

New Criteria for Self-Synchronization of Two Unbalanced Vibro-Exciters

*Original*

New Criteria for Self-Synchronization of Two Unbalanced Vibro-Exciters / Smirnova, V.B., Proskurnikov, A.V., Utina, N.V.. - ELETTRONICO. - (2022), pp. 1-4. (16th International Conference on Stability and Oscillations of Nonlinear Control Systems (Pyatnitskiy's Conference) Moscow, Russian Federation 01-03 June 2022) [10.1109/STAB54858.2022.9807478].

*Availability:*

This version is available at: 11583/2969052 since: 2022-06-30T11:11:18Z

*Publisher:*

IEEE

*Published*

DOI:10.1109/STAB54858.2022.9807478

*Terms of use:*

This article is made available under terms and conditions as specified in the corresponding bibliographic description in the repository

*Publisher copyright*

IEEE postprint/Author's Accepted Manuscript

©2022 IEEE. Personal use of this material is permitted. Permission from IEEE must be obtained for all other uses, in any current or future media, including reprinting/republishing this material for advertising or promotional purposes, creating new collecting works, for resale or lists, or reuse of any copyrighted component of this work in other works.

(Article begins on next page)

# New Criteria for Self-Synchronization of Two Unbalanced Vibro-Exciters

1<sup>st</sup>Vera B. Smirnova  
*dept. of Mathematics*  
*Saint-Petersburg State University*  
*of Architecture and Civil Engineering*  
*and Faculty of Mathematics*  
*and Mechanics*  
*Saint-Petersburg State University*  
 Saint-Petersburg, Russia  
 smirnova\_vera\_b@mail.ru

2<sup>nd</sup>Anton V. Proskurnikov  
*Politecnico di Torino*  
 Turin, Italy  
 anton.p.1982@ieee.org

3<sup>rd</sup>Natalia V. Utina  
*dept. of Mathematics*  
*Saint-Petersburg State University*  
*of Architecture and*  
*Civil Engineering*  
 Saint-Petersburg, Russia  
 unv74@mail.ru

**Abstract**—In this paper, we examine the asymptotic behavior of an equation that describes two rotors installed on a common oscillating platform. Namely, we establish analytic criteria for self-synchronization of the rotors by means of the Popov method of “a priori integral indices”.

**Index Terms**—Synchronization, stability of nonlinear systems, vibrational mechanics

## I. INTRODUCTION

In this paper, we study an equation that originally appeared to explain the phenomenon of self-synchronization of two vibro-exciter (eccentric rotors rotated by asynchronous electric motors) installed on a common rigid platform with one degree of freedom. The rotors can synchronize without additional mechanical couplings between them; this phenomenon was first described in [1] and has inspired numerous engineering applications [2]. This motivates the analysis of mathematical models explaining phenomena of self-synchronization [3]–[5].

In this paper we adopt the mathematical model introduced in [6], which is obtained from the mechanical equations of motion by separating slow and fast dynamics. The problem in question is to find conditions under which asymptotic synchronization of the slow components of the rotors’ frequencies is guaranteed. The deviation between two phases in the model from [6] is governed by an integro-differential Volterra equation with a periodic nonlinearity, which enables the application of Popov’s method of a “p priori integral indices” [7].

Popov’s method, however, is primarily intended to investigate global stability of a unique equilibrium, whereas the system at hand is featured by an infinite set of equilibria. To cope with stability of equilibria set in systems with periodic nonlinearities, Popov’s stability theory has been generalized in our previous works [8]–[10]. In this paper, we employ the nonlocal reduction technique [11], which exploits the information about stability of low-order systems and a special procedure [12] which employs periodic functions with zero mean integral value. Our main results are new analytic criteria for synchronization of two rotors, or, equivalently, convergence of every trajectory to the synchronous manifold.

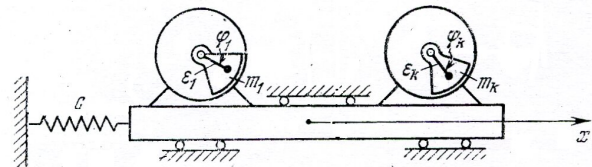


Fig. 1. Rotors situated on a rigid platform

## II. THE STATEMENT OF THE PROBLEM

Consider a system of  $k = 2$  vibro-exciter (rotors) installed on a rigid platform and driven by asynchronous electric motors. The platform with one degree of freedom [1], [2] can move along the axis  $Ox$  and is connected to the stationary support by a spring. The rotors’s axes are orthogonal to  $Ox$ .

The equations of motion of this system are as follows [2]:

$$I_i \ddot{\varphi}_i = L_i(\dot{\varphi}_i) + m_i \varepsilon_i \ddot{x} \sin \varphi_i \quad (i = 1, 2, \dots, n), \quad (1)$$

$$M \ddot{x} = -cx + \sum_{i=1}^n m_i \varepsilon_i (\dot{\varphi}_i^2 \cos \varphi_i + \ddot{\varphi}_i \sin \varphi_i) \quad (2)$$

$$(M = M_0 + \sum_{i=1}^n m_i; I_i = J_i + m_i \varepsilon_i^2).$$

Here  $x$  is the displacement of the platform,  $\varphi_i$  ( $i = 1, 2$ ) is the angle of the  $i$ -th rotor counted from  $Ox$ -axis. The constants  $J_i, m_i, \varepsilon_i$  ( $i = 1, 2$ ) stand for, respectively, the  $i$ -th rotor’s moment of inertia, mass and eccentricity,  $M_0$  is the mass of the platform,  $c$  is the elasticity coefficient of the spring. The rotation torque of the motor  $L_i(\dot{\varphi}_i)$  is computed as

$$L_i = L_i^0 - k_i \dot{\varphi}_i \quad (L_i^0, k_i = \text{const}). \quad (3)$$

To system (1)-(3) the method of “direct partition” of motion can be applied. The method separates the “slow” and the “fast”

components of the motion. According to [6] we suppose that

$$\varphi_i(t) = \Omega t + \alpha_i(t) + \Psi_i(t, \Omega t) \quad (\Omega = \text{const}) \quad (4)$$

$$\frac{1}{2\pi} \int_0^{2\pi} \Psi_i(t, \Omega t) d\Omega t = 0. \quad (5)$$

Here  $\alpha_i$  and  $\Psi_i$  are, respectively, the slow and the fast components of  $\varphi_i$ .

We also assume that  $\Psi_i$  is small and in the equation (2) we have

$$\dot{\varphi}_i \approx \Omega, \quad \ddot{\varphi}_i \approx 0. \quad (6)$$

By virtue of (3) equation (1) transforms into

$$I_i \ddot{\varphi}_i + k_i \dot{\varphi}_i = L_i^0 + m_i \varepsilon_i \ddot{x} \sin \varphi_i \quad (i = 1, 2). \quad (7)$$

Equation (2) is transformed by means of (6) and (4) where  $\Psi_i(t, \Omega t)$  is assumed to be negligibly small. We have

$$M \ddot{x} + cx = \sum_{i=1}^2 f_i \cos(\Omega t + \alpha_i) \quad (f_i = m_i \varepsilon_i \Omega^2). \quad (8)$$

Following [6] we consider the solution of linear non-homogeneous equation (8) and substitute it into (7). Then by virtue of (5) and (4) we obtain from (7) the equations for slow components:

$$I_i \ddot{\alpha}_i + k_i \dot{\alpha}_i = k_i(\Omega_i - \Omega) + V_i \quad (i = 1, 2) \quad (9)$$

with

$$\begin{aligned} \Omega_i &:= \frac{L_i^0}{k_i}, \\ V_i &:= -\frac{A_{xx} f_i}{2\pi} \int_0^{2\pi} \sum_{s=1}^2 f_s \cos(\Omega t + \alpha_s) \sin(\Omega t + \alpha_i) d(\Omega t), \end{aligned} \quad (10)$$

where

$$A_{xx} = \frac{1}{M(\omega^2 - \Omega^2)}, \quad \omega^2 = \frac{c}{M}. \quad (11)$$

Since

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \cos(\Omega t + \alpha_s) \sin(\Omega t + \alpha_i) d(\Omega t) &= \\ &= \frac{1}{2} \sin(\alpha_i - \alpha_s), \end{aligned} \quad (12)$$

one has from (9)

$$\begin{cases} I_1 \ddot{\alpha}_1 + k_1 \dot{\alpha}_1 = k_1(\Omega_1 - \Omega) - \frac{1}{2} A_{xx} f_1 f_2 \sin(\alpha_1 - \alpha_2), \\ I_2 \ddot{\alpha}_2 + k_2 \dot{\alpha}_2 = k_2(\Omega_2 - \Omega) - \frac{1}{2} A_{xx} f_2 f_1 \sin(\alpha_2 - \alpha_1). \end{cases} \quad (13)$$

Note that the equilibrium  $\alpha_i = \text{const}$  ( $i = 1, 2$ ) may exist only if

$$\Omega = \frac{k_1 \Omega_1 + k_2 \Omega_2}{k_1 + k_2}. \quad (14)$$

So for slow components of  $\varphi_i$  we have obtained the system

$$\begin{cases} I_1 \dot{\alpha}_1 + k_1 \alpha_1 + A \Phi(\alpha_1 - \alpha_2) = 0, \\ I_2 \dot{\alpha}_2 + k_2 \alpha_2 - A \Phi(\alpha_1 - \alpha_2) = 0, \end{cases} \quad (15)$$

where

$$\Phi(\sigma) = \sin \sigma - \frac{\beta}{A}, \quad (16)$$

$$A = \frac{1}{2} A_{xx} f_1 f_2, \quad \beta = \frac{k_1 k_2 (\Omega_1 - \Omega_2)}{k_1 + k_2}. \quad (17)$$

We suppose that

$$\left| \frac{\beta}{A} \right| < 1. \quad (18)$$

The self-synchronization of the two vibro-exciter means that

$$\sigma(t) \triangleq \alpha_1(t) - \alpha_2(t) \rightarrow \text{const}, \quad \dot{\sigma}(t) \rightarrow 0 \text{ as } t \rightarrow +\infty. \quad (19)$$

The goal of this paper is to establish conditions on the parameters of (15) which guarantee the self-synchronization of the rotors.

For the purpose we reduce the system (15) to Volterra integro-differential equation with respect to  $\sigma(t)$ :

$$\dot{\sigma}(t) = \sigma_0(t) - \int_0^t \Gamma(t - \tau) \Phi(\sigma(\tau)) d\tau \quad (20)$$

with

$$\sigma_0(t) = \dot{\alpha}_1(0) e^{-\frac{k_1}{I_1} t} - \dot{\alpha}_2(0) e^{-\frac{k_2}{I_2} t}, \quad (21)$$

$$\Gamma(t) = A \left( \frac{1}{I_1} e^{-\frac{k_1}{I_1} t} - \frac{1}{I_2} e^{-\frac{k_2}{I_2} t} \right). \quad (22)$$

There exists a number of frequency-algebraic criteria which are destined for the investigation of asymptotic behavior of Volterra equation (20) (see [10] and references therein). The criteria have been proved by Popov method of a priori integral indices [7]. They are formulated in terms of the transfer function of the equation (20):

$$K_0(p) = A \left( \frac{1}{I_1 p + k_1} + \frac{1}{I_2 p + k_2} \right) \quad (p \in \mathbb{C}). \quad (23)$$

The specific character of equation (20) and of the nonlinear function  $\Phi(\sigma)$  have brought about the employment of special Popov functionals, which have been generated exclusively, for systems with periodic nonlinearities (synchronization systems)

In the succeeding sections we demonstrate the conditions for self-synchronization obtained by the employment of various types of Popov functionals destined for synchronization systems.

### III. THE BAKAEV-GUZH PROCEDURE

The main idea of Bakaev-Guzh procedure [8], [12] is to single out within a Popov functional a periodic function with zero mean integral value.

The advantage of the method for infinite dimensional MIMO system is described in detail in [10]. In [13] the results of [10] are simplified for SISO system. Consider the equation

$$\dot{\sigma}(t) = \alpha(t) - \int_0^t \gamma(t - \tau) \varphi(\sigma) d\tau \quad (24)$$

with

$$\alpha, \gamma : [0, \infty) \rightarrow \mathbb{R}, \varphi : \mathbb{R} \rightarrow \mathbb{R}. \quad (25)$$

Assume that  $\alpha \in C[0, \infty)$ ,  $\gamma$  is piece-wise continuous and

$$|\gamma(t)|, |\alpha(t)| < Me^{-rt}, \quad (M, r > 0). \quad (26)$$

Assume also that

$$\begin{aligned} \varphi \in C^1(\mathbb{R}), \varphi(\sigma) = \varphi(\sigma + \Delta) \quad (\Delta > 0), \\ \mu_1 \leq \varphi'(\sigma) \leq \mu_2 \quad (\mu_1 \cdot \mu_2 < 0). \end{aligned} \quad (27)$$

The transfer function of (24) is as follows

$$K(p) = \int_0^{\infty} \gamma(t)e^{-pt} \quad (p \in \mathbb{C}). \quad (28)$$

**Theorem 1:** [13] Suppose there exist numbers  $\varepsilon, \delta, \tau > 0$ ,  $s_1 \leq \mu_1$ ,  $s_2 \geq \mu_2$  such that

1) the frequency-domain inequality is valid:

$$\begin{aligned} \pi_0(\omega) \triangleq \operatorname{Re}\{K(i\omega) - \tau(K(i\omega) + s_1^{-1}i\omega)^* \cdot \\ \cdot (K(i\omega) + s_2^{-1}i\omega) - \varepsilon|K(i\omega)|^2 - \delta \geq 0, \quad \forall \omega \geq 0, \end{aligned} \quad (29)$$

where symbol  $*$  stands for the complex conjugation;

2)

$$\delta > \frac{\nu_0^2 \nu^2}{4(\varepsilon \nu_0^2 + \tau \nu^2)}, \quad (30)$$

where

$$\begin{aligned} \nu &= \frac{\int_0^{\Delta} \varphi(\sigma) d\sigma}{\int_0^{\Delta} |\varphi(\sigma)| d\sigma}, \\ \nu_0 &= \frac{\int_0^{\Delta} \varphi(\sigma) d\sigma}{\int_0^{\Delta} |\varphi(\sigma)| \sqrt{(1 - s_1^{-1}\varphi'(\sigma))(1 - s_2^{-1}\varphi'(\sigma))} d\sigma}. \end{aligned} \quad (31)$$

Then for every solution of (24) it is true that

$$\sigma(t) \rightarrow q, \quad \varphi(q) = 0; \quad \dot{\sigma}(t) \rightarrow 0 \text{ as } t \rightarrow +\infty. \quad (32)$$

Note that the choice of different couples  $\{s_1, s_2\}$  may result in essentially different conditions for self-synchronization.

In paper [14] Theorem 1 has been applied to system (15) in case  $s_1^{-1} = -1, s_2^{-1} = 1$ . Here we consider the case of  $s_1^{-1} = 0, s_2^{-1} = 1$ . In this case we have

$$\begin{aligned} \nu &= -\frac{\pi\beta}{2(\beta \arcsin \frac{\beta}{A} + \sqrt{A^2 - \beta^2})}, \\ \nu_0 &= -\frac{3\pi\beta}{2\sqrt{2}(3\beta + 2(A - \beta)^{\frac{3}{2}}A^{-\frac{1}{2}})}, \end{aligned} \quad (33)$$

$$\begin{aligned} \pi_0(\omega) &= \omega^4(-\delta I_1^2 I_2^2 + A\tau I_1 I_2 (I_1 + I_2)) + \\ &+ \omega^2(Ak_1 I_2^2 + Ak_2 I_1^2 + A\tau(I_1 k_2^2 + I_2 k_1^2) - \\ &- A^2(\varepsilon + \tau)(I_1 + I_2)^2 - \delta(I_1^2 k_2^2 + I_2^2 k_1^2)) + \\ &+ (Ak_1 k_2^2 + Ak_2 k_1^2 - A^2(\varepsilon + \tau)(k_1 + k_2)^2 - \delta k_1^2 k_2^2). \end{aligned} \quad (34)$$

The inequality (29) is valid for all  $\omega \geq 0$  if the following inequalities are true:

$$\varepsilon + \tau \leq \frac{k_1 k_2}{2A(k_1 + k_2)}, \quad (35)$$

$$\delta \leq \frac{A(k_1 I_2^2 + k_2 I_1^2)}{2(k_1^2 I_2^2 + k_2^2 I_1^2)}, \quad (36)$$

$$\delta \leq \frac{A\tau(I_1 + I_2)}{I_1 I_2}. \quad (37)$$

Introduce the constant

$$P \triangleq \frac{k_1 k_2 (k_1^2 I_2^2 + k_2^2 I_1^2)}{2(k_1 + k_2)(k_1 I_2^2 + k_2 I_1^2)} \cdot \frac{I_1 + I_2}{I_1 I_2}. \quad (38)$$

**Theorem 2:** The limit relations (19) are true for every solution of (20) provided that either

$$A \leq P, \quad \frac{\nu_0^2 \nu^2}{(\nu_0^2 + \nu^2)} < \frac{k_1 k_2 (k_1 I_2^2 + k_2 I_1^2)}{2(k_1 + k_2)(k_1^2 I_2^2 + k_2^2 I_1^2)}, \quad (39)$$

or alternatively

$$A > P, \quad \frac{\nu_0^2 \nu^2}{(\nu_0^2 + \nu^2)} < \frac{k_1^2 k_2^2 (I_2 + I_1)}{4A I_1 I_2 (k_1 + k_2)^2}. \quad (40)$$

*Proof:* We choose

$$\varepsilon = \tau = \frac{k_1 k_2}{4A(k_1 + k_2)}. \quad (41)$$

In the first case, let

$$\delta = \frac{A(k_1 I_2^2 + k_2 I_1^2)}{2(k_1^2 I_2^2 + k_2^2 I_1^2)}. \quad (42)$$

Then due to (39) we obtain that

$$\delta \leq \frac{A\tau(I_1 + I_2)}{I_1 I_2}. \quad (43)$$

So the condition 1) of Theorem 1 is fulfilled. The condition 2) of Theorem 1 takes the form

$$4\varepsilon\delta = 4\tau\delta > \frac{\nu_0^2 \nu^2}{(\nu_0^2 + \nu^2)} \quad (44)$$

which follows from (39).

In the second case let

$$\delta = \frac{A\tau(I_1 + I_2)}{I_1 I_2} \quad (45)$$

Then it follows from (40) that (29) and (36) are true. Thus the Theorem is proved. ■

#### IV. LEONOV'S METHOD OF NONLOCAL REDUCTION

The idea of nonlocal reduction [8], [11] is to “inject” in Popov functionals of infinite dimensional system the trajectories of stable system of low order.

Consider the equation

$$\ddot{\sigma} + a\dot{\sigma} + \varphi(\sigma) = 0 \quad (46)$$

with  $\varphi(\sigma)$  described in previous section.

This equation has been exhaustively investigated (see [8] and references therein ). It is well known that it has a bifurcational value  $a_{cr}(\varphi)$  such that for  $a > a_{cr}$  the limit relations (32) are true for its every solution.

The frequency-algebraic stability criterion exploiting for (24) the nonlocal reduction technique is as follows.

**Theorem 3:** [8] Suppose there exist numbers

$s_1 \leq \mu_1, s_2 \geq \mu_2, \varepsilon, \tau > 0, \lambda \in (0, r)$  such that the conditions are valid:

1) the frequency-domain inequality

$$\pi(\omega, \lambda) \triangleq \operatorname{Re}\{K(i\omega - \lambda) - \tau(K(i\omega - \lambda) + s_1^{-1}(i\omega - \lambda))^* \cdot (K(i\omega - \lambda) + s_2^{-1}(i\omega - \lambda))\} - \varepsilon|K(i\omega - \lambda)|^2 \geq 0 \quad (47)$$

is true for all  $\omega \geq 0$ ,

2)

$$2\sqrt{\varepsilon\lambda} > a_{cr}(\varphi). \quad (48)$$

Then for every solution of (24) the limit relations (32) are true.

Next assertion describes the application of Theorem 3 to equation (20) in case  $s_2 = -s_1 = 1$ .

Introduce the function

$$f(x) = \frac{4x(k_1 - I_1x)(k_2 - I_2x)}{A((k_1 + k_2) - (I_1 + I_2)x)}. \quad (49)$$

Let

$$M \triangleq \sup_{x \in [0, r]} f(x) \quad (50)$$

where

$$r = \min\left\{\frac{k_1}{I_1}, \frac{k_2}{I_2}\right\} \quad (51)$$

**Theorem 4:**

If

$$M > a_{cr}^2(\Phi) \quad (52)$$

then for every solution of (20) the limits (19) are valid.

*Proof:* Note that

$$K(i\omega - \lambda) = A\left(\frac{1}{\varkappa_1 + iI_1\omega} + \frac{1}{\varkappa_2 + iI_2\omega}\right), \quad (53)$$

where

$$\varkappa_i = k_i - \lambda I_i \quad (i = 1, 2). \quad (54)$$

It is not difficult to establish by direct computation that if

$$\varepsilon + \tau = \frac{\varkappa_1 \varkappa_2}{A(\varkappa_1 + \varkappa_2)}, \quad (55)$$

the condition 1) of Theorem 3 is satisfied.

We choose the value of  $\lambda$  such that

$$f(\lambda) = M \quad (56)$$

Then

$$4\lambda(\varepsilon + \tau) = M. \quad (57)$$

It is sufficient to choose

$$\tau < \frac{M - a_{cr}^2}{4\lambda} \quad (58)$$

so that condition 2) of Theorem is fulfilled. Theorem 4 is proved. ■

#### V. CONCLUSION

In this paper the problem of synchronization between two vibro-excitors (eccentric rotors) installed on a common oscillating platform is considered. By means of stability theory for “pendulum-like” systems we establish analytic criteria for self-synchronization of two rotors.

#### ACKNOWLEDGMENT

This work was supported by Russian Foundation for Basic Research, grant 20-01-00619.

#### REFERENCES

- [1] I.I. Blekhan, “Self-synchronization of vibrators in some types of vibrational machines (in Russian),” *Inzhenerny Sbornik*, vol. 15, pp. 49–72, 1953.
- [2] I.I. Blekhan, *Vibrational mechanics* World Scientific Publ. Co., 2000.
- [3] Xiaozhe Chen and Xiangxi Kong and Xueliang Zhang and Lingxuan Li and Bangchun Wen, “On the Synchronization of Two Eccentric Rotors with Common Rotational Axis: Theory and Experiment,” *Shock and Vibration*, vol. 2016, p. 6973597, 2016.
- [4] Pan Fang and Qiming Yang and Yongjun Hou and Ye Chen, “Theoretical study on self-synchronization of two homodromy rotors coupled with a pendulum rod in a far-resonant vibrating system,” *Journal of Vibroengineering*, vol. 16, no. 5, pp. 2188–2204, 2014.
- [5] Pan Fang and Min Zou and Huan Peng and Mingjun Du and Gang Hu and Yongjun Hou, “Spatial synchronization of unbalanced rotors excited with paralleled and counterrotating motors in a far resonance system,” *Journal of Theoretical and Applied Mechanics*, vol. 57, no. 3, pp. 723–738, 2019.
- [6] Sperling L. and Merten F. and Duckstein H., *Rotation und Vibration in Beispilen zur Methode der direkten Bewegungsteilung*, *Technische Mechanik*, vol. 17, no. 3, pp. 231–243, 1977.
- [7] V. Rasvan, “Four lectures on stability. Lecture 3. The frequency domain criterion for absolute stability,” *Control Engineering and Applied Informatics*, vol. 8, no. 2, pp. 13–20, 2006.
- [8] G. A. Leonov, D. Ponomarenko, and V. B. Smirnova, *Frequency-Domain Methods for Nonlinear Analysis. Theory and Applications*, Singapore–New Jersey–London–Hong Kong: World Scientific, 1996.
- [9] G. A. Leonov, V. Reitmann, and V. B. Smirnova, *Non-local methods for pendulum-like feedback systems*, Stuttgart-Leipzig, Teubner, 1992.
- [10] V. B. Smirnova and A. V. Proskurnikov, “Volterra equations with periodic nonlinearities: multistability, oscillations and cycle slipping,” *Int. J. Bifurcation and Chaos*, vol. 29, no. 5, p. 1950068 (26p.), 2019.
- [11] G. A. Leonov, “The nonlocal reduction method in the theory of absolute stability of nonlinear systems,” *Automation and Remote Control*, vol. 45, no. 3, pp. 315–323, 1984.
- [12] Ju.N. Bakaev and A.A. Guzh, “Optimal reception of frequency modulated signals under Doppler effect conditions (in Russian),” *Radiotekhnika i Elektronika*, vol. 10(1), pp. 175–196, 1965.
- [13] A. V. Proskurnikov and V. B. Smirnova, “Constructive estimates of the pull-in range for synchronization circuit described by integro-differential equations,” *Proc. of IEEE Int. Symposium on Circuits and Systems (ISCAS)*, pp. 09180519 (5p.), Seville, Spain, 2020.
- [14] V.B Smirnova and A.V. Proskurnikov, “Self-synchronization of unbalanced rotors and the swing equation,” *IFAC-PapersOnLine*, vol. 54, no. 17, pp. 71–76, 2021.