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Abstract

In this paper we present new quadratures based on both quasi-interpolation and multilevel methods by using bivariate quadratic B-spline functions, defined on simple and multiple knot type-2 triangulations, improving classical quadratures, based on quasi-interpolating splines. We also prove some symmetry properties that simplify their expression, study their approximation performances, propose some numerical results and a comparison with other known multilevel spline quadratures.

Keywords: spline integration, quasi-interpolation, multilevel B-splines, rate of convergence, degree of precision
65D07, 65D15, 41A15, 41A25

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1. Introduction

Spline quasi-interpolation (QI) [1, 2, 3] is a powerful tool to derive local approximation methods that do not require the solution of any linear system and whose resulting quasi-interpolating splines reproduce polynomials of a certain degree to ensure good, and possibly optimal, approximation properties.

Moreover, recently multilevel techniques have been introduced in both univariate [4] and bivariate [5] settings to improve the performances of spline QI results.

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Induced by these researches, in [6] we take advantage of both approaches by submitting spline QI operators, generating quadratic splines on type-2 triangulations on simple and multiple knots [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21], that provide good approximation properties, to multilevel schemes.

Since in [6] multilevel quadratic spline QI seems to perform better than quadratic spline QI as well as in [5] numerical integration based on quadratic spline QI seems to provide better results than quartic spline QI, it is straightforward to wonder if numerical integration based on multilevel quadratic spline QI can be better than the one based on multilevel quartic QI.

So in this paper we study such a question, first facing the problem of completing the theory of numerical integration based on quadratic spline QI, started in [11, 18], and then moving to the multilevel setting.

The paper is organized as follows. Section 2 recalls multilevel quadratic QI splines, citing three cases studied in the literature. In Section 3 we present multilevel spline integration, after some remarks and new results on spline integration. Simmetry properties, degree of precision and approximation order are also studied. Finally in Section 4 some numerical results show the new multilevel spline quadratures can provide better performances than the ones based on just one-level spline.

2. Multilevel spline QI operators

Let $R := [0,1] \times [0,1]$ and let $\{x_i\}_{i=0}^{m}$, $\{y_j\}_{j=0}^{n}$, with $x_i := \frac{i}{m}$, $y_j := \frac{j}{n}$ and m, n two given integers. Let also $X_m^{(1)} \times Y_n^{(1)}$ with

$$X_m^{(1)} : x_{-2} < x_{-1} < x_0 = 0 < \cdots < 1 = x_m < x_{m+1} < x_{m+2},$$

$$Y_n^{(1)} : y_{-2} < y_{-1} < y_0 = 0 < \cdots < 1 = y_n < y_{n+1} < y_{n+2},$$

and $X_m^{(2)} \times Y_n^{(2)}$ with

$$X_m^{(2)} : x_{-2} = x_{-1} = x_0 = 0 < x_1 < \cdots < 1 = x_m = x_{m+1} = x_{m+2},$$

$$Y_n^{(2)} : y_{-2} = y_{-1} = y_0 = 0 < y_1 < \cdots < 1 = y_n = y_{n+1} = y_{n+2}$$

be two partitions that divide R into mn rectangular subdomains.

We denote by $\Delta_{mn}^{(2)}$ the uniform type-2 triangulation of R, obtained by drawing both diagonals in each subdomain defined on the uniform and inside-uniform partitions $X_m^{(i)} \times Y_n^{(i)}$, $i = 1, 2$, respectively.

Moreover we define the following sets of points

- $\Theta := \{A_{i,j} = (x_i, y_j), -1 \le i \le m+1, -1 \le j \le n+1\}$,
- $\Phi := \{M_{i,j} = (s_i, t_j), -1 \le i \le m+2, -1 \le j \le n+2\}$,

with $s_i := \frac{2i-1}{2m}$, $t_j := \frac{2j-1}{2n}$, and the set $\mathcal{B}_{mn} := \{B_{ij} : (i,j) \in K_{mn}\}$, where $B_{ij}(x,y) := B\left(mx - i + \frac{1}{2},\ \ ny - j + \frac{1}{2}\right)$ with B introduced in [7] and $K_{mn} := \{(i,j) : 0 \le i \le m+1, 0 \le j \le n+1\}$, of the $(m+2)(n+2)$ B-splines generating the space $S_2^1(\Delta_{mn}^{(2)})$ of all C^1 piecewise polynomials of total degree 2 on the type-2 triangulation $\Delta_{mn}^{(2)}$, associated with the partitions $X_m^{(i)} \times Y_n^{(i)}$, $i = 1, 2$ of the domain R (see Fig. 1, 2).

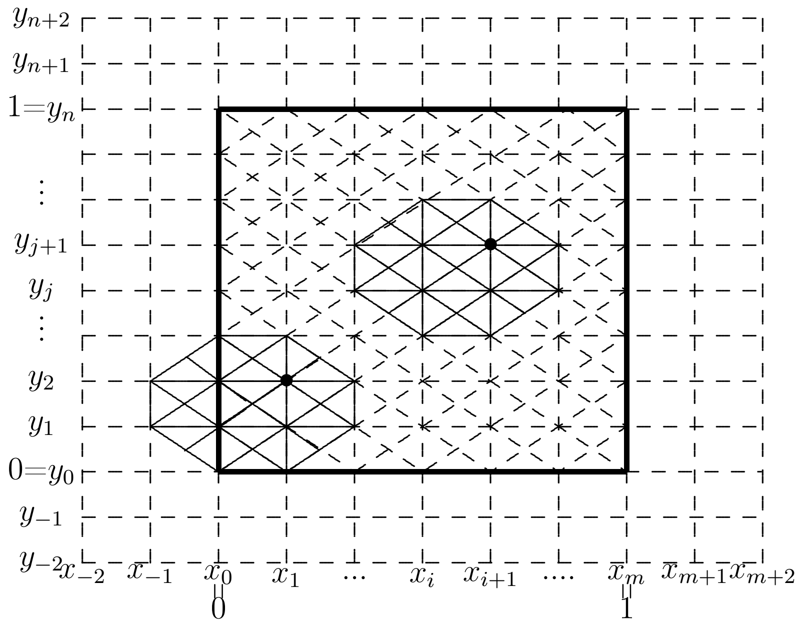

Figure 1: Octagonal supports of simple knot B-splines B_{12} and $B_{i+1,j+1}$.

In order to get better approximation performances with respect to quasi-interpolating spline operators $Q : C(R) \to S_2^1(\Delta_{mn}^{(2)})$ of the form

$$Qf(x,y) := \sum_{ij} \lambda_{ij}^{(Q)}(f) B_{ij}(x,y), \quad (x,y) \in R, \tag{1}$$

with $\lambda_{ij}^{(Q)}(f) := \sum_\ell \nu_\ell^{(ij)} f\left(P_\ell^{(ij)}\right)$, the $P_\ell^{(ij)}$'s triangular mesh-points, the $\nu_\ell^{(ij)}$'s non zero real numbers such that $Qf = f$ for any $f \in \mathbb{P}_r$ for some

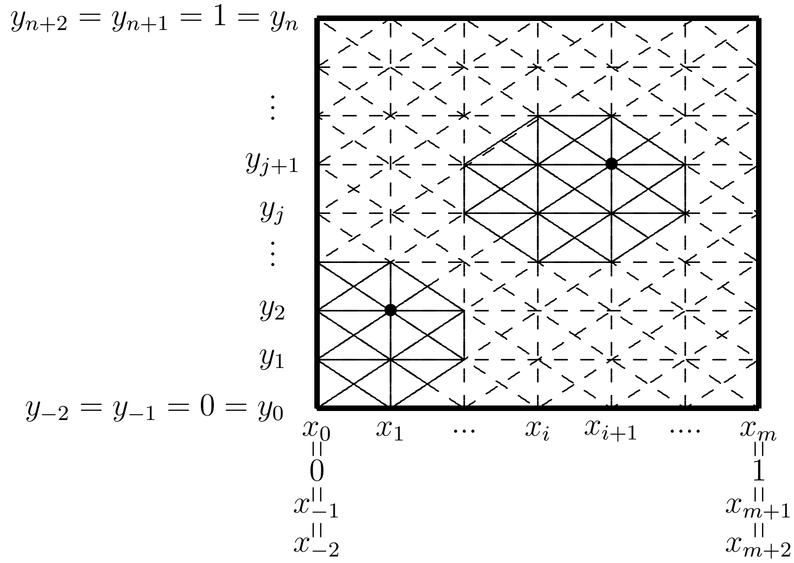

Figure 2: Supports of multiple knot B-splines B_{12} and $B_{i+1,j+1}$.

$0 < r \leq 2$, in [6] we define multilevel spline operators

$$Q^{pL} f := Q^{(p)} f + \sum_{r=1}^{p} Q^{(r-1)} \Delta_r^{p+1-r} f, \quad 0 \leq p \leq \min\{q, s\}, \tag{2}$$

where

- $p+1$, with $p := \min\{q, s\}$, is the largest number of levels it is possible to define for any two even integers m and n such that $m = u \cdot 2^q$, $n = v \cdot 2^s$, $u, v, q, s \in \mathbb{N}$ and u, v odd numbers;
- $Q^{(p)} f := \sum_{i=\kappa}^{\frac{m}{2^p}+\ell} \sum_{j=\kappa}^{\frac{n}{2^p}+\ell} \lambda_{ij}^{(Q,p)}(f) B_{ij}^{(p)}$, with
  - $\lambda_{ij}^{(Q,p)}(f)$ computed at $\Upsilon^{(p)}$;
  - $B_{ij}^{(p)} := B\left(\frac{m}{2^p} \cdot -i + \frac{1}{2}, \frac{n}{2^p} \cdot -j + \frac{1}{2}\right)$ with support centre at $M_{ij}^{(p)}$;
  - $\Upsilon^{(r)} := \Phi^{(r)}$ if $Q = S_1, S_2$ and $\Upsilon^{(r)} := \Phi^{(r)} \cup \Theta^{(r)}$ if $Q = W_2$ where

$$\Phi^{(r)} := \left\{ M_{ij}^{(r)} = \left({s_i}^{(r)}, {t_j}^{(r)}\right) = (2^r s_i, 2^r t_j) \right\}$$

and

$$\Theta^{(r)} := \left\{ A_{ij}^{(r)} = \left({x_i}^{(r)}, {y_j}^{(r)}\right) = (2^r x_i, 2^r y_j) \right\},$$

	$X_m^{(1)} \times Y_n^{(1)}$			$X_m^{(2)} \times Y_n^{(2)}$
	$S_1^{(0)}, S_2^{(0)}, W_2^{(0)}$	$S_1^{(r)}, r = 1, \ldots, p,\ S_2^{(r)}, W_2^{(r)}, r = 1, \ldots, p-1$	$S_2^{(p)}, W_2^{(p)}$	
κ	0	-1	-2	0
ℓ	1	2	3	1

Table 1: Parameters κ and ℓ involved in the variation of the indices (3), where $Q^{(0)} := Q$ and $Q^{(r)}$ appear in the definition of multilevel operators, $Q = S_1, S_2, W_2$.

with

$$\begin{gathered} i = \kappa, ..., \tfrac{m}{2^r} + \ell, j = \kappa, ..., \tfrac{n}{2^r} + \ell, \\ \kappa := \kappa(Q, X_m^{(k)}, Y_n^{(k)}, r),\ \ \ell := \ell(Q, X_m^{(k)}, Y_n^{(k)}, r),\ \ k = 1, 2 \end{gathered} \tag{3}$$

(see Table 1). Such a table shows the indices needed to well define all terms involved in (2), depending on Q, $X_m^{(k)} \times Y_n^{(k)}$, $k = 1, 2$, and r, i.e. in order to take into account all nonzero B-splines at the evaluation points of the set $\varUpsilon^{(r)}$;

- $Q^{(r-1)}\Delta_r^{p+1-r} f := \sum_{i=\kappa}^{\frac{m}{2^{r-1}}+\ell} \sum_{j=\kappa}^{\frac{n}{2^{r-1}}+\ell} \lambda_{ij}^{(Q,r-1)}(\Delta_r^{p+1-r} f) B_{ij}^{(r-1)}$ on the data set $\varUpsilon^{(r-1)}$, $r = p, \ldots, 1$;

- $\Delta_r^{p+1-r} f := \Delta_r^1(\Delta_{r+1}^{p-r} f) = \Delta_{r+1}^{p-r} f - Q^{(r)}\Delta_{r+1}^{p-r} f$, $r = p-1, \ldots, 1$, with $\Delta_p^1 f := f - Q^{(p)} f$, is the $(p+1-r)$-th error function. We remark that all the error functions are well defined since they are based on points where the function evaluations are known. Moreover they are easily computed due to the B-spline local support;

- $Q^{0L} := Q^{(0)} = Q$.

In particular in [6] we study the three following multilevel operators $S_1^{pL}, S_2^{pL}, W_2^{pL}$, coming from

- the Schoenberg-type bivariate variation-diminishing operator S_1 [7, 11, 20, 21], where $\lambda_{ij}^{(S_1)}(f) := f(M_{ij})$ in (1), with $(i, j) \in K_{mn}$;

- the quasi-interpolating spline operator S_2 [20], where $\lambda_{ij}^{(S_2)}(f) := b_{ij} f(M_{ij}) + a_i f(M_{i-1,j}) + c_i f(M_{i+1,j}) + \overline{a}_j f(M_{i,j-1}) + \overline{c}_j f(M_{i,j+1})$ with

$$b_{ij} := 1 - (a_i + c_i + \overline{a}_j + \overline{c}_j),$$

$$a_i := -\frac{\sigma_i^2\sigma'_{i+1}}{\sigma_i+\sigma'_{i+1}}, \quad c_i := -\frac{\sigma_i(\sigma'_{i+1})^2}{\sigma_i+\sigma'_{i+1}}, \quad \overline{a}_j := -\frac{\tau_j^2\tau'_{j+1}}{\tau_j+\tau'_{j+1}}, \quad \overline{c}_j := -\frac{\tau_j(\tau'_{j+1})^2}{\tau_j+\tau'_{j+1}}, \tag{4}$$

$$\begin{aligned} \sigma_i &:= \frac{h_i}{h_{i-1}+h_i}, \quad \sigma'_i := \frac{h_{i-1}}{h_{i-1}+h_i} = 1-\sigma_i\ , \\ \tau_j &:= \frac{k_j}{k_{j-1}+k_j}, \quad \tau'_j := \frac{k_{j-1}}{k_{j-1}+k_j} = 1-\tau_j\ , \end{aligned} \tag{5}$$

$h_i := x_i - x_{i-1} = h = \frac{1}{m}, k_j := y_j - y_{j-1} = k = \frac{1}{n}$ for all i, j, i.e. $b_{ij} = \frac{3}{2}$ and $a_i = c_i = \overline{a}_j = \overline{c}_j = -\frac{1}{8}$ for partitions $X_m^{(1)} \times Y_n^{(1)}$, while $h_0 = h_{m+1} = k_0 = k_{n+1} = 0$ for partitions $X_m^{(2)} \times Y_n^{(2)}$;

- W_2 [7, 15, 21], where $\lambda_{ij}^{(W_2)}(f) := 2f(M_{i,j}) - \frac{1}{4}\sum_{h=-1}^{0}\sum_{k=-1}^{0} f(A_{i+h,j+k})$,

respectively.

For such multilevel QI operators the following results on polynomial reproduction and approximation order are obtained in [6].

Let $\Delta := \max\left\{\frac{1}{m}, \frac{1}{n}\right\}$ and $\|\cdot\|_R$ be the infinity norm over R.

Theorem 2.1. *Let Q^{pL} be a $(p+1)$-level QI operator with $Q = S_1,\ S_2,\ W_2$, $0 \le p \le \min\{q, s\}$. Then*

- $S_1^{pL} f = f \quad if \quad f(x,y) = 1, x, y, xy,$
- $Q^{pL} f = f \quad if \quad f \in \mathbb{P}_2, \quad and \quad Q = S_2, W_2.$

Theorem 2.2. *Let Q^{pL} be a $(p+1)$-level QI operator with $Q = S_1,\ S_2,\ W_2$, $0 \le p \le \min\{q, s\}$. If $f \in C^\mu(R)$, $\mu = 1, 2, 3$, then $Q^{pL}f$ at least has the error estimate*

$$||f - Q^{pL}f||_R \le (1 + ||Q||)^p C_\mu^Q \Delta^\mu,$$

where C_μ^Q, $\mu = 1, 2$ if $Q = S_1$ and $\mu = 1, 2, 3$ if $Q = S_2, W_2$, are positive constants.

We also obtained the following unexpected result for the multilevel operator S_1^{pL}, $0 < p \le \min\{q, s\}$.

Theorem 2.3. *Let $\Delta_{mn}^{(2)}$ be defined on the partition $X_m^{(1)} \times Y_n^{(1)}$. The operator S_1^{pL}, $0 < p \le \min\{q, s\}$, reproduces the polynomial space $\mathbb{P}_2$, i.e.*

$$S_1^{pL} f = f \quad if \quad f \in \mathbb{P}_2. \tag{6}$$

Theorem 2.4. *Let $\Delta_{mn}^{(2)}$ be defined on the partition $X_m^{(1)} \times Y_n^{(1)}$. If $f \in C^{\mu}(R)$, $\mu = 1, 2, 3$, then $S_1^{pL} f$, $0 < p \leq \min\{q, s\}$, at least has the error estimate*

$$||f - S_1^{pL} f||_R \leq 2^p C_{\mu} \Delta^{\mu}$$

with C_{μ} positive constant.

So Theorem 2.3 shows that the multilevel QI operator S_1^{pL}, $0 < p \leq \min\{q, s\}$, gains in polynomial reproduction in case of partitions $\Delta_{mn}^{(2)}$ defined on $X_m^{(1)} \times Y_n^{(1)}$. In this case data points outside the domain R are needed. However either they could be not known or f could be not defined outside R. For partitions $\Delta_{mn}^{(2)}$ defined on $X_m^{(2)} \times Y_n^{(2)}$ all evaluation points are either inside R or on its boundary, but S_1^{pL} only reproduces bilinear polynomials. However numerical evidence in [6] shows that, when m and n increase, partitions $X_m^{(1)} \times Y_n^{(1)}$ and $X_m^{(2)} \times Y_n^{(2)}$ tend to coincide, so that S_1^{pL}, $p \geq 1$ 'tends to reproduce' $\mathbb{P}_2$.

3. Multilevel spline numerical integration

In this section we consider the numerical evaluation of the integral

$$I(f) := I(f; R) = \int_R f(x, y)\, \mathrm{d}x\, \mathrm{d}y, \quad f \in C(R). \tag{7}$$

Before moving to the multilevel setting and defining the corresponding quadratures, we have to sum up some results, concerning the approximation of (7) by replacing f with Qf, $Q = S_1, S_2, W_2$, obtained in [11, 18], and state new ones, all useful later.

3.1. Spline integration

Some theory of numerical integration based on bivariate quadratic spline QI's is already developed in [11] for $Q = S_1$ on triangulations $\Delta_{mn}^{(2)}$ based on partitions $X_m^{(1)} \times Y_n^{(1)}$ and in [18] for $Q = S_1, S_2, W_2$ on triangulations $\Delta_{mn}^{(2)}$ based on partitions $X_m^{(2)} \times Y_n^{(2)}$. However some new results on quadratures based on $Q = S_2, W_2$ on triangulations $\Delta_{mn}^{(2)}$ based on partitions $X_m^{(1)} \times Y_n^{(1)}$ still have to be obtained.

If f in (7) is approximated by Qf, to adopt an as compact as possible notation, we can write

$$
\begin{aligned}
I(Qf) := I(Qf;R) &= \sum_i \sum_j w_{ij}^{(Q)} f(P_{ij}) \\
&= \sum_{i=\kappa_1}^{m+\ell_1} \sum_{j=\kappa_1}^{n+\ell_1} \overline{w}_{ij}^{(Q)} f(M_{ij}) + \sum_{i=\kappa_2}^{m+\ell_2} \sum_{j=\kappa_2}^{n+\ell_2} \overline{\overline{w}}_{ij}^{(Q)} f(A_{ij}),
\end{aligned} \tag{8}
$$

where

- $Q = S_1, S_2, W_2$;
- $P_{ij} \in \Delta_{mn}^{(2)}$, with $P_{ij} = A_{ij}, M_{ij}$;
- $w_{ij}^{(Q)} := \int_{\Psi_{ij}} N_{ij}(x,y)\mathrm{d}x\mathrm{d}y$ are the weights of the cubature, depending on $X_m^{(k)} \times Y_n^{(k)}$, $k = 1,2$ and on the operator Q. In fact $\Psi_{ij} := \Xi_{ij} \cap R$ for $X_m^{(1)} \times Y_n^{(1)}$ and $\Psi_{ij} := \Xi_{ij}$ for $X_m^{(2)} \times Y_n^{(2)}$, where Ξ_{ij} is the support of the ij-th either B-spline or fundamental function, that is a suitable linear combination of B-splines, N_{ij}, depending on the definition of Q:

  - if $Q = S_1$, then $w_{ij}^{(S_1)} = \overline{w}_{ij}^{(S_1)}$ with $N_{ij} := B_{ij}$;

  - if $Q = S_2$, then $w_{ij}^{(S_2)} = \overline{w}_{ij}^{(S_2)}$ with

  $$N_{ij} := \widetilde{B}_{ij} = b_{ij}B_{ij} + a_{i+1}B_{i+1,j} + c_{i-1}B_{i-1,j} + \overline{a}_{j+1}B_{i,j+1} + \overline{c}_{j-1}B_{i,j-1};$$

  - if $Q = W_2$, then

  $$w_{ij}^{(W_2)} = \begin{cases} \overline{w}_{ij}^{(W_2)} \text{ with } N_{ij} := \overline{B}_{ij} = 2B_{ij}, \\ \overline{\overline{w}}_{ij}^{(W_2)} \text{ with } N_{ij} := \overline{\overline{B}}_{ij} = -\dfrac{1}{4}\left[B_{ij} + B_{i,j+1} + B_{i+1,j} + B_{i+1,j+1}\right]; \end{cases}$$

  - the indices κ_i, ℓ_i, $i = 1,2$ are reported in Table 2, depending on Q and on the partition $X_m^{(k)} \times Y_n^{(k)}$, $k = 1,2$.

Now, considering the uniform and inside-uniform partitions $X_m^{(k)} \times Y_n^{(k)}$, $k = 1,2$, then the weights can be written as $\overline{w}_{ij}^{(Q)} = \overline{c}_{ij}^{(Q)} \frac{1}{m}\frac{1}{n}$ and $\overline{\overline{w}}_{ij}^{(Q)} = \overline{\overline{c}}_{ij}^{(Q)} \frac{1}{m}\frac{1}{n}$, for all i and j for which h_i and k_j are not zero, i.e. inside R.

In particular in [11] some symmetry properties of the weights are studied, leading to a quadrature formula, depending on only six weights (see Table 3), when $Q = S_1$ for $X_m^{(1)} \times Y_n^{(1)}$.

	$X_m^{(1)} \times Y_n^{(1)}$				$X_m^{(2)} \times Y_n^{(2)}$			
	κ_1	ℓ_1	κ_2	ℓ_2	κ_1	ℓ_1	κ_2	ℓ_2
S_1	0	1	–	–	0	1	–	–
S_2	-1	2	–	–	0	1	–	–
W_2	0	1	-1	1	0	1	0	0

Table 2: Parameters κ_i and ℓ_i, $i = 1, 2$, involved in the variation of indices in (8).

$\overline{c}_{00}^{(S_1)}$	$\overline{c}_{10}^{(S_1)}$	$\overline{c}_{11}^{(S_1)}$	$\overline{c}_{20}^{(S_1)}$	$\overline{c}_{21}^{(S_1)}$	$\overline{c}_{22}^{(S_1)}$
$\frac{1}{48}$	$\frac{7}{48}$	$\frac{33}{48}$	$\frac{1}{6}$	$\frac{5}{6}$	1

Table 3: The six weights $\overline{c}_{ij}^{(S_1)}$, computed in [11].

Similar symmetry properties are studied in [18], providing new quadrature formulas based on $Q = S_1, S_2, W_2$ for $X_m^{(2)} \times Y_n^{(2)}$ (see Table 4,5,6, respectively).

$\overline{c}_{00}^{(S_1)}$	$\overline{c}_{10}^{(S_1)}$	$\overline{c}_{11}^{(S_1)}$	$\overline{c}_{20}^{(S_1)}$	$\overline{c}_{21}^{(S_1)}$	$\overline{c}_{22}^{(S_1)}$
$\frac{1}{12}$	$\frac{1}{4}$	$\frac{5}{12}$	$\frac{1}{3}$	$\frac{2}{3}$	1

Table 4: The six weights $\overline{c}_{ij}^{(S_1)}$, computed in [18].

In order to complete the theory on this topic, we still need to compute the weights related to $Q = S_2, W_2$ for $X_m^{(1)} \times Y_n^{(1)}$ (see Table 7,8, respectively). Their expressions are obtained through similar tedious computations, as the ones of Theorem 2 in [11], i.e. by directly examining by hand the symmetries related to the supports of either B-splines or fundamental functions involved in the integrals generating the weights.

Further details on this topic can be found in [11, 18]. Here we just reported what is necessary for the following multilevel treatment.

$\overline{c}_{00}^{(S_2)}$	$\overline{c}_{10}^{(S_2)}$	$\overline{c}_{11}^{(S_2)}$	$\overline{c}_{20}^{(S_2)}$	$\overline{c}_{21}^{(S_2)}$	$\overline{c}_{22}^{(S_2)}$	$\overline{c}_{30}^{(S_2)}$	$\overline{c}_{31}^{(S_2)}$	$\overline{c}_{32}^{(S_2)}$	$\overline{c}_{33}^{(S_2)}$
$-\frac{1}{12}$	$\frac{7}{36}$	$\frac{2}{3}$	$\frac{1}{9}$	$\frac{8}{9}$	$\frac{37}{36}$	$\frac{1}{9}$	$\frac{7}{8}$	$\frac{73}{72}$	1

Table 5: The ten weights $\overline{c}_{ij}^{(S_2)}$, computed in [18].

$\overline{\overline{c}}_{00}^{(W_2)}$	$\overline{\overline{c}}_{10}^{(W_2)}$	$\overline{\overline{c}}_{11}^{(W_2)}$	$\overline{\overline{c}}_{20}^{(W_2)}$	$\overline{\overline{c}}_{21}^{(W_2)}$	$\overline{\overline{c}}_{22}^{(W_2)}$
$-\frac{7}{16}$	$-\frac{9}{16}$	$-\frac{11}{16}$	$-\frac{2}{3}$	$-\frac{5}{6}$	-1

Table 6: The six weights $\overline{\overline{c}}_{ij}^{(W_2)}$, computed in [18].

3.2. Multilevel spline integration

Let rewrite (2) as follows

$$Q^{pL}f = \sum_{r=0}^{p} Q^{(r)} \Delta_{r+1}^{p-r} f, \tag{9}$$

where $\Delta_{p+1}^{0} f := f$. Then, if now f is approximated by (9), inserted into (7), we get

$$I(Q^{pL}f) := I(Q^{pL}f; R) = \sum_{r} \sum_{i} \sum_{j} w_{ij}^{(Q,r)} \Delta_{r+1}^{p-r} f(P_{ij}^{(r)}) \tag{10}$$

$$= \sum_{r=0}^{p} \Big(\sum_{i=\kappa_1}^{\frac{m}{2^r}+\ell_1} \sum_{j=\kappa_1}^{\frac{n}{2^r}+\ell_1} \overline{w}_{ij}^{(Q,r)} \Delta_{r+1}^{p-r} f(M_{ij}^{(r)}) + \sum_{i=\kappa_2}^{\frac{m}{2^r}+\ell_2} \sum_{j=\kappa_2}^{\frac{m}{2^r}+\ell_2} \overline{\overline{w}}_{ij}^{(Q,r)} \Delta_{r+1}^{p-r} f(A_{ij}^{(r)}) \Big) \tag{11}$$

where
• $Q^{pL} = S_1^{pL}$, S_2^{pL}, W_2^{pL};
• $P_{ij}^{(r)} \in \Delta_{mn}^{(2)}$, $r = 0, \ldots, p$, with $P_{ij}^{(r)} = A_{ij}^{(r)}, M_{ij}^{(r)}$, according to the different QI's, as in Table 2;
• $w_{ij}^{(Q,r)} := \int_{\Psi_{ij}^{(r)}} N_{ij}^{(r)}(x,y)\mathrm{d}x\mathrm{d}y$ are the weights of the cubature, depending on the operator Q^{pL}, on $X_m^{(k)} \times Y_n^{(k)}$, $k = 1, 2$, and on the level r. Here the superscript (r) of all quantities refers to the level r and the definition of such quantities is similar to the one of those at 0-level. Moreover we let

$\overline{c}^{(S_2)}_{-1,-1}$	$\overline{c}^{(S_2)}_{0,-1}$	$\overline{c}^{(S_2)}_{00}$	$\overline{c}^{(S_2)}_{1,-1}$	$\overline{c}^{(S_2)}_{10}$	$\overline{c}^{(S_2)}_{11}$	$\overline{c}^{(S_2)}_{2,-1}$
0	$-\frac{1}{384}$	$-\frac{1}{192}$	$-\frac{7}{384}$	$\frac{7}{64}$	$\frac{151}{192}$	$-\frac{1}{48}$
$\overline{c}^{(S_2)}_{20}$	$\overline{c}^{(S_2)}_{21}$	$\overline{c}^{(S_2)}_{22}$	$\overline{c}^{(S_2)}_{30}$	$\overline{c}^{(S_2)}_{31}$	$\overline{c}^{(S_2)}_{32}$	$\overline{c}^{(S_2)}_{33}$
$\frac{41}{384}$	$\frac{117}{128}$	$\frac{25}{24}$	$\frac{5}{48}$	$\frac{43}{48}$	$\frac{49}{48}$	1

Table 7: The fourteen weights $\overline{c}^{(S_2)}_{ij}$ with $X^{(1)}_m \times Y^{(1)}_n$.

$\overline{\overline{c}}^{(W_2)}_{-1,-1}$	$\overline{\overline{c}}^{(W_2)}_{0,-1}$	$\overline{\overline{c}}^{(W_2)}_{00}$	$\overline{\overline{c}}^{(W_2)}_{1,-1}$	$\overline{\overline{c}}^{(W_2)}_{10}$	$\overline{\overline{c}}^{(W_2)}_{11}$	$\overline{\overline{c}}^{(W_2)}_{2,-1}$	$\overline{\overline{c}}^{(W_2)}_{20}$	$\overline{\overline{c}}^{(W_2)}_{21}$	$\overline{\overline{c}}^{(W_2)}_{22}$
$-\frac{1}{192}$	$-\frac{1}{24}$	$-\frac{1}{4}$	$-\frac{15}{192}$	$-\frac{11}{24}$	$-\frac{161}{192}$	$-\frac{1}{12}$	$-\frac{1}{2}$	$-\frac{11}{12}$	-1

Table 8: The ten weights $\overline{\overline{c}}^{(W_2)}_{ij}$ with $X^{(1)}_m \times Y^{(1)}_n$.

$w^{(r)}_{ij} := w^{(Q,r)}_{ij}$, when it is not necessary to specify the type of QI;
• the indices κ_i, ℓ_i, $i = 1, 2$ are reported in Table 2.

Now we can provide a more detailed expression of quadrature formulas based on multilevel operators $Q^{pL} = S^{pL}_1$, S^{pL}_2, W^{pL}_2.

The following theorems show the above quadrature (11) can be conveniently simplified by considering some symmetries among the weights for the three QI multilevel operators. Their proofs are carried out similarly to the ones of the particular case $r = 0$, described in Section 3.1 and in [11, 18] and included in them.

Theorem 3.1. *The weights $w^{(r)}_{ij}$, $r = 0, \dots, p$, in (11) satisfy the following symmetry properties:*

- *if $\frac{m}{2^r}, \frac{n}{2^r} \geq \alpha_2$, then*
  (i) $w^{(r)}_{ij} = w^{(r)}_{\frac{m}{2^r}-i+\beta,j} = w^{(r)}_{i,\frac{n}{2^r}-j+\beta} = w^{(r)}_{\frac{m}{2^r}-i+\beta,\frac{n}{2^r}-j+\beta}$
  $= w^{(r)}_{ji} = w^{(r)}_{\frac{m}{2^r}-j+\beta,i} = w^{(r)}_{j,\frac{n}{2^r}-i+\beta} = w^{(r)}_{\frac{m}{2^r}-j+\beta,\frac{n}{2^r}-i+\beta}$,
  $i = \gamma, ..., \delta$, $j = \gamma, ..., i$;
  (ii) $w^{(r)}_{ij} = w^{(r)}_{i,\frac{n}{2^r}-j+\beta} = w^{(r)}_{\delta j}$, $\quad j = \gamma, ..., \epsilon$,
  $w^{(r)}_{i\delta} = w^{(r)}_{i,\frac{n}{2^r}-\rho} = w^{(r)}_{\delta\delta}$, $\quad i = \sigma, ..., \frac{m}{2^r} - \tau$;

(iii) $w_{ij}^{(r)} = w_{\frac{m}{2^r}-i+\beta,j}^{(r)} = w_{i\delta}^{(r)}$, $\quad i = \gamma, ..., \epsilon,\ j = \sigma, ..., \frac{n}{2^r} - \tau$;
(iv) $w_{ij}^{(r)} = w_{\delta\delta}^{(r)}$, $\quad i = \delta, ..., \frac{m}{2^r} - \rho,\ j = \sigma, ..., \frac{n}{2^r} - \tau$;

- *if* $\alpha_1 \leq \frac{m}{2^r} < \alpha_2, \frac{n}{2^r} \geq \alpha_2$, *then (i), (iii), (iv) hold;*
- *if* $\frac{m}{2^r} \geq \alpha_2, \alpha_1 \leq \frac{n}{2^r} < \alpha_2$, *then (i), (ii) hold;*
- *if* $\alpha_1 \leq \frac{m}{2^r}, \frac{n}{2^r} < \alpha_2$, *then (i) holds,*

where the greek parameters are reported in Table 9, according to the different QI's.

	α_1	α_2	β	γ	δ	ϵ	ρ	σ	τ	η
S_1	3	5	1	0	2	1	1	3	2	0
S_2	5	7	1	-1	3	2	2	4	3	-1
W_2	4	6	0	-1	2	1	2	3	3	0

Table 9: Parameters involved in the symmetry properties of Theorem 3.1.

Remark 3.1. *Since the following relations between the* r*-level,* $r = 0, \dots, p$, *and the* 0*-level hold:*

$$h_i^{(r)} := x_i^{(r)} - x_{i-1}^{(r)} = h^{(r)} = 2^r h = \frac{2^r}{m},\ i = -1, ..., \frac{m}{2^r} + 2,$$

$$k_j^{(r)} := y_j^{(r)} - y_{j-1}^{(r)} = k^{(r)} = 2^r k = \frac{2^r}{n},\ j = -1, ..., \frac{n}{2^p} + 2,$$

except for $X_m^{(2)} \times Y_n^{(2)}$ *where*

$$h_i^{(r)} = 0,\ i = -1, 0, \frac{m}{2^r} + 1, \frac{m}{2^r} + 2,$$

$$k_j^{(r)} = 0,\ j = -1, 0, \frac{n}{2^r} + 1, \frac{n}{2^r} + 2,$$

then we can write

$$w_{ij}^{(r)} = c_{ij} h_i^{(r)} k_j^{(r)} = 4^r c_{ij} h_i k_j = 4^r w_{ij}^{(0)},$$

with:
- i *and* j *given in Table 10, according to the type of QI and* $X_m^{(k)} \times Y_n^{(k)}$, $k = 1, 2$. *It is important to keep in mind that* $w_{ij}^{(r)}$, $r = 0, \dots, p$, *is splitted into* $\overline{w}_{ij}^{(r)}$ *and* $\overline{\overline{w}}_{ij}^{(r)}$, *as in (11), according to the type of QI;*
- $w_{ij}^{(0)} := w_{ij}^{(Q,0)} = w_{ij}^{(Q)}$, *as in (8) and according to Tables 3-8.*

	$X_m^{(1)} \times Y_n^{(1)}$				$X_m^{(2)} \times Y_n^{(2)}$			
	$\overline{w}_{ij}^{(r)}$		$\overline{\overline{w}}_{ij}^{(r)}$		$\overline{w}_{ij}^{(r)}$		$\overline{\overline{w}}_{ij}^{(r)}$	
	i	j	i	j	i	j	i	j
S_1	$0,\dots,2$	$0,\dots,i$	$-$	$-$	$0,\dots,2$	$0,\dots,i$	$-$	$-$
S_2	$-1,\dots,3$	$-1,\dots,i$	$-$	$-$	$0,\dots,3$	$0,\dots,i$	$-$	$-$
W_2	$0,\dots,2$	$0,\dots,i$	$-1,\dots,2$	$-1,\dots,i$	$0,\dots,2$	$0,\dots,i$	$0,\dots,2$	$0,\dots,i$

Table 10: Variation of indices i and j of the weights $w_{ij}^{(r)}$, $r = 0,\dots,p$.

From the above symmetry properties of Theorem 3.1 and the above Remark 3.1, we can state the following

Theorem 3.2. *For any function $f \in C(R)$, for $\frac{m}{2^p}, \frac{n}{2^p} \geq \alpha_1$ and for $X_m^{(1)} \times Y_n^{(1)}$, we can write*

$$\begin{aligned} I(Q^{(p)}f) &= \sum_i \sum_j \overline{w}_{ij}^{(p)} z_{ij}^{(Q)}(f^{(p)}) + \sum_i \sum_j \overline{\overline{w}}_{ij}^{(p)} t_{ij}^{(Q)}(f^{(p)}) \\ &= \sum_i \sum_j 4^p \overline{w}_{ij}^{(Q)} z_{ij}^{(Q)}(f^{(p)}) + \sum_i \sum_j 4^p \overline{\overline{w}}_{ij}^{(Q)} t_{ij}^{(Q)}(f^{(p)}), \end{aligned} \tag{12}$$

where the variation of i and j is described in Table 10 and $z_{ij}^{(Q)}, t_{ij}^{(Q)}$ are listed below, depending on the type of QI:

$$
\begin{aligned}
z_{kk}^{(Q)}(f^{(p)}) &:= f_{kk}^{(p)} + f_{\frac{m}{2^p}-k+1,k}^{(p)} + f_{k,\frac{n}{2^p}-k+1}^{(p)} + f_{\frac{m}{2^p}-k+1,\frac{n}{2^p}-k+1}^{(p)},\\
z_{\delta k}^{(Q)}(f^{(p)}) &:= \sum_{i=\delta}^{\frac{m}{2^p}-\epsilon}(f_{ik}^{(p)} + f_{i,\frac{n}{2^p}-k+1}^{(p)}) + \sum_{j=\delta}^{\frac{n}{2^p}-\epsilon}(f_{kj}^{(p)} + f_{\frac{m}{2^p}-k+1,j}^{(p)}), \quad k=\eta,\dots,\epsilon,\\
z_{2k}^{(S_2)}(f^{(p)}) &:= f_{2k}^{(p)} + f_{\frac{m}{2^p}-1,k}^{(p)} + f_{2,\frac{n}{2^p}-k+1}^{(p)} + f_{\frac{m}{2^p}-1,\frac{n}{2^p}-k+1}^{(p)} + f_{k2}^{(p)} + f_{\frac{m}{2^p}-k+1,2}^{(p)}\\
&\quad + f_{k,\frac{n}{2^p}-1}^{(p)} + f_{\frac{m}{2^p}-k+1,\frac{n}{2^p}-1}^{(p)}, \quad k=-1,0,1,\\
z_{1k}^{(Q)}(f^{(p)}) &:= f_{1k}^{(p)} + f_{\frac{m}{2^p},k}^{(p)} + f_{1,\frac{n}{2^p}-k+1}^{(p)} + f_{\frac{m}{2^p},\frac{n}{2^p}-k+1}^{(p)} + f_{k,1}^{(p)} + f_{\frac{m}{2^p}-k+1,1}^{(p)} + f_{k,\frac{n}{2^p}}^{(p)}\\
&\quad + f_{\frac{m}{2^p}-k+1,\frac{n}{2^p}}^{(p)}, \quad k=-1,0, \text{ for } S_2, \ k=0, \text{ for } S_1, W_2,\\
z_{0,-1}^{(S_2)}(f^{(p)}) &:= f_{0,-1}^{(p)} + f_{\frac{m}{2^p}+1,-1}^{(p)} + f_{0,\frac{n}{2^p}+2}^{(p)} + f_{\frac{m}{2^p}+1,\frac{n}{2^p}+2}^{(p)} + f_{-1,0}^{(p)} + f_{\frac{m}{2^p}+2,0}^{(p)} + f_{-1,\frac{n}{2^p}+1}^{(p)}\\
&\quad + f_{\frac{m}{2^p}+2,\frac{n}{2^p}+1}^{(p)},\\
z_{\delta\delta}^{(Q)}(f^{(p)}) &:= \sum_{i=\delta}^{\frac{m}{2^p}-\epsilon}\sum_{j=\delta}^{\frac{n}{2^p}-\epsilon} f_{ij}^{(p)},\\
t_{kk}^{(W_2)}(f^{(p)}) &:= f_{kk}^{(p)} + f_{\frac{m}{2^p}-k,k}^{(p)} + f_{k,\frac{n}{2^p}-k}^{(p)} + f_{\frac{m}{2^p}-k,\frac{n}{2^p}-k}^{(p)},\\
t_{2k}^{(W_2)}(f^{(p)}) &:= \sum_{i=2}^{\frac{m}{2^p}-2}\left(f_{ik}^{(p)} + f_{i,\frac{n}{2^p}-k}^{(p)}\right) + \sum_{j=2}^{\frac{n}{2^p}-2}\left(f_{kj}^{(p)} + f_{\frac{m}{2^p}-k,j}^{(p)}\right), \quad k=-1,0,1,\\
t_{1k}^{(W_2)}(f^{(p)}) &:= f_{1k}^{(p)} + f_{\frac{m}{2^p}-1,k}^{(p)} + f_{1,\frac{n}{2^p}-k}^{(p)} + f_{\frac{m}{2^p}-1,\frac{n}{2^p}-k}^{(p)}\\
&\quad + f_{k,1}^{(p)} + f_{\frac{m}{2^p}-k,1}^{(p)} + f_{k,\frac{n}{2^p}-1}^{(p)} + f_{\frac{m}{2^p}-k,\frac{n}{2^p}-1}^{(p)}, \quad k=-1,0,\\
t_{0,-1}^{(W_2)}(f^{(p)}) &:= f_{0,-1}^{(p)} + f_{\frac{m}{2^p},-1}^{(p)} + f_{0,\frac{n}{2^p}+1}^{(p)} + f_{\frac{m}{2^p},\frac{n}{2^p}+1}^{(p)}\\
&\quad + f_{-1,0}^{(p)} + f_{\frac{m}{2^p}+1,0}^{(p)} + f_{-1,\frac{n}{2^p}}^{(p)} + f_{\frac{m}{2^p}+1,\frac{n}{2^p}}^{(p)},\\
t_{22}^{(W_2)}(f^{(p)}) &:= \sum_{i=2}^{\frac{m}{2^p}-2}\sum_{j=2}^{\frac{n}{2^p}-2} f_{ij}^{(p)},
\end{aligned}
$$

with

$$
f_{\mu\nu}^{(p)} := \begin{cases} f(M_{\mu\nu}^{(p)}) \text{ if } f^{(p)} \text{ is the argument of } z_{\varphi\psi}^{(Q)},\\ f(A_{\mu\nu}^{(p)}) \text{ if } f^{(p)} \text{ is the argument of } t_{\varphi\psi}^{(Q)}. \end{cases}
$$

Remark 3.2. *For $X_m^{(2)} \times Y_n^{(2)}$ Theorem 3.2 has to be modified as follows:*

- *for* $z_{\delta k}^{(S_2)}(f^{(p)})$, $k=\eta,\dots,\epsilon$ *changes to* $k=0,1,2$*;*
- *for* $z_{2k}^{(S_2)}(f^{(p)})$ *and* $t_{2k}^{(W_2)}(f^{(p)})$, $k=-1,0,1$ *changes to* $k=0,1$*;*
- *for* $z_{1k}^{(S_2)}(f^{(p)})$ *and* $t_{1k}^{(W_2)}(f^{(p)})$, $k=-1,0$ *changes to* $k=0$*;*
- $z_{0,-1}^{(S_2)}(f^{(p)})$ *and* $t_{0,-1}^{(W_2)}(f^{(p)})$ *are no more included in the computation because the corresponding weights are zero.*

Finally from Theorem 3.2 and Remark 3.2 (11) can be written as

$$\begin{aligned} I(Q^{pL}f) &= \sum_i \sum_j \overline{w}_{ij}^{(Q)} \sum_{r=0}^{p} 4^r z_{ij}^{(Q)}(\Delta_{r+1}^{p-r} f^{(r)}) \\ &+ \sum_i \sum_j \overline{\overline{w}}_{ij}^{(Q)} \sum_{r=0}^{p} 4^r t_{ij}^{(Q)}(\Delta_{r+1}^{p-r} f^{(r)}), \end{aligned} \tag{13}$$

where i,j are given in Table 10 and $z_{ij}^{(Q)}(\Delta_{r+1}^{p-r}f^{(r)}), t_{ij}^{(Q)}(\Delta_{r+1}^{p-r}f^{(r)})$, $r=0,\dots,p-1$, are defined similarly to $z_{ij}^{(Q)}(f^{(p)}), t_{ij}^{(Q)}(f^{(p)})$ in Theorem 3.2, with $\Delta_{r+1}^{p-r}f^{(r)}$ the function $\Delta_{r+1}^{p-r}f$, evaluated at the points $P_{\mu\nu}^{(r)} = M_{\mu\nu}^{(r)}, A_{\mu\nu}^{(r)}$, involved in the definition of $z_{ij}^{(Q)}(\Delta_{r+1}^{p-r}f^{(r)}), t_{ij}^{(Q)}(\Delta_{r+1}^{p-r}f^{(r)})$, respectively. An idea of such a computation is shown in the proof of Theorem 3.3 of [6].

3.3. Precision degree, convergence and computational complexity

From Theorems 2.1 and 2.3 it immediately follows that the precision degree of (13) is at least 2. Moreover by Corollary 1 of [18] the precision degree is at least 3, since $X_m^{(k)} \times Y_n^{(k)}$, $k=1,2$ are symmetric partitions, as confirmed by numerical results of next section.

Moreover we can state the following

Theorem 3.3. *Let* Q^{pL} *be a* $(p+1)$*-level QI operator with* $Q = S_1,\ S_2,\ W_2$ *and* $X_m^{(k)} \times Y_n^{(k)}$, $k=1,2$. *If* $f \in C^\mu(R)$, $\mu = 1,\ 2,\ 3$, *then* $I(Q^{pL}f)$ *at least satisfies*

$$|\ E(Q^{pL}f)\ | := |\ I(f) - I(Q^{pL}f)\ | = O(\Delta^\mu), \tag{14}$$

with $\Delta = \max\left\{\frac{1}{m}, \frac{1}{n}\right\}$.

Proof. Since

$$\begin{aligned} \mid E(Q^{pL}f) \mid =\mid I(f) - I(Q^{pL}f) \mid=\mid I(f - Q^{pL}f) \mid \\ \leq I(\mid f - Q^{pL}f \mid), \end{aligned} \tag{15}$$

then from Theorems 2.2 and 2.4 on the approximation power of $Q^{pL}f$ [6], we obtain (14). □

Concerning the computational complexity, we remark that for any $Q^{(k)}$, $k = p, \dots, p-r$, appearing in (9), we can approximatively count a number of $\mathcal{O}\Big(\frac{m}{2^k} \cdot \frac{n}{2^k}\Big)$ operations for function evaluations, for multiplications and for algebraic sums. Then it is clear that, running from p to 0, we sum up a number of operations whose leading term goes

$$\text{from } \mathcal{O}\Big(\frac{mn}{2^{2p}}\Big) \text{ to } \mathcal{O}\Big(\frac{m^{p+1}n^{p+1}}{2^{2(p+(p-1)+(p-2)+\ldots+1+0)}}\Big),$$

so that the total number of operations is of the order of

$$\sum_{i=0}^{p} \mathcal{O}\Big(\frac{m^{i+1}n^{i+1}}{2^{2\sum_{j=0}^{i}(p-j)}}\Big), 0 \leq p \leq \min\{q, s\},$$

which provides the particular classical case when $p = 0$, i.e. $\mathcal{O}(mn)$.

On the other hand, from (13) the computation of the weights is performed just considering the classical ones, unless for a factor 4^r.

Moreover we can remark that, according to the definition of each QI, also symmetries can be taken into account to reduce the computational complexity both in the classical and in the multilevel case.

However we finally underline that, increasing p, the computational complexity increases. That is perhaps why we get the best results when we balance multilevel technique and computational complexity. From numerical results, given in next section, we can conclude this is obtained for S_1^{1L}, not only thanks to this balancing, but also for the simple definition of S_1.

4. Matlab numerical results

For the numerical evaluation of (7) we tested the multilevel QI operators $Q^{pL} = S_1^{pL}, S_2^{pL}, W_2^{pL}$ on the following functions with different values of $0 \leq p \leq \min\{q, s\}$ and for triangulations $\Delta_{mn}^{(2)}$ based on both $X_m^{(i)} \times Y_n^{(i)}$, $i = 1, 2$ partitions for $(x, y) \in R$,

- $f_1(x,y) = \mid x^2 + y^2 - 0.25 \mid$,
- $f_2(x,y) = y^2 \sin x$,
- $f_3(x,y) = \dfrac{1}{9}\sqrt{64 - 81\left(\left(x-\dfrac{1}{2}\right)^2 + \left(y-\dfrac{1}{2}\right)^2\right)} - \dfrac{1}{2}$,
- $f_4(x,y) = e^{x+y}$,
- $f_5(x,y) = x^2 + 2y$,
- $f_6(x,y) = xy^2$,
- $f_7(x,y) = x^3 + 2y^2$.

In the following

- $E_kQ^{pL}(X^i_m, Y^i_n)$ denotes the absolute error $|I(f_k) - I(Q^{pL}f_k)|$, $k = 1,\dots,7$, $i = 1,2$,
- $O_kQ^{pL}(X^i_m, Y^i_n)$ denotes the corresponding observed approximation order, i.e.
$$\log_2 \frac{E_kQ^{pL}(X^i_m, Y^i_n)}{E_kQ^{pL}(X^i_{2m}, Y^i_{2n})},$$

for partitions $\Delta^{(2)}_{mn}$ based on $X^{(i)}_m \times Y^{(i)}_n$, $i = 1,2$, computed for increasing values of m and n.

Tables 11-22 are related to triangulations $\Delta^{(2)}_{mn}$ based on partitions $X^{(1)}_m \times Y^{(1)}_n$, while Tables 23-34 show the results with triangulations $\Delta^{(2)}_{mn}$ based on partitions $X^{(2)}_m \times Y^{(2)}_n$. In particular Table 17 shows a numerical confirmation that the multilevel QI operator S^{pL}_1, $0 < p \le \min\{q,s\}$, gains in precision degree in case of triangulations $\Delta^{(2)}_{mn}$ defined on $X^{(1)}_m \times Y^{(1)}_n$, while Tables 19-22 underline the augmented degree of precision, due to the symmetry of the partitions. Similar argument can be carried out for Tables 29-34 with partitions $X^{(2)}_m \times Y^{(2)}_n$.

$m=n$	E_2S_1	$E_2S_1^{1L}$	$E_2S_1^{2L}$	E_2S_2	$E_2S_2^{1L}$	E_2W_2	$E_2W_2^{1L}$
8	1.49(-3)	2.86(-5)	2.46(-5)	4.39(-6)	5.29(-7)	8.18(-6)	5.81(-6)
16	3.74(-4)	1.79(-6)	1.48(-6)	2.75(-7)	3.17(-8)	5.12(-7)	3.64(-7)
32	9.35(-5)	1.12(-7)	9.14(-8)	1.72(-8)	1.97(-9)	3.20(-8)	2.28(-8)
64	2.34(-5)	7.03(-9)	5.70(-9)	1.08(-9)	1.33(-10)	2.01(-9)	1.41(-9)
128	5.85(-6)	4.49(-10)	3.66(-10)	7.77(-11)	1.83(-11)	1.36(-10)	7.84(-11)
256	1.46(-6)	3.80(-11)	3.28(-11)	1.48(-11)	1.11(-11)	1.84(-11)	5.06(-12)

Table 11: Absolute error $E_2Q^{pL}(X_m^1, Y_n^1)$ for the evaluation of $I(f_2) = 0.1532325647$.

$m=n$	O_2S_1	$O_2S_1^{1L}$	$O_2S_1^{2L}$	O_2S_2	$O_2S_2^{1L}$	O_2W_2	$O_2W_2^{1L}$
8	-	-	-	-	-	-	-
16	1.9988	3.9961	4.0595	3.9980	4.0607	3.9982	3.9956
32	1.9997	3.9989	4.0156	3.9987	4.0081	3.9991	3.9995
64	1.9999	3.9977	4.0015	3.9866	3.8914	3.9927	4.0099
128	2.0000	3.9676	3.9612	3.8022	2.8633	3.8900	4.1725
256	2.0000	3.5622	3.4791	2.3912	-	2.8796	3.9537

Table 12: Observed approximation order $O_2Q^{pL}(X_m^1, Y_n^1)$ in the evaluation of $I(f_2)$.

4.1. Comparing numerical integration based on multilevel both quadratic and quartic quasi-interpolation

Since in [6] multilevel quadratic quasi-interpolation seems to perform better than quadratic quasi-interpolation as well as in [5] numerical integration based on quadratic quasi-interpolation seems to provide better results than quartic quasi-interpolation, it is straightforward to wonder if numerical integration based on multilevel quadratic quasi-interpolation can be better than the one based on multilevel quartic quasi-interpolation. Therefore in the following we present some results obtained by using S_1^{1L} in (7), the most promising multilevel QI operator among the studied ones, comparing them with the ones proposed in [5] and generated by a two-level QI operator of the form

$$W_4^{1L}f := W_4^{(1)}f + W_4^{(0)}\Delta_1^1 f,$$

where the corresponding classical QI operator $W_4 : C(R) \to S_4^{2,3}(\Delta_{mn}^{(2)})$ is defined by

$$W_4 f := \sum_{ij} f(x_i, y_j)B_{ij}$$

$m=n$	E_3S_1	$E_3S_1^{1L}$	$E_3S_1^{2L}$	E_3S_2	$E_3S_2^{1L}$	E_3W_2	$E_3W_2^{1L}$
8	5.93(-3)	1.23(-3)	1.57(-3)	2.63(-4)	3.24(-4)	1.96(-4)	1.60(-4)
16	1.47(-3)	4.99(-5)	8.55(-5)	1.19(-5)	1.94(-5)	9.29(-6)	1.52(-5)
32	3.66(-4)	2.64(-6)	3.77(-6)	7.09(-7)	6.29(-8)	5.56(-7)	1.79(-7)
64	9.14(-5)	1.60(-7)	7.72(-8)	4.38(-8)	5.08(-9)	3.44(-8)	1.05(-8)
128	2.28(-5)	9.95(-9)	4.06(-9)	2.73(-9)	3.20(-10)	2.15(-9)	6.55(-10)
256	5.71(-6)	6.21(-10)	2.43(-10)	1.71(-10)	2.01(-11)	1.34(-10)	4.09(-11)

Table 13: Absolute error $E_3Q^{pL}(X_m^1, Y_n^1)$ for the evaluation of $I(f_3) = 0.2865833317293664$.

$m=n$	O_3S_1	$O_3S_1^{1L}$	$O_3S_1^{2L}$	O_3S_2	$O_3S_2^{1L}$	O_3W_2	$O_3W_2^{1L}$
8	-	-	-	-	-	-	-
16	2.0154	4.6197	4.1980	4.4738	4.0633	4.3984	3.4040
32	2.0037	4.2426	4.5034	4.0638	8.2691	4.0606	6.4077
64	2.0009	4.0403	5.6094	4.0151	3.6305	4.0144	4.0909
128	2.0002	4.0097	4.2506	4.0037	3.9873	4.0036	4.0002
256	2.0001	4.0024	4.0618	4.0009	3.9991	4.0009	3.9993

Table 14: Observed approximation order $O_3Q^{pL}(X_m^1, Y_n^1)$ in the evaluation of $I(f_3)$.

with

- $S_4^{2,3}(\Delta_{mn}^{(2)})$ the space of piecewise polynomials of degree 4 with C^2 smoothness on the rectangular subdomain grid segments and C^3 smoothness on the diagonal grid segments;

- B_{ij} the quartic B-spline with centre at (x_i, y_j).

Both classical W_4 and two-level W_4^{1L} operators possess the property of linear polynomial reproduction and, if $f \in C^\mu(R)$, $\mu = 1, 2$, then $||f - W_4^{jL} f||_R = O(\Delta^\mu)$, $j = 0, 1$.

In Table 35 we show a comparison among the operators S_1, S_1^{1L}, W_4 and W_4^{1L}, when used for the numerical evaluation of $I(f_k)$, $k = 1, 2$.

5. Final remarks

In this paper we obtained new results on the numerical evaluation of integrals, based on quadratic QI, generalizing and collecting them and other known ones in the literature into a whole theory of the multilevel setting.

$m=n$	E_4S_1	$E_4S_1^{1L}$	$E_4S_1^{2L}$	E_4S_2	$E_4S_2^{1L}$	E_4W_2	$E_4W_2^{1L}$
8	1.16(-2)	1.78(-4)	1.73(-5)	4.22(-5)	3.11(-6)	4.22(-5)	3.11(-6)
16	2.88(-3)	1.11(-5)	4.05(-7)	2.63(-6)	1.89(-7)	2.63(-6)	1.89(-7)
32	7.21(-4)	6.92(-7)	1.51(-8)	1.64(-7)	1.18(-8)	1.64(-7)	1.18(-8)
64	1.80(-4)	4.33(-8)	7.86(-10)	1.03(-8)	7.34(-10)	1.03(-8)	7.34(-10)
128	4.51(-5)	2.70(-9)	4.66(-11)	6.41(-10)	4.58(-11)	6.42(-10)	4.58(-11)
256	1.13(-5)	1.69(-10)	2.88(-12)	4.01(-11)	2.86(-12)	4.01(-11)	2.86(-12)

Table 15: Absolute error $E_4Q^{pL}(X_m^1, Y_n^1)$ for the evaluation of $I(f_4) = (e-1)^2$.

$m=n$	O_4S_1	$O_4S_1^{1L}$	$O_4S_1^{2L}$	O_4S_2	$O_4S_2^{1L}$	O_4W_2	$O_4W_2^{1L}$
8	-	-	-	-	-	-	-
16	2.0017	4.0075	5.4159	4.0031	4.0359	4.0031	4.0359
32	2.0004	4.0019	4.7449	4.0008	4.0100	4.0008	4.0100
64	2.0001	4.0005	4.2646	4.0002	4.0026	4.0002	4.0026
128	2.0000	4.0001	4.0195	4.0000	4.0007	4.0000	4.0007
256	2.0000	4.0000	4.0195	3.9999	4.0010	3.9999	4.0010

Table 16: Observed approximation order $O_4Q^{pL}(X_m^1, Y_n^1)$ in the evaluation of $I(f_4)$.

We compared the integration error using multilevel QI spline operators to the one using classical QI spline operators and also among the different multilevel QI spline operators.

We confirm the conclusion of [6], where it is stated that the best improvement from classical to multilevel operators is given by S_1^{1L}, also for numerical integration. It combines the performances of multilevel setting with the simplicity of its definition and this fact let it compete with the other both classical and multilevel operators.

6. Acknowledgement

The authors are grateful to the anonymous reviewers for their suggestions and comments that improved this paper.

References

[1] C. de Boor, K. Höllig, S. Riemenschneider, Box splines, Springer-Verlag, New York, 1993.

$m = n$	E_5S_1	$E_5S_1^{1L}$	$E_5S_1^{2L}$	E_5S_2	$E_5S_2^{1L}$	E_5W_2	$E_5W_2^{1L}$
8	3.91(-3)	8.88(-16)	8.88(-16)	8.88(-16)	4.44(-16)	8.88(-16)	2.22(-16)
16	9.77(-4)	8.88(-16)	6.66(-16)	8.88(-16)	4.44(-16)	6.66(-16)	2.22(-16)
32	2.44(-4)	4.44(-16)	4.44(-16)	4.44(-16)	4.44(-16)	6.66(-16)	2.22(-16)
64	6.10(-5)	4.44(-16)	4.44(-16)	4.44(-16)	4.44(-16)	6.66(-16)	4.44(-16)
128	1.53(-5)	2.22(-16)	2.22(-16)	4.44(-16)	4.44(-16)	4.44(-16)	2.22(-16)
256	3.81(-6)	2.22(-16)	2.22(-16)	6.66(-16)	4.44(-16)	6.66(-16)	2.22(-16)

Table 17: Absolute error $E_5Q^{pL}(X_m^1, Y_n^1)$ for the evaluation of $I(f_5) = 4/3$.

$m = n$	O_5S_1	$O_5S_1^{1L}$	$O_5S_1^{2L}$	O_5S_2	$O_5S_2^{1L}$	O_5W_2	$O_5W_2^{1L}$
8	-	-	-	-	-	-	-
16	2.0000	-	-	-	-	-	-
32	2.0000	-	-	-	-	-	-
64	2.0000	-	-	-	-	-	-
128	2.0000	-	-	-	-	-	-
256	2.0000	-	-	-	-	-	-

Table 18: Observed approximation order $O_5Q^{pL}(X_m^1, Y_n^1)$ in the evaluation of $I(f_5)$.

[2] C. K. Chui, Multivariate splines, CBMS-NSF Regional Conference Series in Applied Mathematics, Vol. 54, SIAM, Philadelphia, 1988.

[3] R.-H. Wang, Multivariate spline functions and their applications, Kluwer, Dordrecht, 2001.

[4] C.-Y. Li, C.-G. Zhu, A multilevel univariate cubic spline quasi - interpolation and application to numerical integration, Mathematical Methods in the Applied Sciences 33 (2010) 1578–1586

[5] R.-H. Wang, J. Wu, X. Zhang, Numerical integration based on multilevel quartic quasi-interpolants operator, Applied Mathematics and Computation 227 (2014) 132–138.

[6] P. Lamberti, A. Saponaro, Multilevel quadratic spline quasi-interpolation, Applied Mathematics and Computation, 373 (2020), https://doi.org/10.1016/j.amc.2020.125047.

$m=n$	E_6S_1	$E_6S_1^{1L}$	$E_6S_1^{2L}$	E_6S_2	$E_6S_2^{1L}$	E_6W_2	$E_6W_2^{1L}$
8	1.95(-3)	2.77(-17)	2.77(-17)	2.77(-17)	2.77(-17)	2.77(-17)	2.77(-17)
16	4.88(-4)	2.77(-17)	2.77(-17)	2.77(-17)	2.77(-17)	2.77(-17)	2.77(-17)
32	1.22(-4)	2.77(-17)	2.77(-17)	2.77(-17)	2.77(-17)	2.77(-17)	2.77(-17)
64	3.05(-5)	2.77(-17)	2.77(-17)	2.77(-17)	2.77(-17)	2.77(-17)	2.77(-17)
128	7.63(-6)	2.77(-17)	2.77(-17)	2.77(-17)	2.77(-17)	2.77(-17)	2.77(-17)
256	1.91(-6)	2.77(-17)	2.77(-17)	2.77(-17)	2.77(-17)	2.77(-17)	2.77(-17)

Table 19: Absolute error $E_6Q^{pL}(X_m^1, Y_n^1)$ for the evaluation of $I(f_6) = 1/6$.

$m=n$	O_6S_1	$O_6S_1^{1L}$	$O_6S_1^{2L}$	O_6S_2	$O_6S_2^{1L}$	O_6W_2	$O_6W_2^{1L}$
8	-	-	-	-	-	-	-
16	1.9999	-	-	-	-	-	-
32	2.0000	-	-	-	-	-	-
64	1.9999	-	-	-	-	-	-
128	2.0000	-	-	-	-	-	-
256	1.9999	-	-	-	-	-	-

Table 20: Observed approximation order $O_6Q^{pL}(X_m^1, Y_n^1)$ in the evaluation of $I(f_6)$.

[7] C. K. Chui, R.-H. Wang, On a bivariate B-spline basis., Sci. Sin. XXVII (1984) 1129–1142.

[8] C. K. Chui, R.-H. Wang, Concerning C^1 B-splines on triangulations of non-uniform rectangular partition, Approx. Theory Appl. 1 (1984) 11–18.

[9] C. Dagnino, P. Lamberti, Some performances of local bivariate quadratic C^1 quasi-interpolating splines on nonuniform type-2 triangulations, J. Comput. Appl. Math. 173, no.1 (2005) 21–37.

[10] C. Dagnino, P. Lamberti, On the approximation power of bivariate quadratic C^1 splines, J. Comput. Appl. Math. 131 (2001) 321–332.

[11] C. Dagnino, P. Lamberti, Numerical integration of 2-D integrals based on local bivariate C^1 quasi-interpolating splines, Adv. Comput. Math. 8, no. 1-2 (1998) 19–31.

[12] C. Dagnino, P. Lamberti, On the construction of local quadratic spline

$m = n$	E_7S_1	$E_7S_1^{1L}$	$E_7S_1^{2L}$	E_7S_2	$E_7S_2^{1L}$	E_7W_2	$E_7W_2^{1L}$
8	1.37(-2)	1.11(-16)	2.22(-16)	1.11(-16)	1.11(-16)	1.11(-16)	1.11(-16)
16	3.42(-3)	1.11(-16)	2.22(-16)	1.11(-16)	1.11(-16)	1.11(-16)	1.11(-16)
32	8.54(-4)	1.11(-16)	1.11(-16)	1.11(-16)	1.11(-16)	1.11(-16)	1.11(-16)
64	2.14(-4)	1.11(-16)	1.11(-16)	1.11(-16)	1.11(-16)	1.11(-16)	1.11(-16)
128	5.34(-5)	1.11(-16)	1.11(-16)	1.11(-16)	1.11(-16)	1.11(-16)	1.11(-16)
256	1.34(-5)	1.11(-16)	1.11(-16)	1.11(-16)	1.11(-16)	1.11(-16)	1.11(-16)

Table 21: Absolute error $E_7Q^{pL}(X_m^1, Y_n^1)$ for the evaluation of $I(f_7) = 11/12$.

$m = n$	O_7S_1	$O_7S_1^{1L}$	$O_7S_1^{2L}$	O_7S_2	$O_7S_2^{1L}$	O_7W_2	$O_7W_2^{1L}$
8	-	-	-	-	-	-	-
16	2.0000	-	-	-	-	-	-
32	2.0000	-	-	-	-	-	-
64	2.0000	-	-	-	-	-	-
128	1.9999	-	-	-	-	-	-
256	2.0000	-	-	-	-	-	-

Table 22: Observed approximation order $O_7Q^{pL}(X_m^1, Y_n^1)$ in the evaluation of $I(f_7)$.

quasi-interpolants on bounded rectangular domains, J. Comput. Appl. Math. 221 (2008) 367–375.

[13] C. Dagnino, P. Lamberti, Spline "quasi-interpolants" with boundary conditions on criss-cross triangulations, in: A. Cohen, J.L. Merrien, L.L. Schumaker (Eds.), Curve and surface fitting: Avignon 2006, Nashboro Press, Brentwood, TN, 2007, pp. 101–110.

[14] C. Dagnino, P. Lamberti, S. Remogna, B-spline bases for unequally smooth quadratic spline spaces on non-uniform criss-cross triangulations, Numer. Algorithms 61 (2012) 209–222

[15] C. Dagnino, S. Remogna, P. Sablonnière, Error bounds on the approximation of functions and partial derivatives by quadratic spline quasi-interpolants on non-uniform criss-cross triangulations of a rectangular domain, BIT Numer. Math. 53 (2013) 87–109

[16] F. Foucher, P. Sablonnière, Approximating partial derivatives of first

$m=n$	E_2S_1	$E_2S_1^{1L}$	$E_2S_1^{2L}$	E_2S_2	$E_2S_2^{1L}$	E_2W_2	$E_2W_2^{1L}$
8	1.37(-3)	1.79(-4)	7.05(-5)	4.39(-6)	1.45(-6)	7.14(-6)	3.27(-6)
16	3.58(-4)	2.14(-5)	6.72(-6)	2.91(-7)	1.37(-7)	4.79(-7)	2.89(-7)
32	9.15(-5)	2.58(-6)	7.21(-7)	1.87(-8)	9.91(-9)	3.10(-8)	2.05(-8)
64	2.31(-5)	3.16(-7)	8.34(-8)	1.19(-9)	6.51(-10)	1.98(-9)	1.34(-9)
128	5.81(-6)	3.91(-8)	1.01(-8)	8.51(-11)	3.20(-11)	1.35(-10)	7.62(-11)
256	1.46(-6)	4.88(-9)	1.24(-9)	1.53(-11)	7.92(-12)	1.84(-11)	5.13(-12)

Table 23: Absolute error $E_2Q^{pL}(X_m^2, Y_n^2)$ for the evaluation of $I(f_2) = 0.1532325647$.

$m=n$	O_2S_1	$O_2S_1^{1L}$	$O_2S_1^{2L}$	O_2S_2	$O_2S_2^{1L}$	O_2W_2	$O_2W_2^{1L}$
8	-	-	-	-	-	-	-
16	1.9337	3.0701	3.3918	2.4768	3.4084	3.8985	3.4986
32	1.9682	3.0485	3.2207	3.8107	3.7883	3.9505	3.8185
64	1.9844	3.0282	3.1109	3.9680	3.9283	3.9687	3.9309
128	1.9923	3.0149	3.0533	3.9590	4.3460	3.8773	4.1395
256	1.9962	3.0051	3.0161	3.9178	2.0157	2.8711	3.8934

Table 24: Observed approximation order $O_2Q^{pL}(X_m^2, Y_n^2)$ in the evaluation of $I(f_2)$.

and second order by quadratic spline quasi-interpolants on uniform meshes, Math. & Computers in Simulation 77 (2008) 202–208

[17] F. Foucher, P. Sablonnière, Superconvergence properties of some bivariate C^1 quadratic spline quasi-interpolants, in: A. Cohen, J.-L. Merrien, L. L. Schumaker (Eds.), Curve and surface fitting: Avignon 2006, Nashboro Press, Brentwood, TN, 2007, pp. 160–169.

[18] P. Lamberti, Numerical integration based on bivariate quadratic spline quasi-interpolants on bounded domains, BIT Numer. Math. 49 (2009) 565–588

[19] P. Sablonnière, On some multivariate quadratic spline quasi-interpolants on bounded domains, in: W. Hausmann & al. (Eds.), Modern developments in multivariate approximations, ISNM Vol. 145, Birkhäuser Verlag, Basel, 2003, pp. 263–278.

[20] P. Sablonnière, Quadratic spline quasi-interpolants on bounded domains

$m = n$	E_3S_1	$E_3S_1^{1L}$	$E_3S_1^{2L}$	E_3S_2	$E_3S_2^{1L}$	E_3W_2	$E_3W_2^{1L}$
8	5.14(-3)	6.06(-4)	1.65(-4)	4.21(-5)	1.82(-5)	9.12(-5)	4.76(-5)
16	1.37(-3)	9.11(-5)	2.89(-5)	3.28(-6)	1.11(-6)	6.97(-6)	3.02(-6)
32	3.54(-4)	1.29(-5)	3.96(-6)	2.31(-7)	6.29(-8)	4.87(-7)	1.82(-7)
64	8.99(-5)	1.73(-6)	4.99(-7)	1.54(-8)	3.60(-9)	3.23(-8)	1.10(-8)
128	2.26(-5)	2.25(-7)	6.12(-8)	9.94(-10)	2.13(-10)	2.08(-9)	6.71(-10)
256	5.69(-6)	2.86(-8)	7.51(-9)	6.32(-11)	1.29(-11)	1.32(-10)	4.14(-11)

Table 25: Absolute error $E_3Q^{pL}(X_m^2, Y_n^2)$ for the evaluation of $I(f_3) = 0.2865833317293664$.

$m = n$	O_3S_1	$O_3S_1^{1L}$	$O_3S_1^{2L}$	O_3S_2	$O_3S_2^{1L}$	O_3W_2	$O_3W_2^{1L}$
8	-	-	-	-	-	-	-
16	1.9090	2.7345	2.5091	3.6796	4.0377	3.7096	3.9790
32	1.9538	2.8232	2.8683	3.8266	4.1368	3.8402	4.0502
64	1.9766	2.8968	2.9899	3.9093	4.1274	3.9151	4.0522
128	1.9882	2.9440	3.0259	3.9535	4.0817	3.9560	4.0332
256	1.9941	2.9707	3.0262	3.9764	4.0443	3.9774	4.0176

Table 26: Observed approximation order $O_3Q^{pL}(X_m^2, Y_n^2)$ in the evaluation of $I(f_3)$.

of $\mathbb{R}^d$, $d = 1, 2, 3$, Rend. Sem. Mat. Univ. Pol. Torino 61, no. 3 (2003) 229–246.

[21] R.-H. Wang, Y. Lu, Quasi-interpolating operators and their applications in hypersingular integrals, J. Comput. Math. 16, no.4 (1998) 337–344.

$m=n$	E_4S_1	$E_4S_1^{1L}$	$E_4S_1^{2L}$	E_4S_2	$E_4S_2^{1L}$	E_4W_2	$E_4W_2^{1L}$
8	1.05(-2)	1.17(-3)	2.55(-4)	1.91(-5)	8.94(-7)	3.62(-5)	3.76(-6)
16	2.75(-3)	1.53(-4)	3.82(-5)	1.28(-6)	6.79(-8)	2.44(-6)	1.44(-8)
32	7.05(-4)	1.97(-5)	5.00(-6)	8.31(-8)	8.03(-9)	1.58(-7)	5.59(-9)
64	1.78(-4)	2.50(-6)	6.33(-7)	5.28(-9)	6.19(-10)	1.01(-8)	5.44(10)
128	4.48(-5)	3.15(-7)	7.93(-8)	3.33(-10)	4.23(-11)	6.36(-10)	3.99(-11)
256	1.12(-5)	3.95(-8)	9.92(-9)	2.12(-11)	2.75(-12)	4.02(-11)	2.68(-12)

Table 27: Absolute error $E_4Q^{pL}(X_m^2, Y_n^2)$ for the evaluation of $I(f_4) = (e-1)^2$.

$m=n$	O_4S_1	$O_4S_1^{1L}$	$O_4S_1^{2L}$	O_4S_2	$O_4S_2^{1L}$	O_4W_2	$O_4W_2^{1L}$
8	-	-	-	-	-	-	-
16	1.9319	2.9316	2.7396	3.8945	3.7179	3.8900	8.0347
32	1.9668	2.9582	2.9329	3.9487	3.0792	3.9465	1.3612
64	1.9836	2.9773	2.9829	3.9747	3.6992	3.9736	3.3603
128	1.9918	2.9882	2.9960	3.9874	3.8717	3.9869	3.7672
256	1.9959	2.9940	2.9992	3.9761	3.9400	3.9834	3.8983

Table 28: Observed approximation order $O_4Q^{pL}(X_m^2, Y_n^2)$ in the evaluation of $I(f_4)$.

$m=n$	E_5S_1	$E_5S_1^{1L}$	$E_5S_1^{2L}$	E_5S_2	$E_5S_2^{1L}$	E_5W_2	$E_5W_2^{1L}$
8	3.58(-3)	4.07(-4)	1.02(-4)	6.66(-16)	8.88(-16)	6.66(-16)	4.44(-16)
16	9.36(-4)	5.09(-5)	1.27(-5)	6.66(-16)	8.88(-16)	4.44(-16)	2.22(-16)
32	2.39(-4)	6.36(-6)	1.59(-6)	4.44(-16)	6.66(-16)	2.22(-16)	2.22(-16)
64	6.04(-5)	7.95(-7)	1.99(-7)	2.22(-16)	6.66(-16)	2.22(-16)	2.22(-16)
128	1.52(-5)	9.93(-8)	2.48(-8)	2.22(-16)	4.44(-16)	2.22(-16)	2.22(-16)
256	3.80(-6)	1.24(-8)	3.10(-9)	2.22(-16)	2.22(-16)	2.22(-16)	2.22(-16)

Table 29: Absolute error $E_5Q^{pL}(X_m^2, Y_n^2)$ for the evaluation of $I(f_5) = 4/3$.

$m = n$	O_5S_1	$O_5S_1^{1L}$	$O_5S_1^{2L}$	O_5S_2	$O_5S_2^{1L}$	O_5W_2	$O_5W_2^{1L}$
8	-	-	-	-	-	-	-
16	1.9359	3.0000	3.0000	-	-	-	-
32	1.9690	3.0000	3.0000	-	-	-	-
64	1.9847	3.0000	3.0000	-	-	-	-
128	1.9924	3.0000	3.0000	-	-	-	-
256	1.9962	3.0000	3.0000	-	-	-	-

Table 30: Observed approximation order $O_5Q^{pL}(X_m^2, Y_n^2)$ in the evaluation of $I(f_5)$.

$m = n$	E_6S_1	$E_6S_1^{1L}$	$E_6S_1^{2L}$	E_6S_2	$E_6S_2^{1L}$	E_6W_2	$E_6W_2^{1L}$
8	1.79(-3)	2.03(-4)	5.09(-5)	2.74(-14)	2.73(-14)	2.73(-14)	2.73(-14)
16	4.68(-4)	2.54(-5)	6.36(-6)	2.74(-14)	2.73(-14)	2.73(-14)	2.73(-14)
32	1.19(-4)	3.18(-6)	7.95(-7)	2.74(-14)	2.73(-14)	2.73(-14)	2.73(-14)
64	3.02(-5)	3.97(-7)	9.93(-8)	2.73(-14)	2.73(-14)	2.73(-14)	2.73(-14)
128	7.59(-6)	4.96(-8)	1.24(-8)	2.73(-14)	2.73(-14)	2.73(-14)	2.73(-14)
256	1.90(-6)	6.21(-9)	1.55(-9)	2.73(-14)	2.73(-14)	2.73(-14)	2.73(-14)

Table 31: Absolute error $E_6Q^{pL}(X_m^2, Y_n^2)$ for the evaluation of $I(f_6) = 1/6$.

$m = n$	O_6S_1	$O_6S_1^{1L}$	$O_6S_1^{2L}$	O_6S_2	$O_6S_2^{1L}$	O_6W_2	$O_6W_2^{1L}$
8	-	-	-	-	-	-	-
16	1.9359	2.9999	2.9999	-	-	-	-
32	1.9690	3.0000	3.0000	-	-	-	-
64	1.9847	3.0000	3.0000	-	-	-	-
128	1.9924	3.0000	3.0000	-	-	-	-
256	1.9962	3.0000	2.9999	-	-	-	-

Table 32: Observed approximation order $O_6Q^{pL}(X_m^2, Y_n^2)$ in the evaluation of $I(f_6)$.

$m=n$	E_7S_1	$E_7S_1^{1L}$	$E_7S_1^{2L}$	E_7S_2	$E_7S_2^{1L}$	E_7W_2	$E_7W_2^{1L}$
8	1.25(-2)	1.42(-3)	3.56(-4)	1.11(-16)	1.11(-16)	1.11(-16)	1.11(-16)
16	3.28(-3)	1.78(-4)	4.45(-5)	1.11(-16)	1.11(-16)	1.11(-16)	1.11(-16)
32	8.37(-4)	2.23(-5)	5.56(-6)	1.11(-16)	1.11(-16)	1.11(-16)	1.11(-16)
64	2.11(-4)	2.78(-6)	6.95(-7)	1.11(-16)	1.11(-16)	1.11(-16)	1.11(-16)
128	5.31(-5)	3.48(-7)	8.69(-8)	1.11(-16)	1.11(-16)	1.11(-16)	1.11(-16)
256	1.33(-5)	1.35(-8)	1.09(-8)	1.11(-16)	1.11(-16)	1.11(-16)	1.11(-16)

Table 33: Absolute error $E_7Q^{pL}(X_m^2, Y_n^2)$ for the evaluation of $I(f_7) = 11/12$.

$m=n$	O_7S_1	$O_7S_1^{1L}$	$O_7S_1^{2L}$	O_7S_2	$O_7S_2^{1L}$	O_7W_2	$O_7W_2^{1L}$
8	-	-	-	-	-	-	-
16	1.9359	2.9999	3.0000	-	-	-	-
32	1.9690	3.0000	3.0000	-	-	-	-
64	1.9847	2.9999	3.0000	-	-	-	-
128	1.9924	3.0000	2.9999	-	-	-	-
256	1.9962	3.0000	3.0000	-	-	-	-

Table 34: Observed approximation order $O_7Q^{pL}(X_m^2, Y_n^2)$ in the evaluation of $I(f_7)$.

	(m,n)	E_kW_4	E_kS_1	$E_kW_4^{1L}$	$E_kS_1^{1L}$
f_1	(8,6)	1.36(-2)	1.11(-2)	1.09(-4)	2.29(-4)
	(10,10)	6.23(-3)	5.12(-3)	3.05(-4)	1.26(-4)
	(20,20)	3.96(-3)	3.18(-3)	9.22(-5)	1.89(-5)
	(40,30)	5.61(-4)	4.38(-4)	6.35(-5)	4.19(-6)
f_2	(8,6)	3.85(-3)	2.89(-3)	3.16(-4)	5.25(-5)
	(10,10)	1.27(-3)	9.56(-4)	1.20(-4)	1.18(-5)
	(20,20)	3.19(-4)	2.39(-4)	4.13(-5)	7.36(-7)
	(40,30)	1.54(-4)	1.15(-4)	2.31(-5)	8.43(-8)

Table 35: Absolute error $E_kQ^{pL}(X_m^1, Y_n^1)$ with $Q = W_4, S_1, W_4^{1L}, S_1^{1L}$, $p = 0, 1$ and $k = 1, 2$.