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# Zeckendorf representation of multiplicative inverses modulo a Fibonacci number

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## Abstract

Prempreesuk, Noppakaew, and Pongsriiam determined the Zeckendorf representation of the multiplicative inverse of 2 modulo  $F_n$ , for every positive integer  $n$  not divisible by 3, where  $F_n$  denotes the  $n$ th Fibonacci number. We determine the Zeckendorf representation of the multiplicative inverse of  $a$  modulo  $F_n$ , for every fixed integer  $a \geq 3$  and for all positive integers  $n$  with  $\gcd(a, F_n) = 1$ . Our proof makes use of the so-called base- $\varphi$  expansion of real numbers.

**Keywords** Base- $\varphi$  expansion · Fibonacci number · Multiplicative inverse · Zeckendorf representation

**Mathematics Subject Classification** Primary 11B39 · Secondary 11A67, 11A99

## 1 Introduction

Let  $(F_n)_{n \geq 1}$  be the sequence of Fibonacci numbers, which is defined by the initial conditions  $F_1 = F_2 = 1$  and by the linear recurrence  $F_n = F_{n-1} + F_{n-2}$  for  $n \geq 3$ . It is well known [22] that every positive integer  $n$  can be written as a sum of distinct non-

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consecutive Fibonacci numbers, that is,  $n = \sum_{i=1}^m d_i F_i$ , where  $m \in \mathbb{N}$ ,  $d_i \in \{0, 1\}$ , and  $d_i d_{i+1} = 0$  for all  $i \in \{1, \dots, m-1\}$ . This is called the *Zeckendorf representation* of  $n$  and, apart from the equivalent use of  $F_1$  instead of  $F_2$  or vice versa, is unique.

The Zeckendorf representation of integer sequences has been studied in several works. For instance, Filipponi and Freitag [6, 7] studied the Zeckendorf representation of numbers of the form  $F_{kn}/F_n$ ,  $F_n^2/d$  and  $L_n^2/d$ , where  $L_n$  are the Lucas numbers and  $d$  is a Lucas or Fibonacci number. Filipponi, Hart, and Sanchis [8, 13, 14] analyzed the Zeckendorf representation of numbers of the form  $mF_n$ . Filipponi [8] determined the Zeckendorf representation of  $mF_n F_{n+k}$  and  $mL_n L_{n+k}$  for  $m \in \{1, 2, 3, 4\}$ . Bugeaud [3] studied the Zeckendorf representation of smooth numbers. The study of Zeckendorf representations has been also approached from a combinatorial point of view [1, 9, 12, 21]. Moreover, generalizations of the Zeckendorf representation to linear recurrences other than the sequence of Fibonacci numbers have been considered [4, 5, 10, 11, 16].

For all integers  $a$  and  $m \geq 1$  with  $\gcd(a, m) = 1$ , let  $(a^{-1} \bmod m)$  denote the least positive multiplicative inverse of  $a$  modulo  $m$ , that is, the unique  $b \in \{1, \dots, m\}$  such that  $ab \equiv 1 \pmod{m}$ . Prempreesuk, Noppakaew, and Pongsriam [17] determined the Zeckendorf representation of  $(2^{-1} \bmod F_n)$ , for every positive integer  $n$  that is not divisible by 3. (The condition  $3 \nmid n$  is necessary and sufficient to have  $\gcd(2, F_n) = 1$ .) In particular, they showed [17, Theorem 3.2] that

$$(2^{-1} \bmod F_n) = \begin{cases} \sum_{k=0}^{(n-7)/2} F_{n-3k-2} + F_3 & \text{if } n \equiv 1 \pmod{3}; \\ \sum_{k=0}^{(n-8)/2} F_{n-3k-2} + F_4 & \text{if } n \equiv 2 \pmod{3}; \end{cases}$$

for every integer  $n \geq 8$ . We extend their result by determining the Zeckendorf representation of the multiplicative inverse of  $a$  modulo  $F_n$ , for every fixed integer  $a \geq 3$  and every positive integer  $n$  with  $\gcd(a, F_n) = 1$ . Precisely, we prove the following result.

**Theorem 1.1** *Let  $a \geq 3$  be an integer. Then there exist integers  $M, n_0, i_0 \geq 1$  and periodic sequences  $\mathbf{z}^{(0)}, \dots, \mathbf{z}^{(M-1)}$  and  $\mathbf{w}^{(1)}, \dots, \mathbf{w}^{(i_0)}$  with values in  $\{0, 1\}$  such that, for all integers  $n \geq n_0$  with  $\gcd(a, F_n) = 1$ , the Zeckendorf representation of  $(a^{-1} \bmod F_n)$  is given by*

$$(a^{-1} \bmod F_n) = \sum_{i=i_0}^{n-1} z_{n-i}^{(n \bmod M)} F_i + \sum_{i=1}^{i_0-1} w_n^{(i)} F_i.$$

From the proof of Theorem 1.1 it follows that  $M, n_0, i_0, \mathbf{z}^{(0)}, \dots, \mathbf{z}^{(M-1)}$ , and  $\mathbf{w}^{(1)}, \dots, \mathbf{w}^{(i_0)}$  can be computed from  $a$  (see also Remark 4.1 at the end of the paper).

## 2 Preliminaries on Fibonacci numbers

Let us recall that for every integer  $n \geq 1$  it holds the *Binet formula*

$$F_n = \frac{\varphi^n - \bar{\varphi}^n}{\sqrt{5}},$$

where  $\varphi := (1 + \sqrt{5})/2$  is the Golden ratio and  $\bar{\varphi} := (1 - \sqrt{5})/2$  is its algebraic conjugate. Furthermore, it is well known that for every integer  $m \geq 1$  the Fibonacci sequence  $(F_n)_{n \geq 1}$  is (purely) periodic modulo  $m$ . Let  $\pi(m)$  denote its period length, or the so-called *Pisano period*.

The next lemma gives a formula for the inverse of  $a$  modulo  $F_n$ .

**Lemma 2.1** *For all integers  $a \geq 1$  and  $n \geq 3$  with  $\gcd(a, F_n) = 1$ , we have that*

$$(a^{-1} \bmod F_n) = \frac{bF_n + 1}{a},$$

where  $b := (-F_r^{-1} \bmod a)$  and  $r := (n \bmod \pi(a))$ .

**Proof** Since  $r \equiv n \pmod{\pi(a)}$ , we have that  $F_r \equiv F_n \pmod{a}$ . In particular, it follows that  $\gcd(a, F_r) = \gcd(a, F_n) = 1$ . Hence,  $F_r$  is invertible modulo  $a$ , and consequently  $b$  is well defined. Moreover, we have that

$$bF_n + 1 \equiv -F_r^{-1}F_r + 1 \equiv 0 \pmod{a},$$

and thus  $c := (bF_n + 1)/a$  is an integer. On the one hand, we have that

$$ac \equiv bF_n + 1 \equiv 1 \pmod{F_n}.$$

On the other hand, since  $b \leq a - 1$  and  $n \geq 3$ , we have that

$$0 \leq c \leq \frac{(a - 1)F_n + 1}{a} = F_n - \frac{F_n - 1}{a} < F_n.$$

Therefore, we get that  $c = (a^{-1} \bmod F_n)$ , as desired.

## 3 Preliminaries on base- $\varphi$ expansion

We need some basic results regarding the so-called *base- $\varphi$  expansion* of real numbers, which was introduced by Bergman [2] in 1957 (see also [19]), and which is a particular case of non-integer base expansion (see, e.g., [15, 18]). Let  $\mathfrak{D}$  be the set of sequences in  $\{0, 1\}$  that have no two consecutive terms equal to 1, and that are not ultimately equal to the periodic sequence  $0, 1, 0, 1, \dots$ . Then for every  $x \in [0, 1)$  there exists a unique sequence  $\delta(x) = (\delta_i(x))_{i \in \mathbb{N}}$  in  $\mathfrak{D}$  such that  $x = \sum_{i=1}^{\infty} \delta_i(x)\varphi^{-i}$ . Precisely,  $\delta_i(x) = \lfloor T^{(i)}(x) \rfloor$  for every  $i \in \mathbb{N}$ , where  $T^{(i)}$  denotes the  $i$ th iterate of the map

$T : [0, 1) \rightarrow [0, 1)$  defined by  $T(\hat{x}) := (\varphi\hat{x} \bmod 1)$  for every  $\hat{x} \in [0, 1)$ . Furthermore, letting  $\mathcal{F} := \mathbb{Q}(\varphi) \cap [0, 1)$ , if  $x \in \mathcal{F}$  then  $\delta(x)$  is ultimately periodic. In particular, if  $x \in \mathcal{F}$  is given as  $x = x_1 + x_2\varphi$ , where  $x_1, x_2 \in \mathbb{Q}$ , then the preperiod and the period of  $\delta(x)$  can be effectively computed by finding the smallest  $i \in \mathbb{N}$  such that  $T^{(i)}(x) = T^{(j)}(x)$  for some  $j \in \mathbb{N}$  with  $j < i$ . Conversely, for every ultimately periodic sequence  $\mathbf{d} = (d_i)_{i \in \mathbb{N}}$  in  $\mathfrak{D}$  we have that the number  $x = \sum_{i=1}^{\infty} d_i \varphi^{-i}$  belongs to  $\mathcal{F}$ , and  $x_1, x_2 \in \mathbb{Q}$  such that  $x = x_1 + x_2\varphi$  can be effectively computed in terms of the preperiod and period of  $\mathbf{d}$  by using the formula for the sum of the geometric series. Moreover, in the case that  $x$  is a rational number in  $[0, 1)$  then  $\delta(x)$  is purely periodic [20].

The next lemma collects two easy inequalities for sums involving sequences in  $\mathfrak{D}$ .

**Lemma 3.1** *For every sequence  $(d_i)_{i \in \mathbb{N}}$  in  $\mathfrak{D}$  and for every  $m \in \mathbb{N} \cup \{\infty\}$ , we have:*

- (1)  $\sum_{i=1}^m d_i \varphi^{-i} \in [0, 1)$  and
- (2)  $\sum_{i=1}^m d_i (-\varphi)^{-i} \in (-1, \varphi^{-1})$ .

**Proof** Since  $(d_i)_{i \in \mathbb{N}}$  belongs to  $\mathfrak{D}$ , there exists  $k \in \mathbb{N}$  such that  $d_k = d_{k+1} = 0$ . Let  $k$  be the minimum integer with such property. Then

$$\begin{aligned} \sum_{i=1}^{\infty} d_i \varphi^{-i} &= \sum_{i=1}^{k-1} d_i \varphi^{-i} + \sum_{i=k+2}^{\infty} d_i \varphi^{-i} < \sum_{j=1}^{\lfloor k/2 \rfloor} \varphi^{-(2j-1)} + \sum_{i=k+2}^{\infty} \varphi^{-i} \\ &= \left(1 - \varphi^{-2\lfloor k/2 \rfloor}\right) + \varphi^{-k} \leq 1, \end{aligned}$$

and (1) is proved. Let us prove (2). On the one hand, we have

$$\sum_{i=1}^m d_i (-\varphi)^{-i} \leq \sum_{j=1}^m d_{2j} \varphi^{-2j} < \sum_{j=1}^{\infty} \varphi^{-2j} = \varphi^{-1},$$

where the second inequality is strict because  $\mathfrak{D}$  does not contain sequences that are ultimately equal to  $(0, 1, 0, 1, \dots)$ . On the other hand, similarly, we have

$$\sum_{i=1}^m d_i (-\varphi)^{-i} \geq -\sum_{j=1}^m d_{2j-1} \varphi^{-(2j-1)} > -\sum_{j=1}^{\infty} \varphi^{-(2j-1)} = -1.$$

Thus (2) is proved.

The following lemma relates base- $\varphi$  expansion and Zeckendorf representation.

**Lemma 3.2** *Let  $N$  be a positive integer and write  $N = x\varphi^m / \sqrt{5}$  for some  $x \in \mathcal{F}$  and some integer  $m \geq 2$ . Then the Zeckendorf representation of  $N$  is given by*

$$N = \sum_{i=1}^{m-1} \delta_{m-i}(x) F_i.$$

Moreover, we have  $\delta_m(x) = 0$ .

**Proof** Let  $R := N - \sum_{i=1}^{m-1} \delta_{m-i}(x)F_i$ . We have to prove that  $R = 0$ . Since  $R$  is an integer, it suffices to show that  $|R| < 1$ . We have

$$\begin{aligned} \sqrt{5}N &= x\varphi^m = \sum_{i=1}^{\infty} \delta_i(x)\varphi^{m-i} = \sum_{i=1}^m \delta_i(x)\varphi^{m-i} + \sum_{i=m+1}^{\infty} \delta_i(x)\varphi^{m-i} \\ &= \sum_{i=0}^{m-1} \delta_{m-i}(x)\varphi^i + \sum_{i=1}^{\infty} \delta_{i+m}(x)\varphi^{-i} \\ &= \sum_{i=0}^{m-1} \delta_{m-i}(x)(\varphi^i - \bar{\varphi}^i) + \sum_{i=0}^{m-1} \delta_{m-i}(x)\bar{\varphi}^i + \sum_{i=1}^{\infty} \delta_{i+m}(x)\varphi^{-i} \\ &= \sqrt{5} \sum_{i=1}^{m-1} \delta_{m-i}(x)F_i + \sum_{i=0}^{m-1} \delta_{m-i}(x)(-\varphi)^{-i} + \sum_{i=1}^{\infty} \delta_{i+m}(x)\varphi^{-i}. \end{aligned}$$

Hence, we get that

$$\sqrt{5}R = \sum_{i=0}^{m-1} \delta_{m-i}(x)(-\varphi)^{-i} + \sum_{i=1}^{\infty} \delta_{i+m}(x)\varphi^{-i}.$$

For the sake of contradiction, suppose that  $\delta_m(x) = 1$ . Then  $\delta_{m+1}(x) = 0$  and, by Lemma 3.1, it follows that

$$\sqrt{5}R = 1 + \sum_{i=1}^{m-1} \delta_{m-i}(x)(-\varphi)^{-i} + \sum_{i=2}^{\infty} \delta_{i+m}(x)\varphi^{-i} \in (1 - 1 + 0, 1 + \varphi^{-1} + \varphi^{-1}) = (0, \sqrt{5}),$$

which is a contradiction, since  $R$  is an integer.

Therefore,  $\delta_m(x) = 0$  and, again by Lemma 3.1, we have

$$\sqrt{5}R = \sum_{i=1}^{m-1} \delta_{m-i}(x)(-\varphi)^{-i} + \sum_{i=1}^{\infty} \delta_{i+m}(x)\varphi^{-i} \in (-1 + 0, \varphi^{-1} + 1) \subseteq (-\sqrt{5}, \sqrt{5}),$$

so that  $|R| < 1$ , as desired.

The next lemma regards the base- $\varphi$  expansions of the sum of two numbers.

**Lemma 3.3** *Let  $x, y \in [0, 1)$ ,  $m \in \mathbb{N}$ , and put  $v := x + y\varphi^{-m}$ . Suppose that there exists  $\lambda \in \mathbb{N}$  such that  $\lambda + 2 \leq m$  and  $\delta_\lambda(x) = \delta_{\lambda+1}(x) = 0$ . Then, putting*

$$w := \sum_{i=\lambda+2}^{\infty} \delta_i(x)\varphi^{-i} + \sum_{i=m+1}^{\infty} \delta_{i-m}(y)\varphi^{-i},$$

we have that  $v, w \in [0, 1)$  and

$$\delta_i(v) = \begin{cases} \delta_i(x) & \text{if } i \leq \lambda, \\ \delta_i(w) & \text{if } i > \lambda, \end{cases} \quad (1)$$

for every  $i \in \mathbb{N}$ .

**Proof** From Lemma 3.1(1), we have that

$$0 \leq w < \varphi^{-(\lambda+1)} + \varphi^{-m} < \varphi^{-(\lambda+1)} + \varphi^{-(\lambda+2)} = \varphi^{-\lambda}.$$

Hence,  $w \in [0, \varphi^{-\lambda}) \subseteq [0, 1)$  and so  $w = \sum_{i=\lambda+1}^{\infty} \delta_i(w)\varphi^{-i}$ . Therefore, recalling that  $\delta_{\lambda+1}(x) = 0$ , we get that

$$\begin{aligned} v &= x + y\varphi^{-m} = \sum_{i=1}^{\infty} \delta_i(x)\varphi^{-i} + \sum_{i=1}^{\infty} \delta_i(y)\varphi^{-i-m} = \sum_{i=1}^{\infty} \delta_i(x)\varphi^{-i} + \sum_{i=m+1}^{\infty} \delta_{i-m}(y)\varphi^{-i} \\ &= \sum_{i=1}^{\lambda} \delta_i(x)\varphi^{-i} + w = \sum_{i=1}^{\lambda} \delta_i(x)\varphi^{-i} + \sum_{i=\lambda+1}^{\infty} \delta_i(w)\varphi^{-i}, \end{aligned}$$

which is the base- $\varphi$  expansion of  $v$ . (Note that  $\delta_{\lambda}(x) = 0$ .) In particular, by Lemma 3.1(1), we have that  $v \in [0, 1)$ . Thus (1) follows.

#### 4 Proof of Theorem 1.1

Fix an integer  $a \geq 3$ . Let us begin by defining  $M, n_0, i_0$ , and  $z^{(0)}, \dots, z^{(M-1)}$ . Put  $M := \pi(a)$ . For each  $r \in \{0, \dots, M-1\}$  with  $\gcd(a, F_r) = 1$ , let  $b_r := (-F_r^{-1} \bmod a)$ ,  $x_r := b_r/a$ , and  $z^{(r)} := \delta(x_r)$ . Note that  $x_r \in (0, 1)$ . Since  $x_r$  is a positive rational number, we have that  $z^{(r)}$  is a (purely) periodic sequence belonging to  $\mathfrak{D}$ . Let  $\ell$  be the least common multiple of the period lengths of  $z^{(0)}, \dots, z^{(M-1)}$ , and put  $i_0 := \ell + 3$ . Finally, let  $n_0 := \max\{i_0 + 1, \lceil \log(2a)/\log \varphi \rceil\}$ .

Pick an integer  $n \geq n_0$  with  $\gcd(a, F_n) = 1$  and, for the sake of brevity, put  $r := (n \bmod M)$ . From Lemma 2.1 and Binet's formula (2), we get that

$$(a^{-1} \bmod F_n) = \frac{b_r F_n + 1}{a} = \frac{b_r(\varphi^n - \bar{\varphi}^n)}{\sqrt{5}a} + \frac{1}{a} = (x_r + y_n \varphi^{-n}) \frac{\varphi^n}{\sqrt{5}}, \quad (2)$$

where

$$y_n := \frac{\sqrt{5}}{a} - x_r(-\varphi)^{-n}.$$

Since  $n \geq n_0$ , it follows that  $y_n \in (0, 1)$  and  $x_r + y_n\varphi^{-n} \in (0, 1)$ . Therefore, from (2) and Lemma 3.2, we get that

$$(a^{-1} \bmod F_n) = \sum_{i=1}^{n-1} \delta_{n-i}(x_r + y_n\varphi^{-n})F_i.$$

Since  $\delta(x_r)$  is (purely) periodic and belongs to  $\mathfrak{D}$ , we have that  $\delta(x_r)$  contains infinitely many pairs of consecutive zeros. Furthermore, since the period length of  $\delta(x_r)$  is at most  $\ell$ , we have that among every  $\ell + 1$  consecutive terms of  $\delta(x_r)$  there are two consecutive zero. In particular, there exists  $\lambda = \lambda(r)$  such that  $n - \ell - 3 \leq \lambda \leq n - 2$  and  $\delta_\lambda(x_r) = \delta_{\lambda+1}(x_r) = 0$ . Consequently, by Lemma 3.3, we get that  $\delta_i(x_r + y_n\varphi^{-n}) = \delta_i(x_r)$  for each positive integer  $i \leq \lambda$  and, a fortiori, for each positive integer  $i \leq n - i_0$ . Therefore, we have that

$$\begin{aligned} (a^{-1} \bmod F_n) &= \sum_{i=i_0}^{n-1} \delta_{n-i}(x_r)F_i + \sum_{i=1}^{i_0-1} \delta_{n-i}(x_r + y_n\varphi^{-n})F_i \tag{3} \\ &= \sum_{i=i_0}^{n-1} z_{n-i}^{(r)}F_i + \sum_{i=1}^{i_0-1} w_n^{(i)}F_i, \end{aligned}$$

where  $w^{(1)}, \dots, w^{(i_0)}$  are the sequences defined by  $w_n^{(i)} := \delta_{n-i}(x_r + y_n\varphi^{-n})$ . Note that, by construction,

$$z_1^{(r)}, z_2^{(r)}, \dots, z_{n-i_0}^{(r)}, w_n^{(i_0-1)}, w_n^{(i_0-2)}, \dots, w_n^{(1)}$$

is a string in  $\{0, 1\}$  with no consecutive zeros. Hence, (3) is the Zeckendorf representation of  $(a^{-1} \bmod F_n)$ .

It remains only to prove that  $w^{(1)}, \dots, w^{(i_0)}$  are periodic. By (3) and the uniqueness of the Zeckendorf representation, it suffices to prove that

$$R(n) := (a^{-1} \bmod F_n) - \sum_{i=i_0}^{n-1} z_{n-i}^{(r)}F_i = \sum_{i=1}^{i_0-1} w_n^{(i)}F_i \tag{4}$$

is a periodic function of  $n$ . From the last equality in (4), we have that  $0 \leq R(n) < \sum_{i=1}^{i_0-1} F_i$ . (Actually, one can prove that  $0 \leq R(n) < F_{i_0}$ , but this is not necessary for our proof.) Fix a prime number  $p > \max\{a, \sum_{i=1}^{i_0-1} F_i\}$ . It suffices to prove that  $R(n)$  is periodic modulo  $p$ . Recalling that  $(a^{-1} \bmod F_n) = (b_r F_n + 1)/a$  and that the sequence of Fibonacci numbers is periodic modulo  $p$ , it follows that  $(a^{-1} \bmod F_n)$  is periodic modulo  $p$ . Hence, it suffices to prove that  $R'(n) := \sum_{i=i_0}^{n-1} z_{n-i}^{(r)}F_i$  is periodic modulo  $p$ . Using that  $z^{(r)}$  has period length dividing  $\ell$ , we get that

$$\begin{aligned}
R'(n + \ell M) - R'(n) &= \sum_{i=i_0}^{n+\ell M-1} z_{n+\ell M-i}^{((n+\ell M) \bmod M)} F_i - \sum_{i=i_0}^{n-1} z_{n-i}^{(r)} F_i \\
&= \sum_{i=i_0}^{n+\ell M-1} z_{n+\ell M-i}^{(r)} F_i - \sum_{i=i_0}^{n-1} z_{n-i}^{(r)} F_i \\
&= \sum_{i=n}^{n+\ell M-1} z_{n+\ell M-i}^{(r)} F_i + \sum_{i=i_0}^{n-1} (z_{n+\ell M-i}^{(r)} - z_{n-i}^{(r)}) F_i \\
&= \sum_{j=1}^{\ell M} z_j^{(r)} F_{n+\ell M-j},
\end{aligned}$$

which is a linear combination of sequences that are periodic modulo  $p$ . Hence  $R'(n)$  is periodic modulo  $p$ . The proof is complete.

**Remark 4.1** The proof of Theorem 1.1 provides a way to compute the positive integers  $M, i_0, n_0$  and the periods of the periodic sequences  $z^{(0)}, \dots, z^{(M-1)}$  and  $w^{(1)}, \dots, w^{(i_0)}$ . Indeed, going through the proof, we have that:  $M = \pi(a)$  is the Pisano period of  $a$ , which can be computed in an obvious way;  $z^{(r)} = \delta((-F_r^{-1} \bmod a)/a)$  and so the period of  $z^{(r)}$  can be computed as explained at the beginning of Section 3;  $i_0$  and  $n_0$  have simple formulas in terms of  $\ell$ , which is the least common multiple of the period lengths of  $z^{(0)}, \dots, z^{(M-1)}$ . Finally, the periods of  $w^{(1)}, \dots, w^{(i_0)}$  can be computed from (4) and the fact that  $R(n)$  is periodic with period length at most  $\pi(p)^2 \ell M$ , which follows from the arguments after (4). However, note that proceeding in this way might be impractical, since  $\ell$  might be exponential in  $M$ , and thus  $p$  might be double exponential in  $M$ ; making the search for the periods of  $w^{(1)}, \dots, w^{(i_0)}$  extremely long.

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