THE SOBOLEV EMBEDDING CONSTANT ON LIE GROUPS

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Abstract. In this paper we estimate the Sobolev embedding constant on general noncompact Lie groups, for sub-Riemannian inhomogeneous Sobolev spaces endowed with a left invariant measure. The bound that we obtain, up to a constant depending only on the group and its sub-Riemannian structure, reduces to the best known bound for the classical inhomogeneous Sobolev embedding constant on $\mathbb{R}^d$. As an application, we prove local and global Moser–Trudinger inequalities.

1. Introduction

The aim of this paper is to investigate the behaviour of the Sobolev embedding constant in a sub-Riemannian setting, in particular on noncommutative Lie groups.

In the Euclidean space $\mathbb{R}^d$, if Δ denotes the classical positive Laplacian and $\dot{L}^p_\alpha = \Delta^{\alpha/2}L^p$ the homogeneous Sobolev space, it is well known that $\dot{L}^p_\alpha \hookrightarrow L^q$ when $1 < p < \infty$, $0 \leqslant \alpha < d/p$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$. The best constant and the extremal functions for this embedding have a long history and a multitude of applications, and they can be obtained from the analysis of the Hardy–Littlewood–Sobolev inequality. Lieb [19] determined the best constant in the "diagonal case" $p = q'$, and found an estimate in the other cases; see also earlier works by Aubin [3] and Talenti [30]. If $L^p_\alpha = (I+\Delta)^{\alpha/2}L^p$ is the inhomogeneous Sobolev space, then it is also well known that $L^p_\alpha \hookrightarrow L^q$ when $1 < p, q < \infty$, $0 \leqslant \alpha < d/p$ and $\frac{1}{q} \geqslant \frac{1}{p} - \frac{\alpha}{d}$. The related best embedding constant is not known, though it can be bounded by the best constant for the embedding of homogeneous spaces, up to a dependence on the dimension d.

On a general noncompact Lie group G, the natural substitutes of the Laplacian are sub-Laplacians with drift $\mathcal{L}$, see [4], which are symmetric with respect to the left Haar measure λ. This setting, and this type of operators in particular, were studied in [14, 2], and an associated theory of Sobolev spaces, that we shall denote by $L^p_\alpha(\lambda)$, was developed in [4]. Since the Riesz transforms are not known to be bounded on L^p when $1 < p < \infty$ in such generality, while it is known that the appropriately shifted ones are bounded, see [4], it seems more natural to consider Sobolev spaces endowed with an inhomogeneous norm, which reduces to the Sobolev norm of L^p_α in the Euclidean case.

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Our main result is an estimate for the constant of the embedding $L^p_\alpha(\lambda) \hookrightarrow L^q(\lambda)$, when $1 < p < \infty$, $0 \leqslant \alpha < d/p$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$, of the form $C\, S(p,q)$, where

$$S(p,q) := \min\left(\frac{q^{1/p'}}{p-1}, \frac{{p'}^{1/q}}{q'-1}\right) \tag{1.1}$$

and C depends only on the group and its chosen sub-Riemannian structure. Here and throughout the paper, given any $p \in (1,\infty)$ we denote by p' its conjugate exponent, that is, $p' = p/(p-1)$. In terms of the dependence on p and q, such a bound is comparable to the best known bound in $\mathbb{R}^d$ for the Sobolev embedding constant for inhomogeneous spaces associated with the Laplacian, while it is new in noncommutative groups. In addition to this, we shall also discuss the more general case of relatively invariant measures where, despite the Sobolev embeddings in general fail [4], we are able to prove alternative results.

A well-established application of the Sobolev embedding theorem, both in the homogeneous and inhomogeneous case, is the classical Moser–Trudinger inequality [31, 22], which arises as a substitute of boundedness for functions in the Sobolev space $L^p_{d/p}$, as this does not embed in L^∞. By means of our quantitative Sobolev embedding, we prove quantitative versions of local and global Moser–Trudinger inequalities. Our approach is close in spirit, and inspired by, [23]. We refer the reader also to the recent work [27].

The analysis of sub-Laplacians and more generally of subelliptic differential operators has attracted a great deal of attention since their appearance in the study of Kohn-Laplacians and the renowned sum-of-squares theorem of Hörmander. It appears then very natural to extend geometric and functional inequalities from the Euclidean, elliptic case to a subelliptic setting, also in a quantitative form. Earlier breakthroughs were, e.g., Sobolev embeddings on stratified Lie groups [12] and the Poincaré inequality for sums of squares on $\mathbb{R}^d$ [16]. Among more recent works, we mention the Sobolev embedding theorem on unimodular Lie groups [8], a lower bound for the Hausdorff–Young constant on general Lie groups [10], the best constants for Sobolev and Gagliardo–Nirenberg inequalities on graded groups [27], and Poincaré inequalities on Lie groups [25, 7]. This paper fits into this order of ideas and line of research; we refer the reader also to [11, 26, 4] and the references therein. We emphasize that our setting is a general (connected) Lie group, endowed with a left Haar measure which, in general, has exponential volume growth and is non-doubling.

The structure of the paper is as follows. In Section 2, we describe the setting and all the preliminary results we shall need. Section 3 is the core of the paper, and contains the proof of the quantitative Sobolev embedding, whose constant is compared in Section 4 with the Euclidean ones. In Section 5 we prove a quantitative Moser–Trudinger inequality, and in Section 6 we discuss the case of more general measures.

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2. Setting and Preliminaries

Let G be a noncompact connected Lie group with identity e. Let λ be a left Haar measure on G, and δ be the modular function.

Let $\mathbf{X} = \{X_1, \ldots, X_\ell\}$ be a family of left-invariant linearly independent vector fields which satisfy Hörmander's condition. Let $d_C(\,\cdot\,,\,\cdot\,)$ be its associated left-invariant Carnot–Carathéodory distance. We let $|x| = d_C(x,e)$, and denote by B_r the ball centred at e of radius r. We denote by $V(r) = \lambda(B_r)$ the measure of of the ball B_r with respect to λ. We recall (cf. [13, 32]) that there exist two constants $d \in \mathbb{N}^*$ and $D > 0$ such that

$$C^{-1} r^d \leqslant V(r) \leqslant C r^d \quad \forall r \in (0,1], \qquad V(r) \leqslant C \mathrm{e}^{Dr} \quad \forall r \in (1,\infty), \tag{2.1}$$

where $C > 0$ is independent of r. We emphasize that d is uniquely determined by G and $\mathbf{X}$, while the set of $D > 0$ such that (2.1) holds is independent of $\mathbf{X}$ but does not have a minimum in general; consider, e.g., the case when G has polynomial growth. From this point on, we fix a $D > 0$ for which (2.1) holds, and observe that the metric measure space (G, d_C, λ) is locally doubling, but not doubling in general.

If $p \in [1,\infty)$, the spaces of (equivalent classes of) measurable functions whose p-power is integrable with respect to λ will be denoted by $L^p(\lambda)$, or simply L^p, and endowed with the usual norm which we shall denote by $\|\cdot\|_{L^p(\lambda)}$. The space L^∞ is defined analogously. The convolution between two functions f and g, when it exists, is defined by

$$f * g(x) = \int_G f(xy) g(y^{-1}) \,\mathrm{d}\lambda(y), \qquad x \in G\,.$$

We recall Young's inequality, which has the following form [15]: if $1 \leqslant p \leqslant q \leqslant \infty$ and $r \geqslant 1$ is such that $\frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q}$, then

$$\begin{aligned} \|f * g\|_{L^q(\lambda)} &\leqslant \|f\|_{L^p(\lambda)} \|\breve{g}\|_{L^r(\lambda)}^{r/p'} \|g\|_{L^r(\lambda)}^{r/q}, \qquad (q < \infty) \\ \|f * g\|_{L^\infty} &\leqslant \|f\|_{L^p(\lambda)} \|\breve{g}\|_{L^{p'}(\lambda)}, \end{aligned} \tag{2.2}$$

where $\breve{g}(x) = g(x^{-1})$. We denote by $\mathcal{L}$ the intrinsic sub-Laplacian on G associated with $\mathbf{X}$, see [2],

$$\mathcal{L} = -\sum_{j=1}^{\ell} (X_j^2 + (X_j\delta)(e) X_j),$$

which is symmetric on $L^2(\lambda)$, and essentially self-adjoint on $C_c^\infty(G)$, see [14]. We shall denote by $\mathcal{L}$ as well its unique self-adjoint extension.

The operator $\mathcal{L}$ generates a diffusion semigroup, i.e. $(\mathrm{e}^{-t\mathcal{L}})_{t>0}$ extends to a contraction semigroup on $L^p(\lambda)$ for every $p \in [1,\infty]$ (see [14]) whose infinitesimal generator, with a slight abuse of notation, we still denote by $\mathcal{L}$. We denote by p_t^δ the convolution kernel of $\mathrm{e}^{-t\mathcal{L}}$, and we recall that by [33, Theorems VIII.2.9, VIII.4.3 and IX.1.3] there exist constants $b, c > 0$ depending only on G and $\mathbf{X}$ such that

$$p_t^\delta(x) \leqslant c\,(1 \wedge t)^{-\frac{d}{2}} \,\mathrm{e}^{-\frac{1}{4} t \mathfrak{c}(\delta)^2}\, \mathrm{e}^{-b\frac{|x|^2}{t}}, \qquad x \in G,\ t > 0, \tag{2.3}$$

where $\mathfrak{c}(\delta) = (|X_1\delta(e)|^2 + \cdots + |X_\ell\delta(e)|^2)^{1/2}$. Let $b_0 = \sqrt{b}/2$, and define

$$\tau_\delta = \max\left\{ \frac{2}{b} \left[2D + b_0\right]^2 - \frac{1}{4}\mathfrak{c}(\delta)^2, 1 \right\}. \tag{2.4}$$

Following [4], when $p \in (1,\infty)$ and $\alpha > 0$ we define the Sobolev spaces $L^p_\alpha(\lambda)$ as the set of functions $f \in L^p(\lambda)$ such that $(\tau_\delta I + \mathcal{L})^{\alpha/2} f \in L^p(\lambda)$, endowed with the norm

$$\|f\|_{L^p_\alpha(\lambda)} = \|(\tau_\delta I + \mathcal{L})^{\alpha/2} f\|_{L^p(\lambda)}. \tag{2.5}$$

If $\alpha = 0$, we let $L^p_0(\lambda) = L^p(\lambda)$. We recall that (2.5) is equivalent to the norm $\|f\|_{L^p(\lambda)} + \|\mathcal{L}^{\alpha/2} f\|_{L^p(\lambda)}$, see [4]. The reason for choosing the shift τ_δ in the definition of $L^p_\alpha(\lambda)$ will be clarified later on; for more details about τ_δ, we refer the reader to the beginning of Section 4 below.

In [4] the Sobolev embeddings $L^p_\alpha(\lambda) \hookrightarrow L^q(\lambda)$ when $0 < \alpha < d/p$ and $q > p$ are such that $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$, were established. In this paper we find an explicit bound for the embedding constants, in the spirit which we now explain.

Throughout the paper, we shall disregard any dependence of the embedding constants on G and $\mathbf{X}$, which are assumed to be fixed once and for all from this point on. We shall, instead, obtain explicit results in terms of the dependence on p, q and α. A generic constant depending only on G and $\mathbf{X}$ will be denoted by C or $C(G, \mathbf{X})$, and its value may vary from line to line. Recall in particular that $d = d(G, \mathbf{X})$.

For $\alpha > 0$, let G^α_δ be the convolution kernel of $(\tau_\delta I + \mathcal{L})^{-\alpha/2}$. Let

$$G^{\alpha,\mathrm{loc}}_\delta = G^\alpha_\delta \mathbf{1}_{B_1}, \qquad G^{\alpha,\mathrm{glob}}_\delta = G^\alpha_\delta \mathbf{1}_{B_1^c}. \tag{2.6}$$

The following is a refined version of [4, Lemma 4.1].

Lemma 2.1. *There exists $C = C(G, \mathbf{X}) > 0$ such that, for $\alpha \in (0, d)$ and $x \in G$,*

$$|G^{\alpha,\mathrm{loc}}_\delta(x)| \leqslant C\,\frac{\alpha}{d-\alpha}|x|^{\alpha-d}\mathbf{1}_{B_1}(x),$$
$$|G^{\alpha,\mathrm{glob}}_\delta(x)| \leqslant C\,\mathrm{e}^{-(2D+b_0)|x|}\mathbf{1}_{B_1^c}(x).$$

Proof. We recall that the convolution kernel G^α_δ can be written as

$$G^\alpha_\delta = \frac{1}{\Gamma(\alpha/2)}\int_0^\infty t^{\alpha/2-1}\mathrm{e}^{-\tau_\delta\, t} p^\delta_t \,\mathrm{d}t,$$

so that by (2.3)

$$G^\alpha_\delta(x) \leqslant \frac{C}{\Gamma(\alpha/2)}\int_0^\infty t^{\alpha/2-1}(1\wedge t)^{-d/2}\mathrm{e}^{-(\tau_\delta+\frac{1}{4}\mathfrak{c}(\delta)^2)t}\mathrm{e}^{-b|x|^2/t}\,\mathrm{d}t\,.$$

Set $a = \tau_\delta + \frac{1}{4}\mathfrak{c}(\delta)^2$. Since $at + b|x|^2/t \geqslant \frac{1}{2}(at + b|x|^2/t + \sqrt{2ab}|x|)$, we see that when $|x| \geqslant 1$,

$$G^\alpha_\delta(x) \leqslant \frac{C}{\Gamma(\alpha/2)}\mathrm{e}^{-\frac{1}{2}\sqrt{2ab}|x|}\int_0^\infty t^{\alpha/2-1}(1\wedge t)^{-d/2}\mathrm{e}^{-\frac{at}{2}-\frac{b}{2t}}\,\mathrm{d}t \leqslant C\,\mathrm{e}^{-(2D+b_0)|x|}\,.$$

On the other hand, when $|x| \leqslant 1$, splitting the integral we have

$$\begin{aligned} G^\alpha_\delta(x) &\leqslant C\,\alpha\bigg(\int_0^1 t^{(\alpha-d)/2-1}\mathrm{e}^{-b|x|^2/t}\,\mathrm{d}t + \int_1^\infty t^{\alpha/2-1}\mathrm{e}^{-at}\,\mathrm{d}t\bigg) \\ &=: C\,\alpha\,\left(G_1(x) + G_2(x)\right). \end{aligned}$$

It is clear, since $\alpha \in (0, d)$ and $a \geqslant 1$, that $G_2(x) \leqslant C$. Since $\alpha \in (0, d)$, we also have

$$G_1(x) = |x|^{\alpha-d}\bigg(\int_{|x|^2}^1 + \int_1^\infty\bigg)u^{(d-\alpha)/2-1}\mathrm{e}^{-bu}\,\mathrm{d}u \leqslant C\,|x|^{\alpha-d}\bigg(\frac{1}{d-\alpha}(1-|x|^{d-\alpha})+1\bigg),$$

and the conclusion follows. ∎

3. The Sobolev embedding constant

We are now ready to state our main result. Recall that the constant $S(p,q)$ is defined in (1.1).

Theorem 3.1. *Let $p \in (1,\infty)$, $\alpha \in [0,d/p)$ and $q \in [p,\infty)$ be such that $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$. Then there exists $A_1 = A_1(G,\mathbf{X}) > 0$ such that for all $f \in L^p_\alpha(\lambda)$*

$$\|f\|_{L^q(\lambda)} \leqslant A_1\, S(p,q) \|f\|_{L^p_\alpha(\lambda)}.$$

Proof. When $\alpha = 0$ and hence $q = p$, the statement is the trivial embedding $L^p \hookrightarrow L^p$. Since the function $x \mapsto x^{1-1/x}/(x-1)$ is bounded from below for $x > 1$, one sees that $S(p,p) \geqslant 1/c$ for some $c > 0$. Then

$$\|f\|_{L^p(\lambda)} = \|f\|_{L^p_0(\lambda)} \leqslant c\, S(p,p) \|f\|_{L^p_0(\lambda)},$$

and from this point on we may then assume $\alpha > 0$ and $q > p$. Define

$$K_\alpha(x) = |x|^{\alpha - d} \mathbf{1}_{B_1}(x), \qquad \tilde{K}_\alpha(x) = \mathrm{e}^{-(2D+b_0)|x|} \mathbf{1}_{B_1^c}(x).$$

We claim that

$$\|f * K_\alpha\|_{L^q(\lambda)} \leqslant C(G,\mathbf{X}) \frac{d-\alpha}{\alpha} \frac{q^{1/p'}}{p-1} \|f\|_{L^p(\lambda)}, \tag{3.1}$$

$$\|f * \tilde{K}_\alpha\|_{L^q(\lambda)} \leqslant C(G,\mathbf{X}) \|f\|_{L^p(\lambda)}. \tag{3.2}$$

By combining these bounds and Lemma 2.1, we obtain that

$$\|(\tau_\delta I + \mathcal{L})^{-\alpha/2} f\|_{L^q(\lambda)} \leqslant A_1(G,\mathbf{X}) \frac{q^{1/p'}}{p-1} \|f\|_{L^p(\lambda)}. \tag{3.3}$$

Observe that $q^{1/p'}/(p-1)$ is bounded away from zero when $q \geqslant p > 1$. Assuming the claims for a moment, we complete the proof. Observe that the condition $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$ is invariant under the involution $(p,q) \mapsto (q',p')$. Set $Q(p,q) = \frac{q^{1/p'}}{p-1}$. By duality, from (3.3) we have

$$\|(\tau_\delta I + \mathcal{L})^{-\alpha/2} f\|_{L^{p'}(\lambda)} \leqslant A_1 Q(p,q) \|f\|_{L^{q'}(\lambda)},$$

that is, switching the roles of the pairs (p,q) and (q',p'),

$$\|(\tau_\delta I + \mathcal{L})^{-\alpha/2} f\|_{L^q(\lambda)} \leqslant A_1 Q(q',p') \|f\|_{L^p(\lambda)}.$$

This inequality, together with (3.3) gives

$$\|(\tau_\delta I + \mathcal{L})^{-\alpha/2} f\|_{L^q(\lambda)} \leqslant A_1 \min\big(Q(p,q), Q(q',p')\big) \|f\|_{L^p(\lambda)},$$

which implies

$$\|f\|_{L^q(\lambda)} \leqslant A_1\, S(p,q) \|f\|_{L^p_\alpha(\lambda)}.$$

Thus, it remains to prove the claims. The bound (3.2) follows by observing that $\tilde{K}_\alpha = (\tilde{K}_\alpha)\check{}$ and by applying Young's inequality (2.2)

$$\|f * \tilde{K}_\alpha\|_{L^q(\lambda)} \leqslant \|f\|_{L^p(\lambda)} \|\tilde{K}_\alpha\|^{r(1/p'+1/q)}_{L^r(\lambda)}, \tag{3.4}$$

where $r \in (1, \infty)$ is such that $\frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q}$. We then have

$$\begin{aligned}
\|\tilde{K}_\alpha\|^r_{L^r(\lambda)} &\leqslant C \int_{B_1^c} \mathrm{e}^{-r(2D+b_0)|x|}\, \mathrm{d}\lambda(x) \\
&\leqslant C \sum_{k=0}^{\infty} \int_{2^k \leqslant |x| < 2^{k+1}} \mathrm{e}^{-r(2D+b_0)|x|}\, \mathrm{d}\lambda(x) \leqslant C \sum_{k=0}^{\infty} \mathrm{e}^{-r(2D+b_0)2^k + D2^{k+1}} \leqslant C,
\end{aligned}$$

which combined with (3.4) implies (3.2). The remainder of the proof will be devoted to show (3.1).

For $0 < s \leqslant 1$, define $K^{(1)}_{\alpha,s} = K_\alpha \mathbf{1}_{B_s}$ and $K^{(2)}_{\alpha,s} = K_\alpha \mathbf{1}_{B_s^c}$. Notice that $K^{(1)}_{\alpha,s} = \check{K}^{(1)}_{\alpha,s}$ and that the same holds for $K^{(2)}_{\alpha,s}$. Let now $\tilde{p} \in (1, \infty)$ and $\tilde{q} \in (\tilde{p}, \infty)$ be such that $\frac{1}{\tilde{q}} = \frac{1}{\tilde{p}} - \frac{\alpha}{d}$, and observe that

$$(\alpha - d)\tilde{p}' + d = -\frac{d\tilde{p}'}{\tilde{q}}, \qquad \frac{\tilde{p}}{\tilde{q}} = 1 - \tilde{p}\frac{\alpha}{d}, \qquad \frac{1}{\tilde{p}'}\Big(1 - \frac{\tilde{p}}{\tilde{q}}\Big) = (\tilde{p} - 1)\frac{\alpha}{d}. \tag{3.5}$$

By Young's inequality (2.2), there exists $C > 0$ depending only on G and $\mathbf{X}$ such that

$$\|f * K^{(1)}_{\alpha,s}\|_{L^{\tilde{p}}(\lambda)} \leqslant \|f\|_{L^{\tilde{p}}(\lambda)} \|K^{(1)}_{\alpha,s}\|^{1/\tilde{p}}_{L^1(\lambda)} \|\check{K}^{(1)}_{\alpha,s}\|^{1/\tilde{p}'}_{L^1(\lambda)} \leqslant C\frac{1}{\alpha} s^\alpha \|f\|_{L^{\tilde{p}}(\lambda)} \tag{3.6}$$

and

$$\|f * K^{(2)}_{\alpha,s}\|_{L^\infty} \leqslant \|f\|_{L^{\tilde{p}}(\lambda)} \|\check{K}^{(2)}_{\alpha,s}\|_{L^{\tilde{p}'}(\lambda)} \leqslant C \left(\frac{\tilde{q}}{d\tilde{p}'}\right)^{1/\tilde{p}'} (s^{-d\tilde{p}'/\tilde{q}} - 1)^{1/\tilde{p}'} \|f\|_{L^{\tilde{p}}(\lambda)}. \tag{3.7}$$

For $t > 0$ we now set

$$s(t) = \left[1 + \frac{d\tilde{p}'}{\tilde{q}} \left(\frac{t}{2}\right)^{\tilde{p}'}\right]^{-\frac{\tilde{q}}{d\tilde{p}'}},$$

and observe that $s(t) \leqslant 1$ for every $t > 0$. By (3.7),

$$\|f * K^{(2)}_{\alpha,s(t)}\|_{L^\infty} \leqslant C\frac{t}{2}\|f\|_{L^{\tilde{p}}(\lambda)} \qquad \forall t > 0\,. \tag{3.8}$$

Thus, with C the same constant as in (3.6) and (3.7),

$$\begin{aligned}
&\sup_{t>0} t\, \lambda(\{x\colon\ |f * K_\alpha(x)| > t\})^{1/\tilde{q}} \\
&\quad = C\|f\|_{L^{\tilde{p}}(\lambda)} \sup_{t>0} t\, \lambda\Big(\Big\{x\colon\ |f * K_\alpha(x)| > Ct\|f\|_{L^{\tilde{p}}(\lambda)}\Big\}\Big)^{1/\tilde{q}} \\
&\quad \leqslant C\|f\|_{L^{\tilde{p}}(\lambda)} \sup_{t>0} t\, \lambda\bigg(\bigg\{x\colon\ |f * K^{(1)}_{\alpha,s(t)}(x)| > C\frac{t}{2}\|f\|_{L^{\tilde{p}}(\lambda)}\bigg\}\bigg)^{1/\tilde{q}} \\
&\qquad + C\|f\|_{L^{\tilde{p}}(\lambda)} \sup_{t>0} t\, \lambda\bigg(\bigg\{x\colon\ |f * K^{(2)}_{\alpha,s(t)}(x)| > C\frac{t}{2}\|f\|_{L^{\tilde{p}}(\lambda)}\bigg\}\bigg)^{1/\tilde{q}} \\
&\quad = C\|f\|_{L^{\tilde{p}}(\lambda)} \sup_{t>0} t\, \lambda\bigg(\bigg\{x\colon\ |f * K^{(1)}_{\alpha,s(t)}(x)| > C\frac{t}{2}\|f\|_{L^{\tilde{p}}(\lambda)}\bigg\}\bigg)^{1/\tilde{q}},
\end{aligned}$$

since $s(t)$ was chosen so that the second super-level set was empty. By (3.6), we get

$$\begin{aligned}
&\sup_{t>0} t\,\lambda\bigg(\Big\{x\colon\ |f*K^{(1)}_{\alpha,s(t)}(x)|>C\frac{t}{2}\|f\|_{L^{\tilde p}(\lambda)}\Big\}\bigg)^{1/\tilde q}\\
&\leqslant \sup_{t>0} t\left[\left(\frac{2}{Ct\|f\|_{L^{\tilde p}(\lambda)}}\right)^{\tilde p}\|f*K^{(1)}_{\alpha,s(t)}\|^{\tilde p}_{L^{\tilde p}(\lambda)}\right]^{1/\tilde q}\\
&\leqslant \sup_{t>0} t\left(\frac{Ct\|f\|_{L^{\tilde p}(\lambda)}}{2}\right)^{-\tilde p/\tilde q}\left(\frac{s(t)^\alpha}{\alpha}\right)^{\tilde p/\tilde q} C^{\tilde p/\tilde q}\|f\|^{\tilde p/\tilde q}_{L^{\tilde p}(\lambda)}\\
&=\left(\frac{2}{\alpha}\right)^{\tilde p/\tilde q}\sup_{t>0} t^{1-\tilde p/\tilde q}\left[1+\frac{d\tilde p'}{\tilde q}\left(\frac{t}{2}\right)^{\tilde p'}\right]^{-\frac{1}{\tilde p'}(1-\frac{\tilde p}{\tilde q})}\\
&=\frac{2}{\alpha^{\tilde p/\tilde q}}\left(\frac{\tilde q}{d\tilde p'}\right)^{\frac{1}{\tilde p'}(1-\frac{\tilde p}{\tilde q})}\sup_{u>0} u^{1-\tilde p/\tilde q}(1+u^{\tilde p'})^{-\frac{1}{\tilde p'}(1-\frac{\tilde p}{\tilde q})}.
\end{aligned}$$

It is now easy to see that, for every $\tilde p$ and $\tilde q$,

$$\sup_{u>0} u^{1-\tilde p/\tilde q}(1+u^{\tilde p'})^{-\frac{1}{\tilde p'}(1-\frac{\tilde p}{\tilde q})}=\sup_{v>0}\big[v/(1+v)\big]^{\frac{1}{\tilde p'}(1-\frac{\tilde p}{\tilde q})}=1.$$

Moreover, by (3.5) we end up with the inequality

$$\begin{aligned}
\|f*K_\alpha\|_{L^{\tilde q,\infty}(\lambda)}&=\sup_{t>0} t\,\lambda(\{x\colon\ |f*K_\alpha(x)|>t\})^{\frac{1}{\tilde q}}\\
&\leqslant C\alpha^{\tilde p\alpha/d-1}\left(\frac{\tilde q}{d\tilde p'}\right)^{(\tilde p-1)\alpha/d}\|f\|_{L^{\tilde p}(\lambda)}.
\end{aligned}\tag{3.9}$$

In other words, the operator defined by $\mathcal{K}_\alpha f=f*K_\alpha$ is of weak type $(\tilde p,\tilde q)$ for every $\tilde p,\tilde q$ such that $\frac{1}{\tilde q}=\frac{1}{\tilde p}-\frac{\alpha}{d}$, $1<\tilde p<\tilde q<\infty$, $0<\alpha<d$.

In a similar way we can also prove that $\mathcal{K}_\alpha$ is of weak type $(1,\tilde q)$ for $\frac{1}{\tilde q}=1-\frac{\alpha}{d}$ and $0<\alpha<d$. Indeed, the estimate (3.6) holds also for $\tilde p=1$ and

$$\|f*K^{(2)}_{\alpha,s}\|_{L^\infty}\leqslant C\|f\|_{L^1(\lambda)}\times\begin{cases}s^{\alpha-d} & \text{if } s<1\\ 0 & \text{if } s\geqslant 1.\end{cases}\tag{3.10}$$

We now set

$$s(t)=\begin{cases}\left(1+\frac{t}{2}\right)^{1/(\alpha-d)} & t\geqslant 2\\ 1 & 0<t<2\,,\end{cases}$$

which is $\leqslant 1$. Then (3.8) holds also in this case and we obtain as above that

$$\begin{aligned}
&\sup_{t>0} t\,\lambda(\{x\colon\ |f*K_\alpha(x)|>t\})^{1/\tilde q}\\
&\quad\leqslant C\|f\|_{L^1(\lambda)}\sup_{t>0} t\,\lambda\bigg(\Big\{x\colon\ |f*K^{(1)}_{\alpha,s(t)}(x)|>C\frac{t}{2}\|f\|_{L^{\tilde p}(\lambda)}\Big\}\bigg)^{1/\tilde q}\\
&\quad\leqslant C\|f\|_{L^1(\lambda)}\sup_{t>0} t\left(\frac{2}{Ct\|f\|_{L^1(\lambda)}}\|f*K^{(1)}_{\alpha,s(t)}\|_{L^1(\lambda)}\right)^{1/\tilde q}.
\end{aligned}$$

We now notice that

$$\begin{aligned}\sup_{0<t<2} t\left(\frac{2}{Ct\|f\|_{L^1(\lambda)}}\|f*K^{(1)}_{\alpha,s(t)}\|_{L^1(\lambda)}\right)^{1/\tilde q} &\leqslant \sup_{0<t<2} t\left(\frac{t\|f\|_{L^1(\lambda)}}{2}\right)^{-1/\tilde q}\left(\frac{1}{\alpha}\right)^{1/\tilde q}\|f\|^{1/\tilde q}_{L^1(\lambda)}\\ &= 2\alpha^{-1/\tilde q}\,,\end{aligned}$$

while

$$\begin{aligned}\sup_{t\geqslant 2} t\left(\frac{2}{Ct\|f\|_{L^1(\lambda)}}\|f*K^{(1)}_{\alpha,s(t)}\|_{L^1(\lambda)}\right)^{1/\tilde q} &\leqslant \sup_{t\geqslant 2} t\left(\frac{t\|f\|_{L^1(\lambda)}}{2}\right)^{-1/\tilde q}\left(\frac{s(t)^\alpha}{\alpha}\right)^{1/\tilde q}\|f\|^{1/\tilde q}_{L^1(\lambda)}\\ &\leqslant C\sup_{t\geqslant 2} t^{1-\frac{1}{\tilde q}}\left(\frac{2}{\alpha}\right)^{1/\tilde q}\left(\frac{t}{2}\right)^{-1/d} = C\,\alpha^{-1/\tilde q}\,.\end{aligned}$$

This proves that

$$\|f*K_\alpha\|_{L^{\tilde q,\infty}(\lambda)} \leqslant C\alpha^{-1/\tilde q}\|f\|_{L^1(\lambda)}. \tag{3.11}$$

We shall now use the Marcinkiewicz interpolation theorem for two specific choices of the couple $(\tilde p,\tilde q)$. Being $p\in(1,\infty)$, $q\in(p,\infty)$, and $\alpha/d = 1/p - 1/q$ as in the statement, we define

$$\left(\frac{1}{p_1},\frac{1}{q_1}\right) = \left(1, 1-\frac{\alpha}{d}\right), \qquad \left(\frac{1}{p_2},\frac{1}{q_2}\right) = \left(\frac{\alpha}{d}+\frac{1}{q+1},\frac{1}{q+1}\right). \tag{3.12}$$

By the above, $\mathcal{K}_\alpha$ is both of weak type $(1,q_1)$ and (p_2,q_2) with norms $M(1,q_1)$ and $M(p_2,q_2)$ respectively, given by

$$\begin{aligned}M(1,q_1) &= \alpha^{-(1-\alpha/d)},\\ M(p_2,q_2) &= \left(\frac{d^{\alpha/d}}{\alpha}\right)\left(\frac{\alpha}{d}\right)^{\frac{\alpha/d}{\alpha/d+1/(q+1)}}\left[\left(1-\frac{\alpha}{d}-\frac{1}{q+1}\right)(q+1)\right]^{\frac{1}{1+d/(\alpha(q+1))}-\frac{\alpha}{d}}.\end{aligned}$$

We select

$$\theta = \frac{1-\frac{1}{p}}{1-\frac{\alpha}{d}-\frac{1}{q+1}}.$$

Notice that we indeed have $0<\theta<1$, $1/p = (1-\theta)/p_1 + \theta/p_2$ and $1/q = (1-\theta)/q_1+\theta/q_2$. Thus, $\mathcal{K}_\alpha$ is of strong type (p,q), i.e. bounded from $L^p(\lambda)$ to $L^q(\lambda)$, with norm bounded by

$$CM_0(1,q_1,p_2,q_2)^{1/q}M(1,q_1)^{1-\theta}M(p_2,q_2)^\theta,$$

see e.g. [34, Ch. XII, (4.18)], where

$$M_0(1,q_1,p_2,q_2) = \frac{q(p_2/p)^{q_2/p_2}}{q_2-q}+\frac{q/p^{q_1}}{q-q_1}.$$

If we observe that

$$M_0(1,q_1,p_2,q_2)^{1/q}M(1,q_1)^{1-\theta}M(p_2,q_2)^\theta \leqslant C\,\frac{d-\alpha}{\alpha}\,\frac{q^{1/p'}}{p-1}, \tag{3.13}$$

then we get (3.1), which concludes the proof of the theorem.

We now prove (3.13). First we consider $M_1 = M(1,q_1)$, and simply observe that

$$M_1 = \alpha^{-1}d^{\alpha/d}(\alpha/d)^{\alpha/d} \leqslant d\,\alpha^{-1}$$

as $\alpha/d\leqslant 1$ and $x^x\leqslant 1$ for $x\in(0,1]$.

Then we consider $M_0 = M_0(1, q_1, p_2, q_2)$, and define

$$C(p,q) = p^{-p'q/(q+p')}\Big(1 + \frac{p'}{q}\Big), \qquad y = \frac{\alpha}{d}(q+1).$$

Since

$$\frac{p_2}{q_2} = 1 + y, \qquad \frac{p_2}{p} = y + 1 + \frac{1}{q},$$

we get

$$M_0 = q\left(y + 1 + \frac{1}{q}\right)^{1+y} (1+y)^{-(1+y)} + C(p,q).$$

Moreover

$$\left(y + 1 + \frac{1}{q}\right)^{1+y} (1+y)^{-(1+y)} = \left[\left(1 + \frac{1}{q(1+y)}\right)^{q(1+y)}\right]^{1/q} \leqslant \mathrm{e}$$

since $q(1+y) \geqslant 1$ and by the estimate $(1+\frac{1}{x})^x \leqslant \mathrm{e}$ for $x \geqslant 1$. Thus $M_0 \leqslant \mathrm{e}\, q + C(p,q)$.

We then consider $M_2 = M(p_2, q_2)$, and estimate M_2^θ. We first observe that

$$M_2^\theta \leqslant d^\theta \alpha^{-\theta} \left(\frac{\alpha}{d}\right)^{\theta\frac{\alpha/d}{\alpha/d+1/(q+1)}} \left[\Big(1 - \frac{\alpha}{d} - \frac{1}{q+1}\Big)(q+1)\right]^{\theta\frac{\alpha/d}{\alpha/d+1/(q+1)} - \theta\frac{\alpha}{d}}$$

and that

$$\begin{aligned}\left(\frac{\alpha}{d}\right)^{\theta\frac{\alpha/d}{\alpha/d+1/(q+1)}} &\left[\Big(1 - \frac{\alpha}{d} - \frac{1}{q+1}\Big)(q+1)\right]^{\theta\frac{\alpha/d}{\alpha/d+1/(q+1)} - \theta\frac{\alpha}{d}} \\ &= \left[\left(\frac{\alpha}{d}\right)^{\frac{1}{1-z}}(q+1)\right]^{(1-1/p)\frac{\alpha/d}{z}} (1-z)^{(1-1/p)\frac{\alpha/d}{z}} \qquad (3.14)\end{aligned}$$

where $z = \frac{\alpha}{d} + \frac{1}{q+1}$. Observe that $0 < z < 1/p < 1$, and that

$$\frac{\alpha/d}{\alpha/d + 1/(q+1)} = \frac{\alpha}{dz} = \frac{(q-p)(q+1)}{q(q+1)-p} \leqslant 1. \qquad (3.15)$$

Therefore

$$\left(\frac{\alpha}{d}\right)^{\frac{1}{1-z}} \leqslant \frac{\alpha}{d}, \qquad (1-z)^{(1-1/p)\frac{\alpha/d}{z}} \leqslant 1.$$

Observe now that

$$\left[\left(\frac{\alpha}{d}\right)(q+1)\right]^{(1-1/p)\frac{\alpha/d}{z}} = \left[\frac{(q-p)(q+1)}{q(q+1)-p}\right]^{\frac{1}{p'}\frac{(q-p)(q+1)}{q(q+1)-p}} \left[\frac{q(q+1)-p}{pq}\right]^{\frac{1}{p'}\frac{(q-p)(q+1)}{q(q+1)-p}},$$

and that, by (3.15) and since

$$2\frac{q}{p} \geqslant \frac{q(q+1)-p}{pq} \geqslant \frac{q}{p} \geqslant 1,$$

one gets

$$\left[\left(\frac{\alpha}{d}\right)(q+1)\right]^{(1-1/p)\frac{\alpha/d}{z}} \leqslant 2\left(\frac{q}{p}\right)^{1/p'}.$$

This proves that $M_2^\theta \leqslant 2\, d^\theta\, (q/p)^{1-1/p} \alpha^{-\theta}$.

Putting everything together, we proved that

$$M_0^{1/q} M_1^{1-\theta} M_2^{\theta} \leqslant 2\, d\, \alpha^{-1} (\mathrm{e}\, q + C(p,q))^{1/q} (q/p)^{1-1/p}.$$

It remains to estimate the term in the parenthesis in the right hand side. Observe first that

$$(\mathrm{e}q + C(p,q))^{1/q} \leqslant (\mathrm{e}\, q)^{1/q} + C(p,q)^{1/q} \leqslant 2\mathrm{e} + C(p,q)^{1/q},$$

and then that

$$C(p,q)^{1/q} \leqslant \Big(1 + \frac{p'}{q}\Big)^{1/q} = \frac{d-\alpha}{d}\, p' \Big(1 + \frac{p'}{q}\Big)^{1/q-1} \leqslant \frac{d-\alpha}{d}\, p'.$$

After observing that $(d-\alpha)\, p'/d \geqslant 1$, the proof of (3.13) is complete. This implies (3.1) and completes the proof. □

4. Comparison with the Euclidean case

In this section we compare our embedding constant $A_1 S(p,q)$ with the known embedding constant in the Euclidean case. As a preliminary remark, observe that if G has polynomial growth, then $\delta = 1$, and $\mathcal{L} = \Delta$ is the sum-of-squares sub-Laplacian associated with $\mathbf{X}$. Since the exponential dimension D can be taken arbitrarily small, one obtains $\tau_\delta = 1$. Thus, in this case the Sobolev norm $\|\cdot\|_{L^p_\alpha(\lambda)}$ is the graph norm of $(I+\Delta)^{\alpha/2}$ in $L^p(\lambda)$.

This in particular holds in $\mathbb{R}^d$, where $\mathbf{X} = \{\partial_1, \dots, \partial_d\}$, Δ is the Laplacian, λ is the Lebesgue measure and $L^p_\alpha = L^p_\alpha(\lambda)$ is the classical inhomogeneous Sobolev space. Theorem 3.1 in the Euclidean setting then reads as

$$\|f\|_{L^q} \leqslant A_1\, S(p,q) \|f\|_{L^p_\alpha},$$

where A_1 depends only on the dimension d.

Let $0 < \alpha < d$ and $p, q \in (1,\infty)$ be such that $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$. Denote respectively by $E(p,q,d)$ and $E_H(p,q,d)$ the best embedding constants of L^p_α into L^q, and of $\dot{L}^p_\alpha$ into L^q, where $\dot{L}^p_\alpha$ is the homogeneous Sobolev space given by the closure of the Schwartz functions with respect to the norm $\|f\|_{\dot{L}^p_\alpha} = \|\Delta^{\alpha/2} f\|_{L^p}$. Equivalently, $E(p,q,d)$ and $E_H(p,q,d)$ are respectively the infimum of the constants $C_I, C_H > 0$ such that

$$\|(I+\Delta)^{-\alpha/2} f\|_{L^q} \leqslant C_I \|f\|_{L^p} \qquad \text{and} \qquad \|\Delta^{-\alpha/2} f\|_{L^q} \leqslant C_H \|f\|_{L^p}.$$

Now, $E_H(p,q,d)$ equals

$$E_H(p,q,d) = \frac{1}{(2\pi)^\alpha} \frac{\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)} C_L(p,q,d) \tag{4.1}$$

where $C_L(p,q,d)$ is the best constant for the Hardy–Littlewood–Sobolev inequality, which by [20, Theorem 4.3] can be estimated as follows:

$$C_L(p,q,d) \leqslant \frac{d}{\alpha} \Big(\frac{\omega_{d-1}}{d}\Big)^{1-\frac{\alpha}{d}} \Big(1 - \frac{\alpha}{d}\Big)^{1-\frac{\alpha}{d}} \frac{1}{pq'} \Big({p'}^{\frac{1}{p'}+\frac{1}{q}} + q^{\frac{1}{p'}+\frac{1}{q}} \Big), \tag{4.2}$$

where ω_{d-1} is the surface measure of the unit sphere in $\mathbb{R}^d$. Notice that $\frac{1}{p'} + \frac{1}{q} = 1 - \frac{\alpha}{d}$. In other words, by (4.1) and (4.2) the best known bound for $E_H(p,q,d)$ is given by

$E_H(p,q,d) \leqslant \widetilde{E}_H(p,q,d)$, where

$$\widetilde{E}_H(p,q,d) = \frac{1}{(2\pi)^\alpha}\frac{\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)}\frac{d}{\alpha}\left(\frac{\omega_{d-1}}{d}\right)^{1-\frac{\alpha}{d}}\left(1-\frac{\alpha}{d}\right)^{1-\frac{\alpha}{d}}\frac{1}{pq'}\left(p'^{\frac{1}{p'}+\frac{1}{q}}+q^{\frac{1}{p'}+\frac{1}{q}}\right).$$

To the best of our knowledge, the best known bound for $E(p,q,d)$ is in turn given in terms of $E_H(p,q,d)$, hence in terms of $\widetilde{E}_H(p,q,d)$; in particular, we have the following result. For $p,q \in (1,\infty)$, $q \geqslant p$, set

$$F(p,q) := \frac{1}{\frac{1}{p'}+\frac{1}{q}}\frac{1}{pq'}\left(p'^{\frac{1}{q}}+q^{\frac{1}{p'}}\right). \tag{4.3}$$

Proposition 4.1. *For all $p \in (1,\infty)$, $\alpha \in [0,d/p)$ and $q \in [p,\infty)$ such that $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{d}$,*

$$E(p,q,d) = E_H(p,q,d) \leqslant \widetilde{E}_H(p,q,d). \tag{4.4}$$

Moreover, there exists a positive constant B_1 depending only on d such that, for all p,q,α as above,

$$B_1^{-1}F(p,q) \leqslant \widetilde{E}_H(p,q,d) \leqslant B_1 F(p,q). \tag{4.5}$$

Proof. The equality in (4.4) follows by observing that the best constant in the inequality $\|f\|_q \leqslant C\|(a+\Delta)^{\alpha/2}f\|_p$, with f Schwartz, does not depend on $a>0$ by rescaling; and then by using a limit argument (we thank one of the anonymous referees for pointing this out to us). The inequality in (4.4) follows instead from the discussion preceding the proposition.

We now prove (4.5). Using the conditions $0<\alpha<d$ and $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{d}$, we have

$$\begin{aligned}\frac{1}{(2\pi)^\alpha}&\frac{\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)}\frac{d}{\alpha}\left(\frac{\omega_{d-1}}{d}\right)^{1-\frac{\alpha}{d}}\left(1-\frac{\alpha}{d}\right)^{1-\frac{\alpha}{d}}\\ &= \frac{1}{(2\pi)^\alpha}\frac{\Gamma(1+(d-\alpha)/2)}{\Gamma(1+\alpha/2)}\frac{1}{1-\frac{\alpha}{d}}\left(\frac{\omega_{d-1}}{d}\right)^{1-\frac{\alpha}{d}}\left(1-\frac{\alpha}{d}\right)^{1-\frac{\alpha}{d}}\\ &= \frac{1}{(2\pi)^\alpha}\frac{\Gamma(1+(d-\alpha)/2)}{\Gamma(1+\alpha/2)}\left(\frac{\omega_{d-1}}{d}\right)^{1-\frac{\alpha}{d}}\left(1-\frac{\alpha}{d}\right)^{1-\frac{\alpha}{d}}\frac{1}{\frac{1}{p'}+\frac{1}{q}},\end{aligned}$$

and

$$\frac{1}{B(d)} \leqslant \frac{1}{(2\pi)^\alpha}\frac{\Gamma(1+(d-\alpha)/2)}{\Gamma(1+\alpha/2)}\left(\frac{\omega_{d-1}}{d}\right)^{1-\frac{\alpha}{d}}\left(1-\frac{\alpha}{d}\right)^{1-\frac{\alpha}{d}} \leqslant B(d),$$

where $B(d)$ is a constant depending only on d; observe indeed that each factor in the product above is bounded from above and below by a constant that depends only on d. Hence,

$$\frac{1}{B(d)}\frac{1}{\frac{1}{p'}+\frac{1}{q}}\frac{1}{pq'}\left(p'^{\frac{1}{q}}+q^{\frac{1}{p'}}\right) \leqslant \widetilde{E}_H(p,q,d) \leqslant B(d)e^{1/e}\frac{1}{\frac{1}{p'}+\frac{1}{q}}\frac{1}{pq'}\left(p'^{\frac{1}{q}}+q^{\frac{1}{p'}}\right),$$

since $1 \leqslant x^{1/x} \leqslant e^{1/e}$ when $x \geqslant 1$. Hence, (4.5) follows. ☐

We now show that similar estimates hold in our case, namely that the constant $S(p,q)$ is comparable to $\widetilde{E}_H(p,q,d)$, up to a constant depending only on d. In other words, we show that we recover the best known result, in terms of dependence on p and q, when G is a Euclidean space.

Theorem 4.2. *There exists a constant B_3, depending only on d, such that for all $p \in (1,\infty)$, $\alpha \in [0, d/p)$ and $q \in [p,\infty)$ such that $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$ we have*

$$\frac{1}{B_3} S(p,q) \leqslant \widetilde{E}_H(p,q,d) \leqslant B_3 S(p,q). \tag{4.6}$$

Proof. We are going to show that $S(p,q)$ is bounded above and below by absolute constants times $F(p,q)$, and in view of (4.5) this will suffice. Since $F(p,q) = F(q',p')$ and $S(p,q) = S(q',p')$, it suffices in turn to consider the case $q \geqslant p'$.

We claim that in this regime

$$\frac{1}{4} Q(p,q) \leqslant F(p,q) \leqslant 4Q(p,q),$$

where $Q(p,q) = \frac{q^{1/p'}}{p-1}$. Since $q \geqslant p'$, we also have $\frac{1}{p'} \geqslant \frac{1}{q}$ and $p'^{\frac{1}{q}} \leqslant q^{\frac{1}{p'}}$ (since $x \mapsto x^x$ is increasing on $[1,\infty)$). Then, since as before $1 \leqslant q' \leqslant 2$,

$$F(p,q) \leqslant 2 \frac{1}{\frac{1}{p'} p q'} q^{\frac{1}{p'}} = \frac{2}{q'(p-1)} q^{\frac{1}{p'}} \leqslant 2 \frac{q^{\frac{1}{p'}}}{p-1} = 2Q(p,q).$$

On the other hand,

$$F(p,q) \geqslant \frac{1}{\frac{2}{p'} q' p} q^{\frac{1}{p'}} = \frac{q^{\frac{1}{p'}}}{2q'(p-1)} \geqslant \frac{1}{4} Q(p,q).$$

This proves the claim. It remains to show that if $q \geqslant p'$ then $S(p,q) = Q(p,q)$, namely $Q(p,q) \leqslant Q(q',p')$. The latter inequality is

$$\frac{q^{\frac{1}{p'}}}{p-1} \leqslant \frac{p'^{\frac{1}{q}}}{q'-1}.$$

Multiplying both sides by pq', it becomes

$$q' p'^{\frac{1}{q'}} \leqslant p q^{\frac{1}{p}}.$$

Since $q \geqslant p'$, hence $q' \leqslant p$, it suffices to show that $p'^{\frac{1}{q'}} \leqslant q^{\frac{1}{p}}$, that is, $p'^p \leqslant q^{q'}$. But this follows since $p' \leqslant q$ and the function $x \mapsto e^{\frac{x}{x-1} \log x}$ is increasing in $[1,\infty)$. This concludes the proof. $\square$

5. A Moser–Trudinger inequality

As an application of Theorem 3.1, we shall prove a quantitative Moser–Trudinger inequality. To do this, we will need a precise version of the interpolation inequality [5, eq. (6.1)] associated to the interpolation space $(L^p(\lambda), L^p_\alpha(\lambda))_{[\theta]} = L^p_{\theta\alpha}(\lambda)$ with respect to the complex method. To prove this refined estimate, we follow some ideas developed in [1]; see also [24].

Proposition 5.1. *Let $p \in (1,\infty)$ and define*

$$\mathcal{C}_p = \inf_{\sigma>0} \sup_{t\in\mathbb{R}} \mathrm{e}^{\sigma(1-t^2)} \|(\tau_\delta I + \mathcal{L})^{it}\|_{L^p(\lambda)\to L^p(\lambda)}.$$

Then $1 \leqslant \mathcal{C}_p < \infty$ and for all $f \in L^p_\alpha(\lambda)$, $\alpha \geqslant 0$, and $\theta \in (0,1)$ we have

$$\|f\|_{L^p_{\theta\alpha}(\lambda)} \leqslant \mathcal{C}_p \|f\|^{1-\theta}_{L^p(\lambda)} \|f\|^{\theta}_{L^p_\alpha(\lambda)}. \tag{5.1}$$

Proof. For $\sigma > 0$, let

$$\mathcal{C}_{p,\sigma} = \sup_{t\in\mathbb{R}} \mathrm{e}^{\sigma(1-t^2)} \|(\tau_\delta I + \mathcal{L})^{it}\|_{L^p(\lambda)\to L^p(\lambda)}.$$

Since $\mathcal{C}_{p,\sigma}$ is finite for all $\sigma > 0$ by [9, Corollary 1], see also [21], it follows that $\mathcal{C}_p$ is finite. Moreover, since $(\tau_\delta I + \mathcal{L})^{it} = I$ for $t = 0$, one gets $\mathcal{C}_{p,\sigma} \geqslant \mathrm{e}^\sigma \geqslant 1$, hence also $\mathcal{C}_p \geqslant 1$.

Suppose that $f = \sum_{j=1}^N a_j \chi_{E_j}$, $h = \sum_{k=1}^{N'} a'_k \chi_{E'_k}$ are two simple functions on G. Let $S = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$, and let $\overline{S}$ denote its closure. For every $z \in \overline{S}$ we define

$$w(z) = \mathrm{e}^{\sigma z^2} \int_G (\tau_\delta I + \mathcal{L})^{-\alpha z/2} f(x) h(x) \,\mathrm{d}\lambda(x).$$

Then w is holomorphic on S, continuous on $\overline{S}$ and w is bounded on $\overline{S}$. Indeed,

$$\begin{aligned}\sup_{z\in\overline{S}} |w(z)| &\leqslant \sum_{j=1}^N \sum_{k=1}^{N'} |a_j||a'_k| \sup_{z\in\overline{S}} \left| \mathrm{e}^{\sigma z^2} \int_{E'_k} (\tau_\delta I + \mathcal{L})^{-\alpha z/2} \chi_{E_j}(x) \,\mathrm{d}\lambda(x) \right| \\ &\leqslant \mathcal{C}_{p,\sigma} \sum_{j=1}^N \sum_{k=1}^{N'} |a_j||a'_k| \lambda(E'_k)^{1/p'} \sup_{0\leqslant x\leqslant 1} \|(\tau_\delta I + \mathcal{L})^{-\alpha x/2}\|_{L^p(\lambda)\to L^p(\lambda)} \lambda(E_j)^{1/p} < \infty\,.\end{aligned}$$

We now observe that for every $t \in \mathbb{R}$

$$|w(it)| \leqslant \mathcal{C}_{p,\sigma} \|f\|_{L^p(\lambda)} \|h\|_{L^{p'}(\lambda)}$$

and

$$|w(1 + it)| \leqslant \mathcal{C}_{p,\sigma} \|(\tau_\delta I + \mathcal{L})^{-\alpha/2} f\|_{L^p(\lambda)} \|h\|_{L^{p'}(\lambda)}\,.$$

By the classical three lines theorem it follows that

$$|w(1-\theta)| \leqslant \mathcal{C}_{p,\sigma} \|f\|^\theta_{L^p(\lambda)} \|(\tau_\delta I + \mathcal{L})^{-\alpha/2} f\|^{1-\theta}_{L^p(\lambda)} \|h\|_{L^{p'}(\lambda)}\,.$$

By taking the supremum over all simple functions h such that $\|h\|_{L^{p'}(\lambda)} \leqslant 1$ we have

$$\|(\tau_\delta I + \mathcal{L})^{-(1-\theta)\alpha/2} f\|_{L^p(\lambda)} \leqslant \mathcal{C}_{p,\sigma} \|f\|^\theta_{L^p(\lambda)} \|(\tau_\delta I + \mathcal{L})^{-\alpha/2} f\|^{1-\theta}_{L^p(\lambda)}.$$

By using the density of simple functions in $L^p(\lambda)$ and choosing $g = (\tau_\delta I + \mathcal{L})^{-\alpha/2} f$ we get

$$\|(\tau_\delta I + \mathcal{L})^{\theta\alpha/2} g\|_{L^p(\lambda)} \leqslant \mathcal{C}_{p,\sigma} \|(\tau_\delta I + \mathcal{L})^{\alpha/2} g\|^\theta_{L^p(\lambda)} \|g\|^{1-\theta}_{L^p(\lambda)},$$

which is equivalent to

$$\|g\|_{L^p_{\theta\alpha}(\lambda)} \leqslant C_{p,\sigma} \|g\|^\theta_{L^p_\alpha(\lambda)} \|g\|^{1-\theta}_{L^p(\lambda)}\,.$$

By taking the infimum over all $\sigma > 0$, the inequality (5.1) follows. ☐

As a corollary of the estimate of Theorem 3.1 and Proposition 5.1, we obtain the following global Moser–Trudinger inequality. Keeping the notation therein, we define

$$\gamma_1 = [\mathrm{e}\,(\mathcal{C}_p A_1 (p'-1))^{p'} p']^{-1}.$$

Theorem 5.2. *Let $p \in (1,\infty)$. For $\gamma \in [0, \gamma_1)$ and $f \in L^p_{d/p}(\lambda)$ with $\|f\|_{L^p_{d/p}(\lambda)} \leqslant 1$,*

$$\int_G \Big(\exp(\gamma |f|^{p'}) - \sum_{0\leqslant k<p-1} \frac{\gamma^k}{k!} |f|^{p'k} \Big) \,\mathrm{d}\lambda \leqslant C(G, \mathbf{X}, p) \|f\|^p_{L^p(\lambda)}. \tag{5.2}$$

We point out that, even in the case of the Laplacian in $\mathbb{R}^d$, the best constant γ_1 for which (5.2) holds is not known, other than in the cases $d/p = 1$ [18] and $d/p = 2$ [17].

Proof. By Theorem 3.1 and the interpolation inequality (5.1), when $q > p$ we obtain

$$\|f\|_{L^q(\lambda)} \leqslant A_1\, S(p,q)\, \mathcal{C}_p \|f\|^{1-p/q}_{L^p_{d/p}(\lambda)} \|f\|^{p/q}_{L^p(\lambda)}. \tag{5.3}$$

Then, if $\|f\|_{L^p_{d/p}(\lambda)} \leqslant 1$,

$$\begin{aligned}\int_G \Big(\exp(\gamma |f|^{p'}) - \sum_{0\leqslant k < p-1} \frac{\gamma^k}{k!} |f|^{p'k} \Big)\, \mathrm{d}\lambda &= \sum_{k \geqslant p-1} \frac{\gamma^k}{k!} \|f\|^{p'k}_{L^{p'k}(\lambda)} \\ &\leqslant \|f\|^p_{L^p(\lambda)} \sum_{k\geqslant p-1} \frac{\gamma^k}{k!} (\mathcal{C}_p A_1)^{p'k} S(p, p'k)^{p'k} . \end{aligned} \tag{5.4}$$

Observe that since $S(p,q) = Q(p,q)$ when $q \geqslant p'$ and $p'k \geqslant p'$,

$$S(p,p'k)^{p'k} = \min\Big(\frac{(p'k)^{1/p'}}{p-1}, \frac{p'^{1/(p'k)}}{(p'k)'-1} \Big)^{p'k} = \frac{(p'k)^k}{(p-1)^{p'k}} .$$

Plugging this estimate into (5.4) we obtain

$$\begin{aligned}\int_G \Big(\exp(\gamma |f|^{p'}) - \sum_{0\leqslant k < p-1} \frac{\gamma^k}{k!} |f|^{p'k} \Big)\, \mathrm{d}\lambda &\leqslant \|f\|^p_{L^p(\lambda)} \sum_{k\geqslant p-1} \frac{\gamma^k}{k!} \big(\mathcal{C}_p A_1 (p'-1)\big)^{p'k} (p'k)^k \\ &\leqslant C(G, \mathbf{X}, p) \|f\|^p_{L^p(\lambda)} \end{aligned}$$

if $\gamma < \gamma_1$. The proof of the theorem is complete. □

6. The case of general measures

In this final section we consider the case of more general sub-Laplacians and relatively invariant measures, as in [4], where different phenomena appear. We denote by ρ the right Haar measure such that $\mathrm{d}\lambda = \delta^{-1}\,\mathrm{d}\rho$, and by χ a continuous positive character of G. We then let μ_χ be the measure with density χ with respect to ρ. As δ is a continuous positive character, $\mu_\delta = \lambda$. Since

$$\sup_{|x|\leqslant r} \chi(x) = \mathrm{e}^{\mathfrak{c}(\chi) r}, \quad \text{where} \quad \mathfrak{c}(\chi) = (|X_1\chi(e)|^2 + \cdots + |X_\ell \chi(e)|^2)^{1/2},$$

cf. [14], and $V(r) = \rho(B_r)$, the metric measure space (G, d_C, μ_χ) is locally doubling, though not doubling in general.

The spaces $L^p(\mu_\chi)$ are defined classically and in the same way as the spaces $L^p(\lambda)$ described above. We denote by Δ_χ the sub-Laplacian with drift

$$\Delta_\chi = -\sum_{j=1}^{\ell} (X_j^2 + (X_j\chi)(e) X_j),$$

and recall that it is symmetric on $L^2(\mu_\chi)$. Observe that $\Delta_\delta = \mathcal{L}$ and Δ_1 is the standard left-invariant sum-of-squares sub-Laplacian. The operator Δ_χ generates a diffusion semigroup, namely $(\mathrm{e}^{-t\Delta_\chi})_{t>0}$ extends to a contraction semigroup on $L^p(\mu_\chi)$ for every $p \in [1,\infty]$ whose infinitesimal generator we still denote by Δ_χ; see [14, 4, 5, 6] for more on these matters.

When $p \in (1,\infty)$ and $\alpha > 0$, we define the Sobolev spaces $L^p_\alpha(\mu_\chi)$ as the space of functions $f \in L^p(\mu_\chi)$ such that $(\tau_\chi I + \Delta_\chi)^{\alpha/2} f \in L^p(\mu_\chi)$, endowed with the norm

$$\|f\|_{L^p_\alpha(\mu_\chi)} = \|(\tau_\chi I + \Delta_\chi)^{\alpha/2} f\|_{L^p(\mu_\chi)},$$

where

$$\tau_\chi = \max\left\{\frac{2}{b}\left[\mathfrak{c}(\delta\chi^{-1}) + 2D + b_0\right]^2 - \frac{1}{4}\mathfrak{c}(\chi)^2, 1\right\} \tag{6.1}$$

is the counterpart (or generalized version) of (2.4). Observe that $\mathfrak{c}(\delta\chi^{-1}) = 0$ if $\chi = \delta$ or, equivalently, if $\mu_\chi = \lambda$, so our notation is coherent with the one used in previous sections.

We recall from [4] that an embedding as the one of Theorem 3.1 fails if λ is replaced by any other measure μ_χ; and as we show below in Remark 6.4, a global Moser–Trudinger inequality as Theorem 5.2 also does not hold if $\mu_\chi \neq \lambda$. Nevertheless, we can prove an alternative version of Sobolev embedding, and a local Moser–Trudinger inequality (that is, for compactly supported functions). We shall first need to extend some definitions and results, given above in the case of the left measure λ, to the case of μ_χ.

We denote by p_t^χ the convolution kernel of $\mathrm{e}^{-t\Delta_\chi}$, and we recall that by [33, Theorem IX.1.3], equivalently (2.3), and [4, eq. (2.8)],

$$p_t^\chi(x) \leqslant c\,(\delta\chi^{-1})^{1/2}(x)\,(1\wedge t)^{-\frac{d}{2}}\,\mathrm{e}^{-\frac{1}{4}t\mathfrak{c}(\chi)^2}\,\mathrm{e}^{-b\frac{|x|^2}{t}}, \qquad x \in G,\ t > 0 \tag{6.2}$$

where b and c are those of (2.3).

For $\alpha > 0$, let G_χ^α be the convolution kernel of $(\tau_\chi I + \Delta_\chi)^{-\alpha/2}$, and define $G_\chi^{\alpha,\mathrm{loc}} = G_\chi^\alpha \mathbf{1}_{B_1}$ and $G_\chi^{\alpha,\mathrm{glob}} = G_\chi^\alpha \mathbf{1}_{B_1^c}$. The following result can be proved exactly in the same way as Lemma 2.1, and its proof is omitted.

Lemma 6.1. *There exists $C = C(G, \mathbf{X}) > 0$ such that, for $\alpha \in (0,d)$ and $x \in G$,*

$$\begin{aligned}
|G_\chi^{\alpha,\mathrm{loc}}(x)| &\leqslant C\,\frac{\alpha}{d-\alpha}(\delta\chi^{-1})^{1/2}(x)|x|^{\alpha-d}\mathbf{1}_{B(e,1)}(x),\\
|G_\chi^{\alpha,\mathrm{glob}}(x)| &\leqslant C\,(\delta\chi^{-1})^{1/2}(x)\,\mathrm{e}^{-(2D+\mathfrak{c}(\delta\chi^{-1})+b_0)|x|}\mathbf{1}_{B(e,1)^c}(x).
\end{aligned}$$

Define now $\mathfrak{s}(\chi) = \max_{B_1}\chi\delta^{-1} = \mathrm{e}^{\mathfrak{c}(\chi\delta^{-1})}$, and observe that $\mathfrak{s}(\chi) \geqslant 1$ for all χ's.

Proposition 6.2. *Let $p \in (1,\infty)$ and $q \in [p,\infty)$. There exists $A_2 = A_2(G,\mathbf{X}) > 0$ such that*

$$\|f\|_{L^q(\mu_{\chi^{q/p}\delta^{1-q/p}})} \leqslant \frac{A_2\,\mathfrak{s}(\chi)}{p-1}\left(1+\frac{q}{p'}\right)^{\frac{1}{q}+\frac{1}{p'}}\|f\|_{L^p_{d/p}(\mu_\chi)} \tag{6.3}$$

for all $f \in L^p_{d/p}(\mu_\chi)$.

Proof. By Young's inequality (2.2), we obtain that

$$\begin{aligned}
&\|(\tau_\chi I + \Delta_\chi)^{-d/2p}g\|_{L^q(\mu_{\chi^{q/p}\delta^{1-q/p}})}\\
&\quad= \|(\chi\delta^{-1})^{1/p}g * (\chi\delta^{-1})^{1/p}G_\chi^{d/p}\|_{L^q(\lambda)}\\
&\quad\leqslant \|(\chi\delta^{-1})^{1/p}g\|_{L^p(\lambda)}\|(\chi^{-1}\delta)^{1/p}\check{G}_\chi^{d/p}\|_{L^r(\lambda)}^{r/p'}\,\|(\chi\delta^{-1})^{1/p}G_\chi^{d/p}\|_{L^r(\lambda)}^{r/q}\\
&\quad= \|g\|_{L^p(\mu_\chi)}\|(\chi^{-1}\delta)^{1/p}\check{G}_\chi^{d/p}\|_{L^r(\lambda)}^{r/p'}\,\|(\chi\delta^{-1})^{1/p}G_\chi^{d/p}\|_{L^r(\lambda)}^{r/q},
\end{aligned} \tag{6.4}$$

where $r \in (1,\infty)$ is such that $\frac{1}{p}+\frac{1}{r} = 1+\frac{1}{q}$. We split $G_\chi^{d/p}$ into $G_\chi^{d/p,\mathrm{loc}}$ and $G_\chi^{d/p,\mathrm{glob}}$, and estimate the integrals of the two terms separately.

By Lemma 6.1, we obtain

$$\begin{aligned}\|(\chi\delta^{-1})^{1/p}G_\chi^{d/p,\mathrm{loc}}\|_{L^r(\lambda)} &\leqslant \frac{C}{p-1}\Big(\sum_{k=0}^\infty \int_{2^{-k-1}<|x|\leqslant 2^{-k}} (\delta\chi^{-1})^{r(\frac12-\frac1p)}(x)|x|^{r(d/p-d)}\,\mathrm{d}\lambda(x)\Big)^{1/r}\\ &\leqslant \frac{C}{p-1}\,\mathfrak{s}(\chi)\Big(\sum_{k=0}^\infty 2^{-kr(d/p-d)-kd}\Big)^{1/r}\\ &\leqslant \frac{C}{p-1}\,\mathfrak{s}(\chi)\Big(\int_0^1 u^{(d/p-d)r}u^{d-1}\,\mathrm{d}u\Big)^{1/r} = \frac{C\,\mathfrak{s}(\chi)}{p-1}\left(1+\frac{q}{p'}\right)^{\frac1q+\frac1{p'}},\end{aligned}$$

where we used that

$$\sup_{y\leqslant|x|}(\delta\chi^{-1})^{1/2-1/p}(y) = \sup_{y\leqslant|x|}(\delta\chi^{-1})^{|1/2-1/p|}(x) = \mathrm{e}^{\mathfrak{c}(\chi\delta^{-1})|x|}, \tag{6.5}$$

and that $|1/2-1/p|\leqslant 1$.

As for the global part of the kernel, using again (6.5),

$$\begin{aligned}\|(\chi\delta^{-1})^{1/p}G_\chi^{d/p,\mathrm{glob}}\|_{L^r(\lambda)} &\leqslant C\Big(\int_0^\infty (\chi\delta^{-1})^{r(1/p-1/2)}\mathrm{e}^{-r(2D+\mathfrak{c}(\chi\delta^{-1})+b_0)|x|}\,\mathrm{d}\lambda\Big)^{1/r}\\ &\leqslant C\Big(\int_0^\infty \mathrm{e}^{-r(2D+b_0)|x|}\,\mathrm{d}\lambda\Big)^{1/r}\\ &\leqslant C\left(\sum_{k=0}^\infty \mathrm{e}^{-r(2D+b_0)2^k+D2^{k+1}}\right)^{1/r}\leqslant C. \end{aligned}\tag{6.6}$$

The term $\|\check{G}^c_{d/p}\|_{L^r(\lambda)}$ can be estimated in the same way, in view of (6.5) and by the radiality of the other terms appearing in the bound of Lemma 6.1. $\square$

Keeping the notation of Proposition 6.2, for $1<p<\infty$ we define

$$\gamma_2 = \left[\mathrm{e}\left(\frac{A_2\mathfrak{s}(\chi)^2}{p-1}\right)^{p'}\right]^{-1}.$$

The following result is inspired by [28].

Theorem 6.3. *Let $p\in(1,\infty)$. For $\gamma\in[0,\gamma_2)$,*

$$\sup_{\|f\|_{L^p_{d/p}(\mu_\chi)}\leqslant 1,\,\operatorname{supp} f\subseteq B(e,1)}\int_G\Big(\exp(\gamma|f|^{p'})-1\Big)\,\mathrm{d}\mu_\chi<\infty.$$

Proof. We first notice that if f is supported in B_1 and $q>p$, then

$$\|f\|_{L^q(\mu_\chi)} = \|(\chi\delta^{-1})^{\frac1q-\frac1p}f\|_{L^q(\mu_{\chi^{q/p}\delta^{1-q/p}})}\leqslant\mathfrak{s}(\chi)\|f\|_{L^q(\mu_{\chi^{q/p}\delta^{1-q/p}})},$$

so by Proposition 6.2

$$\|f\|_{L^q(\mu_\chi)}\leqslant\frac{A_2\,\mathfrak{s}(\chi)^2}{p-1}\left(1+\frac{q}{p'}\right)^{\frac1q+\frac1{p'}}\|f\|_{L^p_{d/p}(\mu_\chi)}. \tag{6.7}$$

If f is supported in B_1 and $\|f\|_{L^p_{d/p}(\mu_\chi)}\leqslant 1$, then

$$\|f\|_{L^p(\mu_\chi)}\leqslant\|(\tau_\chi I+\Delta_\chi)^{-d/2p}\|_{L^p(\mu_\chi)\to L^p(\mu_\chi)} = C(\chi,p),$$

and

$$\int_G \Big(\exp(\gamma|f|^{p'}) - 1\Big) \, \mathrm{d}\mu_\chi = \sum_{k=1}^{\infty} \frac{\gamma^k}{k!} \|f\|^{p'k}_{L^{p'k}(\mu_\chi)}$$

$$\leqslant C(\chi,p) \sum_{1\leqslant k<p/p'} \frac{\gamma^k}{k!} \mu_\chi(B(e,1))^{1-k(p'-1)} + \sum_{k\geqslant p/p'} \frac{\gamma^k}{k!} \left(\frac{A_2\,\mathfrak{s}(\chi)^2}{p-1}\right)^{p'k} (k+1)^{k+1},$$

where we applied (6.7) when $kp' \geqslant p$, and Hölder's inequality and the support condition of f if $kp' < p$. If $\gamma \in [0,\gamma_2)$, then the latter series is convergent and the theorem is proved. □

Remark 6.4. Theorem 5.2 does not hold with any other μ_χ in place of λ. Indeed, if there exist $p \in (1,\infty)$, $C > 0$ and $\gamma > 0$ such that for all $f \in L^p_{d/p}(\mu_\chi)$, $\|f\|_{L^p_{d/p}(\mu_\chi)} \leqslant 1$,

$$\int_G \Big(\exp(\gamma|f|^{p'}) - \sum_{0\leqslant k<p-1} \frac{\gamma^k}{k!}|f|^{p'k}\Big) \, \mathrm{d}\mu_\chi \leqslant C\|f\|^p_{L^p(\mu_\chi)}, \tag{6.8}$$

then necessarily $\mu_\chi = \lambda$.

To see this, assume that (6.8) holds for all $f \in L^p_{d/p}(\mu_\chi)$, $\|f\|_{L^p_{d/p}(\mu_\chi)} \leqslant 1$, with $\mu_\chi \neq \lambda$, i.e. $\chi \neq \delta$. We first prove that then (6.8) holds for all $f \in L^p_{d/p}(\mu_\chi)$, with no restriction on its norm (other than being finite). Recall, indeed, that for any $y \in G$ and $f \in L^p_{d/p}(\mu_\chi)$, denoting by L_y the left translation by $y \in G$, one has

$$\|L_y f\|_{L^p_{d/p}(\mu_\chi)} = (\chi\delta^{-1})^{1/p}(y)\|f\|_{L^p_{d/p}(\mu_\chi)}.$$

Since $(\chi\delta^{-1})^{-1/p}$ is a positive nonconstant character, it is unbounded; thus there exists $y \in G$ such that

$$(\chi\delta^{-1})^{-1/p}(y) \geqslant \|f\|_{L^p_{d/p}(\mu_\chi)}.$$

Equivalently, $(\chi\delta^{-1})^{1/p}(y)\|f\|_{L^p_{d/p}(\mu_\chi)} \leqslant 1$, hence $\|L_y f\|_{L^p_{d/p}(\mu_\chi)} \leqslant 1$. Thus, we may apply (6.8) to $L_y f$; and by a change of variable, one obtains (6.8) for f where the constant C does not depend on the norm of f.

But (6.8) cannot hold without restriction on the norm of $f \in L^p_{d/p}(\mu_\chi)$. Indeed, let $\sigma \geqslant 1$ and consider σf, which still belongs to $L^p_{d/p}(\mu_\chi)$ for any σ. Then, by (6.8) applied to σf,

$$\int_G \sum_{k\geqslant p-1} \frac{\gamma^k}{k!} \sigma^{p'k}|f|^{p'k} \, \mathrm{d}\mu_\chi \leqslant C\,\sigma^p \|f\|^p_{L^p(\mu_\chi)}.$$

Since

$$\int_G \sum_{k\geqslant p-1} \frac{\gamma^k}{k!} \sigma^{p'k}|f|^{p'k} \, \mathrm{d}\mu_\chi \geqslant \int_G \sum_{k\geqslant p} \frac{\gamma^k}{k!} \sigma^{p'k}|f|^{p'k} \, \mathrm{d}\mu_\chi \geqslant \sigma^{pp'} \int_G \sum_{k\geqslant p} \frac{\gamma^k}{k!} |f|^{p'k} \, \mathrm{d}\mu_\chi,$$

one obtains

$$\sigma^{p(p'-1)} \int_G \sum_{k\geqslant p} \frac{\gamma^k}{k!} |f|^{p'k} \, \mathrm{d}\mu_\chi \leqslant C\|f\|^p_{L^p(\mu_\chi)}$$

for all $\sigma \geqslant 1$, which is a contradiction since $p(p'-1) > 0$.

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