1}^h \sum_{k=1}^{m_j} \frac{|c_{-k}^{(j)}|}{(r_P - \omega)^k} \end{aligned}$$

and the theorem is completely proved. \square

Now we address the case when $P(x) = x^2 - 2 \operatorname{Re}(q) + |q|^2$ for some $q \in Q_{\mathbb{A}}$ which is related to part (c) of Lemma 4.4. We first present a simple lemma whose proof is a trivial calculus exercise.

Lemma 4.5. *For every fixed $q \in Q_{\mathbb{A}}$, the unique solution of the Cauchy problem*

$$g'' + 2 \operatorname{Re}(q)g' + |q|^2 g = 0, \quad g(0) = 0, \quad g'(0) = 1. \quad (4.12)$$

is the function $g_q : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$g_q(t) := te^{-\operatorname{Re}(q)t} \operatorname{sinc}(t|\operatorname{Im}(q)|), \quad t \in \mathbb{R}, \quad (4.13)$$

where we recall that $\operatorname{sinc} : \mathbb{R} \rightarrow \mathbb{R}$ is the unnormalized sinc function, i.e. the only continuous real function ξ on \mathbb{R} such that $\xi(r) = (\sin r)/r$ for all $r \neq 0$. Moreover we have that $g_q, g'_q, g''_q \in L_T([0, \infty[; \mathbb{A})$.

Now we are in position to find the integral representations of the quadratic resolvent and of the spherical resolvent operators as a simple consequence of our main Theorem 3.4.

Proof of Theorem 3.5. In order to prove (1.8) it is enough to apply part (a) of Theorem 3.4 with $P(x) = x^2 - 2 \operatorname{Re}(q)x + |q|^2$ taking into account of Lemma 4.5. Formula (1.9) is a straightforward application of part (b) of Theorem 3.4 with $p_0 = 0$, $p_1 = 1$ and $p_2 = 0$. Instead (1.10) is obtained taking in part (b) of Theorem 3.4 $p_0 = q^c$, $p_1 = -1$, and $p_2 = 0$. Finally

$$\|\mathbf{Q}_q(\mathbf{A})\| \leq \int_0^\infty \|\mathbb{T}(t)\| |g_p(t)| dt \leq \int_0^\infty Mte^{(\omega - \operatorname{Re}(q))t} dt = \frac{M}{(\operatorname{Re}(q) - \omega)^2},$$

i.e. (1.11) holds. Finally estimate (1.12) can be obtained in a similar way. \square

Remark 4.6. By exploiting (4.13) of our Lemma 4.5 we can write the integral representation (1.8) in a more explicit way for every $q \in Q_{\mathbb{A}}$ such that $\operatorname{Re}(q) > \omega$:

$$\mathbf{Q}_q(\mathbf{A})x = - \int_0^\infty \mathbb{T}(t) \frac{e^{-\operatorname{Re}(q)t} \sin(|\operatorname{Im}(q)|t)}{|\operatorname{Im}(q)|} x dt \quad \forall x \in X, \quad q \notin \mathbb{R}, \quad (4.14)$$

$$\mathbf{Q}_q(\mathbf{A})x = - \int_0^\infty \mathbb{T}(t) te^{-qt} x dt \quad \forall x \in X, \quad q \in \mathbb{R}. \quad (4.15)$$

The following lemma will connect the integral representation of $\mathbf{Q}_q(\mathbf{A})$ to the so called *spherical derivative* of $q \mapsto e^{-tq}$ (cf. [25]).

Lemma 4.7. *For every $t \in \mathbb{R}$ let $\exp^t : Q_{\mathbb{A}} \rightarrow \mathbb{A}$ be the function defined by*

$$\exp^t(q) := \sum_{n=0}^\infty \frac{t^n}{n!} q^n = \sum_{n=0}^\infty \frac{(tq)^n}{n!}, \quad q \in Q_{\mathbb{A}}. \quad (4.16)$$

If $(\exp^t)'_s : Q_{\mathbb{A}} \setminus \mathbb{R} \rightarrow \mathbb{A}$ denotes the function defined by

$$(\exp^t)'_s(q) := (q - q^c)^{-1} (\exp^t(q) - \exp^t(q^c)), \quad q \in Q_{\mathbb{A}} \setminus \mathbb{R}, \quad (4.17)$$

which is also called *spherical derivative* of \exp^t , then $(\exp^t)'_s$ extends to a unique continuous function on $Q_{\mathbb{A}}$, which we still denote by $(\exp^t)'_s : Q_{\mathbb{A}} \rightarrow \mathbb{A}$, and we have

$$(\exp^t)'_s(q) = e^{t \operatorname{Re}(q)} \sum_{n=0}^\infty \frac{t^{2n+1} \operatorname{Im}(q)^{2n}}{(2n+1)!} \in \mathbb{R} \quad \forall q \in Q_{\mathbb{A}} \setminus \mathbb{R} \quad (4.18)$$

and $(\exp^t)'_s(q) = te^{t \operatorname{Re}(q)}$ for every $q \in \mathbb{R}$. In particular $(\exp^t)'_s$ is a real-valued. By abuse of notation, we write $\exp'_s(t, q)$ to indicate the element $(\exp^t)'_s(q)$ of \mathbb{A} for every $t \in \mathbb{R}$ and for every $q \in Q_{\mathbb{A}}$, respectively.

Proof. For every $q \in Q_{\mathbb{A}} \setminus \mathbb{R}$ there exists $\mathbf{j} \in \mathbb{S}_{\mathbb{A}}$ and $a, b \in \mathbb{R}$ such that $b > 0$ and $q = a + b\mathbf{j}$. Hence $q^c = a^c - \mathbf{j}^c b^c = a - b\mathbf{j}$, $\operatorname{Re}(q) = a$, $\operatorname{Im}(q) = b\mathbf{j}$, and $|\operatorname{Im}(q)| = \sqrt{(b\mathbf{j})(b\mathbf{j})^c} = b$. Since $\mathbb{C}_{\mathbf{j}}$ and \mathbb{C} are isomorphic real algebras, we find that $\exp^t(q) = e^{tq} = e^{ta}(\cos(tb) + \sin(tb)\mathbf{j})$ and $\exp^t(q^c) = e^{tq^c} = e^{ta}(\cos(tb) - \sin(tb)\mathbf{j}) = (e^{tq})^c$, therefore

$$(\exp^t)'_s(q) = (q - q^c)^{-1}(e^{tq} - e^{tq^c}) = e^{t\operatorname{Re}(q)} \sin(t|\operatorname{Im}(q)|) |\operatorname{Im}(q)|^{-1} \quad \forall q \in Q_{\mathbb{A}} \setminus \mathbb{R}, \quad (4.19)$$

which proves the first equality in (4.18). The right-hand side of (4.19) immediately extends by continuity to te^{ta} for $q \in \mathbb{R}$, thus $(\exp^t)'_s$ is a real-valued function. As $|\operatorname{Im}(q)|^2 = -\operatorname{Im}(q)^2$, by (4.19) we have

$$(\exp^t)'_s(q) = e^{t\operatorname{Re}(q)} \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1} |\operatorname{Im}(q)|^{2n}}{(2n+1)!} = e^{t\operatorname{Re}(q)} \sum_{n=0}^{\infty} \frac{t^{2n+1} \operatorname{Im}(q)^{2n}}{(2n+1)!} \quad \forall q \in Q_{\mathbb{A}} \setminus \mathbb{R},$$

which implies the second equality of (4.18). \square

Corollary 4.8. *Under the assumption of Theorem 3.5, for every $q \in Q_{\mathbb{A}}$ such that $\operatorname{Re}(q) > \omega$ we have*

$$Q_q(\mathbf{A})x = - \int_0^{\infty} \mathbb{T}(t) \exp'_s(-t, q) x \, dt \quad \forall x \in X. \quad (4.20)$$

5. INTEGRAL REPRESENTATION OF THE POWERS OF $Q_q(\mathbf{A})$

In this section we look for an integral representation of the integer powers of the quadratic resolvent operator $Q_q(\mathbf{A})$. In order to find this representation we need the following simple lemma.

Lemma 5.1. *If $f, g \in L_{\mathbb{T}}([0, \infty[; \mathbb{A})$ and f is real-valued, then*

$$\mathbb{L}(f)\mathbb{L}(g)x = \mathbb{L}(f \star g)x = \mathbb{L}(g \star f)x \quad \forall x \in X,$$

where we recall that $f \star g : [0, \infty[\rightarrow \mathbb{A}$, the convolution of f and g , is defined by $(f \star g)(t) := \int_0^t f(t-s)g(s) \, ds$.

Proof. Using the fact that f is real-valued, we see at once that $f \star g = g \star f$. In addition, bearing in mind the semigroup law for \mathbb{T} , we find

$$\begin{aligned} \mathbb{L}(f)\mathbb{L}(g)x &= \int_0^{\infty} \mathbb{T}(t)f(t) \int_0^{\infty} \mathbb{T}(s)g(s)x \, ds \, dt \\ &= \int_0^{\infty} \int_0^{\infty} \mathbb{T}(t)\mathbb{T}(s)f(t)g(s)x \, ds \, dt \\ &= \int_0^{\infty} \int_0^{\infty} \mathbb{T}(t+s)f(t)g(s)x \, ds \, dt. \end{aligned}$$

Thus a change of variable and an application of Fubini theorem yields

$$\begin{aligned}
\mathsf{L}(f)\mathsf{L}(g)x &= \int_0^\infty \int_t^\infty \mathsf{T}(s)f(t)g(s-t)x \, ds \, dt \\
&= \int_0^\infty \int_0^\infty \chi_{[0,s]}(t)\mathsf{T}(s)f(t)g(s-t)x \, ds \, dt \\
&= \int_0^\infty \int_0^\infty \chi_{[0,s]}(t)\mathsf{T}(s)f(t)g(s-t)x \, dt \, ds \\
&= \int_0^\infty \int_0^t \mathsf{T}(s)f(t)g(s-t)x \, dt \, ds \\
&= \int_0^\infty \mathsf{T}(s) \int_0^t f(t)g(s-t)x \, dt \, ds \\
&= \mathsf{L}(f \star g) = \mathsf{L}(g \star f).
\end{aligned}$$

The proof is complete. \square

Given $n \in \mathbb{N} \setminus \{0\}$ and $f \in L_{\mathsf{T}}([0, \infty[; \mathbb{A})$, we define $f^{\star n} \in L_{\mathsf{T}}([0, \infty[; \mathbb{A})$ by

$$f^{\star n} := \underbrace{f \star f \star \cdots \star f}_{n \text{ times}}.$$

Corollary 5.2. *Let $\mathsf{T} : [0, \infty[\rightarrow \mathcal{L}^r(X)$ be a strongly continuous right linear semigroup, let $\mathsf{A} : D(\mathsf{A}) \rightarrow X$ be its generator, and let $\omega \in \mathbb{R}$ be a real constant such that $M := \sup_{t \in [0, \infty[} \|\mathsf{T}(t)\|e^{-\omega t} < \infty$. Given any $q \in Q_{\mathbb{A}}$ with $\operatorname{Re}(q) > \omega$, we have that $q \in \rho_S(\mathsf{A})$ and*

$$\mathsf{Q}_q(\mathsf{A})^n x = (-1)^n \int_0^\infty \mathsf{T}(t) \exp'_s(-t, q)^{\star n} x \, dt \quad (5.1)$$

where $\exp'_s(-t, q)^{\star n} \in \mathbb{A}$ indicates the value of $((\exp^{-t})'_s)^{\star n}$ at q .

Moreover for every $q \in Q_{\mathbb{A}}$ with $\operatorname{Re}(q) > \omega$ we have

$$\|\mathsf{Q}_q(\mathsf{A})^n\| \leq \frac{M}{(\operatorname{Re}(q) - \omega)^{2n}} \quad \forall n \in \mathbb{N} \setminus \{0\}. \quad (5.2)$$

Proof. Formula (5.1) follows immediately from n applications of Theorem 3.5 and Lemma 5.1. In order to prove (5.2) let us observe that, given $q \in Q_{\mathbb{A}}$, if $a = \operatorname{Re}(q)$ and $b = |\operatorname{Im}(q)|$, and $g(t) = -\exp'_s(-t, q)$ for $t \geq 0$, then

$$\begin{aligned}
|(g \star g)(t)| &\leq \int_0^t |g(t-s)||g(s)| \, ds \\
&\leq \int_0^t (t-s)e^{-(t-s)a} s e^{-sa} \, ds \\
&= e^{-ta} \int_0^t s(t-s) \, ds = \frac{1}{2 \cdot 3} t^3 e^{-ta}.
\end{aligned}$$

Let us assume by induction that $|g^{*(n-1)}| \leq ((2n-3)!)^{-1} t^{2n-3} e^{-at}$. Therefore for every n

$$\begin{aligned} |g^{*n}(t)| &\leq \int_0^t |g(t-s)| |g^{*(n-1)}(s)| \, ds \\ &\leq \frac{1}{(2n-3)!} \int_0^t (t-s) e^{-(t-s)a} s^{2n-3} e^{-as} \, ds \\ &= \frac{e^{-ta}}{(2n-3)!} \int_0^t (t-s) s^{2n-3} \, ds \\ &= \frac{e^{-ta}}{(2n-3)!} \frac{t^{2n-1}}{(2n-2)(2n-1)} = \frac{t^{2n-1} e^{-ta}}{(2n-1)!}. \end{aligned}$$

Thus $|g^{*n}(t)| \leq \frac{1}{(2n-1)!} t^{2n-1} e^{-ta}$ for every $t \geq 0$ and every $n \in \mathbb{N} \setminus \{0\}$ and, recalling that $\int_0^\infty t^{2n-1} e^{-t} \, dt = (2n-1)!$, we have

$$\begin{aligned} \|Q_q(\mathbf{A})^n\| &\leq \int_0^\infty \|T(t)\| |g^{*n}(t)| \, dt \\ &\leq \int_0^\infty M e^{\omega t} \frac{1}{(2n-1)!} t^{2n-1} e^{-ta} \, dt \\ &= \frac{M}{(2n-1)!} \int_0^\infty t^{2n-1} e^{-(a-\omega)t} \, dt \\ &= \frac{M}{(2n-1)!} \int_0^\infty \frac{t^{2n-1}}{(a-\omega)^{2n}} e^{-t} \, dt \\ &= \frac{M}{(a-\omega)^{2n}}, \end{aligned}$$

and we are done. \square

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