

The generalized Wiener-Hopf equations for the elastic wave motion in angular regions

Original

The generalized Wiener-Hopf equations for the elastic wave motion in angular regions / Daniele, V. G.; Lombardi, G.. - In: PROCEEDINGS OF THE ROYAL SOCIETY OF LONDON. SERIES A. - ISSN 1364-5021. - STAMPA. - 478:2257(2022), p. 20210624. [10.1098/rspa.2021.0624]

Availability:

This version is available at: 11583/2958990 since: 2022-03-21T12:29:53Z

Publisher:

Royal Society Publishing

Published

DOI:10.1098/rspa.2021.0624

Terms of use:

This article is made available under terms and conditions as specified in the corresponding bibliographic description in the repository

Publisher copyright

(Article begins on next page)



One contribution to a special feature Advances in Wiener-Hopf type techniques: theory and applications

Subject Areas:

applied mathematics, computational physics, mathematical physics, wave motion, elasticity

Keywords:

Wave motion, Wedge, Scattering, Wiener-Hopf method, Integral equations, Spectral Domain, Elasticity

Author for correspondence:

Guido Lombardi

e-mail: guido.lombardi@polito.it

The Generalized Wiener-Hopf Equations for the elastic wave motion in angular regions

V. G. Daniele¹ and G. Lombardi¹

¹DET-Poltecnico di Torino, 10129 Torino, Italy

In this work, we introduce a general method to deduce spectral functional equations in elasticity and thus, the Generalized Wiener-Hopf Equations (GWHEs), for the wave motion in angular regions filled by arbitrary linear homogeneous media and illuminated by sources localized at infinity. The work extends the methodology used in electromagnetic applications and proposes for the first time a complete theory to get the GWHEs in elasticity. In particular we introduce a vector differential equation of first order characterized by a matrix that depends on the medium filling the angular region. The functional equations are easily obtained by a projection of the reciprocal vectors of this matrix on the elastic field present on the faces of the angular region. The application of the boundary conditions to the functional equations yields GWHEs for practical problems. This paper extends and applies the general theory to the challenging canonical problem of elastic scattering in angular regions.

1. Introduction

In [1], we have applied a general theory to obtain spectral functional equations in electromagnetics and thus Generalized Wiener-Hopf Equations (GWHEs) for scattering problem in angular regions filled by arbitrarily linear media, inspired by [2] and described also in [3]. The monographs [4]- [5] show the efficacy of the generalization of the Wiener-Hopf (WH) technique in practical electromagnetic wave scattering problems in presence of geometries containing angular regions and/or stratified planar regions, see references therein.

In this paper we implement for the first time the methodology to the challenging canonical problem of elastic scattering in angular regions where some physical quantities are tensors. The technique consists of three steps: 1) the deduction of functional equations in spectral domain of sub-regions that constitute the whole geometry of the problem, 2) the imposition of boundary conditions to get the GWHEs and, 3) the solution of the

© The Authors. Published by the Royal Society under the terms of the Creative Commons Attribution License <http://creativecommons.org/licenses/by/4.0/>, which permits unrestricted use, provided the original author and source are credited.

2 system of the WH equations using exact or semianalytical approximate techniques of factorization
3 as the Fredholm factorization technique [6]- [7].

4 This paper is focused on the first fundamental step and introduce the potentialities to develop
5 the other two steps through validations. We follows the method to obtain the WH equations in
6 spectral domain proposed by Jones [8]- [9], with the application of Fourier/Laplace transforms
7 directly to the PDE formulation of the problem avoiding the tricky derivation of the Green's
8 function in the natural domain. In this work we use a first order differential vector formulation for
9 continuous components of the fields, inspired by Bresler and Marcuvitz in [10] for stratified media
10 in electromagnetics. We note that some of theoretical aspects used in electromagnetics (see [1])
11 are not available in elasticity or are cumbersome to be extended. For this reason, the GWHEs
12 derivation for scattering by angular regions in elasticity is more complicated and challenging,
13 although following the same general theory. Indeed, the authors of this paper have preliminary
14 introduced in [4]- [5] an abstract formulation for simplified elastic scattering problem concerning
15 the semi-infinite crack and some initial aspects of wedge problems.

16 In this paper, we first extend the formulation presented in [1] to elastic problems in angular
17 regions using oblique Cartesian coordinates. It yields a matrix differential problem of first order
18 whose unknowns are the field components continuous to the faces of the angular regions. The
19 application of Laplace transform along one face of the angular region and the assumption of
20 problem invariance along the edge profile yield a matrix ordinary differential problem of first
21 order. Following [1] based on [11], we develop a spectral solution before imposing boundary
22 conditions based on the derivation of the dyadic Green's functions in terms of eigenvectors and
23 eigenvalue of the algebraic matrix operator (of the first order differential formulation) .

24 The projection of the solution on reciprocal vectors allows to get a set of functional equations
25 that relate the Laplace transforms of continuous field components along one face of the angular
26 regions to the ones of the other face. The imposition of boundary conditions yields a set of GWHEs
27 for practical angular region problems.

28 For the sake of simplicity, even if challenging, this work is focused on elastic wedge problem
29 filled by an elastic isotropic solid and extendable to anisotropic media. This problem is considered
30 a fundamental problem in the mathematical theory of elastic diffraction and, despite numerous
31 attempts to solve it in closed form, no exact solution exists for arbitrary aperture angle of the
32 wedge region. Three major semi-analytical approaches [12]- [14] have been proposed to solve
33 this problem in the two-dimensional case (i.e. at normal incidence). The first method is presented
34 by Budaev in his monograph [12] that is based on the Sommerfeld integral (SI) representation
35 of the elastic potentials and extends the popular and effective Sommerfeld-Malyuzhinets (SM)
36 method to wedge problems with two concurrent different propagation constants. The difference
37 equations, that initially arise from this formulation, are reduced to singular integral equations
38 that are treated with a regularization method. Further interesting aspects of this formulation are
39 presented also in [15]. A second method to study elastic wedge problems is reported in [13],
40 where the scattered field by the faces of the wedge is related to the Fourier transforms of the
41 displacement field of the faces (the spectral functions). Applying the Fourier transforms to the
42 differential formulation of the elastic field and taking into account the boundary conditions, the
43 authors obtain singular integral equations in terms of the spectral functions, that are numerically
44 solved by using the Galerkin collocation method. An important aspect of this work is the
45 use of recursive equations that provide analytical continuation (propagation of the solution) of
46 the approximate spectral functions obtained by the numerical solution in a certain strip. New
47 development of this method are reported in [16], where double Fourier transforms are introduced
48 to obtain the kernels of the singular integral equations. In [17] the method is extended to 3D
49 problems, however the proposed functional equations in spectral domain are again written
50 in terms of singular integral operators and not in an algebraic form. The concept of spectral
51 representation of the displacements on the wedge faces is applied also by Gautesen's group
52 works [18]- [20], [14] that, according to our opinion, have produced the best practical results in the
53 solution of the two dimensional elastic isotropic wedge problem [14]. The difference with respect

54 to [13] is the use of an integral representation in terms of the displacements in the natural domain.
 55 Substantially, the integral representations of this method are those that in electromagnetism are
 56 called Kirchhoff's representations. The kernel of the integral representations are suitable Green
 57 functions of the free space and the integral does not contain components of the stress tensors.
 58 The traction-free boundary conditions on the faces of the wedge impose this property. Another
 59 important aspect in these works is to resort to an extinction theorem that allows to impose the
 60 vanishing of the displacement outside the elastic wedge. The application of the theorem allows
 61 to use unilateral Fourier transform (or Laplace transform) on the Gautesen (Kirchhoff) integral
 62 representations and it yield functional equations which are algebraic with respect to the Laplace
 63 transforms of the displacements on the two faces of the wedge. We note that the arguments of
 64 the Laplace transforms of the displacements on the two faces are different. Substantially, the
 65 functional equations obtained in [14] are GWHEs¹, although not defined in this way.

66 In this paper we derive with a systematic and efficient method spectral functional equations
 67 in algebraic form useful to derive GWHEs in 3D elastic wedge problems. These equations are
 68 validated by comparison with the ones proposed in [14]. The proposed method has the following
 69 important characteristics:

- 70 (i) The functional equations are easily obtained in terms of eigenvectors and eigenvalues of
 71 a matrix that characterizes the medium filling the angular region.
- 72 (ii) These functional equations hold independently from the boundary conditions of the
 73 angular region.
- 74 (iii) The application of boundary conditions yields a system of GWHEs for a specific problem.
- 75 (iv) The deduction of the GWHEs is general, since the method can be applied to study wave
 76 motion in angular regions filled by arbitrary linear media.

77 We remark that property (i) avoids the introduction of Kirchhoff type representations that require
 78 the computation of the Green's function. This computation can be difficult in elasticity, see
 79 Gautesen's group works [14]. Property (ii) allows the possibility to study complex wave motion
 80 problems constituted of different angular sub-regions or angular regions connected to planar
 81 stratified media, see in electromagnetics [21]- [24]. The third and the fourth characteristics allow
 82 the derivation of GWHEs in isotropic elastic media with plane wave source at skew incidence
 83 and in the general case of an elastic wedge filled by anisotropic medium. Moreover, we note
 84 that it is possible to directly compute from the spectral solution of the GWHEs the field in every
 85 point of the angular regions, avoiding Kirchhoff's representations and Green's function in natural
 86 domain. In particular the diffracted field component can be asymptotically computed with the
 87 saddle point method. A last but not less important property of the GWHE formulations of wedge
 88 problems is constituted by the set of mathematical tools in complex analysis. The Wiener-Hopf
 89 technique provides powerful solution methods based on exact and approximate factorization
 90 methods. In their works, Gautesen et al. have proposed a possible original method to deal with
 91 GWHEs of elastic wedge problems, exploiting analytical properties of the unknowns, see [14] and
 92 references therein. We propose, alternatively, the Fredholm factorization method [6]- [7] which is
 93 an effective semi-analytical technique for the solution of arbitrary GWHEs and it is based on
 94 the reduction of the factorization problem to Fredholm integral equations of second kind. We
 95 expect, in a future work, to effectively apply the Fredholm factorization to solve the GWHEs of
 96 elastic wedge problem using the same methodology applied in electromagnetic scattering from
 97 dielectric wedge [5].

98 The paper is organized into eight sections and we assume plane wave sources and/or
 99 sources localized at infinity in time harmonic fields with a time dependence specified by $e^{j\omega t}$
 100 (electrical engineering notation) which is suppressed. In Section 2, we introduce the first order
 101 vector differential formulation for continuous components of the elastic field in an indefinite
 102 homogeneous medium. Note that, while in electromagnetics the continuous components of field
 103 are the transversal ones, in elasticity we have a more complex definition in term of stress tensor

¹The GWHEs differ from the Classical Wiener-Hopf equations (CWHEs) for the definitions of the unknowns in spectral domain. While CWHEs introduce plus and minus functions that are always defined in the same complex plane, the GWHEs present plus and minus functions that are defined in different complex planes but related together.

104 and velocity vector. The theory presented in Section 2 is also useful to study propagation in
 105 stratified media. Using oblique Cartesian coordinates and taking into account the results of
 106 Section 2, Section 3 describes the novel application of the method to angular regions, yielding
 107 the oblique first order vector differential formulation for continuous components of the elastic
 108 field. The application of Laplace transform along one face of the angular region and assumption
 109 of problem with invariance along the edge profile yield a vector ordinary differential problem
 110 of first order (oblique equations). The solution of these oblique equations, projected on the
 111 reciprocal vectors of an algebraic matrix defined in Section 2, provides the functional equations
 112 of an arbitrary angular region (Section 4). It is remarkable that we get functional equations
 113 independently from the materials and the sources that can be present outside of the considered
 114 angular region. Explicit expressions in algebraic form are reported in Section 5 for isotropic media
 115 and arbitrary boundary conditions. Section 6 shows the validation of functional equations in
 116 special simplified cases reported in literature by other authors for the planar problem; and Section
 117 7 reports the validation of functional equations by evaluating the characteristic impedances of
 118 half spaces in planar problem. Finally, conclusions are reported in section 8 and a glossary of the
 119 symbols useful for the readability of the text is provided at the end. We remark that, according
 120 to our opinion, the functional equations for the non planar (3D) general case, are deduced and
 121 reported for the first time in literature in this paper at Section 5. We finally state that the scope of
 122 our paper is to present algebraic spectral functional equations for arbitrary boundary conditions
 123 for 3D wave motion problems in angular regions that are useful for the examination of practical
 124 problems by imposing specific boundary conditions yielding GWHE formulations.

125 2. First order differential equations for continuous components of 126 the elastic field in an indefinite rectangular isotropic medium

127 In this section we study elastic wave propagation in stratified media along a direction (say y) and,
 128 consequently in Section 3, we use these results to develop the theory for angular regions.

The evaluation of the physical fields in an elastic linear medium can be generally described
 by a system of partial differential equation of first order. In absence of sources localized at finite
 or in presence of plane wave sources, the system is constituted of the translational equation of
 motion and the stress-displacement equation [25]- [26], i.e. considering dyadic notation and time
 harmonic regime we have

$$\nabla \cdot \underline{T} = -\rho\omega^2 \mathbf{u}, \quad (2.1)$$

$$\underline{S} = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})'), \quad (2.2)$$

where \underline{T} , \underline{S} , \mathbf{u} are respectively the stress tensor, the strain tensor and the displacement vector and,
 ρ is the mass density ($'$ stands for transpose). In a general media the stress and strain tensors have
 constitutive relation given by the Hooke's law

$$\underline{T} = \underline{C} : \underline{S}, \quad (2.3)$$

where \underline{C} is a fourth order stiffness tensor that in isotropic media simplifies to

$$\underline{C} = \lambda \underline{I} \underline{I} + 2\mu \underline{I}^{sym}, \quad (2.4)$$

129 where λ and μ are the Lamé's constants of the elastic medium and, \underline{I} and \underline{I}^{sym} are respectively
 130 the unit dyadic and the symmetric fourth order unit dyadic (tetradic).

Using vector (Voigt) representation for tensor quantities [25] we re-write (2.1) as

$$\nabla_T \mathbf{T} = j\omega \mathbf{p}, \quad (2.5)$$

$$\nabla_v \mathbf{v} = j\omega \mathbf{S}, \quad (2.6)$$

with

$$\nabla_T = \begin{pmatrix} \frac{\partial}{\partial x} & 0 & 0 & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{pmatrix}, \quad \nabla_v = \begin{pmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{pmatrix} = (\nabla_T)', \quad (2.7)$$

131 and where \mathbf{T} , \mathbf{S} , \mathbf{p} and \mathbf{v} are respectively the symmetric stress tensor in six-component vector
132 form (2.8), the symmetric strain tensor in six-component vector form (2.8), the vector momentum
133 density $\mathbf{p} = \rho\mathbf{v}$ and the vector particle velocity $\mathbf{v} = j\omega\mathbf{u}$:

$$\mathbf{T} = (T_{xx}, T_{yy}, T_{zz}, T_{yz}, T_{xz}, T_{xy})', \quad \mathbf{S} = (S_{xx}, S_{yy}, S_{zz}, 2S_{yz}, 2S_{xz}, 2S_{xy})'. \quad (2.8)$$

134 Inspired by [1] for electromagnetic applications, to effectively study wave motion problems in
135 elasticity, it is convenient to introduce the concept of transverse equations using abstract notation.
The homogeneous abstract form of (2.5) and (2.6), see section 2.9 of [4], is

$$\Gamma_{\nabla} \psi = j\omega\theta, \quad (2.9)$$

where Γ_{∇} is a matrix differential operator of first order that relates the fields ψ and θ :

$$\psi = \begin{pmatrix} \mathbf{T} \\ \mathbf{v} \end{pmatrix}, \quad \theta = \begin{pmatrix} \mathbf{S} \\ \mathbf{p} \end{pmatrix}, \quad \Gamma_{\nabla} = \begin{pmatrix} 0 & \nabla_v \\ \nabla_T & 0 \end{pmatrix}. \quad (2.10)$$

The vectors ψ and θ have constitutive relation defined by the equation

$$\theta = \mathbb{W} \psi, \quad (2.11)$$

136 where the matrix \mathbb{W} depends on the medium that is considered.

137 In order to close the mathematical problem (2.9)-(2.11), we need to enforce the geometrical
138 domain of the problem, its boundaries conditions and the radiation condition.

139 For simplicity, in the following, we consider isotropic loss-less material, however we claim that
140 transversal elastic equations in a general anisotropic medium assume the same form. Considering
141 the Hooke's law $\mathbf{T} = \mathbb{C}\mathbf{S}$ in lossless isotropic medium we obtain

$$\mathbb{W} = \begin{pmatrix} \mathbb{C}^{-1} & \mathbb{O} \\ \mathbb{O} & \mathbb{R} \end{pmatrix}, \quad \mathbb{C} = \begin{pmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix}, \quad \mathbb{R} = \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho \end{pmatrix}. \quad (2.12)$$

142 In the following we use also alternative parameters to define the medium characteristics with
143 respect to the mass density ρ , and the Lamé's constants λ and μ :

$$k_p = \omega \sqrt{\frac{\rho}{\lambda + 2\mu}}, \quad k_s = \omega \sqrt{\frac{\rho}{\mu}}, \quad Z_o = \frac{k_s \mu}{\omega}, \quad (2.13)$$

144 where k_p is the propagation constant of the longitudinal/principal wave, k_s is the propagation
145 constant of the transversal/secondary wave (vertical or horizontal) and the impedance Z_o is a
146 quantity such that stress components have same dimensions of velocity components time Z_o .

147 Comparing the equations (2.9)-(2.12) to the ones reported in [1] for electromagnetic
148 applications, we note that the stress \mathbf{T} , the particle velocity \mathbf{v} , the strain \mathbf{S} and the momentum
149 density \mathbf{p} are analogous respectively to the electric field \mathbf{E} , the magnetic field \mathbf{H} , the dielectric
150 induction \mathbf{D} and the magnetic induction \mathbf{B} with constitutive relations $\mathbf{T} = \mathbb{C}\mathbf{S}$ and $\mathbf{p} = \rho\mathbf{v}$
151 analogous respectively to $\mathbf{E} = \varepsilon^{-1}\mathbf{D}$ and $\mathbf{B} = \mu\mathbf{H}$ (where ε , μ can be either scalar or tensor).
152 Moreover (2.5)-(2.6) are the elastic analogue of Maxwell's equations in electromagnetism.

153 Substituting (2.11) into (2.9) with (2.12)-(2.13) we get the nine equations that relate the stress \mathbf{T}
154 with the velocity \mathbf{v} [4]:

$$(\Gamma_{\nabla} - j\omega\mathbb{W})\boldsymbol{\psi} = \mathbf{0}, \quad (2.14)$$

155 whose explicit form is

$$\left\{ \begin{array}{l} D_x T_{xx} + D_z T_{xz} + D_y T_{xy} = jk_s Z_o v_x \\ D_y T_{yy} + D_z T_{yz} + D_x T_{xy} = jk_s Z_o v_y \\ D_z T_{zz} + D_y T_{yz} + D_x T_{xz} = jk_s Z_o v_z \\ D_x v_x = \frac{jk_s [2k_p^2 (T_{xx} - T_{yy} - T_{zz}) + k_s^2 (-2T_{xx} + T_{yy} + T_{zz})]}{8k_p^2 Z_o - 6k_s^2 Z_o} \\ D_y v_y = \frac{jk_s [k_s^2 (T_{xx} - 2T_{yy} + T_{zz}) - 2k_p^2 (T_{xx} - T_{yy} + T_{zz})]}{8k_p^2 Z_o - 6k_s^2 Z_o} \\ D_z v_z = \frac{jk_s [k_s^2 (T_{xx} + T_{yy} - 2T_{zz}) - 2k_p^2 (T_{xx} + T_{yy} - T_{zz})]}{8k_p^2 Z_o - 6k_s^2 Z_o} \\ D_z v_y + D_y v_z = \frac{jk_s T_{yz}}{Z_o} \\ D_z v_x + D_x v_z = \frac{jk_s T_{xz}}{Z_o} \\ D_y v_x + D_x v_y = \frac{jk_s T_{xy}}{Z_o} \end{array} \right. , \quad (2.15)$$

156 where $D_x = \frac{\partial}{\partial x}$, $D_y = \frac{\partial}{\partial y}$, $D_z = \frac{\partial}{\partial z}$.

157 While the constitutive parameters change only in one direction, say y , using the divergence
158 theorem [25], it is possible to demonstrate that the continuous components of $\boldsymbol{\psi}$ at interfaces are
159 the ones of \mathbf{v} and $\mathbf{n} \cdot \underline{T}$, where \mathbf{n} is the unit normal at the interface, i.e.

$$\boldsymbol{\psi}_t = (T_{yy}, T_{yz}, T_{xy}, v_x, v_y, v_z)'. \quad (2.16)$$

The transverse equations of a field are equations that involve only the components that remain continuous along the stratification according to the boundary conditions on the interfaces and, starting from (2.15), in general they assume the following form

$$-\frac{\partial}{\partial y} \boldsymbol{\psi}_t = \mathcal{M} \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial x} \right) \boldsymbol{\psi}_t \quad (2.17)$$

160 where we have a first order derivative along y and a matrix differential operator in x and z .

161 The reduction of the elastic differential problems to the transverse equations starts from
162 deriving expressions of the discontinuous components (along y) direction (T_{xx}, T_{zz}, T_{xz}) from
163 the 4th, the 6th and the 8th of (2.15). We get:

$$\left\{ \begin{array}{l} T_{xx} = \frac{k_p^2 (-2k_s T_{yy} + 4jZ_o (D_x v_x + D_z v_z)) + k_s^2 (k_s T_{yy} - 2jZ_o (2D_x v_x + D_z v_z))}{k_s^3} \\ T_{zz} = \frac{k_p^2 (-2k_s T_{yy} + 4jZ_o (D_x v_x + D_z v_z)) + k_s^2 (k_s T_{yy} - 2jZ_o (D_x v_x + 2D_z v_z))}{k_s^3} \\ T_{xz} = -\frac{j(D_z v_x + D_x v_z)Z_o}{k_s} \end{array} \right. . \quad (2.18)$$

164 By substituting (2.18) into the six non used equations of (2.15) (i.e. equations at line 1,2,3,5,7,9) we
165 get the transverse equations (2.17) where

$$\mathcal{M} \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial x} \right) = \begin{pmatrix} 0 & D_z & D_x & 0 & -jk_s Z_o & 0 \\ D_z - \frac{2D_z k_p^2}{k_s^2} & 0 & 0 & \frac{jD_x D_z (4k_p^2 - 3k_s^2)Z_o}{k_s^3} & 0 & M_{26}(D_z, D_x) \\ D_x - \frac{2D_x k_p^2}{k_s^2} & 0 & 0 & M_{34}(D_z, D_x) & 0 & \frac{jD_x D_z (4k_p^2 - 3k_s^2)Z_o}{k_s^3} \\ 0 & 0 & -\frac{jk_s}{Z_o} & 0 & D_x & 0 \\ -\frac{jk_p^2}{k_s Z_o} & 0 & 0 & D_x - \frac{2D_x k_p^2}{k_s^2} & 0 & D_z - \frac{2D_z k_p^2}{k_s^2} \\ 0 & -\frac{jk_s}{Z_o} & 0 & 0 & D_z & 0 \end{pmatrix}, \quad (2.19)$$

$$M_{34}(D_z, D_x) = -\frac{j(k_s^4 + (4D_x^2 + D_z^2)k_s^2 - 4D_x^2k_p^2)Z_o}{k_s^3}, \quad (2.20)$$

$$M_{26}(D_z, D_x) = -\frac{j(k_s^4 + (D_x^2 + 4D_z^2)k_s^2 - 4D_z^2k_p^2)Z_o}{k_s^3}, \quad (2.21)$$

167 and where $D_x = \frac{\partial}{\partial x}$, $D_y = \frac{\partial}{\partial y}$, $D_z = \frac{\partial}{\partial z}$.

168 The transverse equations along y direction takes the form reported in (2.17) where $\mathcal{M}(\frac{\partial}{\partial z}, \frac{\partial}{\partial x})$
 169 is matrix differential operator of arbitrary differential order and dimension that, in case of
 170 electromagnetic and elastic problems, have respectively dimension 4 and 6, both with differential
 171 order 2 in x and z . In the following, we assume that the geometry of the elastic wave-motion
 172 problem as well as the eventual boundary conditions are invariant along the z -direction, thus,
 173 without loss of generality, when a source depends on a $e^{-j\alpha_o z}$ factor, also the total field depends
 174 on the same factor, i.e. $\psi_t = \psi_t(x, y, z) = \mathbf{f}(x, y)e^{-j\alpha_o z}$, see for instance [17] before (2.8). Of
 175 course, the same behavior can be obtained by applying Fourier transform also along z direction
 176 and assuming an incident plane wave with a particular skew direction that yields $e^{-j\alpha_o z}$.
 177 However, for simplicity, we prefer to avoid the use of a double Fourier transform, recalling that
 178 in the present context an arbitrary source can be expanded in a summation of plane waves.

It yields $\frac{\partial}{\partial z}\psi_t(x, y, z) = -j\alpha_o\psi_t(x, y, z)$, i.e. $\frac{\partial}{\partial z} \rightarrow -j\alpha_o$, thus

$$\mathcal{M}(\frac{\partial}{\partial z}, \frac{\partial}{\partial x}) = \mathcal{M}(-j\alpha_o, \frac{\partial}{\partial x}) = \mathbb{M}_0 + \mathbb{M}_1 \frac{\partial}{\partial x} + \mathbb{M}_2 \frac{\partial^2}{\partial x^2}, \quad (2.22)$$

where \mathbb{M}_m with $m = 0, 1, 2$ are 6x6 matrices and do not depend on x , as they are easily derived from (2.19):

$$\mathbb{M}_0 = \begin{pmatrix} 0 & -j\alpha_o & 0 & 0 & -jk_s Z_o & 0 \\ -j\alpha_o \left(1 - \frac{2k_p^2}{k_s^2}\right) & 0 & 0 & 0 & 0 & -\frac{jZ_o(4\alpha_o^2 k_p^2 + k_s^4 - 4\alpha_o^2 k_s^2)}{k_s^3} \\ 0 & 0 & 0 & -\frac{jZ_o(k_s^2 - \alpha_o^2)}{k_s} & 0 & 0 \\ 0 & 0 & -\frac{jk_s}{Z_o} & 0 & 0 & 0 \\ -\frac{jk_p^2}{k_s Z_o} & 0 & 0 & 0 & 0 & -j\alpha_o \left(1 - \frac{2k_p^2}{k_s^2}\right) \\ 0 & -\frac{jk_s}{Z_o} & 0 & 0 & -j\alpha_o & 0 \end{pmatrix}, \quad (2.23)$$

$$\mathbb{M}_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\alpha_o Z_o(4k_p^2 - 3k_s^2)}{k_s^3} & 0 & 0 \\ 1 - \frac{2k_p^2}{k_s^2} & 0 & 0 & 0 & 0 & \frac{\alpha_o Z_o(4k_p^2 - 3k_s^2)}{k_s^3} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 - \frac{2k_p^2}{k_s^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (2.24)$$

$$\mathbb{M}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{jZ_o}{k_s} \\ 0 & 0 & 0 & \frac{4jZ_o(k_p^2 - k_s^2)}{k_s^3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (2.25)$$

179 (a) The eigenvalues and the eigenvectors of \mathcal{M} in spectral domain

By applying Fourier transform along x direction to (2.17) with (2.22)-(2.25) ($\mathbb{M}_m = 0$, $m > 2$) in absence of source, we obtain an ordinary vector first order differential equation

$$-\frac{d}{dy}\Psi_t(\eta) = \mathbb{M}(\eta)\Psi_t(\eta), \quad (2.26)$$

where $\psi_t(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi_t(\eta)e^{-j\eta x} d\eta$ (notation with omission of y, z dependence) and

$$\mathbb{M}(\eta) = \mathcal{M}(-j\alpha_o, -j\eta) = \mathbb{M}_o - j\eta\mathbb{M}_1 - \eta^2\mathbb{M}_2, \quad (2.27)$$

180 where $\frac{\partial}{\partial z} \rightarrow -j\alpha_o$ for the field factor $e^{-j\alpha_o z}$ (see comment before (2.22)) and $\frac{\partial}{\partial x} \rightarrow -j\eta$ for the
181 property of Fourier transforms.

Now, let us investigate the properties of the eigenvalue problem (2.28) associated to (2.26):

$$\mathbb{M}(\eta)\mathbf{u}_i(\eta) = \lambda_i(\eta)\mathbf{u}_i(\eta), \quad (2.28)$$

where $\mathbf{u}_i(\eta)$ and λ_i ($i = 1..n$) are respectively the eigenvectors and the eigenvalues of the 6×6 matrix $\mathbb{M}(\eta)$ (2.27). In presence of a passive medium we observe that three eigenvalues (say $\lambda_1, \lambda_2, \lambda_3$) present non-negative real part and the other three eigenvalues (say $\lambda_4, \lambda_5, \lambda_6$) present non-positive real part. In the following we use also alternative expressions:

$$\lambda_1 = j\xi_p(\eta) = -\lambda_4, \quad \lambda_2 = \lambda_3 = j\xi_s(\eta) = -\lambda_5 = -\lambda_6. \quad (2.29)$$

The explicit form of (2.29) are expressed in terms of $\tau_{op} = \sqrt{k_p^2 - \alpha_o^2}$, $\tau_{os} = \sqrt{k_s^2 - \alpha_o^2}$

$$\xi_p(\eta) = \sqrt{\tau_{op}^2 - \eta^2}, \quad \xi_s(\eta) = \sqrt{\tau_{os}^2 - \eta^2}, \quad (2.30)$$

182 with $Im[k_{p,s}] < 0$, $Im[\tau_{op,os}] < 0$ in lossy media. Since $k_{p,s}^2 = k_x^2 + k_y^2 + k_z^2 = \eta^2 + \xi_{p,s}^2 + \alpha_o^2$,
183 $\xi_{p,s}(\eta)$ are multivalued functions of η . In the following we assume as proper sheets of $\xi_{p,s}(\eta)$, the
184 ones with $\xi_{p,s}(0) = \tau_{op,os}$ and as branch lines of $\xi_{p,s}(\eta)$ the classical line $Im[\xi_{p,s}(\eta)] = 0$ (see in
185 practical engineering estimations Ch. 5.3b of [32]) or the vertical line ($Re[\eta] = Re[\tau_{os,op}]$, $Im[\eta] <$
186 $Im[\tau_{os,op}]$). In (2.29) we have that $\lambda_1, \lambda_2, \lambda_3$ ($\lambda_4, \lambda_5, \lambda_6$) are related to progressive (regressive)
187 waves and, $\xi_{p,s}$ are with non-positive imaginary part. In this framework we associate the direction
188 of propagation to attenuation phenomena.

Since the matrix $\mathbb{M}(\eta)$ is diagonalizable, $\mathbb{M}(\eta)$ is semi-simple² [33], Ch. V.9. The semi-simple property is fundamental to develop the procedure as it yields a set of independent eigenvectors $\mathbf{u}_i(\eta)$ even with same eigenvalues. Although a theory about geometric and mathematical multiplicity of eigenvalues is available, in practice, we checked the diagonalizability of $\mathbb{M}(\eta)$ using Jordan decomposition algorithm that in our case yields $\mathbb{M}(\eta) = \mathbb{U}^{-1}\mathbb{D}\mathbb{U}$ where the matrix \mathbb{U} is a matrix with column elements $\mathbf{u}_i(\eta)$ and \mathbb{D} is a diagonal matrix with diagonal elements the eigenvalues λ_i . In relation to the eigenvectors $\mathbf{u}_i(\eta)$, we introduce the reciprocal vectors $\boldsymbol{\nu}_i(\eta)$ (see chapter 3.16 of [33]) that, in the general elastic case with $\alpha_o \neq 0$, can be computed by inversion of the matrix \mathbb{U} . The vectors $\boldsymbol{\nu}_i(\eta)$ satisfy the bi-orthogonal relations

$$\boldsymbol{\nu}_j \cdot \mathbf{u}_i = \delta_{ji}, \quad i.e. \quad \underline{\mathbb{1}}_t = \sum_{i=1}^6 \mathbf{u}_i \boldsymbol{\nu}_i, \quad (2.31)$$

189 where \cdot is the vector scalar product, δ_{ij} is the Kronecker symbol and, $\underline{\mathbb{1}}_t$ is the unit dyadic defined
190 in terms of dyadic products and such that $\underline{\mathbb{1}}_t \cdot \mathbf{a} = \mathbf{a} \cdot \underline{\mathbb{1}}_t = \mathbf{a}$ for an arbitrary vector \mathbf{a} .

191 From a physics point of view, the eigenvalues $\lambda_1 = -\lambda_4$ are associated to longitudinal P
192 (principal) waves, while $\lambda_2 = -\lambda_5$ and $\lambda_3 = -\lambda_6$ are relevant to the transversal S (secondary)
193 waves of two types: secondary vertical (SV) and secondary horizontal (SH). The P, SV and SH
194 waves are not decoupled when $\alpha_o \neq 0$, while if $\alpha_o = 0$ we have two decoupled problems: one
195 related to P and SV waves (planar problem) and the other to SH waves (antiplanar problem).

²A square matrix of dimension n is called semi-simple iff it has a basis of eigenvectors in \mathbb{R}^n .

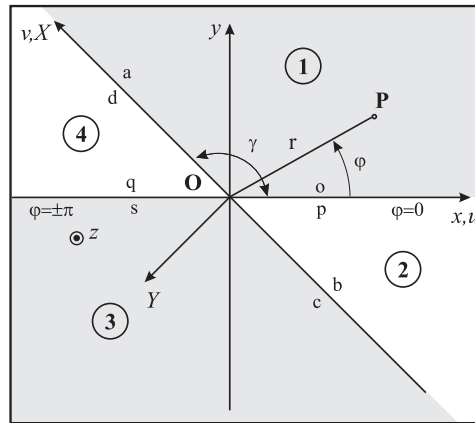


Figure 1. Angular regions and oblique Cartesian coordinates. The figure reports the x, y, z Cartesian coordinates and r, φ, z cylindrical coordinates useful to define the oblique Cartesian coordinate system u, v, z with reference to the angular region 1 $0 < \varphi < \gamma$ with $0 < \gamma < \pi$. In the figure, the space is divided into four angular regions delimited by $\varphi = \pm\gamma, 0, \pi$, and the face boundaries are labeled a, b, c, d, o, p, q, s . The figure reports also the local-to-face-a Cartesian coordinate system $X, Y, Z \equiv z$. Note that $x \equiv u$ and $v \equiv X$.

The computation of eigenvectors in (2.28), using Wolfram Mathematica[®], it yields in compact notation

$$\mathbb{U} = \begin{pmatrix} \frac{Z_o(\alpha_o^2 + \eta^2 - \xi_s^2)}{k_s \alpha_o} & -\frac{2Z_o \xi_s}{k_s} & 0 & \frac{Z_o(\alpha_o^2 + \eta^2 - \xi_s^2)}{k_s} & \frac{2Z_o \xi_s}{k_s} & 0 \\ -\frac{2Z_o \xi_p}{k_s} & -\frac{\alpha_o Z_o}{k_s} & -\frac{Z_o \xi_s}{k_s} & \frac{2Z_o \alpha_o}{k_s} & -\frac{\alpha_o Z_o}{k_s} & \frac{Z_o \xi_s}{k_s} \\ -\frac{2\eta Z_o \xi_p}{k_s \alpha_o} & \frac{Z_o(\xi_s^2 - \eta^2)}{k_s \eta} & \frac{\alpha_o Z_o \xi_s}{k_s \eta} & \frac{2\eta Z_o \xi_p}{k_s \alpha_o} & \frac{Z_o(\xi_s^2 - \eta^2)}{k_s \eta} & -\frac{\alpha_o Z_o \xi_s}{k_s \eta} \\ \frac{\eta}{\alpha_o} & -\frac{\xi_s}{\eta} & -\frac{\alpha_o}{\eta} & \frac{\eta}{\alpha_o} & \frac{\xi_s}{\eta} & -\frac{\alpha_o}{\eta} \\ \frac{\xi_p}{\alpha_o} & 1 & 0 & -\frac{\xi_p}{\alpha_o} & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}, \quad (2.32)$$

¹⁹⁶ whose columns are $\mathbf{u}_i(\eta)$ corresponding to the eigenvalues as defined and ordered in (2.29). The inverse of \mathbb{U} yields in its rows the reciprocal vectors $\mathbf{v}_i(\eta)$:

$$\mathbb{V} = \begin{pmatrix} -\frac{\alpha_o}{2k_s Z_o} & -\frac{\alpha_o^2}{2k_s Z_o \xi_p} & -\frac{\alpha_o \eta}{2k_s Z_o \xi_p} & \frac{\alpha_o \eta}{k_s^2} & -\frac{\alpha_o(\alpha_o^2 + \eta^2 - \xi_s^2)}{2k_s^2 \xi_p} & \frac{\alpha_o^2}{k_s^2} \\ -\frac{\alpha_o^2 + \eta^2}{2k_s Z_o \xi_s} & \frac{\alpha_o}{2k_s Z_o} & \frac{\eta}{2k_s Z_o} & \frac{\eta(\alpha_o^2 + \eta^2 - \xi_s^2)}{2k_s^2 \xi_s} & \frac{\alpha_o^2 + \eta^2}{k_s^2} & \frac{\alpha_o(\alpha_o^2 + \eta^2 - \xi_s^2)}{2k_s^2 \xi_s} \\ \frac{\alpha_o}{2k_s Z_o} & -\frac{(k_s - \alpha_o)(k_s + \alpha_o)}{2k_s Z_o \xi_s} & \frac{\alpha_o \eta}{2k_s Z_o \xi_s} & -\frac{\alpha_o \eta}{k_s^2} & -\frac{\alpha_o \xi_s}{k_s^2} & \frac{1}{2} - \frac{\alpha_o^2}{k_s^2} \\ -\frac{\alpha_o}{2k_s Z_o} & \frac{\alpha_o^2}{2k_s Z_o \xi_p} & \frac{\alpha_o \eta}{2k_s Z_o \xi_p} & \frac{\alpha_o \eta}{k_s^2} & \frac{\alpha_o(\alpha_o^2 + \eta^2 - \xi_s^2)}{2k_s^2 \xi_p} & \frac{\alpha_o^2}{k_s^2} \\ \frac{\alpha_o^2 + \eta^2}{2k_s Z_o \xi_s} & \frac{\alpha_o}{2k_s Z_o} & \frac{\eta}{2k_s Z_o} & -\frac{\eta(\alpha_o^2 + \eta^2 - \xi_s^2)}{2k_s^2 \xi_s} & \frac{\alpha_o^2 + \eta^2}{k_s^2} & -\frac{\alpha_o(\alpha_o^2 + \eta^2 - \xi_s^2)}{2k_s^2 \xi_s} \\ \frac{\alpha_o}{2k_s Z_o} & \frac{(k_s - \alpha_o)(k_s + \alpha_o)}{2k_s Z_o \xi_s} & -\frac{\alpha_o \eta}{2k_s Z_o \xi_s} & -\frac{\alpha_o \eta}{k_s^2} & \frac{\alpha_o \xi_s}{k_s^2} & \frac{1}{2} - \frac{\alpha_o^2}{k_s^2} \end{pmatrix}. \quad (2.33)$$

¹⁹⁷ In the following Sections 3-5, the eigenvectors $\mathbf{u}_i(\eta)$ and the reciprocal vectors $\mathbf{v}_i(\eta)$ will be used
¹⁹⁹ to obtain functional equations that relates spectral quantities in elastic wave motion problems
²⁰⁰ between the two terminal faces of an angular region for an arbitrary α_o , i.e. non planar problems.
²⁰¹ We also note that $\mathbf{u}_i(\eta)$ and $\mathbf{v}_i(\eta)$ can be used to build the solution of the transverse equations
²⁰² (2.26) in Laplace domain for elastic wave motion problems in a rectangular stratified region [31].

²⁰³ 3. First order differential oblique equations for continuous ²⁰⁴ components of the elastic field in an angular region

²⁰⁵ In this section we introduce the oblique equations for continuous components of the elastic field in
²⁰⁶ an angular region using an oblique system of Cartesian axes and applying the properties reported

207 in Section 2 for rectangular regions. In the following sections, first, we deduce spectral functional
208 equations then, by imposing boundary conditions, the GWHs for angular shaped regions.

With reference to Fig. 1 where angular regions are defined thorough the angle γ ($0 < \gamma < \pi$), we introduce the oblique Cartesian coordinates u, v, z in terms of the Cartesian coordinates x, y, z :

$$u = x - y \cot \gamma, v = \frac{y}{\sin \gamma} \text{ or } x = u + v \cos \gamma, y = v \sin \gamma, \quad (3.1)$$

with partial derivatives

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial u}{\partial x} \frac{\partial}{\partial u} + \frac{\partial v}{\partial x} \frac{\partial}{\partial v} = \frac{\partial}{\partial u}, & \frac{\partial}{\partial y} &= \frac{\partial u}{\partial y} \frac{\partial}{\partial u} + \frac{\partial v}{\partial y} \frac{\partial}{\partial v} = -\cot \gamma \frac{\partial}{\partial u} + \frac{1}{\sin \gamma} \frac{\partial}{\partial v}, \\ \frac{\partial}{\partial u} &= \frac{\partial x}{\partial u} \frac{\partial}{\partial x} + \frac{\partial y}{\partial u} \frac{\partial}{\partial y} = \frac{\partial}{\partial x}, & \frac{\partial}{\partial v} &= \frac{\partial x}{\partial v} \frac{\partial}{\partial x} + \frac{\partial y}{\partial v} \frac{\partial}{\partial y} = \cos \gamma \frac{\partial}{\partial x} + \sin \gamma \frac{\partial}{\partial y}. \end{aligned} \quad (3.2)$$

Starting from (2.17) with (2.22) the transverse (with respect to y) equation of dimension $n = 6$ for an elastic problem with invariant geometry along z -direction (i.e. $e^{-j\alpha_0 z}$) is

$$-\frac{\partial}{\partial y} \psi_t = \mathcal{M}(-j\alpha_0, \frac{\partial}{\partial x}) \psi_t = (\mathbb{M}_0 + \mathbb{M}_1 \frac{\partial}{\partial x} + \mathbb{M}_2 \frac{\partial^2}{\partial x^2}) \psi_t. \quad (3.3)$$

Note that for elastic problems, we have second differential order in x . Substituting (3.2), in particular $\frac{\partial}{\partial x} = \frac{\partial}{\partial u}$ and $\frac{\partial}{\partial y} = -\cot \gamma \frac{\partial}{\partial u} + \frac{1}{\sin \gamma} \frac{\partial}{\partial v}$, into (3.3), we obtain

$$-\frac{\partial}{\partial v} \psi_t = \mathcal{M}_e(-j\alpha_0, \frac{\partial}{\partial u}) \psi_t = (\mathbb{M}_{e0} + \mathbb{M}_{e1} \frac{\partial}{\partial u} + \mathbb{M}_{e2} \frac{\partial^2}{\partial u^2}) \psi_t, \quad (3.4)$$

where

$$\mathbb{M}_{e0} = \mathbb{M}_0 \sin \gamma, \quad \mathbb{M}_{e1} = \mathbb{M}_1 \sin \gamma - \mathbb{I} \cos \gamma, \quad \mathbb{M}_{e2} = \mathbb{M}_2 \sin \gamma. \quad (3.5)$$

For the sake of simplicity and in order to get simple explicit expressions, we consider homogeneous isotropic media filling the angular regions. In this case the explicit forms of \mathbb{M}_{em} , $m = 0, 1, 2$ (3.5) are straightforwardly derived from (2.23)-(2.25). By applying the Fourier transform along $x = u$ direction to (3.4), i.e. $\psi_t(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi_t(\eta) e^{-j\eta x} d\eta$ with notation omitting v, z dependence, we obtain the ordinary system of differential equations

$$-\frac{\partial}{\partial v} \Psi_t = \mathbb{M}_e(\gamma, \eta) \Psi_t \quad (3.6)$$

with

$$\mathbb{M}_e(\gamma, \eta) = \mathcal{M}_e(-j\alpha_0, -j\eta) = \mathbb{M}_{e0} - j\eta \mathbb{M}_{e1} - \eta^2 \mathbb{M}_{e2} \quad (3.7)$$

209 since $\frac{\partial}{\partial u} = \frac{\partial}{\partial x} \overset{FT}{\leftrightarrow} -j\eta$.

210 (a) Link between eigenvalues of $\mathbb{M}(\eta)$ and $\mathbb{M}_e(\gamma, \eta)$

In the oblique coordinate system, the solution of (3.6) is related to the eigenvalue problem

$$\mathbb{M}_e(\gamma, \eta) \mathbf{u}_{ei}(\gamma, \eta) = \lambda_{ei}(\gamma, \eta) \mathbf{u}_{ei}(\gamma, \eta), \quad (3.8)$$

where λ_{ei} and $\mathbf{u}_{ei}(\gamma, \eta)$ ($i = 1..n$) are respectively the eigenvalues and the eigenvectors of the 6×6 matrix $\mathbb{M}_e(\gamma, \eta)$. Using (3.6) and (3.7) equation (3.8) becomes

$$(\mathbb{M}_0 \sin \gamma - j\eta \mathbb{M}_1 \sin \gamma - \eta^2 \mathbb{M}_2 \sin \gamma) \mathbf{u}_{ei}(\gamma, \eta) = (\lambda_{ei}(\gamma, \eta) - j\eta \cos \gamma) \mathbf{u}_{ei}(\gamma, \eta) \quad (3.9)$$

and thus

$$\mathbb{M}(\eta) \mathbf{u}_{ei}(\gamma, \eta) = \left(\frac{\lambda_{ei}(\gamma, \eta) - j\eta \cos \gamma}{\sin \gamma} \right) \mathbf{u}_{ei}(\gamma, \eta). \quad (3.10)$$

Comparing (3.10) with (2.28) we observe the relation among the eigenvalues and the eigenvectors of the two problems. The two problems defined by the matrices $\mathbb{M}(\eta)$ and $\mathbb{M}_e(\gamma, \eta)$ have same

eigenvectors

$$\mathbf{u}_{ei}(\gamma, \eta) = \mathbf{u}_i(\eta), \quad (3.11)$$

thus same reciprocal vectors and related eigenvalues

$$\frac{\lambda_{ei}(\gamma, \eta) - j\eta \cos \gamma}{\sin \gamma} = \lambda_i(\eta). \quad (3.12)$$

Since $\mathbb{M}_e(\gamma, \eta)$ and $\mathbb{M}(\eta)$ have same eigenvectors (3.11), i.e. $\mathbf{u}_i(\eta)$ reported in the columns of (2.32), we note the important property that the eigenvectors of $\mathbb{M}_e(\gamma, \eta)$ do not depends on the aperture angle γ of the angular region (Fig. 1). From (3.12), the eigenvalues λ_{ei} of $\mathbb{M}_e(\gamma, \eta)$ can be re-written using the notation (2.29)-(2.30):

$$\begin{aligned} \lambda_{e1}(\gamma, \eta) &= j(\eta \cos \gamma + \sin \gamma \xi_p(\eta)), \\ \lambda_{e2,e3}(\gamma, \eta) &= j(\eta \cos \gamma + \sin \gamma \xi_s(\eta)), \\ \lambda_{e4}(\gamma, \eta) &= j(\eta \cos \gamma - \sin \gamma \xi_p(\eta)), \\ \lambda_{e5,e6}(\gamma, \eta) &= j(\eta \cos \gamma - \sin \gamma \xi_s(\eta)). \end{aligned} \quad (3.13)$$

211 where the first three λ_{ei} are related to progressive waves and the last three to regressive waves
212 according to the definitions reported in Section 2. The corresponding eigenvectors and reciprocal
213 vectors corresponding to λ_{ei} are \mathbf{u}_i and $\boldsymbol{\nu}_i$ reported in (2.32) and (2.33) according to (3.11).

214 As we will see in the next section, the bi-orthogonal basis \mathbf{u}_i and $\boldsymbol{\nu}_i$ can be used to build the
215 solution of the transverse equations (3.6) in Laplace domain for elastic wave motion problems in
216 an angular region with arbitrary α_o , i.e. non planar problems.

217 4. Solution of the oblique equations for angular regions

With reference to Fig. 1, let us introduce the Laplace transforms of $\psi_t(u, v)$ (2.16)

$$\tilde{\psi}_t(\eta, v) = \int_0^{\infty} e^{j\eta u} \psi_t(u, v) du \quad (4.1)$$

for regions 1,2 and $\tilde{\psi}_t(\eta, v) = \int_{-\infty}^0 e^{j\eta u} \psi_t(u, v) du$ for regions 3,4. The Laplace transforms applied to (3.4) yield:

$$-\frac{d}{dv} \tilde{\psi}_t = \mathbb{M}_e(\gamma, \eta) \tilde{\psi}_t + \boldsymbol{\psi}_s(v) \quad (4.2)$$

with

$$\mathbb{M}_e(\gamma, \eta) = \mathbb{M}_{eo} - j\eta \mathbb{M}_{e1} - \eta^2 \mathbb{M}_{e2}. \quad (4.3)$$

218 Note that (4.3) and (3.7) share the same symbol and explicit mathematical expression, however
219 the first is related to a Fourier transform while the second to a Laplace transform, thus obviously
220 they have the same eigenvalues and eigenvectors.

221 The term $\boldsymbol{\psi}_s(v)$ is obtained from the derivative property of the Laplace transform and for each
222 angular region we obtain a different expression. In particular, we indicate with $\psi_{as}(v)$ the value
223 of $\psi_s(v)$ on the face a, see Fig. 1, ($0 \leq v < +\infty, u = 0_+$), with $\psi_{bs}(v)$ the value of $\psi_s(v)$ on the face
224 b ($-\infty \leq v < 0, u = 0_+$), with $\psi_{cs}(v)$ the value of $\psi_s(v)$ on the face c ($-\infty \leq v < 0, u = 0_-$) and
225 with $\psi_{ds}(v)$ the value of $\psi_s(v)$ on the face d ($0 \leq v < +\infty, u = 0_-$).

226 Since (4.2) is a system of six ordinary differential equations of first order with constant
227 coefficients in a semi-infinite interval, we have mainly two methods for its solution: 1) to apply the
228 dyadic Green's function procedure in v domain, 2) to apply the Laplace transform in v that yields
229 a linear system of six algebraic equations from which one can write down the general solution in
230 terms of eigenvalues and eigenfunctions. We note that both methods are effective and in particular
231 the second method is more useful for representing the spectral solution in each point of the
232 considered angular region. However, it initially introduces complex functions of two variables.
233 As proposed in the following subsections, we prefer the first method because, by this way, we

234 get the functional equations of the angular regions that involve directly complex functions of one
235 variable.

236 Using the concept of non-standard Laplace transforms (see section 1.4 of [4]), the validity of
237 (4.2) and (4.3) in absence of sources is extended to the total fields in presence of plane-wave
238 sources or sources located at infinity from any direction yielding isolated poles in spectral domain.

239 With reference to Fig. 1, let us now focus the attention on the angular region 1 in details. The
240 results for the other regions will follow a similar procedure. We observe that the selection of four
241 angular regions as in Fig. 1 related to a unique aperture angle γ does not limit the applicability
242 of the method. In fact all the equations (once derived) can be used with a different appropriate
243 aperture angle just replacing γ with the proper value. The purpose of deriving the functional
244 equations with a unique γ is related to the fact that we formulate and solve the angular region
245 problems by analyzing once and for all the matrix $\mathbb{M}_e(\gamma, \eta)$ (4.3). We recall also that the imposition
246 of boundary conditions and media for each region will be made only while examining a practical
247 problem and it yields GWHs from the functional equations.

248 (a) Region 1: $u > 0, v > 0$

Focusing the attention on region 1 (Fig. 1), i.e. $u > 0, v > 0$, (4.2) holds with

$$\psi_s(v) = \psi_{as}(v) = -\mathbb{M}_{e1} \psi_t(0_+, v) + j\eta \mathbb{M}_{e2} \psi_t(0_+, v) - \mathbb{M}_{e2} \frac{\partial}{\partial u} \psi_t(0_+, v). \quad (4.4)$$

Equation (4.2) is a system of differential equations of first order of dimension six, whose solution $\tilde{\psi}_t$ is obtainable as sum of a particular integral $\tilde{\psi}_p$ with the general solution of the homogeneous equation $\tilde{\psi}_o$ [11]:

$$\tilde{\psi}_t = \tilde{\psi}_o + \tilde{\psi}_p. \quad (4.5)$$

The solution of the homogeneous equation must satisfy

$$-\frac{d}{dv} \tilde{\psi}_o = \mathbb{M}_e(\gamma, \eta) \tilde{\psi}_o. \quad (4.6)$$

Considering the solution form $\tilde{\psi}_o = C e^{-\lambda(\gamma, \eta)v} \mathbf{u}(\eta)$, the most general solution is

$$\tilde{\psi}_o(\gamma, v) = \sum_{i=1}^6 C_i e^{-\lambda_{ei}(\gamma)v} \mathbf{u}_i(\eta), \quad (4.7)$$

249 where λ_{ei} and \mathbf{u}_i ($i=1..6$) are the eigenvalues and the eigenvectors of the matrix $\mathbb{M}_e(\gamma, \eta)$
250 respectively reported at (3.13) and (2.32).

In presence of a passive medium, following the properties described in Section 2(a), we observe that the first three eigenvalues $\lambda_{ei}, i = 1, 2, 3$ present non-negative real part and are related to progressive waves along positive v direction while the last three eigenvalues $\lambda_{ei}, i = 4, 5, 6$ present non-positive real part and are related to regressive waves. The evaluation of the particular integral $\tilde{\psi}_p(\eta, v)$ of (4.2) is easier if carried out in dyadic notation i.e.

$$-\frac{d}{dv} \tilde{\psi}_t = \underline{M}_e(\gamma, \eta) \cdot \tilde{\psi}_t + \psi_s(v), \quad (4.8)$$

where \underline{M}_e is the dyadic counterpart of the matrix \mathbb{M}_e assuming canonical basis³. It yields

$$\tilde{\psi}_p(\eta, v) = - \int_0^\infty \underline{G}(v, v') \cdot \psi_s(v') dv', \quad (4.9)$$

where $\underline{G}(v, v')$ is the dyadic Green's function of (4.8), i.e. solution of

$$\frac{d}{dv} \underline{G}(v, v') + \underline{M}_e(\gamma, \eta) \cdot \underline{G}(v, v') = \delta(v - v') \underline{1}_t \quad (4.10)$$

251 with the unit dyadic $\underline{1}_t$ of dimension six.

³Any dyadic $\underline{A} = \sum_{ij} A_{ij} \mathbf{e}_i \mathbf{e}_j$ can be represented by a matrix \mathbb{A} with elements A_{ij} where \mathbf{e}_i are unit vectors and vice versa.

Based on the theory reported in [31] and [11], we apply the methodology reported in Section 4 and Appendix B of [1], where we build the dyadic Green's function for arbitrary boundary conditions by selecting progressive and regressive waves in indefinite half-space as homogeneous solutions of (4.10). It yields:

$$\underline{G}(v, v') = \begin{cases} \sum_{i=1}^3 \mathbf{u}_i \boldsymbol{\nu}_i e^{-\lambda_{ei}(\gamma, \eta)(v-v')}, & v > v' \\ -\sum_{i=4}^6 \mathbf{u}_i \boldsymbol{\nu}_i e^{-\lambda_{ei}(\gamma, \eta)(v-v')}, & v < v' \end{cases}. \quad (4.11)$$

252 In our framework, we avoid to impose the boundary condition at this step, since we want to
253 find functional equations that are free of this constraint, as described in [1] based on [11]. Only,
254 while investigating a practical problem, we will impose boundary condition to the functional
255 equations (for instance in region 1 at face $\varphi = 0$ i.e. $u > 0, v = 0$ and face $\varphi = \gamma$ i.e. $u = 0, v > 0$)
256 yielding GWHEs of the problem.

By substituting (4.7) and (4.9) with (4.11) into (4.5), it yields

$$\begin{aligned} \tilde{\boldsymbol{\psi}}_t(\eta, v) = & \sum_{i=1}^6 C_i e^{-\lambda_{ei}(\gamma) v} \mathbf{u}_i - \sum_{i=1}^3 \mathbf{u}_i \boldsymbol{\nu}_i \cdot \int_0^v e^{-\lambda_{ei}(\gamma, \eta)(v-v')} \boldsymbol{\psi}_{as}(v') dv' + \\ & + \sum_{i=4}^6 \mathbf{u}_i \boldsymbol{\nu}_i \cdot \int_v^\infty e^{-\lambda_{ei}(\gamma, \eta)(v-v')} \boldsymbol{\psi}_{as}(v') dv'. \end{aligned} \quad (4.12)$$

257 Looking at the asymptotic behavior of (4.12) for $v \rightarrow +\infty$ we have that the divergent terms are
258 the ones in $\sum_{i=4}^6 C_i e^{-\lambda_{ei}(\gamma) v} \mathbf{u}_i$. For this reason we assume $C_i = 0, i = 4, 5, 6$. Note in particular the
259 vanishing of the last three integral terms as $v \rightarrow +\infty$ (last sum in (4.12)).

Setting $v = 0$ in (4.12), we have

$$\tilde{\boldsymbol{\psi}}_t(\eta, 0) = \sum_{i=1}^3 C_i \mathbf{u}_i + \sum_{i=4}^6 \mathbf{u}_i \boldsymbol{\nu}_i \cdot \int_0^\infty e^{\lambda_{ei}(\gamma, \eta)v'} \boldsymbol{\psi}_{as}(v') dv'. \quad (4.13)$$

Multiplying (4.13) by $\boldsymbol{\nu}_i(\eta)$ for $i = 1..6$, using bi-orthogonality, we obtain

$$\begin{cases} \boldsymbol{\nu}_i \cdot \tilde{\boldsymbol{\psi}}_t(\eta, 0) = C_i, & i = 1, 2, 3 \\ \boldsymbol{\nu}_i \cdot \tilde{\boldsymbol{\psi}}_t(\eta, 0) = \boldsymbol{\nu}_i \cdot \tilde{\boldsymbol{\psi}}_{as}(-j\lambda_{ei}(\gamma, \eta)), & i = 4, 5, 6 \end{cases}, \quad (4.14)$$

where $\lambda_{ei}(\gamma, \eta)$ are reported in (3.13) and $\tilde{\boldsymbol{\psi}}_{as}(\chi)$ is the Laplace transform in v along face a ($v = r$ in cylindrical coordinates)

$$\tilde{\boldsymbol{\psi}}_{as}(\chi) = \int_0^\infty e^{j\chi v} \boldsymbol{\psi}_{as}(v) dv. \quad (4.15)$$

We note that in the first three equations of (4.14) we use progressive reciprocal vectors and we obtain C_i that are needed in the computation of the homogeneous portion of the solution $\tilde{\boldsymbol{\psi}}_t(\eta, v)$ (4.12) through the Green's function method. In particular, the unknowns $C_i, i = 1, 2, 3$ are related to the Laplace transform $\tilde{\boldsymbol{\psi}}_t(\eta, 0)$ evaluated in the lower face of the angular region ($v = 0$). We now focus the attention on the last three equations of (4.14) obtained by using regressive reciprocal vectors that yield the three functional equations of the angular region. We re-write them as

$$\boldsymbol{\nu}_i \cdot \tilde{\boldsymbol{\psi}}_t(\eta, 0) = \boldsymbol{\nu}_i \cdot \tilde{\boldsymbol{\psi}}_{as}(-m_{ai}(\gamma, \eta)), \quad i = 4, 5, 6 \quad (4.16)$$

with

$$\begin{aligned} m_{a4}(\gamma, \eta) = m_p(\gamma, \eta) = j\lambda_{e4}(\gamma, \eta) = -\eta \cos \gamma + \xi_p \sin \gamma, \\ m_{a5, a6}(\gamma, \eta) = m_s(\gamma, \eta) = j\lambda_{e5, e6}(\gamma, \eta) = -\eta \cos \gamma + \xi_s \sin \gamma. \end{aligned} \quad (4.17)$$

260 In (4.16) the Laplace transforms of combinations of the field components defined on the
261 boundaries of an angular region, i.e. $v = 0$ (face o) and $u = 0$ (face a) in Fig. 1, are related to each
262 other. These functional equations are the starting point to define the GWHEs of region 1. They are
263 valid for any linear isotropic elastic medium filling the region. Moreover, in (4.16), we note that

the reciprocal vectors and eigenvectors do not appear in the definitions of the Laplace transforms of the field. Only the eigenvalues are used as argument of the Laplace transforms at the right hand side. In the following we apply the notation $+$ to $\tilde{\psi}_t(\eta, 0)$ and $\tilde{\psi}_{as}(-m_{ai}(\gamma, \eta))$, i.e. $\tilde{\psi}_{t+}(\eta, 0)$ and $\tilde{\psi}_{as+}(-m_{ai}(\gamma, \eta))$, to highlight that these Laplace transforms are plus functions respectively in η and $\chi = -m_{ai}(\gamma, \eta)$, i.e. they are regular in the upper half plane of the complex plane η and χ .

Note that the functional equations (4.16) contains spectral unknowns defined into two different complex planes (η and $\chi = -m_{ai}(\gamma, \eta)$) related together via (4.17) and thus, we impose the boundary conditions we get GWHEs and not CWHEs (except in the case $\gamma = \pi$).

Explicit form of functional equations (4.16) are obtained and reported in Section 5 for isotropic media, however the theory reported in this paper can be applied to more complex media.

(b) From Region 1 to the other angular regions

Now, let us repeat the procedure for region 2 (Fig. 1), i.e. $u > 0, v < 0$. The solution $\tilde{\psi}_t(\eta, v)$ of the system of differential equations of first order of dimension six (4.2) is obtainable as sum (4.5) of the general homogeneous solution $\tilde{\psi}_o$ with a particular integral $\tilde{\psi}_p$ defined in terms of

$$\psi_s(v) = \psi_{bs}(v) = -\mathbb{M}_{e1} \psi_t(0+, v) + j\eta \mathbb{M}_{e2} \psi_t(0+, v) - \mathbb{M}_{e2} \frac{\partial}{\partial u} \psi_t(0+, v). \quad (4.18)$$

in region 2 ($v < 0$). We note that (4.18) is equal to (4.4) but with different support in v . The homogeneous solution takes the form (4.7). In presence of a passive medium, we recall that the first three eigenvalues present non-negative real part and are related to progressive waves along positive v while the last three eigenvalues present non-positive real part and are related to regressive waves, thus looking at the asymptotic behavior of (4.7) for $v \rightarrow -\infty$ we have $C_i = 0, i = 1, 2, 3$. Once obtained the dyadic Green's function specialized for region 2, the solution is

$$\begin{aligned} \tilde{\psi}_t(\eta, v) = & \sum_{i=4}^6 C_i \mathbf{u}_i e^{-\lambda_{ei}(\gamma, \eta)v} - \sum_{i=1}^3 \mathbf{u}_i \nu_i \cdot \int_{-\infty}^v e^{-\lambda_{ei}(\gamma, \eta)(v-v')} \psi_{bs}(v') dv' + \\ & + \sum_{i=4}^6 \mathbf{u}_i \nu_i \cdot \int_v^0 e^{-\lambda_{ei}(\gamma, \eta)(v-v')} \psi_{bs}(v') dv' \end{aligned} \quad (4.19)$$

before imposing the boundary conditions. Setting $v = 0$ in (4.19), we have

$$\tilde{\psi}_t(\eta, 0) = \sum_{i=4}^6 C_i \mathbf{u}_i - \sum_{i=1}^3 \mathbf{u}_i \nu_i \cdot \int_{-\infty}^0 e^{\lambda_{ei}(\gamma, \eta)v'} \psi_{bs}(v') dv' \quad (4.20)$$

Multiplying (4.20) by $\nu_i(\eta)$ for $i = 1..6$, using bi-orthogonality, we obtain

$$\begin{cases} \nu_i \cdot \tilde{\psi}_t(\eta, 0) = C_i, & i = 4, 5, 6 \\ \nu_i \cdot \tilde{\psi}_t(\eta, 0) = -\nu_i \cdot \tilde{\psi}_{bs}(j\lambda_{ei}(\gamma, \eta)), & i = 1, 2, 3 \end{cases} \quad (4.21)$$

where $\lambda_{ei}(\gamma, \eta)$ are reported in (3.13) and where

$$\tilde{\psi}_{bs}(\chi) = \int_{-\infty}^0 e^{-j\chi v} \psi_{bs}(v) dv = \int_0^{\infty} e^{j\chi r} \psi_{bs}(-r) dr \quad (4.22)$$

is the left Laplace transform of $\psi_{bs}(v)$ in v along face b (Fig. 1) or the Laplace transform in r of $\psi_{bs}(-r)$ in cylindrical coordinates (r, φ, z) . The properties of (4.21) are the same as for region 1. In particular, we focus the attention on the last three equations obtained by using progressive reciprocal vectors that yield the functional equations of the angular region. We re-write them as

$$\nu_i \cdot \tilde{\psi}_t(\eta, 0) = -\nu_i \cdot \tilde{\psi}_{bs}(-m_{bi}(\gamma, \eta)), \quad i = 1, 2, 3 \quad (4.23)$$

with

$$\begin{aligned} m_{b1}(\gamma, \eta) = m_{pb}(\gamma, \eta) &= -j\lambda_{e1}(\gamma, \eta) = \eta \cos \gamma + \xi_p \sin \gamma, \\ m_{b2, b3}(\gamma, \eta) = m_{sb}(\gamma, \eta) &= -j\lambda_{e2, e3}(\gamma, \eta) = \eta \cos \gamma + \xi_s \sin \gamma. \end{aligned} \quad (4.24)$$

275 In (4.23) the Laplace transforms of combinations of the field components defined on the
 276 boundaries of an angular region, i.e. $v = 0$ (face o) and $u = 0$ (face b) in Fig. 1, are related together.
 277 These functional equations are the starting point to define the GWHEs of region 2 by imposing
 278 boundary conditions and in particular they can be coupled to the ones of region 1 to build a
 279 structure with two angular regions with different elastic properties.

280 Observing (4.23), we note that at the second members we have that, in general,
 281 $\tilde{\psi}_{bs}(-m_{bi}(\gamma, \eta))$ contains discontinuous field components at the boundary $u = 0, v < 0$ of the
 282 angular region, while $\tilde{\psi}_t(\eta, 0)$ (by definition 2.16) is continuous at the boundary $u > 0, v = 0$.

283 Similarly to what has been done in [1] for electromagnetic applications, we can repeat the
 284 procedure to obtain functional equations for regions 3 and 4 (Fig. 1).

285 5. Explicit form of the functional equations for non planar (3D) 286 problems in angular regions

287 In this Section, according to our opinion, we deduce and report for the first time in literature
 288 explicit spectral functional equations in algebraic form for the non planar (3D) elastic scattering
 289 problem in isotropic angular regions with arbitrary boundary conditions.

290 (a) Explicit form for region 1

291 We remark that (4.16) are the functional equations of region 1 for an elastic wave motion problem
 292 in an isotropic medium at skew (non planar) incidence ($\alpha_o \neq 0$). The functional equations for the
 293 2D (planar and antiplanar) problems are a particular case of the general wave motion problem
 294 with $\alpha_o = 0$. In the following we demonstrate for validation that the GWHEs obtained from the
 295 proposed functional equations enforcing the boundary conditions and the functional equations
 296 obtained in [14] using the Gaudesen (Kirchhoff) integral representations in the natural domain are
 297 identical, although the applied notations are different from each other and not immediate in the
 298 comparison.

299 To explicitly represent (4.16) in region 1, we need ν_i reported in the rows of V (2.33), the
 300 Laplace transform of the field $\tilde{\psi}_t(\eta, 0)$ along $x, u > 0, v = 0_+$ (face o, see Fig. 1) and the Laplace
 301 transform $\tilde{\psi}_{as}(-m_{ai}(\gamma, \eta))$ along $x, u = 0_+, v > 0$ (face a, see Fig. 1). An important property of
 302 functional equations is that they report combination of field components that are continuous
 303 on the two boundary of the angular region. This property is fundamental to enforce boundary
 304 conditions in particular while connecting the angular region to a different body. We observe
 305 that, while $\tilde{\psi}_t(\eta, 0)$ is continuous at face o by definition (2.16), we need some mathematical
 306 manipulations to demonstrate that $\tilde{\psi}_{as}(-m_{ai}(\gamma, \eta))$ (4.4) is defined in terms of continuous field
 307 components at face a for an arbitrary aperture angle γ , since its expression contains potential
 308 discontinuous components such as derivatives of the field. The proof follows.

309 According to a local-to-face-a Cartesian coordinate system $X, Y, Z \equiv z$ (see Fig. 1) we have that

the continuous components of the field are $T_{YY}, T_{YZ}, T_{XY}, v_X, v_Y, v_Z$, but $\tilde{\psi}_{as}(-m_{ai}(\gamma, \eta))$ and thus $\psi_s(v) = \psi_{as}(v)$ are originally defined in terms of $T_{yy}, T_{yz}, T_{xy}, v_x, v_y, v_z$ and their derivatives which in general are discontinuous, see (4.15), (4.4) and (2.16). In fact, the explicit form of $\psi_{as}(v)$ (4.4), using (3.5) and (2.23)-(2.25), is:

$$\psi_{as}(v) = \left(\begin{array}{c} \frac{T_{yy} \cos(\gamma) - T_{xy} \sin(\gamma)}{k_s^3 T_{yz} \cos(\gamma) + Z_o \sin(\gamma) (j D_u v_z k_s^2 - 4 \alpha_o k_p^2 v_x + k_s^2 (\eta v_z + 3 \alpha_o v_x))} \\ \frac{k_s^3 T_{xy} \cos(\gamma) + \sin(\gamma) (2 k_p^2 (-2 j D_u v_x Z_o + k_s T_{yy} - 2 Z_o (\alpha_o v_z + \eta v_x)) + k_s^2 (-k_s T_{yy} + Z_o (4 j D_u v_x + 3 \alpha_o v_z + 4 \eta v_x)))}{k_s^3} \\ \frac{v_x \cos(\gamma) - v_y \sin(\gamma)}{v_x \sin(\gamma) \left(\frac{2 k_p^2}{k_s^2} - 1 \right) + v_y \cos(\gamma)} \\ v_z \cos(\gamma) \end{array} \right) \quad (5.1)$$

with $D_u = \frac{\partial}{\partial u} \Big|_{u=0+}$. As a first step to check the properties of (5.1) on face a, we derive expressions for D_u components of the velocity that appears at the 2nd and 3rd components of (5.1). Noting that $D_u = D_x$ and $D_z = -j \alpha_o$, from the 4th and the 8th basic equations reported in (2.15), we have:

$$\begin{aligned} D_u v_x &= \frac{j k_s [2 k_p^2 (T_{xx} - T_{yy} - T_{zz}) + k_s^2 (-2 T_{xx} + T_{yy} + T_{zz})]}{8 k_p^2 Z_o - 6 k_s^2 Z_o}, \\ D_u v_z &= \frac{j k_s T_{xz}}{Z_o} + j \alpha_o v_x. \end{aligned} \quad (5.2)$$

Substituting (5.2) into (5.1), we get an expression of $\psi_{as}(v)$ in terms of \mathbf{T} and \mathbf{v} components without derivatives but still defined in terms of x, y, z . Now, in order to rewrite $\psi_s(v) = \psi_{as}(v) = \psi_s(X, Y = 0)$ only in term of the local continuous components $T_{YY}, T_{YZ}, T_{XY}, v_X, v_Y, v_Z$ (face a, see Fig. 1), we formulate the rotational problem between components along x, y, z with respect to their definition along X, Y, Z . Without loss of generality, assuming $0 < \gamma < \pi$,

$$\mathbf{T} = \mathbb{R}_a^{-1} \mathbb{T}_a \mathbb{R}_a, \quad (5.3)$$

$$\mathbb{T} = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{xy} & T_{yy} & T_{yz} \\ T_{xz} & T_{yz} & T_{zz} \end{pmatrix}, \quad \mathbb{T}_a = \begin{pmatrix} T_{XX} & T_{XY} & T_{XZ} \\ T_{XY} & T_{YY} & T_{YZ} \\ T_{XZ} & T_{YZ} & T_{ZZ} \end{pmatrix}, \quad \mathbb{R}_a = \begin{pmatrix} \cos(\gamma) & \sin(\gamma) & 0 \\ -\sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (5.4)$$

and

$$\mathbf{v} = \mathbb{R}_a^{-1} \mathbf{v}_a, \quad \mathbf{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}, \quad \mathbf{v}_a = \begin{pmatrix} v_X \\ v_Y \\ v_Z \end{pmatrix}. \quad (5.5)$$

Substituting (5.3) and (5.5) into (5.1) after the application of (5.2), it yields an expression of $\psi_{as}(v)$ in terms of the components \mathbb{T}_a and \mathbf{v}_a in X, Y, Z

$$\psi_{as}(v) = \left(\begin{array}{c} \frac{T_{XY} \sin(\gamma) + T_{YY} \cos(\gamma)}{\frac{\alpha_o Z_o (k_s^2 - 2 k_p^2) (v_X \sin(2\gamma) + v_Y \cos(2\gamma))}{k_s^3} - \frac{\alpha_o v_Y Z_o (k_s^2 - 2 k_p^2)}{k_s^3} + \frac{\eta v_Z Z_o \sin(\gamma)}{k_s} + T_{YZ}} \\ \psi_{as3}(v) \\ \frac{v_X \cos(2\gamma) - v_Y \sin(2\gamma)}{v_Y (k_p^2 \cos(2\gamma) - k_p^2 + k_s^2) + k_p^2 v_X \sin(2\gamma)} \\ v_Z \cos(\gamma) \end{array} \right), \quad (5.6)$$

310 where

$$\begin{aligned} \psi_{as3}(v)(4k_p^2 k_s^3 - 3k_s^5) &= k_s^3 T_{XY} \cos(\gamma) (4k_p^2 - 3k_s^2) + \sin(\gamma) \left[\alpha_o (-v_Z) Z_o (4k_p^2 - 3k_s^2)^2 + \right. \\ &+ k_s (4k_p^4 (T_{XX} + T_{YY} - T_{ZZ}) - 2k_p^2 k_s^2 (2T_{XX} + 4T_{YY} - 3T_{ZZ})) + k_s^4 (T_{XX} + 4T_{YY} - 2T_{ZZ}) \left. \right] + \\ &+ 4\eta Z_o (4k_p^4 - 7k_p^2 k_s^2 + 3k_s^4) (v_Y \sin(\gamma) - v_X \cos(\gamma)) \end{aligned} \quad (5.7)$$

We recall that the procedure aims at finding $\psi_{as}(v)$ in terms of the continuous field $T_{YY}, T_{YZ}, T_{XY}, v_X, v_Y, v_Z$. The result of the proposed substitutions is that the components of $\psi_{as}(v)$ (5.6) are expressed all in terms of the continuous field except the component 3. In fact, from the beginning, the component 3 of (5.1) contains $D_u v_x$ that is represented by the 1st of (5.2) where the discontinuous T_{xx}, T_{zz} are present. The subsequent application of (5.3) and (5.5) do not change the properties $\psi_{as}(v)$ in terms of continuous components and in particular the 3rd component contains the discontinuous components T_{XX}, T_{ZZ} as reported in (5.6) with (5.7). Noting that the basic equations (2.15) are invariant for rotations of the coordinate axes, by applying the 6th of (2.15) in X, Y, Z coordinates we get

$$T_{ZZ} = \frac{k_s (k_s^2 - 2k_p^2) (T_{XX} + T_{YY}) + 2\alpha_o v_Z Z_o (4k_p^2 - 3k_s^2)}{2 (k_s^3 - k_p^2 k_s)} \quad (5.8)$$

The substitution of (5.8) into $\psi_{as3}(v)$ (5.7), after mathematical manipulations, yields an expression in terms of continuous field, whose embedding in (5.6) gives a representation of $\psi_{as}(v)$ only in terms of continuous field at face a:

$$\psi_{as}(v) = \left(\begin{array}{c} \frac{T_{XY} \sin(\gamma) + T_{YY} \cos(\gamma)}{k_s^3} - \frac{\alpha_o v_Y Z_o (k_s^2 - 2k_p^2)}{k_s^3} + \frac{\eta v_Z Z_o \sin(\gamma)}{k_s} + T_{YZ} \\ \frac{\sin(\gamma) (4\eta v_Y Z_o \sin(\gamma) (k_p^2 - k_s^2) + k_s^2 (\alpha_o v_Z Z_o - k_s T_{YY})) + 2\eta v_X Z_o \sin(2\gamma) (k_s - k_p) (k_p + k_s) + k_s^3 T_{XY} \cos(\gamma)}{k_s^3} \\ \frac{v_X \cos(2\gamma) - v_Y \sin(2\gamma)}{k_s^2} \\ \frac{v_Y (k_p^2 \cos(2\gamma) - k_p^2 + k_s^2) + k_p^2 v_X \sin(2\gamma)}{k_s^2} \\ v_Z \cos(\gamma) \end{array} \right) \quad (5.9)$$

From (5.9), we note that $\psi_{as}(v)$ is defined only in term of continuous field component at face a. Now, the application of Laplace transform (4.15) to $\psi_{as}(v)$ yields the explicit expression of the spectral functional equations (4.16) for region 1 in terms of continuous components. We remark that this property is fundamental to easily impose impenetrable boundary conditions and to couple region 1 with other penetrable surrounding regions of arbitrary geometry and in general non-homogeneous to region 1.

311 The property of the elastic wave motion problem to be formulated in terms of a differential
312 problem (4.2) with sources $\psi_{as}(v)$ (5.9) defined only in term of continuous field on the boundary
313 represents an *equivalence theorem in elasticity* analogous to the well-known equivalence theorem
314 in electromagnetism. In fact, the solution is given by $\tilde{\psi}_t(\eta, v)$ (4.12) through the Green's function
315 formulation only in terms of continuous components on the two faces of the angular region (C_i
316 on face o and $\psi_{as}(v)$ on face a), see (4.12)-(4.14). This property is corresponding to the well-
317 known Schelkunoff's equivalence theorem together the uniqueness theorem in electromagnetics
318 [36] where the equivalent sources are defined in terms of the components of electromagnetic field
319 \mathbf{E}, \mathbf{H} tangent (continuous) to (at) the boundaries. A tentative text may be the following.

320 **Equivalence theorem in elasticity:** *A field in a lossy region is uniquely specified by the sources*
321 *within the region plus the continuous components of the fields over the boundary.*

In order to avoid trivial identities for $\alpha_o = 0$ and in order to simplify a little the explicit form of functional equations (4.16), we redefine the reciprocal vectors ν_i starting from the rows $\mathbb{V}(i, :)$, $i = 1..6$ of (2.33) according to the following scaling (reciprocal vectors as eigenvectors are defined up to a multiplicative constant):

$$\begin{aligned} \nu_1 &= \frac{2Z_o \xi_p k_s^2 \mathbb{V}(1, :)}{\alpha_o}, \quad \nu_2 = 2Z_o \xi_s k_s^2 \mathbb{V}(2, :), \quad \nu_3 = 2Z_o \xi_s k_s^2 \mathbb{V}(3, :), \\ \nu_4 &= \frac{2Z_o \xi_p k_s^3 \mathbb{V}(4, :)}{\alpha_o}, \quad \nu_5 = 2Z_o \xi_s k_s^2 \mathbb{V}(5, :), \quad \nu_6 = 2Z_o \xi_s k_s^2 \mathbb{V}(6, :). \end{aligned} \quad (5.10)$$

With (5.10), (4.16) take the form (5.11)-(5.13) where the T, v quantities with lowercase subscripts in the LHS of the equations are defined for $u > 0, v = 0_+$ and are Laplace transforms in η , while the T, v quantities with uppercase subscripts are defined for $u = 0_+, v > 0$ and are Laplace transforms in $-m_p, -m_s, -m_s$ respectively in the RHS of (5.11),(5.12),(5.13).

$$\begin{aligned} & k_s (-T_{yy}\xi_p + \eta T_{xy} + \alpha_o T_{yz}) + Z_o [2\xi_p(\eta v_x + \alpha_o v_z) + v_y (\alpha_o^2 + \eta^2 - \xi_s^2)] = \\ & = Z_o [v_Y (\alpha_o^2 + k_p^2 - k_s^2) + v_X \sin(2\gamma) (\eta^2 - \xi_p^2) + 2\xi_p(\eta v_X \cos(2\gamma) + \\ & - \eta v_Y \sin(2\gamma) + \alpha_o v_Z \cos(\gamma)) + v_Y \cos(2\gamma) (\eta^2 - \xi_p^2) + 2\alpha_o \eta v_Z \sin(\gamma)] + \\ & + k_s [-\xi_p(T_{XY} \sin(\gamma) + T_{YY} \cos(\gamma)) + \eta T_{XY} \cos(\gamma) - \eta T_{YY} \sin(\gamma) + \alpha_o T_{YZ}] \end{aligned} \quad (5.11)$$

$$\begin{aligned} & k_s \xi_s (\eta T_{xy} + \alpha_o T_{yz}) + k_s T_{yy} (\alpha_o^2 + \eta^2) + \\ & + Z_o [\xi_s^2 (\eta v_x + \alpha_o v_z) + 2v_y (\alpha_o^2 + \eta^2) \xi_s - (\alpha_o^2 + \eta^2) (\eta v_x + \alpha_o v_z)] = \\ & = k_s \xi_s [\eta T_{XY} \cos(\gamma) - \eta T_{YY} \sin(\gamma) + \alpha_o T_{YZ}] + \\ & + k_s (\alpha_o^2 + \eta^2) [T_{XY} \sin(\gamma) + T_{YY} \cos(\gamma)] + \\ & + Z_o \{ \xi_s [\xi_s (\eta v_X \cos(2\gamma) - \eta v_Y \sin(2\gamma) + \alpha_o v_Z \cos(\gamma)) + v_X (\alpha_o^2 + 2\eta^2) \sin(2\gamma) + \\ & + v_Y (\alpha_o^2 + 2\eta^2) \cos(2\gamma) + \alpha_o^2 v_Z + 2\alpha_o \eta v_Z \sin(\gamma)] + \\ & - (\alpha_o^2 + \eta^2) [\eta v_X \cos(2\gamma) - \eta v_Y \sin(2\gamma) + \alpha_o v_Z \cos(\gamma)] \} \end{aligned} \quad (5.12)$$

$$\begin{aligned} & k_s^3 T_{yz} + \xi_s \{ Z_o [k_s^2 v_z + 2\alpha_o v_y \xi_s - 2\alpha_o (\eta v_x + \alpha_o v_z)] + \alpha_o k_s T_{yy} \} - \alpha_o k_s (\eta T_{xy} + \alpha_o T_{yz}) = \\ & = Z_o \{ \alpha_o \sin(2\gamma) [v_X (-\alpha_o^2 - 2\eta^2 + k_s^2) + 2\eta v_Y \xi_s] - \alpha_o \cos(2\gamma) [v_Y (\alpha_o^2 + 2\eta^2 - k_s^2) + \\ & + 2\eta v_X \xi_s] + v_Z \cos(\gamma) (k_s^2 - 2\alpha_o^2) \xi_s + \eta v_Z \sin(\gamma) (k_s^2 - 2\alpha_o^2) + \alpha_o v_Y (k_s^2 - \alpha_o^2) \} + \\ & + k_s \{ T_{YZ} (k_s^2 - \alpha_o^2) + \alpha_o \xi_s [T_{XY} \sin(\gamma) + T_{YY} \cos(\gamma)] + \alpha_o \eta [T_{YY} \sin(\gamma) - T_{XY} \cos(\gamma)] \} \end{aligned} \quad (5.13)$$

322 We remark that (5.11)-(5.13) are the functional equations of region 1 for an elastic wave motion
323 problem in an isotropic medium at skew (non planar) incidence ($\alpha_o \neq 0$). These equations,
324 according to our opinion, are deduced and reported for the first time in literature.

325 In particular, by applying the traction-free boundary conditions ($T_{xy} = T_{yy} = T_{yz} = T_{XY} =$
326 $T_{YY} = T_{YZ} = 0$), (5.11)-(5.13) becomes GWHEs formulating the 3D elastic wedge problem
327 considered in [17]. This formulation is important because allows to get semi-analytical solutions
328 via Fredholm factorization method as developed by the authors in [4]. According to the authors'
329 opinion, this technique constitutes a very power tool for the accurate approximate solutions of
330 arbitrary WH equations. We remark that the GWHEs are algebraic, while in [17] the solution
331 is obtained by functional equations written in terms of singular integral operators and solved
332 by numerical technique. We assert that the semi-analytic solution using Fredholm factorization
333 method allows physical insights by asymptotics in spectral domain.

334 (b) Explicit form for region 2

335 In this subsection, we repeat the procedure reported in subsection 5.(a) for region 2 (see Fig. 1),
336 i.e. $u > 0, v < 0$, but with different aperture angle as reported in Fig. 2(b): the aperture angle of
337 region 2 is γ instead of $\pi - \gamma$ as originally taken in Fig. 1. This difference is of great utility in
338 the analysis of wedge structures with symmetries. For this purpose, we first start on deriving
339 functional equations of region 2 (4.23) with the original aperture angle γ (Fig. 1 and Fig. 2(a)) for
340 an elastic wave motion problem in an isotropic medium at skew (non planar) incidence ($\alpha_o \neq 0$).
341 Second, we apply the change in the aperture angle and the rotation of local reference system. To
342 explicitly represent (4.23) for region 2, we need ν_i reported in the rows of \mathbb{V} (2.33), the Laplace
343 transform $\tilde{\psi}_t(\eta, 0)$ along $x, u > 0, v = 0$ (face o) and the Laplace transform $\tilde{\psi}_{bs}(-m_{bi}(\gamma, \eta))$ along
344 $x, u = 0, v < 0$ (face b). We observe that, while $\tilde{\psi}_t(\eta, 0)$ is continuous at face p by definition (2.16),
345 we need some mathematical manipulations to demonstrate that $\tilde{\psi}_{bs}(-m_{bi}(\gamma, \eta))$ (4.18) is defined

346 in terms of continuous field components at face b for an arbitrary aperture angle γ , since its
 347 expression contains potential discontinuous components such as derivatives of the field.

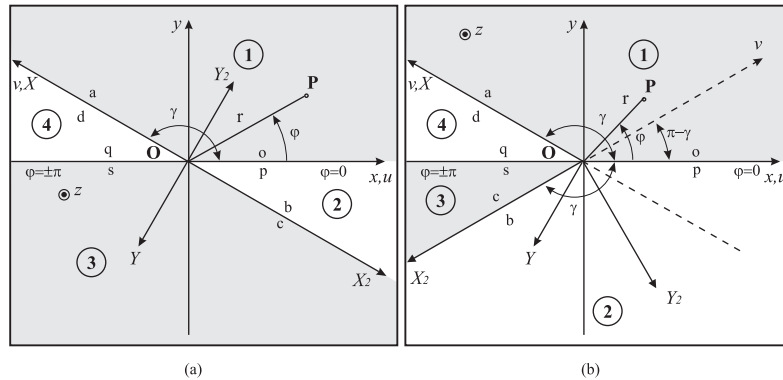


Figure 2. Angular regions and oblique Cartesian coordinates. The left subfigure re-reports Fig. 1 for convenience and it is the reference for the theory developed in the previous sections. The right subfigure shows the new framework of the space divided into four angular regions for wedge structures. We note symmetry between regions 1(3) and 2(4). The figure reports the x, y, z Cartesian coordinates and r, φ, z cylindrical coordinates useful to define the oblique Cartesian coordinate system u, v, z with reference to the angular region 1 $0 < \varphi < \gamma$ with $0 < \gamma < \pi$ and u, v, z with reference to the angular region 2 (only in the right subfigure). The face boundaries are labeled a, b, c, d, o, p, q, s . The figure reports also the local-to-face-a Cartesian coordinate system $X, Y, Z \equiv z$ and the local-to-face-b Cartesian coordinate system $X_2, Y_2, Z_2 \equiv z$ (only in the right subfigure). The $X, Y, Z \equiv z$ and $X_2, Y_2, Z_2 \equiv z$ Cartesian coordinate systems are obtained from x, y, z Cartesian coordinate system by rotation, respectively for a positive γ and $-\gamma$.

According to a local-to-face-b Cartesian coordinate system $X_2, Y_2, Z_2 \equiv z$ (see Fig. 2) we have that the continuous components of the field are $T_{Y_2Y_2}, T_{Y_2Z_2}, T_{X_2Y_2}, v_{X_2}, v_{Y_2}, v_{Z_2}$, but $\tilde{\psi}_{bs}(-m_{bi}(\gamma, \eta))$ and thus $\psi_s(v) = \psi_{bs}(v)$ are defined in terms of $T_{yy}, T_{yz}, T_{xy}, v_x, v_y, v_z$ and their derivatives which in general are discontinuous, see (4.22), (4.18) and (2.16). In fact, the explicit form of $\psi_{bs}(v)$ (4.18), using (3.5) and (2.23)-(2.25), yields the same expression of $\psi_{as}(v)$ given in (5.1), even if $\psi_{bs}(v)$ is defined for $v < 0$ and $\psi_{as}(v)$ for $v > 0$. Following the steps done for $\psi_{as}(v)$ in region 1, we derive expressions for D_u components of the velocity appearing in (5.1). Noting that $D_u = D_x$ and $D_z = -j\alpha_o$, from the 4th and the 8th basic equations reported in (2.15), we have (5.2) that substituted into $\psi_{bs}(v)$ yields an expression in terms of \mathbf{T} and \mathbf{v} components without derivatives but still defined in terms of the coordinate system x, y, z .

Now, in order to rewrite $\psi_s(v) = \psi_{bs}(v) = \psi_s(X_2, Y_2 = 0)$ only in term of the local continuous components $T_{Y_2Y_2}, T_{Y_2Z_2}, T_{X_2Y_2}, v_{X_2}, v_{Y_2}, v_{Z_2}$ (face b), we formulate the rotational problem between components along x, y, z with respect to their definition along X_2, Y_2, Z_2 . The required rotation in Fig. 2(a) is $-\pi + \gamma$. Now, let us introduce also the change of aperture angle from γ to $\pi - \gamma$ as in the right subfigure of Fig. 2. This change of aperture angle impacts on the definitions of \mathbb{M}_{ei} matrices (due to the replacement of γ with $\pi - \gamma$) and then $\psi_{bs}(v)$ that now becomes different from $\psi_{as}(v)$. In the new region 2 (Fig.2(b)) the rotation relations (5.3)-(5.5) of region 1 are replaced by the relations for region 2 where we have performed the substitution $\gamma \rightarrow -\pi + \gamma$ (rotation) and $\gamma \rightarrow \pi - \gamma$ (change of aperture angle), thus $\gamma \rightarrow -\gamma$. It yields:

$$\mathbb{T} = \mathbb{R}_b^{-1} \mathbb{T}_b \mathbb{R}_b, \tag{5.14}$$

$$\mathbb{T} = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{xy} & T_{yy} & T_{yz} \\ T_{xz} & T_{yz} & T_{zz} \end{pmatrix}, \mathbb{T}_b = \begin{pmatrix} T_{X_2X_2} & T_{X_2Y_2} & T_{X_2Z_2} \\ T_{X_2Y_2} & T_{Y_2Y_2} & T_{Y_2Z_2} \\ T_{X_2Z_2} & T_{Y_2Z_2} & T_{Z_2Z_2} \end{pmatrix}, \mathbb{R}_b = \begin{pmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.15}$$

$$\mathbf{v} = \mathbb{R}_b^{-1} \mathbf{v}_b, \quad \mathbf{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}, \quad \mathbf{v}_b = \begin{pmatrix} v_{X2} \\ v_{Y2} \\ v_{Z2} \end{pmatrix}. \quad (5.16)$$

Substituting (5.14) and (5.16) into $\psi_{bs}(v)$ (same expression of $\psi_{as}(v)$ (5.1)) after the application of (5.2) and (5.8) in X_2, Y_2, Z_2 coordinates, it yields an expression of $\psi_{bs}(v)$ in terms of the continuous (at face b) components $T_{Y_2Y_2}, T_{Y_2Z_2}, T_{X_2Y_2}, v_{X_2}, v_{Y_2}, v_{Z_2}$:

$$\psi_{bs}(v) = \begin{pmatrix} \frac{T_{X_2Y_2} \sin(\gamma) - T_{Y_2Y_2} \cos(\gamma)}{\alpha_o v_{X_2} Z_o \sin(2\gamma) (k_s^2 - 2k_p^2) + \alpha_o v_{Y_2} Z_o \cos(2\gamma) (2k_p^2 - k_s^2) + \alpha_o v_{Y_2} Z_o (k_s^2 - 2k_p^2) + \eta k_s^2 v_{Z_2} Z_o \sin(\gamma) - T_{Y_2Z_2}} \\ \frac{\sin(\gamma) [4\eta v_{Y_2} Z_o \sin(\gamma) (k_s^2 - k_p^2) + k_s^2 (\alpha_o v_{Z_2} Z_o - k_s T_{Y_2Y_2})] + 2\eta v_{X_2} Z_o \sin(2\gamma) (k_s^2 - k_p^2) - k_s^3 T_{X_2Y_2} \cos(\gamma)}{k_s^3} \\ \frac{-v_{X_2} \cos(2\gamma) - v_{Y_2} \sin(2\gamma)}{k_p^2 [v_{X_2} \sin(2\gamma) - v_{Y_2} \cos(2\gamma)] + v_{Y_2} (k_p^2 - k_s^2)} \\ -v_{Z_2} \cos(\gamma) \end{pmatrix}. \quad (5.17)$$

Now, the application of Laplace transform (4.22) to $\psi_{bs}(v)$ yields the explicit expression of the spectral functional equations (4.16) for region 2 in terms of continuous components.

Again the property of the elastic wave motion problem to be formulated in terms of a differential problem (4.2) with sources $\psi_{bs}(v)$ (5.17) defined only in term of continuous field on the boundary represents an *equivalence theorem in elasticity* for region 2 as discussed in 5(a).

As done for region 1, in order to avoid trivial identities for $\alpha_o = 0$ and in order to simplify a little the explicit form of (4.23), we redefine the reciprocal vectors as reported in (5.10). With (5.10), (4.23) take the form (5.18)-(5.20) where the T, v quantities with lowercase subscripts in the LHS of the equations are defined for $u > 0, v = 0_-$ and are Laplace transforms in η , while the T, v quantities with uppercase subscripts are defined for $u = 0_+, v < 0$ and are Laplace transforms in $-m_{pb}, -m_{sb}, -m_{sb}$ respectively in the RHS of (5.18),(5.19),(5.20). It yields:

$$\begin{aligned} & Z_o [2\xi_p(\eta v_x + \alpha_o v_z) - v_y (\alpha_o^2 + \eta^2 - \xi_s^2)] - k_s (T_{yy} \xi_p + \eta T_{xy} + \alpha_o T_{yz}) = \\ & = Z_o [-v_{Y2} (\alpha_o^2 + k_p^2 - k_s^2) + v_{X2} \sin(2\gamma) (\eta^2 - \xi_p^2) + 2\xi_p(\eta v_{X2} \cos(2\gamma) + \\ & \quad \eta v_{Y2} \sin(2\gamma) + \alpha_o v_{Z2} \cos(\gamma)) + v_{Y2} \cos(2\gamma) (\xi_p^2 - \eta^2) + 2\alpha_o \eta v_{Z2} \sin(\gamma)] + \\ & -k_s [\xi_p (T_{Y_2Y_2} \cos(\gamma) - T_{X_2Y_2} \sin(\gamma)) + \eta T_{X_2Y_2} \cos(\gamma) + \eta T_{Y_2Y_2} \sin(\gamma) + \alpha_o T_{Y_2Z_2}] \end{aligned} \quad (5.18)$$

$$\begin{aligned} & k_s \xi_s (\eta T_{xy} + \alpha_o T_{yz}) - k_s T_{yy} (\alpha_o^2 + \eta^2) + \\ & + Z_o [\xi_s^2 (-\eta v_x + \alpha_o v_z) + 2v_y (\alpha_o^2 + \eta^2) \xi_s + (\alpha_o^2 + \eta^2) (\eta v_x + \alpha_o v_z)] = \\ & = k_s \xi_s [\eta T_{X_2Y_2} \cos(\gamma) + \eta T_{Y_2Y_2} \sin(\gamma) + \alpha_o T_{Y_2Z_2}] + \\ & -k_s (\alpha_o^2 + \eta^2) [T_{Y_2Y_2} \cos(\gamma) - T_{X_2Y_2} \sin(\gamma)] + \\ & + Z_o \{ \xi_s [-\xi_s (\eta v_{X2} \cos(2\gamma) + \eta v_{Y2} \sin(2\gamma) + \alpha_o v_{Z2} \cos(\gamma)) - v_{X2} (\alpha_o^2 + 2\eta^2) \sin(2\gamma) \\ & \quad + v_{Y2} (\alpha_o^2 + 2\eta^2) \cos(2\gamma) + \alpha_o^2 v_{Z2} - 2 \sin(\gamma) \alpha_o \eta v_{Z2}] + \\ & \quad + (\alpha_o^2 + \eta^2) [\eta v_{X2} \cos(2\gamma) + \eta v_{Y2} \sin(2\gamma) + \alpha_o v_{Z2} \cos(\gamma)] \} \end{aligned} \quad (5.19)$$

$$\begin{aligned} & -k_s^3 T_{yz} + \xi_s \{ Z_o [k_s^2 v_z - 2\alpha_o v_y \xi_s - 2\alpha_o (\eta v_x + \alpha_o v_z)] + \alpha_o k_s T_{yy} \} + \alpha_o k_s (\eta T_{xy} + \alpha_o T_{yz}) = \\ & = Z_o \{ \alpha_o \sin(2\gamma) [v_{X2} (-\alpha_o^2 - 2\eta^2 + k_s^2) - 2\alpha_o \eta v_{Y2} \xi_s] + \alpha_o \cos(2\gamma) [v_{Y2} (\alpha_o^2 + 2\eta^2 - k_s^2) + \\ & \quad - 2\eta v_{X2} \xi_s] + v_{Z2} \cos(\gamma) (k_s^2 - 2\alpha_o^2) \xi_s + \eta v_{Z2} \sin(\gamma) (k_s^2 - 2\alpha_o^2) + \alpha_o v_{Y2} (\alpha_o^2 - k_s^2) \} + \\ & + k_s \{ T_{Y_2Z_2} (\alpha_o^2 - k_s^2) + \alpha_o \xi_s [T_{Y_2Y_2} \cos(\gamma) - T_{X_2Y_2} \sin(\gamma)] + \alpha_o \eta [T_{X_2Y_2} \cos(\gamma) + T_{Y_2Y_2} \sin(\gamma)] \} \end{aligned} \quad (5.20)$$

We remark that (5.18)-(5.20) are the spectral functional equations of region 2 for an elastic wave motion problem in an isotropic medium at skew (non planar) incidence ($\alpha_o \neq 0$). As cross-validation, we note that (5.18)-(5.20) of region 2 are equivalent to (5.11)-(5.13) of region 1,

according to the following replacements dictated by means of symmetry (see Fig. 2):

$$\begin{aligned} \{v_x, v_y, v_z, T_{yy}, T_{xy}, T_{yz}\} &\rightarrow \{v_x, -v_y, v_z, T_{yy}, -T_{xy}, -T_{yz}\}, \\ \{v_{X2}, v_{Y2}, v_{Z2}, T_{Y2Y2}, T_{X2Y2}, T_{Y2Z2}\} &\rightarrow \{v_X, -v_Y, v_Z, T_{YY}, -T_{XY}, -T_{YZ}\}. \end{aligned} \tag{5.21}$$

The procedure reported in this Section can be repeated to get the functional equations for regions 3 and 4 following also the explicit mathematical steps described in [1] for em applications.

6. Validation of functional equations for an isotropic angular region with traction-free boundary conditions in the 2D case

The functional equations for the 2D (planar and antiplanar) problems ($\alpha_o = 0$) are a particular case of the ones obtained for the general 3D problem (5.11)-(5.13) and (5.18)-(5.20) respectively for region 1 and region 2 with reference to the right subfigure of Fig. 2.

Taking into consideration region 1, in the following, we demonstrate that the GWHEs obtained from the proposed functional equations while enforcing the traction-free face boundary conditions in the planar angular problem ($\alpha_o = 0$) and the functional equations obtained in [14] by Gautesen's group are identical, although the applied notations are very different from each other and cumbersome to be compared. Moreover, the functional equation for the anti-planar problem are checked with an independent method, too.

We recall that the explicit functional equations for region 1 reported in (5.11)-(5.13) are derived from (4.16). Since functional equations can be written up to multiplicative constant as eigenvectors, to perform the comparison with compact expressions and to avoid the lack of definition of some eigenvectors/reciprocal vectors for $\alpha_o = 0$, we redefine the reciprocal vectors (2.33) as in the following scaling:

$$\nu_1 = \frac{2\xi_p k_s^2 \mathbb{V}(1, :)}{\alpha_o}, \nu_2 = \frac{2\xi_s k_s^2 \mathbb{V}(2, :)}{\eta}; \nu_3 = 2\mathbb{V}(3, :), \nu_4 = \frac{2\xi_p k_s^2 \mathbb{V}(4, :)}{\alpha_o}, \nu_5 = \frac{2\xi_s k_s^2 \mathbb{V}(5, :)}{\eta}, \nu_6 = 2\mathbb{V}(6, :). \tag{6.1}$$

For readability, we report (6.1) in explicit form for $\alpha_o = 0$ in terms of rows of the following matrix:

$$\mathbb{V}_o = \begin{pmatrix} -\frac{k_s \xi_p}{Z_o} & 0 & -\frac{\eta k_s}{Z_o} & 2\eta \xi_p & \xi_s^2 - \eta^2 & 0 \\ -\frac{\eta k_s}{Z_o} & 0 & \frac{k_s \xi_s}{Z_o} & \eta^2 - \xi_s^2 & 2\eta \xi_s & 0 \\ 0 & -\frac{k_s}{Z_o \xi_s} & 0 & 0 & 0 & 1 \\ -\frac{k_s \xi_p}{Z_o} & 0 & \frac{\eta k_s}{Z_o} & 2\eta \xi_p & \eta^2 - \xi_s^2 & 0 \\ \frac{\eta k_s}{Z_o} & 0 & \frac{k_s \xi_s}{Z_o} & \xi_s^2 - \eta^2 & 2\eta \xi_s & 0 \\ 0 & \frac{k_s}{Z_o \xi_s} & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.2}$$

For $\alpha_o = 0$ we obtain a simplified version of (5.6)

$$\psi_{as}(v) = \begin{pmatrix} T_{XY} \sin(\gamma) + T_{YY} \cos(\gamma) \\ \frac{\eta v_Z \sin(\gamma) Z_o}{k_s} + T_{YZ} \\ \frac{4\eta Z_o \sin(\gamma)(k_p^2 - k_s^2)(v_Y \sin(\gamma) - v_X \cos(\gamma))}{k_s^3} + T_{XY} \cos(\gamma) - T_{YY} \sin(\gamma) \\ \frac{v_X \cos(2\gamma) - v_Y \sin(2\gamma)}{v_Y (k_p^2 \cos(2\gamma) - k_p^2 + k_s^2) + k_p^2 v_X \sin(2\gamma)} \\ v_Z \cos(\gamma) \end{pmatrix}. \tag{6.3}$$

With reference to Fig. 1 we now explicit the functional equations (4.16) of an angular region filled by isotropic elastic medium before imposing face boundary conditions in the 2D case.

With $\alpha_o = 0$, the re-scaled reciprocal vectors (6.2), the Laplace transform $\tilde{\psi}_t(\eta, v = 0)$ (4.1) of the continuous field (2.17) at face o and the Laplace transform $\tilde{\psi}_{as}(\chi)$ (4.15) of the quantity (6.3)

expressed in terms of the continuous field at face a, we obtain the following explicit form of the functional equations (4.16):

$$\frac{k_s(\eta T_{xy} - T_{yy}\xi_p)}{Z_o} + 2\eta v_x \xi_p + v_y (\eta^2 - \xi_s^2) = \sin(2\gamma)[-2\eta \xi_p v_Y - v_X \xi_p^2 + \eta^2 v_X] + v_Y (k_p^2 - k_s^2) + \cos(2\gamma)[- \xi_p^2 v_Y + 2\eta \xi_p v_X + \eta^2 v_Y] - \frac{k_s \xi_p [T_{XY} \sin(\gamma) + T_{YY} \cos(\gamma)] + \eta k_s [T_{XY} \cos(\gamma) - T_{YY} \sin(\gamma)]}{Z_o} \quad (6.4)$$

$$\frac{k_s(T_{xy}\xi_s + \eta T_{yy})}{Z_o} - v_x (\eta^2 - \xi_s^2) + 2\eta v_y \xi_s = \sin(2\gamma)[2\eta v_X \xi_s - v_Y \xi_s^2 + \eta^2 v_Y] + \cos(2\gamma)[v_X \xi_s^2 + 2\eta v_Y \xi_s - \eta^2 v_X] + \frac{k_s \xi_s [T_{XY} \cos(\gamma) - T_{YY} \sin(\gamma)] + k_s \eta [T_{XY} \sin(\gamma) + T_{YY} \cos(\gamma)]}{Z_o} \quad (6.5)$$

$$\frac{k_s T_{yz}}{Z_o \xi_s} + v_z = \frac{k_s T_{YZ}}{Z_o \xi_s} + \frac{\eta v_Z}{\xi_s} \sin(\gamma) + v_Z \cos(\gamma). \quad (6.6)$$

We recall the T, v quantities with lowercase subscripts in the LHS of the equations are defined for $u > 0, v = 0_+$ and are Laplace transforms in η of $\tilde{\psi}_t(\eta, v = 0)$, while the T, v quantities with uppercase subscripts are defined for $u = 0_+, v > 0$ and are Laplace transforms in $-m_p, -m_s, -m_s$ of $\psi_{a,s}(v)$ respectively in the RHS of (6.4),(6.5),(6.6).

We note that (6.4) is related to the complex propagation constant $-m_p$ of the principal wave while (6.5),(6.6) are related to $-m_s$, i.e. the one of the secondary waves.

We note also some sort of symmetry between (6.4) and (6.5) except for the additional term $v_Y (k_p^2 - k_s^2)$ in (6.4).

Eqs. (6.4),(6.5),(6.6) are functional equations for the general 2D wave motion angular problem in elasticity before imposing boundary conditions, i.e. they represent the planar and anti-planar problems.

To complete the validation with the equations proposed at (4.1) of [14], with reference to region 1 of Fig.1, we impose traction-free face boundary conditions at faces o and a, i.e. the traction $\mathbf{t} = \underline{\mathbf{T}} \cdot \mathbf{n} = \mathbf{0}$ where \mathbf{n} is the unit normal to the face:

$$T_{yy}, T_{yz}, T_{yx} = 0 \text{ at face o } (u > 0, v = 0_+), \quad T_{YY}, T_{YZ}, T_{YX} = 0 \text{ at face a } (u = 0_+, v > 0). \quad (6.7)$$

It yields the following GWHEs:

$$2\eta v_x \xi_p + v_y (\eta^2 - \xi_s^2) = \sin(2\gamma)[-2\eta \xi_p v_Y + v_X (\eta^2 - \xi_p^2)] + \cos(2\gamma)[v_Y (\eta^2 - \xi_p^2) + 2\eta \xi_p v_X] + v_Y (k_p^2 - k_s^2) \quad (6.8)$$

$$-v_x (\eta^2 - \xi_s^2) + 2\eta v_y \xi_s = \sin(2\gamma)[2\eta v_X \xi_s - v_Y \xi_s^2 + \eta^2 v_Y] + \cos(2\gamma)[v_X \xi_s^2 + 2\eta v_Y \xi_s - \eta^2 v_X], \quad (6.9)$$

$$v_z = \frac{\eta v_Z}{\xi_s} \sin(\gamma) + v_Z \cos(\gamma). \quad (6.10)$$

where the v quantities with lowercase subscripts in the LHS of (6.8),(6.9),(6.10) are plus functions in η and v quantities with uppercase subscripts in the RHS are minus functions (plus functions) in m_p, m_s, m_s ($-m_p, -m_s, -m_s$). Both minus and plus functions are Laplace transforms. Standard plus(minus) functions are analytic in the upper(lower) half-plane. We extend the theory to non-standard functions when they have isolated poles due to plane wave sources located in the standard regularity half-plane.

Note that (6.10) is independent from (6.8),(6.9). In fact (6.10) is associated to SH wave in the wave motion problem (antiplanar problem), while (6.8),(6.9) model the coupled problem between P and SV waves (planar problem).

Eq. (6.10) can be checked and validated after imposing the traction-free face boundary conditions with (3.15.5) of [4] where a completely different method specialized on antiplanar problems has been used. Now, let us compare (6.8),(6.9) with (4.1) of [14], reported in original

form at (6.11) with (6.12)- (6.13).

$$\begin{aligned} a(\xi)\hat{u}_1(\xi) - b_1(\xi)\hat{u}_2(\xi) + \hat{U}_1(\xi) &= f_1(\xi), \\ b_2(\xi)\hat{u}_1(\xi) + a(\xi)\hat{u}_2(\xi) + \hat{U}_2(\xi) &= f_2(\xi), \end{aligned} \quad (6.11)$$

$$\begin{aligned} \hat{U}_1(\xi) &= (-1)^\ell [-a(\zeta_1)\hat{u}_1(\zeta_1) + \bar{b}_1(\xi)\hat{u}_2(\zeta_1)], \quad \ell = 1, 2, (\text{antisym}, \text{sym}), \\ \hat{U}_2(\xi) &= (-1)^\ell [\bar{b}_2(\xi)\hat{u}_1(\zeta_2) + a(\zeta_2)\hat{u}_2(\zeta_2)], \quad \ell = 1, 2 (\text{antisym}, \text{sym}), \end{aligned} \quad (6.12)$$

$$\begin{aligned} \zeta_{1,2} &= \xi \cos \alpha + \gamma_{1,2}(\xi) \sin \alpha, \\ \eta_{1,2} &= \xi \sin \alpha - \gamma_{1,2}(\xi) \cos \alpha, \\ \bar{b}_{1,2}(\xi) &= 2\zeta_{1,2}\eta_{1,2}. \end{aligned} \quad (6.13)$$

In (6.11) $\hat{u}_1(\xi), \hat{u}_2(\xi)$ are one-sided Fourier transforms of unknown displacements on face o (Fig.1) respectively in x, y , ξ is the spectral variable, $a(\xi), b_1(\xi), b_2(\xi)$ are spectral functions and, $\hat{U}_1(\xi), \hat{U}_2(\xi)$ are one-sided Fourier transforms of quantities defined in terms of unknown displacements on face a (Fig.1) respectively in $X, -Y$. $f_1(\xi), f_2(\xi)$ model the source of the wave motion problem. In order to compare (6.11) with (6.8),(6.9), we scale all the displacements by $j\omega$ to get the velocities, thus (6.11) hold in homogeneous form ($f_1(\xi), f_2(\xi) = 0$) also interpreting $\hat{u}_i(\xi), \hat{U}_i(\xi)$ in terms of velocities. Moreover, we observe that $i = 1, 2$ waves in [14] are respectively associated to SV, P waves, thus we need to compare (6.8),(6.9) respectively with the 2nd and the 1st equation of (6.11). With the help of the definitions given in [14], let us interpret (6.11) in our formalism. Table 1 reports the correspondences for the definition of some quantities in the two works. With Table 1, it is easy to show the equivalence between the LHS of (6.8),(6.9) and the terms in $\hat{u}_i(\xi)$ in (6.11).

Table 1. Translation of definitions between this work and [14]

[14]	ξ	$\kappa_{1,2}$	α	$\hat{u}_{1,2}(\xi)$	$\gamma_{1,2}^2 = \kappa_{1,2}^2 - \xi^2$	$a(\xi) = \kappa_1^2 - 2\xi^2$	$b_{1,2}(\xi) = 2\xi\gamma_{1,2}(\xi)$
this paper	η	$k_{s,p}$	γ	$v_{x,y}(\eta)$	$\xi_{s,p}^2 = k_{s,p}^2 - \eta^2$	$\xi_s^2 - \eta^2$	$2\eta\xi_{s,p}$

To complete the comparison we need to check the 1st equation of (6.11) and (6.9) focusing the attention on $\hat{U}_1(\xi)$ (6.12) and then check the 2nd equation of (6.11) and (6.8) focusing the attention on $\hat{U}_2(\xi)$ (6.12). Starting from (6.13), $\zeta_{1,2}$ play the roles of $-m_{s,p}$ (4.17) and $\eta_{1,2}$ play the role of $n_{s,p}$. In particular we note that, in our notation,

$$\zeta_{1,2} \rightarrow \eta \cos \gamma + \xi_{s,p} \sin \gamma, \quad \eta_{1,2} \rightarrow \eta \sin \gamma - \xi_{s,p} \cos \gamma, \quad (6.14)$$

that apart from a sign in the combination of the two terms are respectively $-m_{s,p}$ (4.17) and $n_{s,p}$:

$$m_{s,p} = -\eta \cos \gamma + \xi_{s,p} \sin \gamma, \quad n_{s,p} = \eta \sin \gamma + \xi_{s,p} \cos \gamma. \quad (6.15)$$

Further sign differences appear also in the combination of the quantities between (6.8)-(6.9) and (6.11). We are convinced that these differences are due to different notations in Fourier transforms between engineering (ours, [7] p.XV) and applied mathematics (as in [14]) and, to the different orientation of local coordinate system on face a between our work and [14] where $(X, -Y)$ are selected (see Fig. 1). We note that $\hat{u}_{1,2}(\zeta_1)$ in $\hat{U}_1(\xi)$ (6.12) for equation (6.11) play the roles of $v_{X,Y}(-m_s)$ for equation (6.9). Let us compare the functional coefficient of $\hat{u}_{1,2}(\zeta_1)$ with the ones of $v_{X,Y}(-m_s)$. With the help of Table 1 and (6.14)-(6.15), for $\hat{u}_1(\zeta_1)$ and $v_X(-m_s)$ we have resp.

$$-a(\zeta_1) = \kappa_1^2 - 2\zeta_1^2 \rightarrow k_s^2 - 2m_s^2, \quad (6.16)$$

$$\sin(2\gamma)2\eta\xi_s + \cos(2\gamma)[\xi_s^2 - \eta^2] = k_s^2 - 2m_s^2 \quad (6.17)$$

after some trigonometric manipulation. Again for $\hat{u}_2(\zeta_1)$ and $v_Y(-m_s)$ we have respectively

$$\bar{b}_1(\xi) = 2\zeta_1\eta_1 \rightarrow 2m_s n_s, \quad (6.18)$$

$$\sin(2\gamma)[- \xi_s^2 + \eta^2] + \cos(2\gamma)[2\eta\xi_s] = 2m_s n_s. \quad (6.19)$$

Now let us complete the comparison between the 2nd equation of (6.11) and (6.8), focusing the attention on $\widehat{U}_2(\xi)$ (6.12) and comparing the functional coefficient of $\hat{u}_{1,2}(\zeta_1)$ in $\widehat{U}_2(\xi)$ with the ones of $v_{X,Y}(-m_p)$. With the help of Table 1 and (6.14)-(6.15), for $\hat{u}_1(\zeta_2)$ and $v_X(-m_p)$ we have respectively

$$\bar{b}_2(\xi) = 2\zeta_2\eta_2 \rightarrow 2m_p n_p, \quad (6.20)$$

$$\sin(2\gamma)[- \xi_p^2 + \eta^2] + \cos(2\gamma)[2\eta\xi_p] = 2m_p n_p \quad (6.21)$$

with same calculus done in (6.18)-(6.19). On the contrary, we note that $\hat{u}_2(\zeta_2)$ and $v_Y(-m_p)$ show different properties with respect to (6.16)-(6.17). Their respective functional coefficients are

$$a(\zeta_2) = \kappa_1^2 - 2\zeta_2^2 \rightarrow k_s^2 - 2m_p^2, \quad (6.22)$$

$$\sin(2\gamma)[-2\eta\xi_p] + \cos(2\gamma)[- \xi_p^2 + \eta^2] + (k_p^2 - k_s^2) = k_s^2 - 2m_p^2 \quad (6.23)$$

402 that are equivalent after some trigonometric manipulation. Note in (6.22)-(6.23) we have the
403 simultaneous presence of SV and P spectral variables and propagation constants and, the presence
404 of additional term $(k_p^2 - k_s^2)$ in the LHS of (6.23) with respect to the LHS of (6.17). This property
405 denotes coupling between SV and P waves.

406 We conclude by affirming that (6.8),(6.9),(6.10) are the GWHEs for the elastic wave motion
407 angular problem in 2D ($\alpha_o = 0$) with traction-free face boundary conditions that model the planar
408 (6.8),(6.9) and antiplanar (6.10) problems in presence of plane-wave sources or sources located at
409 infinity with the help of the concept of non-standard Laplace transforms (see section 1.4 of [5]).

410 7. Validation of functional equations through the estimation of 411 characteristic impedances in half-space planar regions

412 In this Section we further validate the functional equations (5.11)-(5.13) and (5.18)-(5.20) obtained
413 in the general case of 3D angular region problems by computing the characteristic impedances of
414 the half spaces identified as region 1 ($y > 0$) and region 2 ($y < 0$) in Fig. 3 for planar problems.

415 Fig. 3 shows the half-plane problem (crack) where arbitrary boundary condition can be
416 applied. We recall that GWHEs for practical problems can be derived from (5.11)-(5.13) and (5.18)-
417 (5.20) by applying specific boundary conditions (traction-free, clamped, ...). For example, this
418 method can be used to compare with solutions reported in [34]- [35] for the half-plane problem.
419 In this case, we note that, starting from the general functional equations, by imposing $\gamma = \pi$, we
420 model the half-plane problem via GWHEs that reduce to Classical Wiener-Hopf equations due to
421 the definitions of spectral variables m .

Let us start from region 1, considering (5.11)-(5.13). To model the planar problem, we impose
 $\gamma = \pi$, $\alpha_o = 0$ and all the continuous z components of the field \mathbf{T} and \mathbf{v} null: $T_{yz} = T_{YZ} = 0$, $v_z =$
 $v_Z = 0$. From (5.11)-(5.12) ((5.13) is trivially null in this case) we have

$$\begin{aligned} Z_o \left((2\eta^2 - k_s^2)v_y + 2\eta v_x \xi_p \right) + k_s (\eta T_{xy} - T_{yy} \xi_p) &= Z_o \left((2\eta^2 - k_s^2)v_Y + 2\eta v_X \xi_p \right) - k_s (\eta T_{XY} - T_{YY} \xi_p), \\ Z_o \left((k_s^2 - 2\eta^2)v_x + 2\eta v_y \xi_s \right) + k_s (T_{xy} \xi_s + \eta T_{yy}) &= Z_o \left((k_s^2 - 2\eta^2)v_X + 2\eta v_Y \xi_s \right) - k_s (T_{XY} \xi_s + \eta T_{YY}). \end{aligned} \quad (7.1)$$

422 Now let us focus the attention on the non null continuous field component of \mathbf{T} and \mathbf{v} , we have
423 respectively for (2.16) with (4.1) and (5.9) with (4.15):

$$\boldsymbol{\psi}_t = (T_{yy}, T_{xy}, v_x, v_y)', \quad \boldsymbol{\psi}_{as} = (-T_{YY}, -T_{XY}, v_X, v_Y)'. \quad (7.2)$$

424 From the definitions of $\boldsymbol{\psi}_t$ and $\boldsymbol{\psi}_{as}$, respectively defined in $x > 0, y = 0$ in x, y coordinates and in
425 $x < 0, y = 0_+$ in X, Y coordinates, we estimate the total fields for $y = 0_+$ as

$$\boldsymbol{\psi}_{0_+}^{tot} = \boldsymbol{\psi}_t - \boldsymbol{\psi}_{as} = \left(T_{yy}^{tot}, T_{xy}^{tot}, v_x^{tot}, v_y^{tot} \right)'. \quad (7.3)$$

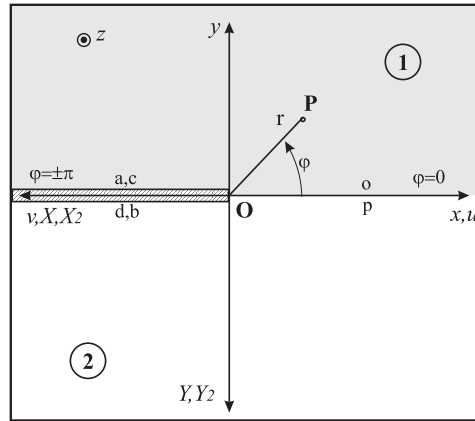


Figure 3. Half-plane planar crack problem with the reference coordinate systems and boundaries adapted from the general configuration reported in Fig. 2 ($X \equiv X_2, Y \equiv Y_2$ local face Cartesian coordinates are reported and are equal in this case due to rotation). The half crack is localized at $x < 0, y = 0$ and the surrounding space is divided into two rectangular regions: region 1 ($y > 0$) and region 2 ($y < 0$). In this section we evaluate the characteristic impedances of the half-space regions 1 and 2 that are independent from the boundary conditions on the half-plane and implicitly assume absence of sources localized at finite.

In fact, we note that the local-to-face-a X, Y coordinates have opposite direction with respect to x, y thus the velocity vectors are measured with opposite directions while the tensorial stress components have same directions because of the double inversion.

With the definition of total fields at $y = 0_+$ (7.3), from (7.1) we derive expressions of $T_{yy}^{tot}, T_{xy}^{tot}$ in terms of v_x^{tot}, v_y^{tot} that in matrix form yield the matrix characteristic impedance of region 1:

$$\begin{pmatrix} T_{yy}^{tot} \\ T_{xy}^{tot} \end{pmatrix} = \mathbb{Z}_c^+ \begin{pmatrix} v_x^{tot} \\ v_y^{tot} \end{pmatrix}, \quad \mathbb{Z}_c^+ = \begin{pmatrix} \frac{\eta Z_o}{k_s} \left(2 - \frac{k_s^2}{\eta^2 + \xi_p \xi_s} \right) & -\frac{k_s Z_o \xi_s}{\eta^2 + \xi_p \xi_s} \\ -\frac{k_s Z_o \xi_p}{\eta^2 + \xi_p \xi_s} & \frac{\eta Z_o}{k_s} \left(\frac{k_s^2}{\eta^2 + \xi_p \xi_s} - 2 \right) \end{pmatrix}. \quad (7.4)$$

Note that the definition of the characteristic impedance is independent from boundary conditions on the half-plane and implicitly assumes absence of sources localized at finite. The impedance (7.4) is validated with the admittance $\mathbb{Y}_c^+ = (\mathbb{Z}_c^+)^{-1}$ reported in (2.12.5)-(2.12.8) of [4] where, by mistake, a coefficient 2 is missing in (2.12.7) and (2.12.8). We note that while in section 2.12 of [4] the characteristic impedance is evaluated from the homogeneous solution of transverse equations in Fourier domain, in the present work we have used Laplace transforms with boundary conditions that results in a completely different and independent proof.

Now, let us consider region 2 (Fig. 3) and the related functional equations (5.18),(5.19),(5.20) and (5.17) with (4.22). To model the planar problem, we impose $\gamma = \pi, \alpha_o = 0$ and all the continuous z components of the field \mathbf{T} and \mathbf{v} null: $T_{yz} = T_{YZ} = 0, v_z = v_Z = 0$. From (5.18)-(5.19) ((5.20) is trivially null in this case) we have

$$\begin{aligned} Z_o \left(v_y (k_s^2 - 2\eta^2) + 2\eta v_x \xi_p \right) - k_s (T_{yy} \xi_p + \eta T_{xy}) &= Z_o \left(v_Y (k_s^2 - 2\eta^2) + 2\eta v_X \xi_p \right) + k_s (T_{YY} \xi_p + \eta T_{XY}), \\ Z_o \left(v_x (2\eta^2 - k_s^2) + 2\eta v_y \xi_s \right) + k_s (T_{xy} \xi_s - \eta T_{yy}) &= Z_o \left(v_X (2\eta^2 - k_s^2) + 2\eta v_Y \xi_s \right) - k_s (T_{XY} \xi_s - \eta T_{YY}). \end{aligned} \quad (7.5)$$

Now let us focus the attention on the non null continuous field component of \mathbf{T} and \mathbf{v} , we have respectively for (2.16) with (4.1) and (5.17) with (4.22):

$$\boldsymbol{\psi}_t = (T_{yy}, T_{xy}, v_x, v_y)', \quad \boldsymbol{\psi}_{bs} = (T_{YY}, T_{XY}, -v_X, -v_Y)'. \quad (7.6)$$

From the definitions of $\boldsymbol{\psi}_t$ and $\boldsymbol{\psi}_{bs}$, respectively defined in $x > 0, y = 0$ in x, y coordinates and in $x < 0, y = 0_-$ in X, Y coordinates, we estimate the total fields for $y = 0_-$ as

$$\boldsymbol{\psi}_{0-}^{tot} = \boldsymbol{\psi}_t + \boldsymbol{\psi}_{bs} = \left(T_{yy}^{tot}, T_{xy}^{tot}, v_x^{tot}, v_y^{tot} \right)'. \quad (7.7)$$

Due to the expressions (7.6), the total field in region 2 (7.7) show a different sign with respect to the expression of region 1 (7.3) to maintain the same physical meaning. With the definition of total fields at $y = 0_-$ (7.7), from (7.5) we derive expressions of $T_{yy}^{tot}, T_{xy}^{tot}$ in terms of v_x^{tot}, v_y^{tot} that in matrix form yield the matrix characteristic impedance of region 2:

$$\begin{pmatrix} T_{yy}^{tot} \\ T_{xy}^{tot} \end{pmatrix} = \mathbb{Z}_c^- \begin{pmatrix} -v_x^{tot} \\ -v_y^{tot} \end{pmatrix}, \quad \mathbb{Z}_c^- = \begin{pmatrix} \frac{\eta Z_o}{k_s} \left(\frac{k_s^2}{\eta^2 + \xi_p \xi_s} - 2 \right) & -\frac{k_s Z_o \xi_s}{\eta^2 + \xi_p \xi_s} \\ -\frac{k_s Z_o \xi_p}{\eta^2 + \xi_p \xi_s} & \frac{\eta Z_o}{k_s} \left(2 - \frac{k_s^2}{\eta^2 + \xi_p \xi_s} \right) \end{pmatrix}. \quad (7.8)$$

The impedance (7.8) is validated with the admittance $\mathbb{Y}_c^- = (\mathbb{Z}_c^-)^{-1}$ reported in section 12 at (2.12.5)-(2.12.8) of [4] as discussed for region 1. Note that in (7.8) we have assumed different sign in the velocity with respect to (7.4) of region 1 due to the different direction of propagation in the two regions. Finally, we recall that the method presented in this paper for the calculation of the characteristic impedances is more general and independent from the one reported in [4].

8. Remarks and Conclusions

In this work, we have introduced a general method for the deduction of spectral functional equations and thus GWHEs in angular regions filled by arbitrary linear isotropic homogeneous media in elasticity. The importance to formulate wedge problems with GWHEs in Electromagnetism has been showed in [4]- [5]. We remark that these equations are important also for elastic wedge problems. In particular the functional equations obtained and solved in [14] by Gautesen's group for the planar elastic wedge are GWHEs, although not defined in this way.

The method is based on the original solution of vector differential equations of first order via dyadic Green's function method and on the projection of this solution along the boundaries of the angular region using reciprocal vectors of the pertinent algebraic matrix related to the matrix differential operator. The application of the boundary conditions to the functional equations yields GWHEs for practical problems. We observe that the functional equations are the starting point to develop solutions using WH technique for complex scattering problems.

Using the concept of non-standard Laplace transforms (see section 1.4 of [5]), the validity of the functional equations and of the GWHEs obtained in absence of sources is extended to the total fields in presence of plane-wave sources or in general of sources located at infinity. We observe that the GWHEs in elasticity contains unknowns defined in multiple complex planes $\eta, -m_p, -m_s$ related to P and S waves and this property recall electromagnetic applications (and related solution methods) in media with multiple propagation constants as reported in [27]- [30]. In fact, in this case the reduction of GWHEs to classical WH equations is not possible. Explicit expressions of spectral functional equations in algebraic form are provided in the text in the general case of non planar elastic problems in angular regions with isotropic media and arbitrary boundary conditions and, we remark that, according to our opinion, this is the first time in literature. Validation of the GWHE formulation has been demonstrated by comparison with prestigious literature references reporting special simplified cases in anti-planar and planar problems. The paper demonstrates the flexibility and the advantages of the proposed method, based on first order differential formulation, that is useful for the analysis of complex scattering problem containing angular regions in arbitrarily linear media by changing the matrix operator defined through the fundamental matrices $\mathbb{M}_o, \mathbb{M}_1, \mathbb{M}_2$. The paper shows systematic procedural steps that can be used for arbitrary wave motion problems in different physics.

Data Accessibility. This article has no additional data.

Authors' Contributions. V.D and G.L. co-developed the mathematics, performed the numerical implementation, provided physical interpretations, produced the figures and wrote the article. Both authors approved the final version and agree to be accountable for all aspects of the work.

Competing Interests. We declare we have no competing interests.

486 **Funding.** This work was supported by the Italian Ministry of University and Research under PRIN Grant
487 2017NT5W7Z GREEN TAGS.

488 **Acknowledgements.** Both authors thank the Isaac Newton Institute for Mathematical Sciences, Cambridge,
489 for organization, support and hospitality during the programme *Bringing pure and applied analysis together via*
490 *the Wiener-Hopf technique, its generalizations and applications* where some work on this paper was undertaken.

491 References

- 492 1. Daniele VG, Lombardi G. 2021. The Generalized Wiener-Hopf Equations for the wave motion
493 in angular regions. Part 1: general theory and electromagnetic application. *Proceedings of the*
494 *Royal Society A*. accepted July 28, 2021, doi:10.1098/rspa.2021.0040
- 495 2. Daniele VG. 2004. The Wiener-Hopf technique for wedge problems. *Internal Rep. ELT-2004-2,*
496 *DET, Politecnico di Torino*, See <http://personal.det.polito.it/vito.daniele>
- 497 3. Daniele VG, Lombardi G. 2005. The Wiener-Hopf technique for impenetrable wedge
498 problems. *Proc. Days Diffraction Internat. Conf.*, Saint Petersburg, Russia, Jun. 2005, 50-61. (doi:
499 10.1109/DD.2005.204879)
- 500 4. Daniele VG, Lombardi G. 2020. *Scattering and Diffraction by Wedges 1: The Wiener-Hopf Solution*
501 *- Theory*. Hoboken, NJ: John Wiley & Sons, Inc. London, UK: ISTE.
- 502 5. Daniele VG, Lombardi G. 2020. *Scattering and Diffraction by Wedges 2: The Wiener-Hopf Solution*
503 *- Advanced Applications*. Hoboken, NJ: John Wiley & Sons, Inc. London, UK: ISTE.
- 504 6. Daniele VG, Lombardi G. 2007. Fredholm Factorization of Wiener-Hopf scalar and matrix
505 kernels. *Radio Science*. **42**, RS6S01, 1-19.
- 506 7. Daniele VG, Zich RS. 2014. *The Wiener-Hopf method in electromagnetics*. Raleigh, NC: SciTech Pub.
- 507 8. Jones DS. 1952. Diffraction by a waveguide of finite length. *Proc. Camb. Phil. Soc.* **48**, 118-134.
- 508 9. Noble B. 1958. *Methods Based on the Wiener-Hopf Technique For the Solution of Partial Differential*
509 *Equations*. Belfast, Northern Ireland: Pergamon Press.
- 510 10. Bresler AD, Marcuvitz N. 1956. Operator methods in electromagnetic field theory. *Report R-*
511 *495,56, PIB-425, MRI Polytechnic Institute of Brooklyn*
- 512 11. Friedman B. 1956. *Principles and Techniques of Applied Mathematics*. New York, NY: John Wiley
513 & Sons. Ch. 3.
- 514 12. Budaev B. 1995. *Diffraction by Wedges*. Harlow, UK: Longman Scientific and Technical.
- 515 13. Croisille JP, Lebeau G. 1999. *Diffraction by an Immersed Elastic Wedge, ser. Lecture notes*
516 *Mathematics n. 1723*. Berlin, Germany: Springer-Verlag.
- 517 14. Gautesen AK, Fradkin LJ. 2010. Diffraction by a two-dimensional traction-free elastic wedge.
518 *SIAM, J. Appl. Math.* **70**, 3065-3085.
- 519 15. Kamotski VV, Fradkin LJ, Borovikov VA, Babich VM, Samokish BA. 2006. The diffraction of a
520 plane wave by a 2D traction free isotropic wedge. *AIP Conference Proceedings*. **834**, 167-174.
- 521 16. Chehade A, Darmon M, Lebeau G. 2019. 2D elastic plane-wave diffraction by a stress-free
522 wedge of arbitrary angle. *Journal of Computational Physics*. **394**, 532-558.
- 523 17. Chehade S, Darmon M, Lebeau G. 2021. 3D elastic plane-wave diffraction by a stress-free
524 wedge for incident skew angles below the critical angle in diffraction. *Journal of Computational*
525 *Physics*. **427**, 110062, 1-33.
- 526 18. Gautesen AK. 1985. Scattering of a plane longitudinal wave by an elastic quarter space. *Wave*
527 *Motion* **7**, 557-568.
- 528 19. Gautesen AK. 1985. Scattering of a Rayleigh wave by an elastic quarter space. *J. Appl. Mech*
529 **52**, 664-668.
- 530 20. Gautesen AK. 2002. Scattering of a Rayleigh wave by an elastic wedge whose angle is less
531 than 180. *Wave Motion* **36**, 417-424.
- 532 21. Daniele VG, Lombardi G, Zich RS. 2017. The electromagnetic field for a PEC wedge over a
533 grounded dielectric slab: 1. Formulation and validation. *Radio Science*. **52**, 1472-1491.
- 534 22. Daniele VG, Lombardi G, Zich RS. 2017. The electromagnetic field for a PEC wedge over a
535 grounded dielectric slab: 2. Diffraction, Modal Field, Surface Waves, and Leaky Waves. *Radio*
536 *Science*. **52**, 1492-1509.
- 537 23. Daniele VG, Lombardi G, Zich RS. 2018. The Double PEC Wedge Problem: Diffraction and
538 Total Far Field. *IEEE Trans. Antennas Propag.* **66**:12, 6482-6499. (doi: 10.1109/TAP.2018.2877260)
- 539 24. Daniele VG, Lombardi G, Zich RS. 2019. Radiation and Scattering of an Arbitrarily

- 540 Flanged Dielectric-Loaded Waveguide. *IEEE Trans. Antennas Propag.* **67**:12, 7569-7584. (doi:
541 10.1109/TAP.2019.2948494)
- 542 25. Auld BA. 1973. *Acoustic fields and waves in solids-Volume 1*. New York, NY: Wiley.
- 543 26. Slaughter WS. 2002. *The linearized theory of elasticity*. New York: Birkhauser Boston.
- 544 27. Daniele VG. 2010. The Wiener-Hopf formulation of the dielectric wedge problem. Part I.
545 *Electromagnetics.* **30**:8, 625-643.
- 546 28. Daniele VG. 2011. The Wiener-Hopf formulation of the dielectric wedge problem. Part II.
547 *Electromagnetics.* **31**:1, 1-17.
- 548 29. Daniele V, Lombardi G. 2011. The Wiener-Hopf solution of the isotropic penetrable wedge
549 problem: Diffraction and total field. *IEEE Trans. Antennas Propag.* **59**:10, 3797-3818.
- 550 30. Daniele VG. 2011. The Wiener-Hopf formulation of the dielectric wedge problem. Part III.
551 *Electromagnetics.* **31**:8, 550-570.
- 552 31. Daniele V, Zich R. 1973. Radiation by arbitrary sources in anisotropic stratified media. *Radio*
553 *Science.* **8**, 63-70.
- 554 32. Felsen LB, Marcuvitz N. 1973. *Radiation and Scattering of Waves*. Englewood Cliffs, NJ: Prentice-
555 Hall.
- 556 33. Pease MC. 1965. *Methods of Matrix Algebra*. New York, NY: Academic Press. Section 3.16.
- 557 34. Achenbach JD, Gautesen AK, McMaken H. 1982. *Ray Methods for Waves in Elastic Solids: With*
558 *Applications to Scattering by Cracks*. Pitman Advanced Publishing Program.
- 559 35. Achenbach JD, Gautesen AK. 1977. Elastodynamic Stress-Intensity Factors for a Semi-Infinite
560 Crack Under 3-D Loading. *ASME. J. Appl. Mech.* **44**:2, 243-249.
- 561 36. Schelkunoff SA. 1936. Some Equivalence Theorems of Electromagnetics and Their Application
562 to Radiation Problems. *Bell Syst. Tech. J.* **15**, 92-112.

563 Glossary

Table 2. Symbols introduced in the paper

Notation	Description
$(x, y, z), (r, \varphi, z), (u, v, z), (X, Y, Z)$	Cartesian, cylindrical, oblique Cartesian, local to face Cartesian coordinates
$A, \mathbf{A}, \underline{A}, \mathbb{A}, \mathcal{A}(\cdot, \cdot)$	scalar, column vector, dyadic, matrix, matrix differential operator
k_p, k_s	propagation constants of P and S waves
$\underline{T}(\mathbf{T}), \underline{S}(\mathbf{S})$	stress tensor (Voigt notation), strain tensor (Voigt notation)
\mathbf{p}, \mathbf{v}	vector momentum density, vector particle velocity
ρ, λ, μ	material density and Lamé's constants
\underline{C}	Hooke's law as fourth order stiffness tensor
$\nabla_T, \nabla_v, \Gamma\nabla$	matrix differential operators
ψ, θ	vector fields in abstract notation
\mathbb{W}	matrix constitutive parameters of media
ψ_t	transverse field for a stratification along the y direction
$\mathcal{M}(\frac{\partial}{\partial z}, \frac{\partial}{\partial x})$	transversal matrix differential operator for elastic equations
$D_x = \frac{\partial}{\partial x}$	alternative partial derivative notation
α_o	field dependence specified by the factor $e^{-j\alpha_o z}$ due to invariance along z
η	Fourier or Laplace spectral variable according to the position on the text
$\Psi_i(\eta)$	Fourier transform along $x = u$ direction (y, z or v, z dependence is omitted)
$\mathbb{M}(\eta)$	matrix operator in Fourier/Laplace domain in indefinite rectangular region
$\lambda_i, \mathbf{u}_i, \mathbf{v}_i$	eigenvalues, eigenvector and reciprocal vectors of $\mathbb{M}(\eta)$
ξ_i	different notation of λ_i for propagation's properties, multivalued function
γ	aperture angle of angular regions (Fig. 1)
$\mathbb{M}_e(\gamma, \eta)$	matrix operator in Fourier/Laplace domain in indefinite angular region
λ_{ei}	eigenvalues of $\mathbb{M}_e(\gamma, \eta)$
$\tilde{\psi}_t(\eta, v)$	Laplace transform along $x \equiv u$ of $\psi_t(u, v)$ (omitting z dependence)
$\psi_s(v)$	field components on the face of an angular region in Laplace domain
$\psi_{as}(v), \tilde{\psi}_{as}(\chi)$	specialized expression of $\psi_s(v)$ on face a and its Laplace transform
$\underline{G}(v, v')$	dyadic Green's function in Laplace domain for an angular region
m_{ai}	spectral variable for the evaluation of $\tilde{\psi}_{as}(\chi)$ along face a in functional eqs