On imitation dynamics in population games with Markov switching

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Abstract— **Imitation dynamics in population games are a class of evolutionary game-theoretic models, widely used to study decision-making processes in social groups. Different from other models, imitation dynamics allow players to have minimal information on the structure of the game they are playing, and are thus suitable for many different applications, including traffic management, marketing, and disease control. In this work, we study a general case of imitation dynamics where the structure of the game and the imitation mechanisms change in time due to external factors —such as weather conditions or social trends. These changes are modeled using a continuous-time Markov jump process. We present tools to identify the dominant strategy that emerges from the dynamics through methodological analysis of the function parameters. Numerical simulations are provided to support our theoretical findings.**

I. INTRODUCTION

Evolutionary game theory is a powerful mathematical paradigm, which was originally proposed to model and study the evolution of behaviors in economic and biological settings [1]–[5]. In the last few decades, evolutionary game theory found many applications in different fields, ranging from the modeling of business cycles, GDP growth, and interest rates [6] to predicting the decision on whether to vaccinate or not in social communities [7]. Imitation dynamics are a specific class of evolutionary game-theoretic models [3], [4]. Different from other classes of models, such as best-response dynamics [8] and logit choices [9], imitation dynamics require minimal information on the structure of the game, making them suitable to model many realistic scenarios in which the players have no complete information on all the strategies they can choose and the corresponding rewards. In fact, in imitation dynamics, players compare their current strategy and reward with those of the others players, and possibly imitate them, according to an imitation mechanisms that characterizes the dynamics. Specific choices for the imitation mechanism give rise to the well-known replicator equation [10] and pairwise proportional imitation [11], [12], which have been extensively studied.

The recent interest on imitation dynamics has produced a substantial body of theoretical findings on this class of models. Besides the studies of the specific cases of imitation dynamics mentioned above, theoretical analyses of more general cases of imitation dynamics have been performed for specific classes of games, including games with strategic substitutes and complements [13], public good games [14], and potential games [4], [15]. While the majority of such literature focuses on scenarios in which the players interact on a fully-mixed structure, promising theoretical extensions have been reported for networked structures [16], [17] and multi-population scenarios [4].

Most of the literature on imitation dynamics deals with scenarios in which the structure of the game is deterministic and time-invariant, and thus the reward that an individual receive is uniquely determined by the strategy that the individual chooses and the strategies selected by the rest of the population. However, in many real-world scenarios, this assumption is quite simplistic, since external events that are independent of the population dynamics and of stochastic nature —such as weather conditions or social trends— may influence the reward that individuals receive for their strategies and, consequently, the imitation mechanisms. Switched systems, in which the payoff structure of the game (termed *mode*) may instantly change to another mode, according to a continuous-time Markov jump process, have emerged as a valuable modeling framework to capture these phenomena [18]. However, to the best of our knowledge, theoretical results on imitation dynamics with Markov switching are few, and limited to the analysis of specific cases, such as the replicator equation [19].

In this paper, we aim at extending the understanding of stochastic imitation dynamics by studying the scenario in which a population is anonymously engaged in a game, whose reward structure stochastically switches according to a continuous-time Markov jump process. The players in the population revise their strategies according to generic imitation dynamics. Besides the problem formulation, our main contribution is the extensive analysis of the scenario in which players can choose between two actions, and the payoff matrix switches between two possible modes. In our analysis, we focus on the characterization of the asymptotic behavior of the system, establishing conditions for almost sure global asymptotic stability for the system's equilibrium points. This work contributes to extending the understanding of stochastic imitation dynamics.

The rest of the paper is organized as follows. In Section II, we introduce continuous-time Markov jump systems, the concept of stochastic stability, population games, imitation mechanism, and strategies in game theory. In Section III we present the problem formulation, where we study a two population game problem as a particular case of the imitation population dynamics with continuous-time Markov switching. The main results are presented in Section IV, where a theorem is provided giving the conditions to characterize the long-run strategy behavior of the class of problems

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under study. Then, in Section V we show some simulation examples. The conclusion is presented in Section VI.

II. Preliminaries

A. Notation

We gather here the notational conventions used throughout the paper. Let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{N}_+ = \{1, 2, \dots\}$, $\mathbb{N} = \{0\} \cup \mathbb{N}_+$, and $\Gamma = \{1, 2, \dots, r\}$, where $r \in \mathbb{N}_+$ is a fixed positive integer. Given a set $\mathcal{A}$, we denote by $R^{\mathcal{A}}$ ($R^{\mathcal{A}\times\mathcal{A}}$) the space of real vectors (matrices), whose components are indexed by the elements in $\mathcal{A}$. For a vector $x \in \mathbb{R}^d$, $|x|$ denotes its Euclidean norm. The sign function is denoted by sgn.

B. Markov Jump Systems

Consider the following generic nonlinear switched system

$$\dot{x}(t) = g^{\sigma(t)}\big(x(t)\big)\,, \qquad t \geq 0\,, \tag{1}$$

where $x(t) \in \mathbb{R}^d$. The functions $g^n : \mathbb{R}^d \to \mathbb{R}^d$ are locally Lipschitz for $n \in \Gamma$. The switching signal $\sigma : \mathbb{R}_+ \to \Gamma$ is a piecewise constant function, which is continuous from the right, specifying the index of the active subsystem (that is, the mode), i.e., $\sigma(t) = n \in \Gamma$ means that the system is in its nth mode at time $t \in \mathbb{R}_+$. In this paper, we consider switching signals that are governed by stochastic processes, and the corresponding switched systems are termed *randomly switched systems*. Specifically, we consider a class of Markov signals, under which the future switches depend on the past history of the process only through the current state. This is formally defined in the following.

Definition 1 (See [20]): The switching signal $\sigma(t)$ is said to be Markov, if for $\forall n \in \Gamma$ and $\Delta > 0$,

$$P\big(\sigma(t+\Delta) = n|\{\sigma(s)\}_{s\leq t}\big) = P\big(\sigma(t+\Delta) = n|\sigma(t)\big)\,. \tag{2}$$

A Markov switching signal $\sigma(t) \in \Gamma,\ t \geq 0$ is unequivocally defined by its initial condition $\sigma(0) = \sigma_0 \in \Gamma$, and its generator $Q \in \mathbb{R}_+^{\Gamma\times\Gamma}$, such that

$$P\big(\sigma(t+\Delta){=}m|\sigma(t){=}n\big) = \begin{cases} q_{nm}\Delta + o(\Delta), & n \neq m, \\ 1 + q_{nm}\Delta + o(\Delta), & n = m, \end{cases} \tag{3}$$

for any $\Delta > 0$, where $q_{nn} = -\sum_{m\neq n} q_{nm}$. If the matrix Q is irreducible, then the Markov switching signal has a unique stationary distribution, denoted by $\pi = (\pi_1, \pi_2, \dots, \pi_r)$ [20]. Set $q_n = |q_{nm}|, n \in \Gamma$.

Definition 2 (See [18]): Let $V^n(x) : \mathbb{R}^d \times \Gamma \to \mathbb{R}^+$ be a function that is differentiable with respect to the variable x, where x evolves according to Equation (1), and $\sigma(t) = n \in \Gamma$ is a Markov switching signal with generator Q. We define the infinitesimal operator L as

$$LV^n(x) = \frac{\partial V^n(x)}{\partial x}\dot{x} + \sum_{m\in\Gamma} q_{nm}V^m(x). \tag{4}$$

Finally, we provide now some notions of stability for the switched system in (1), which will be used throughout this paper.

Definition 3 (See [21], [22]): The randomly switched system in (1) is said to be asymptotically stable almost surely, if the following two properties hold simultaneously:

i) for any $\varepsilon \in (0, 1)$, there is a $\delta = \delta(\varepsilon) > 0$ such that when $|x_0| < \delta$

$$P\left(\sup_{t\geq 0} |x(t)| < \varepsilon\right) > 1 - \varepsilon\,; \tag{5}$$

ii) for any $h > 0$ and $\varepsilon' > 0$, there is a positive random variable $T(h, \varepsilon')$ such that $P(\sup_{t\geq T(h,\varepsilon')} |x(t)| < \varepsilon') = 1$, provided $|x_0| < h$.

C. Population games

Consider a continuum of players engaged in an anonymous game whereby players choose actions from a finite set $\mathcal{A}$. Without any loss of generality we shall assume that the total mass of the players' population is unitary. Let

$$\mathcal{X} := \Big\{x \in \mathbb{R}_+^{\mathcal{A}} : \sum_{i\in\mathcal{A}} x_i = 1\Big\} \tag{6}$$

denote the unitary simplex over $\mathcal{A}$, whose elements $x \in \mathcal{X}$ represent the state of the system. Specifically, the entry x_i represents the fraction of players playing action $i \in \mathcal{A}$.

In population games, we assume that individuals interact anonymously and that the reward for a player that plays action i depends only on the state of the system [4]. Hence, we define the (Lipschitz-continuous) reward function

$$r : \mathcal{X} \to \mathbb{R}^{\mathcal{A}}\,, \tag{7}$$

with the interpretation that its ith entry $r_i(x)$ is the reward of any player playing action i when the system is in state $x \in \mathcal{X}$. We will refer to the pair $(\mathcal{A}; r)$ as a (continuous) population game.

While our theoretical findings hold for the general case of any Lipschitz-continuous reward function, a classical choice, which will be used in the examples presented in this paper, are matrix games [23], where the reward function is

$$r(x) = Ax, \tag{8}$$

for some payoff matrix $A \in \mathbb{R}^{\mathcal{A}\times\mathcal{A}}$.

D. Imitation mechanism

In a fully-mixed scenario, if the system is in state $x \in \mathcal{X}$, for any two actions $i, j \in \mathcal{A}$, the product

$$x_i x_j \tag{9}$$

describes the rate at which players playing action i meet players playing action j. When such an interaction takes place, the player that is currently choosing action i gets informed of the existence of action j and its reward $r_j(x)$. The player compares it with their own reward $r_i(x)$, and decides whether to imitate, modifying their action from i to j, with a rate $f_{ij}(x)$. The functions $f_{ij}(x)$ are assumed to be nonnegative-valued and Lipschitz-continuous in x. They are assembled in a matrix function $f(x)$ called the imitation mechanism. The imitation rates $f_{ij}(x)$ are often but not necessarily assumed to be dependent on the reward difference $r_i(x) - r_j(x)$. We now are able to properly define the imitation dynamics.

Definition 4 (Imitation dynamics): A continuous-time imitation dynamics, for a population game $(\mathcal{A}; r)$, and imitation mechanism f is the dynamical system

$$\dot{x}_i = x_i \sum_{j \in \mathcal{A}} x_j \big(f_{ji}(x) - f_{ij}(x)\big), \tag{10}$$

for $i \in \mathcal{A}$.

A natural assumption that is often made for imitation dynamics is the following [17], which captures the tendency to revise strategies toward maximizing the reward by enforcing that imitation rates are greater in the direction of increasing rewards.

Assumption 1: Let us assume that the imitation rates $f_{ij}(x)$ satisfy the following property

$$sgn\big(f_{ij}(x) - f_{ji}(x)\big) = sgn\big(r_j(x) - r_i(x)\big). \tag{11}$$

We conclude this section by presenting a well-known example of imitation dynamics that satisfy Assumption 1.

Example 1 (Replicator dynamics): Define the imitation rates as the affine functions of the reward, that is,

$$f_{ij}(x) = c + r_j(x), \tag{12}$$

for some constant $c > -\min\{r_i(x) : x \in \mathcal{X}, i \in \mathcal{A}\}$, the imitation dynamics (10) read

$$\begin{aligned}\dot{x}_i &= x_i \sum_{j \in \mathcal{A}} x_j \big(f_{ji}(x) - f_{ij}(x)\big) \\ &= x_i \sum_{j \in \mathcal{A}} x_j \big(r_i(x) - r_j(x)\big).\end{aligned} \tag{13}$$

For $\mathcal{A} = \{1, 2\}$, the dynamics become

$$\dot{x}_i = x_i(1 - x_i)\big(r_i(x) - r_j(x)\big). \tag{14}$$

Equation (14) represents the replicator dynamics. Considering the payoff matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{15}$$

and recalling that $x_1 + x_2 = 1$, the dynamics above reduce to the first-order ordinary differential equation

$$\dot{x}_1 = x_1 x_2 \big((a - c)x_1 + (b - d)x_2\big). \tag{16}$$

As showed in [19], four different outcomes are possible:

1) **Strategy** 1 **dominates**: $a > c$ and $b > d$; then the system has two equilibria: $x_1 = 1$ is asymptotically stable and $x_1 = 0$ is unstable.
2) **Strategy** 2 **dominates**: $a < c$ and $b < d$; then the system has two equilibria: $x_1 = 1$ is unstable and $x_1 = 0$ is asymptotically stable.
3) **Coordination game**: $a > c$ and $d > b$; then the system has two equilibria $x_1 = 0$ and $x_1 = 1$, both are stable. Convergence to either $x_1 = 0$ or $x_1 = 1$ depends on the initial conditions;
4) **Mixed strategy dominates**: $a < c$ and $b > d$; then the system has three equilibria: $x_1 = 0$ and $x_1 = 1$, which are both unstable, and $x_1 = (\frac{a-c}{a-c+d-b}, \frac{d-b}{a-c+d-b})$, which is asymptotically stable.

Studying the conditions necessary to each outcome is a key step towards better understanding the dynamics related to different imitation mechanisms. We will see that it is also possible to identify one of the four outcomes described above for a more general case where the imitation rates change according to Markov jumps.

III. Problem Statement

We now generalize the imitation dynamics in Equation (10), by considering applications where the reward function $r(x)$ and, consequently, the imitation rates function $f(x)$ switch according to a Markov switching signal.

Definition 5 (Imitation dynamics with Markov switching): A (stochastic, continuous-time) imitation dynamics, for a population game $(\mathcal{A}; r)$, imitation mechanism f, and Markov switching signal $\sigma(t)$ is the dynamical system

$$\dot{x}_i = x_i \sum_{j \in \mathcal{A}} x_j \big(f_{ji}^{\sigma(t)}(x) - f_{ij}^{\sigma(t)}(x)\big), \tag{17}$$

for $i \in \mathcal{A}$ and $\sigma(t)$ defined as in (3).

In this paper, we perform a preliminary analysis of imitation dynamics with Markov switching. Specifically, we will study the asymptotic behavior of the system for the specific case of a 2-mode Markov process, that is, $\Gamma = \{1, 2\}$, and a 2-action population game, that is, $\mathcal{A} = \{1, 2\}$. Equation (17) becomes

$$\dot{x}_i = x_i x_j \big(f_{ji}^{\sigma(t)}(x) - f_{ij}^{\sigma(t)}(x)\big). \tag{18}$$

Observe that $x_1 + x_2 = 1$ and without loss of generality, we will focus our analysis on the dynamics of the fraction of 1-players, since the dynamics for the fraction of 2-players can be easily derived using the relation above. The main equation of study becomes

$$\dot{x}_1 = x_1(1 - x_1)\big(f_{21}^{\sigma(t)}(x) - f_{12}^{\sigma(t)}(x)\big). \tag{19}$$

From (19), we intuitively observe that $x_1 = 0$ and $x_1 = 1$ are always equilibria of the system, while the last term of the equation may yield mixed-strategy equilibria. However, under Assumption 1, it is clear that the last term can be equal to 0 only if $r_1(x) = r_2(x)$. Hence, the four asymptotic outcomes described for the deterministic replicator equation can still be observed. However, the stochastic nature of the dynamical process and the presence of generic imitation rates complicate the analysis of the dynamics, since other outcomes including oscillations due to the fact that switching behavior may emerge. In this paper, we provide sufficient conditions for the model parameters to ensure the emergence of the four different convergence outcomes described in the previous section for the deterministic replicator dynamics. To the authors' knowledge, a similar analysis has been performed only for the particular case of the replicator equation in [19]. Thus, this work generalizes the existing results in the literature, providing theoretical tools to properly analyze Equation (19), shedding light on the role of the imitation rate function f and of the Markov transition rate matrix Q in determining the behavior of the system.

IV. Results

This section presents the main results of this work. Our first contribution lies in a stability analysis theorem. Then, this general result will be used to derive a corollary that gives conditions on how to forecast the steady-state behavior of the imitation dynamics in population games with Markov switching.

The first result is summarized in the following theorem, which generalizes the results in [19] by discussing stability or instability of the two equilibria $x_1 = 0$ and $x_1 = 1$.

Theorem 1: Consider the imitation dynamics with Markov switching in Equation (19), where $\mathcal{A} = \{1, 2\}$ and $\Gamma = \{1, 2\}$. Let q_{nm} be the elements of the transition rate matrix. Then, for each mode $n \in \Gamma$, we denote by $m = 3 - n$ the other mode and we define the following four conditions:

i) $1 - \alpha c_n > 0$ and $f^n_{21}(0, 1) - f^n_{12}(0, 1) + q_{nm}\frac{c_n - c_m}{1 - \alpha c_n} < 0$
ii) $1 + \alpha c_n > 0$ and $f^n_{21}(0, 1) - f^n_{12}(0, 1) + q_{nm}\frac{c_n - c_m}{1 + \alpha c_n} > 0$
iii) $1 + \alpha c_n > 0$ and $f^n_{21}(1, 0) - f^n_{12}(1, 0) + q_{nm}\frac{c_n - c_m}{1 + \alpha c_n} > 0$
iv) $1 - \alpha c_n > 0$ and $f^n_{21}(1, 0) - f^n_{12}(1, 0) + q_{nm}\frac{c_n - c_m}{1 - \alpha c_n} < 0$,

with $0 < \alpha < 1$, and c_1 and c_2 are two constants.

Then, the following holds true:

1) if condition i) holds for all $n \in \Gamma$, then $x_1 = 0$ is asymptotically stable almost surely;
2) if condition ii) holds for all $n \in \Gamma$, then $x_1 = 0$ is unstable almost surely;
3) if condition iii) holds for all $n \in \Gamma$, then $x_1 = 1$ is asymptotically stable almost surely;
4) if condition iv) holds for all $n \in \Gamma$, then $x_1 = 1$ is unstable almost surely.

Proof: The analysis will be carried using the Lyapunov method for Markov jump functions. This is similar to the work of [19], but here we are applying the idea to a more general case. Thus, from Definition 2, and observing that $m = 3 - n$, for $n = 1, 2$, we have

$$LV^n(x) = V^n_x(x)\dot{x} + q_{nm}\big(V^n(x) - V^m(x)\big). \tag{20}$$

Substituting (19) in (20), we have

$$\begin{aligned} LV^n(x_1) =& V^n_x(x_1)x_1(1 - x_1)\big(f^n_{21}(x) - f^n_{12}(x)\big) \\ &+ q_{nm}\big(V^n(x_1) - V^m(x_1)\big). \end{aligned} \tag{21}$$

Borrowing and adjusting the Lyapunov function defined in [24], [25], we give conditions for 0 and 1 to be stochastically stable or unstable. Take $0 < \alpha < 1$, and constants c_1 and c_2. These constants are not unique and will be taken accordingly in order to establish stability or instability of a particular vertex. Define the four positive Lyapunov functions as $V^{n\pm}_0(x) = (1 \mp \alpha c_n)x^{\pm\alpha}$, and $V^{n\pm}_1(x) = (1 \pm \alpha c_n)(1 - x)^{\pm\alpha}$.

Applying the infinitesimal generator for each function as defined in (21) results in

$$\begin{aligned} LV^{n+}_0(x_1) =& \alpha(1 - \alpha c_n)x^\alpha_1\Big\{(1 - x - 1)\big(f^n_{21}(x) - f^n_{12}(x)\big) \\ &+ q_{nm}\frac{c_n - c_m}{1 - \alpha c_n}\Big\} \end{aligned} \tag{22}$$

$$\begin{aligned} LV^{n-}_0(x_1) =& -\alpha(1 + \alpha c_n)x^{-\alpha}_1\Big\{(1 - x_1)\big(f^n_{21}(x) - f^n_{12}(x)\big) \\ &+ q_{nm}\frac{c_n - c_m}{1 + \alpha c_n}\Big\} \end{aligned} \tag{23}$$

$$\begin{aligned} LV^{n+}_1(x_1) =& -\alpha(1 + \alpha c_n)(1 - x_1)^\alpha\Big\{x_1\big(f^n_{21}(x) - f^n_{12}(x)\big) \\ &+ q_{nm}\frac{c_n - c_m}{1 + \alpha c_n}\Big\} \end{aligned} \tag{24}$$

$$\begin{aligned} LV^{n-}_1(x_1) =& \alpha(1 - \alpha c_n)(1 - x_1)^{-\alpha}\Big\{x_1\big(f^n_{21}(x) - f^n_{12}(x)\big) \\ &+ q_{nm}\frac{c_n - c_m}{1 - \alpha c_n}\Big\}. \end{aligned} \tag{25}$$

If there exist a neighborhood (within the simplex) around $x_1 = 0$ or $x_1 = 1$ such that $LV^{n\pm}_0(x_1) \leq 0$ or $LV^{n\pm}_1(x_1) \leq 0$, then $V\big(X(t)\big)$ is a supermartingale in these neighborhoods [19]. Thus, the objective is to find conditions in Equations (22)-(25) around $x_1 = 0$ and $x_1 = 1$ that makes possible that the function is a supermatingale.

Thus, if we analyze at the vertices $0, 1$, we have that one of the four conditions stated in Theorem 1 should hold for each Markov mode.

Let us examine if there is any set of variables where condition (i) holds. First, since $0 < \alpha < 1$, there exist c_n such that $1 - \alpha c_n > 0$. Now, since $q_{nm} > 0$, we need to show that $\frac{(f^n_{12}(0,1) - f^n_{21}(0,1))}{c_n - c_m} > 0$ is possible to occur in every Markov mode. Hence

$$\frac{\big(f^n_{12}(0, 1) - f^n_{21}(0, 1)\big)}{c_n - c_m} > 0 \tag{26}$$

$$\frac{\big(f^m_{12}(0, 1) - f^m_{21}(0, 1)\big)}{c_m - c_n} > 0. \tag{27}$$

Thus, we need

$$\begin{aligned} sgn(c_n - c_m) &= sgn\big(f^n_{12}(0, 1) - f^n_{21}(0, 1)\big) \\ &= -sgn\big(f^m_{12}(0, 1) - f^m_{21}(0, 1)\big). \end{aligned} \tag{28}$$

This show that there exist α, c_n, $f^n_{12}(0, 1)$, $f^n_{21}(0, 1)$, for $n = \{1, 2\}$ such that condition (i) is feasible.

The feasibility analysis of conditions (ii) and (iii) are trivial, since $q_{nm} > 0$ is the only mandatory condition, and thus it is always possible to find a set of values in which

$$q_{nm} > \big(f^n_{21}(x) - f^n_{12}(x)\big)\frac{1 + \alpha c_n}{c_n - c_m}, \tag{29}$$

with $x = \{(0, 1), (1, 0)\}$. Similarly to the analysis of item (i), the feasibility conditions for item (iv) are

$$\begin{aligned} sgn(c_n - c_m) &= sgn\big(f^n_{12}(1, 0) - f^n_{21}(1, 0)\big) \\ &= -sgn\big(f^m_{12}(1, 0) - f^m_{21}(1, 0)\big). \end{aligned} \tag{30}$$

Thus, conditions (i)-(iv) are applicable to any function of type (19) and attend the Lyapunov stability criterion. This completes the proof. ■

Taking the conditions established in Theorem 1 we derive some conclusions on the long-run behavior of the Markov jump imitation dynamics. We can now state a corollary that give information, similar to what happens in the replicator dynamics, about the four different behaviors that can emerge in this more general scenario.

Corollary 1: The following strategy arises depending on which conditions of Theorem 1 are satisfied

1) **Strategy 2 dominates**: if conditions (i) and (iv) hold, then the system has two equilibria: $x_1 = 1$ is unstable and $x_1 = 0$ is asymptotically stable a.s.
2) **Strategy 1 dominates**: if conditions (ii) and (iii) hold, then the system has two equilibria: $x_1 = 1$ is asymptotically stable a.s. and $x_1 = 0$ is unstable.
3) **Coordination game**: if conditions (i) and (iii) hold, then the system has two equilibria $x_1 = 0$ and $x_1 = 1$, both are a.s. asymptotically stable.
4) **Mixed strategy dominates**: if conditions (ii) and (iv) hold, then both $x_1 = 0$ and $x_1 = 1$ are unstable.

Proof: The proof is a direct consequence of Theorem 1 and the proving argument is very similar to the one adopted for the replicator dynamics in [19] and is omitted here. ■

Corollary 1 give the conditions for analyzing a Markov jump imitation dynamics equation and predicting the asymptotic behavior of the system. In fact, if Assumption 1 is verified, then we can use Corollary 1 to conclude that, in scenario 1), the system converges to $x_1 = 0$ a.s.; in 2), it converges to $x_1 = 1$ a.s.; in scenario 3), both these events are possible, depending on the initial condition; while in scenario 4), the imitation system does not converge to a pure strategy a.s., and both strategies will be played in the steady-state. This is a new result that generalizes the analyses of the resulting steady-state of the system for a broader class of problems, where the activity function can be chosen according to the particular applications of each problem.

V. Examples

This section will give some examples on how to analyze some imitation dynamics according to its activity function. We will present the already mentioned replicator dynamics, the pairwise proportional imitation, and the sigmoid imitation mechanism. In all the cases, the mode determines a change in the reward function, while the imitation mechanism is the same in the two modes.

A. Replicator equation

Define the imitation rates as affine functions of the reward, that is,

$$f_{ij}^{\sigma(t)}(x) = c^{\sigma(t)} + r_j^{\sigma(t)}(x) \tag{31}$$

for some constant $c^{\sigma(t)} > -\min\{r_i^{\sigma(t)}(x) : x \in \mathcal{X}, i \in \mathcal{A}\}$. The imitation dynamics (17) read

$$\begin{aligned} \dot{x}_i &= x_i \sum_{j\in\mathcal{A}} x_j \big(f_{ji}^{\sigma(t)}(x) - f_{ij}^{\sigma(t)}(x)\big) \\ &= x_i \sum_{j\in\mathcal{A}} x_j \big(r_i^{\sigma(t)}(x) - r_j^{\sigma(t)}(x)\big). \end{aligned} \tag{32}$$

For $\mathcal{A} = \{1, 2\}$, the dynamics become

$$\dot{x}_i = x_i(1 - x_i)\big(r_i^{\sigma(t)}(x) - r_j^{\sigma(t)}(x)\big). \tag{33}$$

The work of [19] showed under which conditions the system (33) is asymptotically stable in probability, as a particular case of Theorem 1.

B. Pairwise proportional imitation

Consider the imitation mechanism

$$f_{ij}^{\sigma(t)}(x) = \max\{r_j^{\sigma(t)}(x) - r_i^{\sigma(t)}(x),\ 0\}, \tag{34}$$

proposed in [4]. For a single fully mixed population, (34) simplifies to the replicator equation (14) and can also be studied using Theorem (1).

C. Sigmoid imitation

Let

$$f_{ij}^{\sigma(t)}(x) = \frac{1}{1 + exp\{-K_{ij}^{\sigma(t)}\big(r_j^{\sigma(t)}(x) - r_i^{\sigma(t)}(x)\big)\}}, \tag{35}$$

where $K_{ij}^{\sigma(t)} > 0$ are constants possibly different for each pair of actions (i, j) in $\mathcal{A}\times\mathcal{A}$. This is an extension including Markov jumps in the logistic function often used in the literature to model learning curves and adoption of innovation [26]. This framework can also be used with other sigmoid functions, such as the hyperbolic tangent or the arctangent.

Define $\alpha_{ij}^{\sigma(t)}(x) := exp\{-K_{ij}^{\sigma(t)}(r_j^{\sigma(t)}(x) - r_i^{\sigma(t)}(x))\}$ and observe that

$$\begin{aligned} f_{ji}^{\sigma(t)}(x) - f_{ij}^{\sigma(t)}(x) &= \frac{1}{1 + \alpha_{ji}^{\sigma(t)}(x)} - \frac{1}{1 + \alpha_{ij}^{\sigma(t)}(x)} \\ &= \frac{\alpha_{ij}^{\sigma(t)}(x) - \alpha_{ji}^{\sigma(t)}(x)}{\big(1 + \alpha_{ji}^{\sigma(t)}(x)\big)\big(1 + \alpha_{ij}^{\sigma(t)}(x)\big)}. \end{aligned} \tag{36}$$

Since the denominator is always positive, we have that $sgn(f_{ji}^{\sigma(t)}(x) - f_{ij}^{\sigma(t)}(x)) = sgn(\alpha_{ij}^{\sigma(t)}(x) - \alpha_{ji}^{\sigma(t)}(x))$.

Because $\alpha_{ij}^{\sigma(t)}(x) - \alpha_{ji}^{\sigma(t)}(x)$ can be positive or negative, depending on $K_{ij}^{\sigma(t)}, r_j^{\sigma(t)}$, and $r_i^{\sigma(t)}$, we know that the set of solutions that holds for the conditions (i)-(iv) may not be empty. Thus, for example, in condition (i), we should have $sgn(c_n - c_m) = sgn\big(\alpha_{12}^n(0, 1) - \alpha_{21}^n(0, 1)\big) = -sgn\big(\alpha_{12}^m(0, 1) - \alpha_{21}^m(0, 1)\big)$. It becomes clear that Theorem 1 can be used to study the stability of the sigmoid imitation equation.

For the special case where $K_{12}^n = K_{21}^n = K^n$, define $\beta_{ij}^n(x) = exp\{K^n\big(r_i^n(x) - r_j^n(x)\big)\}$. Thus,

$$f_{ij}^n(x) = \frac{1}{1 + \beta_{ij}^n(x)} \tag{37}$$

$$f_{ji}^n(x) = \frac{1}{1 + \beta_{ji}^n(x)} = \frac{\beta_{ij}^n(x)}{1 + \beta_{ij}^n(x)}. \tag{38}$$

This implies that

$$f_{ij}^n(x) - f_{ji}^n(x) = \frac{1 - \beta_{ij}^n(x)}{1 + \beta_{ij}^n(x)} := \gamma_{ij}^n(x). \tag{39}$$

For a two Markov mode matrix game, consider

$$A^n = \begin{pmatrix} a^n & b^n \\ c^n & d^n \end{pmatrix}. \quad (40)$$

Thus

$$\begin{aligned} \beta_{12}^n(x_1) &= exp\{K^n\big((b^n - d^n) + (a^n - c^n - b^n + d^n)x_1\big)\} \\ \beta_{12}^n(0) &= exp\{K^n(b^n - d^n)\} \\ \beta_{12}^n(1) &= exp\{K^n(a^n - c^n)\}. \end{aligned} \quad (41)$$

The conditions become

$$(i)\begin{cases} 1 - \alpha c_n > 0 \\ 0 < q_{nm} < \gamma_{ij}^n(0)\frac{1-\alpha c_n}{c_n - c_m} \end{cases} \quad (42)$$

$$(ii)\begin{cases} 1 + \alpha c_n > 0 \\ \gamma_{ij}^n(0)\frac{1+\alpha c_n}{c_n - c_m} < q_{nm} \end{cases} \quad (43)$$

$$(iii)\begin{cases} 1 + \alpha c_n > 0 \\ \gamma_{ij}^n(1)\frac{1+\alpha c_n}{c_n - c_n} < q_{nm} \end{cases} \quad (44)$$

$$(iv)\begin{cases} 1 - \alpha c_n > 0 \\ 0 < q_{nm} < \gamma_{ij}^n(1)\frac{1-\alpha c_n}{c_n - c_m} \end{cases} \quad (45)$$

Consider the following theoretical numerical example

$$\begin{aligned} &A^1 = \begin{pmatrix} 2 & 5 \\ 8 & 10 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 9 & 7 \\ 1 & 3 \end{pmatrix}, \\ &c_1 = 8, \quad c_2 = 3, \quad \alpha = 0.1, \quad K^n = 1. \end{aligned} \quad (46)$$

Then, we have

$$\begin{bmatrix} \gamma_{12}^1(0) & \gamma_{12}^2(0) \\ \gamma_{12}^1(1) & \gamma_{12}^2(1) \end{bmatrix} = \begin{bmatrix} 0.9866 & -0.9640 \\ 0.9951 & -0.9993 \end{bmatrix}. \quad (47)$$

Applying to the conditions

$$(i)\begin{cases} 0 < q_{12} < 0.0395 \\ 0 < q_{21} < 0.1350 \end{cases} \quad (48)$$

$$(ii)\begin{cases} 0.3552 < q_{12} \\ 0.2506 < q_{21} \end{cases} \quad (49)$$

$$(iii)\begin{cases} 0.3582 < q_{12} \\ 0.2598 < q_{21} \end{cases} \quad (50)$$

$$(iv)\begin{cases} 0 < q_{12} < 0.0398 \\ 0 < q_{21} < 0.1399 \end{cases}, \quad (51)$$

for a transition rate matrix

$$Q = \begin{pmatrix} -0.03 & 0.03 \\ 0.1 & -0.1 \end{pmatrix}, \quad (52)$$

the system satisfies condition (i) and (iv) and strategy 2 dominates. Figure 1 shows a time simulation of $t = 50$ where the figure above describes the $x_1(t)$ and the figure below shows in which mode the system is at each time. Observe that even with many transitions governed by a Markov jump chain between modes 1 and 2, the state $x_1(t)$ converges to 0, as expected since the system is strategy 2 dominant.

On the other hand, if the transition rate matrix is given by

$$Q = \begin{pmatrix} -0.8 & 0.8 \\ 0.5 & -0.5 \end{pmatrix}, \quad (53)$$

the system satisfies condition (ii) and (iii) and strategy 1 dominates. Figure 2 shows the numerical simulation for this case.

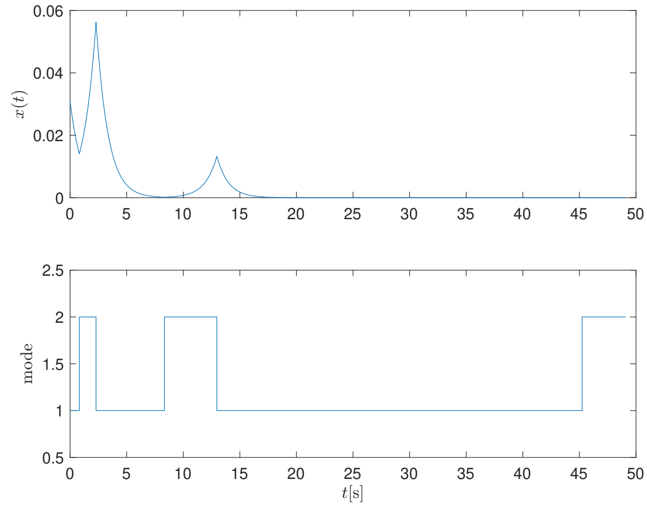

Fig. 1. Simulation of the imitation dynamics with Markov switching with transition rate matrix Q in (52) supports the analytical predictions from Corollary 1 that strategy 2 dominates and the system converges to the equilibrium $x_1^* = 0$. The upper panel shows the temporal evolution of the state of the system $x_1(t)$; the lower panel depicts the temporal evolution of the mode $\sigma(t)$.

VI. Conclusion

This paper presents tools to identify the steady-state behavior of imitation dynamics with Markov switching applied to a two population game. This is an extension of previous works where the rate function represents the replicator dynamics. The proof is made considering the Lyapunov method applied to Markov jump systems. To identify which strategy will be the outcome of the system, it is important to better understand how to use this tool in applications such as traffic and disease control. Future work will focus on extending the results to multi-population games [4] and also to network imitation dynamics [17] with Markov jumps in population games.

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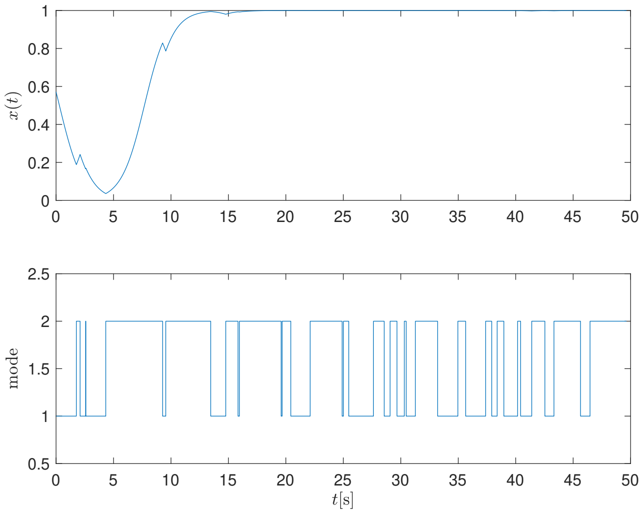

Fig. 2. Simulation of the imitation dynamics with Markov switching with transition rate matrix Q in (53), where Corollary 1 predicts that strategy 2 dominates and the system converges to the equilibrium $x_1^* = 1$. The numerical simulation is consistent with the analytical predictions. The upper panel shows the temporal evolution of the state of the system $x_1(t)$; the lower panel depicts the temporal evolution of the mode $\sigma(t)$.

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