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***Abstract*—In this paper, we examine the asymptotic behavior of an equation that describes two rotors installed on a common oscillating platform. Namely, we establish analytic criteria for self-synchronization of the rotors by means of the Popov method of "a priori integral indices".**

***Index Terms*—Synchronization, stability of nonlinear systems, vibrational mechanics**

I. Introduction

In this paper, we study an equation that originally appeared to explain the phenomenon of self-synchronization of two vibro-exciters (eccentric rotors rotated by asynchronous electric motors) installed on a common rigid platform with one degree of freedom. The rotors can synchronize without additional mechanical couplings between them; this phenomenon was first described in [1] and has inspired numerous engineering applications [2]. This motivates the analysis of mathematical models explaining phenomena of self-synchronization [3]–[5].

In this paper we adopt the mathematical model introduced in [6], which is obtained from the mechanical equations of motion by separating slow and fast dynamics. The problem in question is to find conditions under which asymptotic synchronization of the slow components of the rotors' frequencies is guaranteed. The deviation between two phases in the model from [6] is governed by an integro-differential Volterra equation with a periodic nonlinearity, which enables the application of Popov's method of a "priori integral indices" [7].

Popov's method, however, is primarily intended to investigate global stability of a unique equilibrium, whereas the system at hand is featured by an infinite set of equilibria. To cope with stability of equilibria set in systems with periodic nonlinearities, Popov's stability theory has been generalized in our previous works [8]–[10]. In this paper, we employ the nonlocal reduction technique [11], which exploits the information about stability of low-order systems and a special procedure [12] which employs periodic functions with zero mean integral value. Our main results are new analytic criteria for synchronization of two rotors, or, equivalently, convergence of every trajectory to the synchronous manifold.

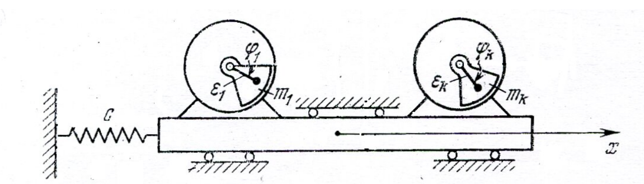

Fig. 1. Rotors situated on a rigid platform

II. The statement of the problem

Consider a system of $k = 2$ vibro-exciters (rotors) installed on a rigid platform and driven by asynchronous electric motors. The platform with one degree of freedom [1], [2] can move along the axis Ox and is connected to the stationary support by a spring. The rotors's axes are orthogonal to Ox.

The equations of motion of this system are as follows [2]:

$$I_i\ddot{\varphi}_i = L_i(\dot{\varphi}_i) + m_i\varepsilon_i\ddot{x}\sin\varphi_i \quad (i = 1, 2, \dots, n), \qquad (1)$$

$$M\ddot{x} = -cx + \sum_{i=1}^{n} m_i\varepsilon_i(\dot{\varphi}_i^2\cos\varphi_i + \ddot{\varphi}_i\sin\varphi_i)$$
$$(M = M_0 + \sum_{i=1}^{n} m_i;\ I_i = J_i + m_i\varepsilon_i^2). \qquad (2)$$

Here x is the displacement of the platform, φ_i $(i = 1, 2)$ is the angle of the i-th rotor counted from Ox-axis. The constants J_i, m_i, ε_i $(i = 1, 2)$ stand for, respectively, the i-th rotor's moment of inertia, mass and eccentricity, M_0 is the mass of the platform, c is the elasticity coefficient of the spring. The the rotation torque of the motor $L_i(\dot{\varphi}_i)$ is computed as

$$L_i = L_i^0 - k_i\dot{\varphi}_i \quad (L_i^0, k_i = const). \qquad (3)$$

To system (1)-(3) the method of "direct partition" of motion can be applied. The method separates the "slow" and the "fast" components of the motion. According to [6] we suppose that

$$\varphi_i(t) = \Omega t + \alpha_i(t) + \Psi_i(t, \Omega t) \quad (\Omega = const) \tag{4}$$

$$\frac{1}{2\pi} \int_0^{2\pi} \Psi_i(t, \Omega t)\, d\Omega t = 0. \tag{5}$$

Here α_i and Ψ_i are, respectively, the slow and the fast components of φ_i.

We also assume that Ψ_i is small and in the equation (2) we have

$$\dot{\varphi}_i \approx \Omega, \quad \ddot{\varphi}_i \approx 0. \tag{6}$$

By virtue of (3) equation (1) transforms into

$$I_i \ddot{\varphi}_i + k_i \dot{\varphi}_i = L_i^0 + m_i \varepsilon_i \ddot{x} \sin \varphi_i \quad (i = 1, 2). \tag{7}$$

Equation (2) is transformed by means of (6) and (4) where $\Psi_i(t, \Omega t)$ is assumed to be negligibly small. We have

$$M\ddot{x} + cx = \sum_{i=1}^{2} f_i \cos(\Omega t + \alpha_i) \quad (f_i = m_i \varepsilon_i \Omega^2). \tag{8}$$

Following [6] we consider the solution of linear non-homogeneous equation (8) and substitute it into (7). Then by virtue of (5) and (4) we obtain from (7) the equations for slow components:

$$I_i \ddot{\alpha}_i + k_i \dot{\alpha}_i = k_i(\Omega_i - \Omega) + V_i \ \ (i = 1, 2) \tag{9}$$

with

$$\Omega_i := \frac{L_i^0}{k_i},$$

$$V_i := -\frac{A_{xx} f_i}{2\pi} \int_0^{2\pi} \sum_{s=1}^{2} f_s \cos(\Omega t + \alpha_s) \sin(\Omega t + \alpha_i)\, d(\Omega t), \tag{10}$$

where

$$A_{xx} = \frac{1}{M(\omega^2 - \Omega^2)}, \quad \omega^2 = \frac{c}{M}. \tag{11}$$

Since

$$\frac{1}{2\pi} \int_0^{2\pi} \cos(\Omega t + \alpha_s) \sin(\Omega t + \alpha_i)\, d(\Omega t) = \\ = \frac{1}{2} \sin(\alpha_i - \alpha_s), \tag{12}$$

one has from (9)

$$\begin{cases} I_1 \ddot{\alpha}_1 + k_1 \dot{\alpha}_1 = k_1(\Omega_1 - \Omega) - \frac{1}{2} A_{xx} f_1 f_2 \sin(\alpha_1 - \alpha_2), \\ I_2 \ddot{\alpha}_2 + k_2 \dot{\alpha}_2 = k_2(\Omega_2 - \Omega) - \frac{1}{2} A_{xx} f_2 f_1 \sin(\alpha_2 - \alpha_1). \end{cases} \tag{13}$$

Note that the equilibrium $\alpha_i = const \, (i = 1, 2)$ may exist only if

$$\Omega = \frac{k_1 \Omega_1 + k_2 \Omega_2}{k_1 + k_2}. \tag{14}$$

So for slow components of φ_i we have obtained the system

$$\begin{cases} I_1 \ddot{\alpha}_1 + k_1 \dot{\alpha}_1 + A\Phi(\alpha_1 - \alpha_2) = 0, \\ I_2 \ddot{\alpha}_2 + k_2 \dot{\alpha}_2 - A\Phi(\alpha_1 - \alpha_2) = 0, \end{cases} \tag{15}$$

where

$$\Phi(\sigma) = \sin \sigma - \frac{\beta}{A}, \tag{16}$$

$$A = \frac{1}{2} A_{xx} f_1 f_2, \quad \beta = \frac{k_1 k_2 (\Omega_1 - \Omega_2)}{k_1 + k_2}. \tag{17}$$

We suppose that

$$|\frac{\beta}{A}| < 1. \tag{18}$$

The self-synchronization of the two vibro-exciters means that

$$\sigma(t) \triangleq \alpha_1(t) - \alpha_2(t) \to const, \ \dot{\sigma}(t) \to 0 \text{ as } t \to +\infty. \tag{19}$$

The goal of this paper is to establish conditions on the parameters of (15) which guarantee the self-synchronization of the rotors.

For the purpose we reduce the system (15) to Volterra integro-differential equation with respect to $\sigma(t)$:

$$\dot{\sigma}(t) = \sigma_0(t) - \int_0^t \Gamma(t - \tau) \Phi(\sigma(\tau))\, d\tau \tag{20}$$

with

$$\sigma_0(t) = \dot{\alpha}_1(0) e^{-\frac{k_1}{I_1} t} - \dot{\alpha}_2(0) e^{-\frac{k_2}{I_2} t}, \tag{21}$$

$$\Gamma(t) = A \left(\frac{1}{I_1} e^{-\frac{k_1}{I_1} t} - \frac{1}{I_2} e^{-\frac{k_2}{I_2} t} \right). \tag{22}$$

There exists a number of frequency-algebraic criteria which are destined for the investigation of asymptotic behavior of Volterra equation (20) (see [10] and references therein). The criteria have been proved by Popov method of a priori integral indices [7]. They are formulated in terms of the transfer function of the equation (20):

$$K_0(p) = A \left(\frac{1}{I_1 p + k_1} + \frac{1}{I_2 p + k_2} \right) \quad (p \in \mathbb{C}). \tag{23}$$

The specific character of equation (20) and of the nonlinear function $\Phi(\sigma)$ have brought about the employment of special Popov functionals, which have been generated exclusively, for systems with periodic nonlinearities (synchronization systems)

In the succeeding sections we demonstrate the conditions for self-synchronization obtained by the employment of various types of Popov functionals destined for synchronization systems.

III. The Bakaev-Guzh procedure

The main idea of Bakaev-Guzh procedure [8], [12] is to single out within a Popov functional a periodic function with zero mean integral value.

The advantage of the method for infinite dimensional MIMO system is described in detail in [10]. In [13] the results of [10] are simplified for SISO system. Consider the equation

$$\dot{\sigma}(t) = \alpha(t) - \int_0^t \gamma(t - \tau) \varphi(\sigma)\, d\tau \tag{24}$$

with

$$\alpha, \gamma : [0, \infty) \to \mathbb{R},\ \varphi : \mathbb{R} \to \mathbb{R}. \tag{25}$$

Assume that $\alpha \in C[0, \infty)$, γ is piece-wise continuous and

$$|\gamma(t)|, |\alpha(t)| < Me^{-rt}, \quad (M, r > 0). \tag{26}$$

Assume also that

$$\begin{gathered} \varphi \in C^1(\mathbb{R}),\ \varphi(\sigma) = \varphi(\sigma + \Delta) \quad (\Delta > 0), \\ \mu_1 \le \varphi'(\sigma) \le \mu_2 \ \ (\mu_1 \cdot \mu_2 < 0). \end{gathered} \tag{27}$$

The transfer function of (24) is as follows

$$K(p) = \int\limits_0^\infty \gamma(t) e^{-pt} \quad (p \in \mathbb{C}). \tag{28}$$

Theorem 1: [13] Suppose there exist numbers ε, δ, $\tau > 0$, $s_1 \le \mu_1$, $s_2 \ge \mu_2$ such that

1) the frequency-domain inequality is valid:

$$\begin{gathered} \pi_0(\omega) \triangleq Re\{K(i\omega) - \tau(K(i\omega) + s_1^{-1} i\omega)^* \cdot \\ \cdot (K(i\omega) + s_2^{-1} i\omega) - \varepsilon |K(i\omega)|^2 - \delta \ge 0, \ \ \forall \omega \ge 0, \end{gathered} \tag{29}$$

where symbol $*$ stands for the complex conjugation;

2)

$$\delta > \frac{\nu_0^2 \nu^2}{4(\varepsilon \nu_0^2 + \tau \nu^2)}, \tag{30}$$

where

$$\begin{gathered} \nu = \frac{\int\limits_0^\Delta \varphi(\sigma)\, d\sigma}{\int\limits_0^\Delta |\varphi(\sigma)|\, d\sigma}, \\ \nu_0 = \frac{\int\limits_0^\Delta \varphi(\sigma)\, d\sigma}{\int\limits_0^\Delta |\varphi(\sigma)| \sqrt{\left(1 - s_1^{-1}\varphi'(\sigma)\right)\left(1 - s_2^{-1}\varphi'(\sigma)\right)}\, d\sigma}. \end{gathered} \tag{31}$$

Then for every solution of (24) it is true that

$$\sigma(t) \to q, \ \ \varphi(q) = 0; \ \dot{\sigma}(t) \to 0 \text{ as } t \to +\infty. \tag{32}$$

Note that the choice of different couples $\{s_1, s_2\}$ may result in essentially different conditions for self-synchronization.

In paper [14] Theorem 1 has been applied to system (15) in case $s_1^{-1} = -1, s_2^{-1} = 1$. Here we consider the case of $s_1^{-1} = 0, s_2^{-1} = 1$. In this case we have

$$\begin{gathered} \nu = -\frac{\pi\beta}{2\left(\beta \arcsin \frac{\beta}{A} + \sqrt{A^2 - \beta^2}\right)}, \\ \nu_0 = -\frac{3\pi\beta}{2\sqrt{2}(3\beta + 2(A - \beta)^{\frac{3}{2}} A^{-\frac{1}{2}})}, \end{gathered} \tag{33}$$

$$\begin{gathered} \pi_0(\omega) = \omega^4(-\delta I_1^2 I_2^2 + A\tau I_1 I_2 (I_1 + I_2)) + \\ + \omega^2(A k_1 I_2^2 + A k_2 I_1^2 + A\tau(I_1 k_2^2 + I_2 k_1^2) - \\ - A^2(\varepsilon + \tau)(I_1 + I_2)^2 - \delta(I_1^2 k_2^2 + I_2^2 k_1^2)) + \\ + (A k_1 k_2^2 + A k_2 k_1^2 - A^2(\varepsilon + \tau)(k_1 + k_2)^2 - \delta k_1^2 k_2^2). \end{gathered} \tag{34}$$

The inequality (29) is valid for all $\omega \ge 0$ if the following inequalities are true:

$$\varepsilon + \tau \le \frac{k_1 k_2}{2A(k_1 + k_2)}, \tag{35}$$

$$\delta \le \frac{A(k_1 I_2^2 + k_2 I_1^2)}{2(k_1^2 I_2^2 + k_2^2 I_1^2)}, \tag{36}$$

$$\delta \le \frac{A\tau(I_1 + I_2)}{I_1 I_2}. \tag{37}$$

Introduce the constant

$$P \triangleq \frac{k_1 k_2 (k_1^2 I_2^2 + k_2^2 I_1^2)}{2(k_1 + k_2)(k_1 I_2^2 + k_2 I_1^2)} \cdot \frac{I_1 + I_2}{I_1 I_2}. \tag{38}$$

Theorem 2: The limit relations (19) are true for every solution of (20) provided that ether

$$A \le P, \ \frac{\nu_0^2 \nu^2}{(\nu_0^2 + \nu^2)} < \frac{k_1 k_2 (k_1 I_2^2 + k_2 I_1^2)}{2(k_1 + k_2)(k_1^2 I_2^2 + k_2^2 I_1^2)}, \tag{39}$$

or alternatively

$$A > P, \ \frac{\nu_0^2 \nu^2}{(\nu_0^2 + \nu^2)} < \frac{k_1^2 k_2^2 (I_2 + I_1)}{4A I_1 I_2 (k_1 + k_2)^2}. \tag{40}$$

Proof: We choose

$$\varepsilon = \tau = \frac{k_1 k_2}{4A(k_1 + k_2)}. \tag{41}$$

In the first case, let

$$\delta = \frac{A(k_1 I_2^2 + k_2 I_1^2)}{2(k_1^2 I_2^2 + k_2^2 I_1^2)}. \tag{42}$$

Then due to (39) we obtain that

$$\delta \le \frac{A\tau(I_1 + I_2)}{I_1 I_2}. \tag{43}$$

So the condition 1) of Theorem 1 is fulfilled. The condition 2) of Theorem 1 takes the form

$$4\varepsilon\delta = 4\tau\delta > \frac{\nu_0^2 \nu^2}{(\nu_0^2 + \nu^2)} \tag{44}$$

which follows from (39).

In the second case let

$$\delta = \frac{A\tau(I_1 + I_2)}{I_1 I_2} \tag{45}$$

Then it follows from (40) that (29) and (36) are true. Thus the Theorem is proved. ■

IV. Leonov's method of nonlocal reduction

The idea of nonlocal reduction [8], [11] is to "inject" in Popov functionals of infinite dimensional system the trajectories of stable system of low order.

Consider the equation

$$\ddot{\sigma} + a\dot{\sigma} + \varphi(\sigma) = 0 \tag{46}$$

with $\varphi(\sigma)$ described in previous section.

This equation has been exhaustively investigated (see [8] and references therein). It is well known that it has a bifurcational value $a_{cr}(\varphi)$ such that for $a > a_{cr}$ the limit relations (32) are true for its every solution.

The frequency-algebraic stability criterion exploiting for (24) the nonlocal reduction technique is as follows.

Theorem 3: [8] Suppose there exist numbers $s_1 \le \mu_1, s_2 \ge \mu_2,\ \varepsilon, \tau > 0,\ \lambda \in (0, r)$ such that the conditions are valid:

1) the frequency-domain inequality

$$\pi(\omega,\lambda) \triangleq Re\{K(i\omega-\lambda) - \tau(K(i\omega-\lambda) + s_1^{-1}(i\omega-\lambda))^* \cdot \\ \cdot (K(i\omega-\lambda) + s_2^{-1}(i\omega-\lambda))\} - \varepsilon|K(i\omega-\lambda)|^2 \ge 0 \tag{47}$$

is true for all $\omega \ge 0$,

2)

$$2\sqrt{\varepsilon\lambda} > a_{cr}(\varphi). \tag{48}$$

Then for every solution of (24) the limit relations (32) are true.

Next assertion describes the application of Theorem 3 to equation (20) in case $s_2 = -s_1 = 1$.

Introduce the function

$$f(x) = \frac{4x(k_1 - I_1x)(k_2 - I_2x)}{A\left((k_1 + k_2) - (I_1 + I_2)x\right)}. \tag{49}$$

Let

$$M \triangleq \sup_{x\in[0,r)} f(x) \tag{50}$$

where

$$r = \min\{\frac{k_1}{I_1}, \frac{k_2}{I_2}\} \tag{51}$$

Theorem 4:
If

$$M > a_{cr}^2(\Phi) \tag{52}$$

then for every solution of (20) the limits (19) are valid.

Proof: Note that

$$K(i\omega-\lambda) = A\Big(\frac{1}{\varkappa_1 + iI_1\omega} + \frac{1}{\varkappa_2 + iI_2\omega}\Big), \tag{53}$$

where

$$\varkappa_i = k_i - \lambda I_i \quad (i = 1, 2). \tag{54}$$

It is not difficult to establish by direct computation that if

$$\varepsilon + \tau = \frac{\varkappa_1\varkappa_2}{A(\varkappa_1 + \varkappa_2)}, \tag{55}$$

the condition 1) of Theorem 3 is satisfied.

We choose the value of λ such that

$$f(\lambda) = M \tag{56}$$

Then

$$4\lambda(\varepsilon + \tau) = M. \tag{57}$$

It is sufficient to choose

$$\tau < \frac{M - a_{cr}^2}{4\lambda} \tag{58}$$

so that condition 2) of Theorem is fulfilled. Theorem 4 is proved. ■

V. Conclusion

In this paper the problem of synchronization between two vibro-exciters (eccentric rotors) installed on a common oscillating platform is considered. By means of stability theory for "pendulum-like" systems we establish analytic criteria for self-synchronization of two rotors.

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