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Weighted Spectral Cluster Bounds and a Sharp Multiplier Theorem for Ultraspherical Grushin Operators

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We study degenerate elliptic operators of Grushin type on the d-dimensional sphere, which are singular on a k-dimensional sphere for some $k < d$. For these operators we prove a spectral multiplier theorem of Mihlin–Hörmander type, which is optimal whenever $2k \leq d$, and a corresponding Bochner–Riesz summability result. The proof hinges on suitable weighted spectral cluster bounds, which in turn depend on precise estimates for ultraspherical polynomials.

1 Introduction

In this paper we continue the study of spherical Grushin-type operators started in [13] with the case of the 2-dimensional sphere. The focus here is on a family of hypoelliptic operators $\{\mathcal{L}_{d,k}\}_{1 \leq k < d}$, acting on functions defined on the unit sphere $\mathbb{S}^d$ in $\mathbb{R}^{1+d}$, that is, on

$$\mathbb{S}^d = \{(z_0, \dots, z_d) \in \mathbb{R}^{1+d} : z_0^2 + \cdots + z_d^2 = 1\}, \tag{1.1}$$

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for some $d \geq 2$. The special orthogonal groups $\mathrm{SO}(1+k)$ with $1 \leq k \leq d$ can be naturally identified with a sequence of nested subgroups of $\mathrm{SO}(1+d)$, so that $\mathrm{SO}(1+k)$ acts trivially on the coordinates $z_{k+1}, \dots, z_d$. Correspondingly each of these groups acts on $\mathbb{S}^d$ by rotations. We denote by Δ_k the (positive semidefinite) 2nd-order differential operator on $\mathbb{S}^d$ corresponding through this action to the Casimir operator on $\mathrm{SO}(1+k)$. The operators Δ_k for $k = 1, \dots, d$ commute pairwise and Δ_d turns out to be the Laplace–Beltrami operator on $\mathbb{S}^d$. The operators we are interested in are defined as

$$\mathcal{L}_{d,k} = \Delta_d - \Delta_k, \tag{1.2}$$

with $k = 1, \dots, d-1$.

While it may not be immediately apparent from the above formula, each operator $\mathcal{L}_{d,k}$ can be written as minus the sum of squares of a system of smooth divergence-free vector fields on $\mathbb{S}^d$ (see Section 3.2 below), and in particular is a positive semidefinite operator too. These vector fields span the tangent space of $\mathbb{S}^d$ at each point not lying in the k-submanifold $\mathbb{S}^k \times \{0\}$, and consequently $\mathcal{L}_{d,k}$ is elliptic away from this submanifold. Moreover, by introducing a suitable system of "spherical coordinates" (ω, ψ) on $\mathbb{S}^d$, where $\omega \in \mathbb{S}^k$ and $\psi = (\psi_{k+1}, \dots, \psi_d) \in (-\pi/2, \pi/2)^{d-k}$ (see Section 3.3 below for details), one can write $\mathcal{L}_{d,k}$ in a neighbourhood of the submanifold $\mathbb{S}^k \times \{0\}$ more explicitly as

$$\mathcal{L}_{d,k} = \sum_{r=k+1}^{d} Y_r^+ Y_r + \mathfrak{V}(\psi)\, \Delta_k, \tag{1.3}$$

where the Y_r and their formal adjoints Y_r^+ (with respect to the standard rotation-invariant measure σ on $\mathbb{S}^d$) are vector fields only depending on ψ, to wit,

$$Y_r = \frac{1}{\cos\psi_{r+1} \cdots \cos\psi_d} \frac{\partial}{\partial \psi_r}, \tag{1.4}$$

and $\mathfrak{V} : (-\pi/2, \pi/2)^{d-k} \to \mathbb{R}$ is given by

$$\mathfrak{V}(\psi) = \frac{1}{\cos^2\psi_{k+1} \cdots \cos^2\psi_d} - 1 = \prod_{j=k+1}^{d} (1 + \tan^2\psi_j) - 1. \tag{1.5}$$

The vanishing of $\mathfrak{V}(\psi)$ for $\psi = 0$ corresponds to the fact that $\mathcal{L}_{d,k}$ is not elliptic at any point of the submanifold $\mathbb{S}^k \times \{0\}$. The loss of global ellipticity is anyway compensated by the fact that $\mathcal{L}_{d,k}$ is hypoelliptic and satisfies subelliptic estimates, as shown by an application of Hörmander's theorem for sums of squares of vector fields

[32]. Indeed the expression (1.3) reveals the analogy of the operators $\mathcal{L}_{d,k}$ with certain degenerate elliptic operators $\mathcal{G}_{d,k}$ on $\mathbb{R}^d$, given by

$$\mathcal{G}_{d,k} = \Delta_x + |x|_2^2 \Delta_y, \tag{1.6}$$

where x, y are the components of a point in $\mathbb{R}^{d-k}_x \times \mathbb{R}^k_y$ and Δ_x, Δ_y denote the corresponding (positive definite) partial Laplacians, while $|x|_2$ is the Euclidean norm of x.

In light of [28, 29], the operators $\mathcal{G}_{d,k}$ are often called Grushin operators; sometimes they are also called Baouendi–Grushin operators, since shortly before the papers by V. V. Grushin appeared, M. S. Baouendi introduced a more general class of operators containing also the $\mathcal{G}_{d,k}$ [5]. In these and other works (see, e.g., [18, 23, 49]), the coefficient $|x|_2^2$ in (1.6) may be replaced by a more general function $V(x)$. As prototypical examples of differential operators with mixed homogeneity, operators of the form (1.6) have attracted increasing interest in the past 50 years; we refer to [13] for a brief list of the main results, focused on the field of harmonic analysis. More recently, the study of Grushin-type operators began to develop also on more general manifolds than $\mathbb{R}^n$, from both a geometric and an analytic perspective [6, 7, 9, 10, 25, 26, 48].

In this article, we investigate L^p boundedness properties of operators of the form $F(\sqrt{\mathcal{L}_{d,k}})$ in connection with size and smoothness properties of the spectral multiplier $F : \mathbb{R} \to \mathbb{C}$; here L^p spaces on the sphere $\mathbb{S}^d$ are defined in terms of the spherical measure σ, and the operators $F(\sqrt{\mathcal{L}_{d,k}})$ are initially defined on $L^2(\mathbb{S}^d)$ via the Borel functional calculus for the self-adjoint operator $\mathcal{L}_{d,k}$. The study of the L^p boundedness of functions of Laplace-like operators is a classical and very active area of harmonic analysis, with a number of celebrated results and open questions, already in the case of the classical Laplacian in Euclidean space (think, e.g., of the Bochner–Riesz conjecture). Regarding the spherical Grushin operators $\mathcal{L}_{d,k}$, in the case $d = 2$ and $k = 1$, a sharp multiplier theorem of Mihlin–Hörmander type and a Bochner–Riesz summability result for $\mathcal{L}_{d,k}$ were obtained in [13]. Here we treat the general case $d \geq 2$, $1 \leq k < d$, and obtain the following result.

Let $\eta \in C_c^\infty((0,\infty))$ be any nontrivial cutoff, and denote by $L^q_s(\mathbb{R})$ the L^q Sobolev space of (fractional) order s on $\mathbb{R}$.

Theorem 1.1. Let $D = \max\{d, 2k\}$ and $s > D/2$.

(i) For all continuous functions supported in $[-1,1]$,

$$\sup_{t>0} \|F(t\sqrt{\mathcal{L}_{d,k}})\|_{L^1(\mathbb{S}^d)\to L^1(\mathbb{S}^d)} \lesssim_s \|F\|_{L^2_s}.$$

(ii) For all bounded Borel functions $F:\mathbb{R}\to\mathbb{C}$ such that $F|_{(0,\infty)}$ is continuous,

$$\|F(\sqrt{\mathcal{L}_{d,k}})\|_{L^1(\mathbb{S}^d)\to L^{1,\infty}(\mathbb{S}^d)} \lesssim_s \sup_{t\geq 0} \|F(t\cdot)\,\eta\|_{L^2_s}. \tag{1.7}$$

Hence, whenever the right-hand side of (1.7) is finite, the operator $F(\sqrt{\mathcal{L}_{d,k}})$ is of weak type $(1,1)$ and bounded on $L^p(\mathbb{S}^d)$ for all $p\in(1,\infty)$.

Part (i) of the above theorem and a standard interpolation technique imply the following Bochner–Riesz summability result.

Corollary 1.2. Let $D=\max\{d,2k\}$ and $p\in[1,\infty]$. If $\delta>(D-1)|1/2-1/p|$, then the Bochner–Riesz means $(1-t\mathcal{L}_{d,k})_+^\delta$ of order δ associated with $\mathcal{L}_{d,k}$ are bounded on $L^p(\mathbb{S}^d)$ uniformly in $t\in(0,\infty)$.

It is important to point out that weaker versions of the above results, involving more restrictive requirements on the smoothness parameters s and δ, could be readily obtained by standard techniques. Indeed the sphere $\mathbb{S}^d$, with the measure σ and the sub-Riemannian distance associated to $\mathcal{L}_{d,k}$ (see Sections 3.2 to 3.4 below for details), is a doubling metric measure space of "homogeneous dimension" $Q=d+k$, and the operator $\mathcal{L}_{d,k}$ satisfies Gaussian-type heat kernel bounds. As a consequence (see, e.g., [17, 20, 21, 31]), one would obtain the analogue of Theorem 1.1 with smoothness requirement $s>Q/2$, measured in terms of an L^∞ Sobolev norm, and the corresponding result for Bochner–Riesz means would give L^p boundedness only for $\delta>Q|1/p-1/2|$. Since $Q>D>D-1$, the results in this paper yield an improvement on the standard result for all values of d and k.

As a matter of fact, in the case $k\leq d/2$, the above multiplier theorem is sharp, in the sense that the lower bound $D/2$ to the order of smoothness s required in Theorem 1.1 cannot be replaced by any smaller quantity. Since $\mathcal{L}_{d,k}$ is elliptic away from a negligible subset of $\mathbb{S}^d$, and $D=d$ is the topological dimension of $\mathbb{S}^d$ when $k\leq d/2$, the sharpness of the above result can be seen by comparison to the Euclidean case via a transplantation technique [34, 42].

The fact that for subelliptic nonelliptic operators one can often obtain "improved" multiplier theorems, by replacing the relevant homogeneous dimension with the topological dimension in the smoothness requirement, was first noticed in the case of sub-Laplacians on Heisenberg and related groups by D. Müller and E. M. Stein [44] and independently by W. Hebisch [30] and has since been verified in multiple cases. However, despite a flurry of recent progress (see, e.g., [13, 18, 38, 39] for more detailed accounts and further references), the question whether such an improvement is always possible remains open. The results in the present paper can therefore be considered as part of a wider programme, attempting to gain an understanding of the general problem by tackling particularly significant particular cases.

In these respects, it it relevant to point out that Theorem 1.1 above can be considered as a strengthening of the multiplier theorem for the Grushin operators $\mathcal{G}_{d,k}$ on $\mathbb{R}^d$ proved in [40]: indeed a "nonisotropic transplantation" technique (see, e.g., [36, Theorem 5.2]) allows one to deduce from Theorem 1.1 the analogous result where $\mathbb{S}^d$ and $\mathcal{L}_{d,k}$ are replaced by $\mathbb{R}^d$ and $\mathcal{G}_{d,k}$.

The structure of the proof of Theorem 1.1 broadly follows that of the analogous result in [13], but additional difficulties need to be overcome here. An especially delicate point is the proof of the "weighted spectral cluster estimates" stated as Propositions 4.2 and 4.3 below, essentially consisting in suitable weighted $L^1 \to L^2$ norm bounds for "weighted spectral projections"

$$(\mathcal{L}_{d,k}/\Delta_d)^{\alpha/2} \chi_{[i,i+1]}(\sqrt{\mathcal{L}_{d,k}}) \tag{1.8}$$

associated with bands of unit width of the spectrum of $\sqrt{\mathcal{L}_{d,k}}$. These can be thought of as subelliptic analogues of the Agmon–Avakumovič–Hörmander spectral cluster estimates

$$\|\chi_{[i,i+1]}(\sqrt{\Delta_d})\|_{L^1 \to L^2} \lesssim i^{(d-1)/2} \tag{1.9}$$

for the elliptic Laplacian Δ_d, which are valid more generally when $\sqrt{\Delta_d}$ is replaced with an elliptic pseudodifferential operator of order one on a compact d-manifold [33] and are the basic building block for a sharp multiplier theorem for elliptic operators on compact manifolds and related restriction-type estimates [24, 50, 52, 53]. Thanks to pseudodifferential and Fourier integral operator techniques, estimates of the form (1.9) can be proved for elliptic operators in great generality, but these techniques break down when the ellipticity assumption is weakened. Nevertheless alternative *ad hoc* methods may be developed in many cases, based on a detailed analysis of the spectral decomposition of the operator under consideration, often made possible by underlying symmetries.

In the case of the spherical Grushin operator $\mathcal{L}_{d,k}$, as a consequence of its spectral decomposition in terms of joint eigenfunctions of the operators $\Delta_d, \ldots, \Delta_k$, the integral kernel of the "weighted projection" in (1.8) involves sums of $(d-k)$-fold tensor products of ultraspherical polynomials. This is a substantial difference from the case considered in [13] (where $d-k=1$) and requires new ideas and greater care. Section 6 of this paper is devoted to the proof of these estimates. As in [13], here we make fundamental use of precise estimates for ultraspherical polynomials, which are uniform in suitable ranges of indices. These estimates, which are consequences of the asymptotic approximations of [11, 45–47], could be of independent interest, and their derivation is presented in an auxiliary paper [14].

In the context of subelliptic operators on compact manifolds, "weighted spectral cluster estimates" were first obtained in the seminal work by Cowling and Sikora [17] for a distinguished sub-Laplacian on SU(2), leading to a sharp multiplier theorem in that case; their technique was then applied to many different frameworks [2, 15, 16, 36]. However, the general theory developed in [17], based on spectral cluster estimates involving a single weight function, does not seem to be directly applicable to the spherical Grushin operator $\mathcal{L}_{d,k}$ (which, differently from the sub-Laplacian of [17], is not invariant under a transitive group of isometries of the underlying manifold). For this reason, here we take the opportunity to establish an "abstract" multiplier theorem, which applies to a rather general setting of self-adjoint operators on bounded metric measure spaces, satisfying the volume doubling property, and extends the analogous result in [17] to the framework of a family of scale-dependent weights.

It would be of great interest to establish whether Theorem 1.1 is sharp when $k > d/2$ or alternatively improve on it. The corresponding question for the Grushin operators $\mathcal{G}_{d,k}$ on $\mathbb{R}^d$ has been settled in [37]; based on that result, one may expect that Theorem 1.1 and Corollary 1.2 actually hold with D replaced by d. However, when the dimension k of the singular set is larger than the codimension, the approach developed in this paper, which is based on a "weighted Plancherel estimate with weights on the first layer," does not suffice to obtain such result and new methods (inspired, for instance, to those in [37] and involving the "second layer" as well) appear to be necessary.

We point out that the method of [37] for the Grushin operators $\mathcal{G}_{d,k}$ is based on a delicate inductive argument, which relates multiplication by a weight on the space side to differentiation on the spectral side; in turn, this inductive argument crucially hinges on special identities for Hermite functions. While setting up an analogous inductive scheme in the case of the spherical Grushin operators $\mathcal{L}_{d,k}$ may be possible, a number of nontrivial additional technical challenges would need to be tackled, due to the discreteness of the spectrum of $\mathcal{L}_{d,k}$ and to the different nature of the available identities for spherical harmonics compared to those for Hermite functions. We hope to be able to investigate these matters in the future.

Structure of the paper

In Section 2 we state our abstract multiplier theorem, of which Theorem 1.1 will be a direct consequence; in order not to burden the exposition, we postpone the proof of the abstract theorem to Section 7.

In Section 3 we introduce the spherical Laplacians and the Grushin operators on $\mathbb{S}^d$. A precise estimate for the sub-Riemannian distance ϱ associated with the Grushin operator $\mathcal{L}_{d,k}$ is also given. Moreover, we introduce the system of spherical coordinates on $\mathbb{S}^d$, which is key to our approach.

In Section 4 we recall the construction of a complete system of joint eigenfunctions of $\Delta_d, \dots, \Delta_k$ on $\mathbb{S}^d$, in terms of which we explicitly write down the spectral decomposition of the Grushin operator $\mathcal{L}_{d,k} = \Delta_d - \Delta_k$. We also prove some Riesz-type bounds for $\mathcal{L}_{d,k}$. Moreover we state the crucial "weighted spectral cluster estimates" for the Grushin operators $\mathcal{L}_{d,k}$; due to its technical nature, the proof of these estimates is deferred to Section 6.

In Section 5 we use the Riesz-type bounds and the weighted spectral cluster estimates to prove "weighted Plancherel-type estimates" for the Grushin operator $\mathcal{L}_{d,k}$. After this preparatory work, the proof of Theorem 1.1, which boils down to verifying the assumptions of the abstract theorem, concludes the section.

Notation

Throughout the paper, for any two nonnegative quantities X and Y, we use $X \lesssim Y$ or $Y \gtrsim X$ to denote the estimate $X \leq CY$ for a positive constant C. The symbol $X \simeq Y$ is shorthand for $X \lesssim Y$ and $Y \gtrsim X$. We use variants such as $\lesssim_{a,b}$ and $\simeq_{a,b}$ to indicate that the implicit constants may depend on the parameters a and b.

2 An Abstract Multiplier Theorem

We state an abstract multiplier theorem, which is a refinement of [17, Theorem 3.6] and [20, Theorem 3.2]. The proof of our main result, Theorem 1.1, for the operator $\mathcal{L}_{d,k}$ will follow from this result.

As in [17, 20], for all $q \in [2,\infty]$, $N \in \mathbb{N} \setminus \{0\}$ and $F : \mathbb{R} \to \mathbb{C}$ supported in $[0,1]$, we define the norm $\|F\|_{N,q}$ by

$$\|F\|_{N,q} = \begin{cases} \left(\frac{1}{N}\sum_{i=1}^{N} \sup_{\lambda\in[(i-1)/N,i/N]} |F(\lambda)|^q\right)^{1/q} & \text{if} \quad q<\infty, \\ \sup_{\lambda\in[0,1]} |F(\lambda)| & \text{if} \quad q=\infty. \end{cases} \tag{2.1}$$

Moreover, by $\mathcal{K}_T$ we denote the integral kernel of an operator T. Further, let $\eta \in C_c^\infty((0,\infty))$ be any nontrivial cutoff.

Theorem 2.1. Let (X,ϱ) be a bounded metric space, equipped with a regular Borel measure μ. Let $\mathfrak{L}$ be a nonnegative self-adjoint operator on $L^2(X)$. Let $q \in [2,\infty]$. Suppose that there exist a family of weight functions $\pi_r\colon X \times X \to [0,\infty)$, where $r \in (0,1]$, and a constant $\mathfrak{d} \in [1,\infty)$ such that the following conditions are satisfied:

(a) the doubling condition:

$$\mu(B(x,2r)) \lesssim \mu(B(x,r)) \qquad \forall x \in X \quad \forall r > 0;$$

(b) heat kernel bounds:

$$|\mathcal{K}_{\exp(-t\mathfrak{L})}(x,y)| \lesssim_N \mu(B(y,t^{1/2}))^{-1}(1+\varrho(x,y)/t^{1/2})^{-N}$$

for all $N \geq 0$, for all $t \in (0,\infty)$ and $x,y \in X$;

(c) the growth condition:

$$1 \lesssim \pi_r(x,y) \lesssim (1+\varrho(x,y)/r)^{M_0} \tag{2.2}$$

for some $M_0 \geq 0$, for all $r \in (0,1]$ and $x,y \in X$;

(d) the integrability condition

$$\int_X (1+\varrho(x,y)/r)^{-\beta}(\pi_r(x,y))^{-1}\,\mathrm{d}\mu(x) \lesssim_\beta \mu(B(y,r)) \tag{2.3}$$

for all $r \in (0,1]$, $\beta > \mathfrak{d}$ and for all $y \in X$;

(e) weighted Plancherel-type estimates:

$$\operatorname{ess\,sup}_{y\in X} \mu(B(y,1/N)) \int_X \pi_{1/N}(x,y)\,|\mathcal{K}_{F(\sqrt{\mathfrak{L}})}(x,y)|^2\,\mathrm{d}\mu(x) \lesssim \|F(N\cdot)\|_{N,q}^2 \tag{2.4}$$

for all $N \in \mathbb{N} \setminus \{0\}$ and for all bounded Borel functions $F : \mathbb{R} \to \mathbb{C}$ supported in $[0, N]$.

Finally, assume that $s > \eth/2$. Then the following hold.

(i) For continuous functions $F : \mathbb{R} \to \mathbb{C}$ supported in $[-1, 1]$,

$$\sup_{t>0} \|F(t\sqrt{\mathfrak{L}})\|_{L^1(X)\to L^1(X)} \lesssim_s \|F\|_{L^q_s}.$$

(ii) For all bounded Borel functions $F : \mathbb{R} \to \mathbb{C}$ continuous on $(0, \infty)$,

$$\|F(\sqrt{\mathfrak{L}})\|_{L^1(X)\to L^{1,\infty}(X)} \lesssim_s \sup_{t\geq 0} \|F(t\cdot)\,\eta\|_{L^q_s}. \tag{2.5}$$

Hence, whenever the right-hand side of (2.5) is finite, the operator $F(\sqrt{\mathfrak{L}})$ is of weak type $(1, 1)$ and bounded on $L^p(X)$ for all $p \in (1, \infty)$.

Since the subject is replete with technicalities, which could weigh on the discussion, we defer the proof of the abstract theorem to Section 7.

Let us just observe that Assumption (b) only requires a polynomial decay in space (of arbitrary large order) for the heat kernel; hence this assumption is weaker than the corresponding ones in [20], where Gaussian-type (i.e., superexponential) decay is required, and in [17], where finite propagation speed for the associated wave equation is required (which, under the "on-diagonal bound" implied by (2.4), is equivalent to "second order" Gaussian-type decay [51]) and matches instead the assumption in [31] (see also [36, Section 6]).

Another important feature of the above result, which is crucial for the applicability to the spherical Grushin operators $\mathcal{L}_{d,k}$ considered in this paper, is the use of a family of weight functions, where the weight π_r may depend on the scale r in a nontrivial way; this constitutes another important difference to [17], where the weights considered are effectively scalar multiples of a single weight function (compare Assumptions (d) and (e) above with [17, Assumptions 2.2 and 2.5]).

The attentive reader will have noticed that it is actually enough to verify Assumptions (c) and (d) for scales $r = 1/N$ for $N \in \mathbb{N} \setminus \{0\}$ (indeed, one can redefine π_r as $\pi_{1/\lfloor 1/r \rfloor}$ when $1/r \notin \mathbb{N}$); the slightly redundant form of the above assumptions is just due to notational convenience.

3 Spherical Laplacians and Grushin Operators

3.1 The Laplace–Beltrami operator on the unit sphere

For $d \in \mathbb{N}$, $d \geq 1$, let $\mathbb{S}^d$ denote the unit sphere in $\mathbb{R}^{1+d}$, as in (1.1). The Euclidean structure on $\mathbb{R}^{1+d}$ induces a natural, rotation-invariant Riemannian structure on $\mathbb{S}^d$. Let σ denote the corresponding Riemannian measure, and Δ_d the Laplace–Beltrami operator on the unit sphere $\mathbb{S}^d$ in $\mathbb{R}^{1+d}$.

It is possible (see, e.g., [27]) to give a more explicit expression for Δ_d as a sum of squares of vector fields, namely,

$$\Delta_d = -\sum_{0 \leq j < r \leq d} Z_{j,r}^2; \tag{3.1}$$

here the vector fields $Z_{j,s}$ are the restrictions to the sphere of the vector fields on $\mathbb{R}^{1+d}$ given by

$$Z_{j,r} = z_j \frac{\partial}{\partial z_r} - z_r \frac{\partial}{\partial z_j},$$

where $(z_0, \ldots, z_d)$ are the coordinates of $\mathbb{R}^{1+d}$.

Indeed the rotation group $\mathrm{SO}(1+d)$ acts naturally on $\mathbb{R}^{1+d}$ and $\mathbb{S}^d$; via this action, the vector fields $Z_{j,r}$ $(0 \leq j < r \leq d)$ correspond to the standard orthonormal basis of the Lie algebra of $\mathrm{SO}(1+d)$, and Δ_d corresponds to the Casimir operator. The commutation relations

$$[Z_{j,r}, Z_{j',r'}] = \delta_{r,j'} Z_{j,r'} + \delta_{j,r'} Z_{r,j'} - \delta_{j,j'} Z_{r,r'} - \delta_{r,r'} Z_{j,j'} \tag{3.2}$$

are easily checked and correspond to those of the Lie algebra of $\mathrm{SO}(1+d)$.

3.2 A family of commuting Laplacians and spherical Grushin operators

By (3.2), the operator Δ_d commutes with all the vector fields $Z_{j,r}$ (this corresponds to the fact that the Casimir operator is in the centre of the universal enveloping algebra of the Lie algebra of $\mathrm{SO}(1+d)$); in particular it commutes with each of the "partial Laplacians"

$$\Delta_r = -\sum_{0 \leq j < s \leq r} Z_{j,s}^2 \tag{3.3}$$

for $r = 1, \ldots, d$.

Assume that $d \geq 2$. We now observe that, for $r = 1, \ldots, d-1$, we can identify $\mathrm{SO}(1+r)$ with a subgroup of $\mathrm{SO}(1+d)$, by associating to each A in $\mathrm{SO}(1+r)$ the element

$$\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$$

of $\mathrm{SO}(1+d)$. Via this identification, the operator Δ_r corresponds to the Casimir operator of $\mathrm{SO}(1+r)$, and therefore it commutes with all the operators Δ_s for $s = 1, \ldots, r$.

In conclusion, the operators $\Delta_1, \ldots, \Delta_d$ commute pairwise and admit a joint spectral decomposition. In what follows we will be interested in the study of the Grushin-type operator

$$\mathcal{L}_{d,k} = \Delta_d - \Delta_k = -\sum_{r=k+1}^{d} \sum_{j=0}^{r-1} Z_{j,r}^2. \tag{3.4}$$

for $k = 1, \ldots, d-1$.

The operator $\mathcal{L}_{d,k}$ is not uniformly elliptic: indeed it degenerates on the k-submanifold $\mathbb{E}_{d,k} = \mathbb{S}^k \times \{0\}$ of $\mathbb{S}^d$. More precisely, if

$$\mathcal{Z}_{d,k} = \{Z_{j,r} \,:\, k+1 \leq r \leq d,\, 0 \leq j < r\}$$

is the family of vector fields appearing in the sum (3.4), then it is easily checked that, for all $z \in \mathbb{S}^d$,

$$\mathsf{H}_z^{d,k} := \operatorname{span}\{X|_z \,:\, X \in \mathcal{Z}_{d,k}\} = \begin{cases} T_z\mathbb{S}^d & \text{if} \quad z \notin \mathbb{E}_{d,k}, \\ (T_z\mathbb{E}_{d,k})^{\perp} & \text{if} \quad z \in \mathbb{E}_{d,k}. \end{cases} \tag{3.5}$$

On the other hand, the commutation relations (3.2) give that

$$[Z_{j,d}, Z_{j',d}] = -Z_{j,j'}$$

for all $j, j' = 0, \ldots, d-1$; in particular the vector fields in $\mathcal{Z}_{d,k}$, together with their Lie brackets, span the tangent space of $\mathbb{S}^d$ at each point. In other words, the family of vector fields $\mathcal{Z}_{d,k}$ satisfies Hörmander's condition and (together with the Riemannian measure σ) determines a (non-equiregular) 2-step sub-Riemannian structure on $\mathbb{S}^d$ with the horizontal distribution $\mathsf{H}^{d,k}$ described in (3.5). The corresponding sub-Riemannian norm on the fibres of $\mathsf{H}^{d,k}$ is given, for all $p \in \mathbb{S}^d$ and $v \in \mathsf{H}^{d,k}_p$, by

$$|v|_{\mathcal{L}_{d,k}} = \inf \left\{ \sqrt{\sum_{X \in \mathcal{Z}_{d,k}} a_X^2} \,:\, (a_X)_X \in \mathbb{R}^{\mathcal{Z}_{d,k}},\ v = \sum_{X \in \mathcal{Z}_{d,k}} a_X X|_p \right\}. \tag{3.6}$$

For more details on sub-Riemannian geometry we refer the reader to [1, 8, 12, 43].

We remark that from (3.4) it follows that $\mathcal{L}_{d,k}$ is positive semidefinite; moreover, from the fact that $\mathcal{Z}_{d,k}$ satisfies Hörmander's condition, it follows that $\mathcal{L}_{d,k}$ is hypoelliptic and moreover $\mathcal{L}_{d,k} f = 0$ if and only if f is constant, whence

$$\ker \mathcal{L}_{d,k} = \ker \Delta_d. \tag{3.7}$$

3.3 Spherical coordinates

In order to study the operator $\mathcal{L}_{d,k}$, it is useful to introduce a system of "spherical coordinates" on $\mathbb{S}^d$ that will provide a particularly revealing expression for $\mathcal{L}_{d,k}$ in a neighbourhood of the singular set $\mathbb{E}_{d,k}$.

For all $\omega \in \mathbb{S}^{d-1}$ and $\psi \in [-\pi/2, \pi/2]$, let us define the point $\lfloor \omega, \psi \rceil \in \mathbb{S}^d$ by

$$\lfloor \omega, \psi \rceil = ((\cos\psi)\omega, \sin\psi). \tag{3.8}$$

Away from $\psi = \pm\pi/2$, the map $(\omega, \psi) \mapsto \lfloor \omega, \psi \rceil$ is a diffeomorphism onto its image, which is the sphere without the two poles; so (3.8) can be thought of as a "system of coordinates" on $\mathbb{S}^d$, up to null sets. In these coordinates, the spherical measure σ on $\mathbb{S}^d$ is given by

$$d\sigma(\lfloor \omega, \psi \rceil) = \cos^{d-1}\psi\, d\psi\, d\sigma_{d-1}(\omega),$$

where σ_{d-1} is the spherical measure on $\mathbb{S}^{d-1}$. Moreover, the Laplace–Beltrami operator may be written in these coordinates as

$$\Delta_d = -\frac{1}{\cos^{d-1}\psi} \frac{\partial}{\partial\psi} \cos^{d-1}\psi \frac{\partial}{\partial\psi} + \frac{1}{\cos^2\psi} \Delta_{d-1}, \tag{3.9}$$

where Δ_{d-1}, given by (3.3), corresponds to the Laplace–Beltrami operator on $\mathbb{S}^{d-1}$ (see, e.g., [56, §IX.5]).

We now iterate the previous construction. Let $k \in \mathbb{N}$ such that $1 \le k < d$ be fixed. Starting from (3.8), we can inductively define the point

$$\lfloor \omega, \psi \rceil = \lfloor \dots \lfloor \lfloor \omega, \psi_{k+1} \rceil, \psi_{k+2} \rceil \dots, \psi_d \rceil \tag{3.10}$$

of $\mathbb{S}^d$ for all $\psi = (\psi_{k+1}, \dots, \psi_d) \in [-\pi/2, \pi/2]^{d-k}$ and $\omega \in \mathbb{S}^k$; if we restrict ψ to $(-\pi/2, \pi/2)^{d-k}$, then (3.10) defines a "system of coordinates" for an open subset $\Omega_{d,k}$ of $\mathbb{S}^d$ of full measure, namely,

$$\Omega_{d,k} = \{\lfloor \omega, \psi \rceil : \omega \in \mathbb{S}^k,\ \psi \in (-\pi/2, \pi/2)^{d-k}\}.$$

In these coordinates, the spherical measure σ on $\mathbb{S}^d$ is given by

$$d\sigma(\lfloor \omega, \psi \rceil) = \cos^{d-1} \psi_d \cdots \cos^k \psi_{k+1} \, d\psi_d \cdots d\psi_{k+1} \, d\sigma_k(\omega), \tag{3.11}$$

where σ_k is the spherical measure on $\mathbb{S}^k$. Moreover, starting from (3.9), we get inductively that

$$\Delta_d = -\sum_{r=k+1}^{d} \frac{1}{\cos^2 \psi_{r+1} \cdots \cos^2 \psi_d} \frac{1}{\cos^{r-1} \psi_r} \frac{\partial}{\partial \psi_r} \cos^{r-1} \psi_r \frac{\partial}{\partial \psi_r} + \frac{1}{\cos^2 \psi_{k+1} \cdots \cos^2 \psi_d} \Delta_k,$$

where again Δ_k is the operator given by (3.3).

In particular, the Grushin operator $\mathcal{L}_{d,k} = \Delta_d - \Delta_k$ on $\mathbb{S}^d$ may be written in these coordinates as in (1.3), where the vector fields Y_r and the function $\mathfrak{V} : (-\pi/2, \pi/2)^{d-k} \to \mathbb{R}$ are defined by (1.4) and (1.5), respectively. Note that $\mathfrak{V}(\psi)$ vanishes only for $\psi = 0$, corresponding to the singular set $\mathbb{E}_{d,k}$. We also remark that

$$\frac{1}{\cos \psi_{r+1} \cdots \cos \psi_d} \simeq 1, \qquad \mathfrak{V}(\psi) \simeq |\psi|^2 \tag{3.12}$$

for $r = k+1, \dots, d$, uniformly for $|\psi| \le \varepsilon$, for any given $\varepsilon \in (0, \pi/2)$; here

$$|\psi| = |\psi|_\infty = \max_{j \in \{k+1, \dots, d\}} |\psi_j|. \tag{3.13}$$

The formula (1.3) for the sub-Laplacian corresponds to a somewhat more explicit expression for the sub-Riemannian norm (3.6) on the fibres of the horizontal distribution, which is better written by identifying, via the "coordinates" (3.10), the tangent space $T_{\lfloor\omega,\psi\rceil}\mathbb{S}^d$ with $T_\omega\mathbb{S}^k \times \mathbb{R}^{d-k}$ for all $\lfloor\omega,\psi\rceil \in \Omega_{d,k}$. Under this identification,

$$\mathsf{H}^{d,k}_{\lfloor\omega,\psi\rceil} = \begin{cases} T_\omega\mathbb{S}^k \times \mathbb{R}^{d-k} & \text{if} \quad \psi \neq 0, \\ \{0\} \times \mathbb{R}^{d-k} & \text{if} \quad \psi = 0 \end{cases} \tag{3.14}$$

and, for all $(v,w) \in \mathsf{H}^{d,k}_{\lfloor\omega,\psi\rceil}$, its sub-Riemannian norm satisfies

$$|(v,w)|^2_{\mathcal{L}_{d,k}} = \begin{cases} \sum_{r=k+1}^d (\cos\psi_{r+1}\cdots\cos\psi_d)^2 |w_r|_2^2 + \mathfrak{V}(\psi)^{-1}|v|_2^2 & \text{if} \quad \psi \neq 0, \\ \sum_{r=k+1}^d |w_r|_2^2 & \text{if} \quad \psi = 0, \end{cases} \tag{3.15}$$

where $w = (w_{k+1}, \dots, w_d) \in \mathbb{R}^{d-k}$, $|w|_2$ is its Euclidean norm, and $|v|_2$ is the Riemannian norm of $v \in T_\omega\mathbb{S}^k$.

3.4 The sub-Riemannian distance

Thanks to (3.15), we can obtain a precise estimate for the sub-Riemannian distance ϱ associated with the Grushin operator $\mathcal{L}_{d,k}$. This is the analogue of [49, Proposition 5.1] that treats the case of "flat" Grushin operators on $\mathbb{R}^n$, and [13, Proposition 2.1], that treats the case of $\mathcal{L}_{2,1}$ on $\mathbb{S}^2$.

In the statement below we represent the points of the sphere in the form $\lfloor\omega,\psi\rceil$ for $\omega \in \mathbb{S}_k$, $\psi \in [-\pi/2,\pi/2]^{d-k}$, as in (3.10); moreover, $|\psi|$ has the same meaning as in (3.13). We also denote by $\varrho_{R,\mathbb{S}^k}$ and $\varrho_{R,\mathbb{S}^d}$ the Riemannian distances on the spheres $\mathbb{S}^k$ and $\mathbb{S}^d$.

Proposition 3.1. Let $\varepsilon \in (0,\pi/2)$. The sub-Riemannian distance ϱ on $\mathbb{S}^d$ associated with $\mathcal{L}_{d,k}$ satisfies

$$\varrho(\lfloor\omega,\psi\rceil, \lfloor\omega',\psi'\rceil) \simeq |\psi-\psi'| + \min\left\{\varrho_{R,\mathbb{S}^k}(\omega,\omega')^{1/2}, \frac{\varrho_{R,\mathbb{S}^k}(\omega,\omega')}{\max\{|\psi|,|\psi'|\}}\right\}, \tag{3.16}$$

if $\max\{|\psi|,|\psi'|\} \leq \varepsilon$; if instead $\max\{|\psi|,|\psi'|\} \geq \varepsilon$, then

$$\varrho(\lfloor\omega,\psi\rceil, \lfloor\omega',\psi'\rceil) \simeq \varrho_{R,\mathbb{S}^d}(\lfloor\omega,\psi\rceil, \lfloor\omega',\psi'\rceil). \tag{3.17}$$

Consequently, the σ-measure $V(\lfloor\omega,\psi\rceil,r)$ of the ϱ-ball centred at $\lfloor\omega,\psi\rceil$ with radius $r\geq 0$ satisfies

$$V(\lfloor\omega,\psi\rceil,r)\simeq\min\{1,r^d\max\{r,|\psi|\}^k\}. \tag{3.18}$$

The implicit constants may depend on ε.

Proof. Note that the sub-Riemannian distance ϱ and the Riemannian distance $\varrho_{R,\mathbb{S}^d}$ are locally equivalent far from the singular set $\mathbb{E}_{d,k}$: since $\mathsf{H}^{d,k}_p=T_pM$ for all $p\in\mathbb{S}^d\setminus\mathbb{E}_{d,k}$ (see (3.5)), and the Riemannian and sub-Riemannian inner products on T_pM depend continuously on p, it is enough to apply [13, Lemma 2.3] by choosing as M and N the Riemannian and sub-Riemannian $\mathbb{S}^d$, respectively, and as F the identity map restricted to any open subset U of $\mathbb{S}$ with compact closure not intersecting $\mathbb{E}$. Then [13, Lemma 2.2], applied with $K=\{(\lfloor\omega,\psi\rceil,\lfloor\omega',\psi'\rceil)\in\mathbb{S}^d\times\mathbb{S}^d\,:\,\max\{|\psi|,|\psi'|\}\geq\varepsilon\}$, yields (3.17).

Note now that the expression in the right-hand side of (3.16) defines a continuous function $\Phi:\Omega_{d,k}\times\Omega_{d,k}\to[0,\infty)$, which is nondegenerate in the sense of [13, Lemma 2.2]. Hence, in order to prove the equivalence (3.16), it is enough to show that Φ and ϱ are locally equivalent at each point $p_0\in\Omega_{d,k}$, and then apply [13, Lemma 2.2] with $K=\{(\lfloor\omega,\psi\rceil,\lfloor\omega',\psi'\rceil)\in\mathbb{S}^d\times\mathbb{S}^d\,:\,\max\{|\psi|,|\psi'|\}\leq\varepsilon\}$.

Consider now the Grushin operator $\mathcal{G}=\mathcal{G}_{d,k}$ on $\mathbb{R}^{d-k}_x\times\mathbb{R}^k_y$ defined in (1.6). The associated horizontal distribution $\mathsf{H}^{\mathcal{G}}$ and sub-Riemannian metric are given by

$$\mathsf{H}^{\mathcal{G}}_{(x,y)}=\begin{cases}\mathbb{R}^{d-k}\times\mathbb{R}^k & \text{if}\quad x\neq 0,\\ \mathbb{R}^{d-k}\times\{0\} & \text{if}\quad x=0,\end{cases}\qquad |(w,v)|^2_{\mathcal{G}}=\begin{cases}|w|_2^2+|x|_2^{-2}|v|_2^2 & \text{if}\quad x\neq 0,\\ |w|_2^2 & \text{if}\quad x=0,\end{cases} \tag{3.19}$$

for all $(x,y)\in\mathbb{R}^{d-k}\times\mathbb{R}^k$ and $(w,v)\in\mathsf{H}^{\mathcal{G}}_{(x,y)}$. Moreover, according to [49, Proposition 5.1], the associated sub-Riemannian distance $\varrho_{\mathcal{G}}$ satisfies

$$\varrho_{\mathcal{G}}((x,y),(x',y'))\simeq|x-x'|+\min\left\{|y-y'|^{1/2},\frac{|y-y'|}{\max\{|x|,|x'|\}}\right\}. \tag{3.20}$$

Let $p_0=\lfloor\omega_0,\psi_0\rceil\in\Omega_{d,k}$. Choose coordinates for $\mathbb{S}^k$ centred at ω_0, thus determining a diffeomorphism f from an open neighbourhood A of 0 in $\mathbb{R}^k$ to a neighbourhood $f(A)$ of p in $\mathbb{S}^k$. By the equivalence of norms, up to shrinking A, we may assume that

$$|df_y(w)|_2\simeq|w|_2 \tag{3.21}$$

for all $y \in A$, $w \in \mathbb{R}^k \cong T_y\mathbb{R}^k$, where the norms in (3.21) are those determined by the Riemannian structures of $\mathbb{S}^k$ and $\mathbb{R}^k$; similarly, we may also assume that

$$\varrho_{R,\mathbb{S}^k}(f(y), f(y')) \simeq |y - y'| \tag{3.22}$$

for all $y, y' \in A$. Let now $U = B \times A$, where B is a neighbourhood of ψ_0 with compact closure in $(-\pi/2, \pi/2)^{d-k}$, and define $F : U \to \mathbb{S}^d$ by $F(x, y) = \lfloor f(y), x\rceil$. A comparison of (3.14) and (3.15) with (3.19), taking (3.12) and (3.21) into account, immediately shows that [13, Lemma 2.3] can be applied to the map F and the sub-Riemannian structures associated with $\mathcal{G}$ and $\mathcal{L}_{d,k}$; consequently, up to shrinking U, we obtain that

$$\varrho(F(p), F(p')) \simeq \varrho_{\mathcal{G}}(p, p') \simeq \Phi(F(p), F(p'))$$

for all $p, p' \in U$, where the latter equivalence readily follows from (3.20) and (3.22).

Finally, the estimate (3.18) for the volume of balls follows from (3.11), (3.16), and (3.17) by considering separately the cases $|\psi|$ small and $|\psi|$ large. ■

4 A Complete System of Joint Eigenfunctions

Let $d, k \in \mathbb{N}$ with $1 \le k < d$. In this section we briefly recall the construction of a complete system of joint eigenfunctions of $\Delta_d, \dots, \Delta_k$ on $\mathbb{S}^d$. This will give in particular the spectral decomposition of the Grushin operator $\mathcal{L}_{d,k} = \Delta_d - \Delta_k$.

This construction is classical and can be found in several places in the literature (see, e.g., [56, Ch. IX] or [22, Ch. XI]), where explicit formulas for spherical harmonics on spheres of arbitrary dimension are given, in terms of ultraspherical (Gegenbauer) polynomials. The discussion below is essentially meant to fix the notation that will be used later.

By the symbol $P_j^{(\alpha,\beta)}$ we shall denote the Jacobi polynomial of degree $j \in \mathbb{N}$ and indices $\alpha, \beta > -1$, defined by means of Rodrigues' formula:

$$P_j^{(\alpha,\beta)}(x) = \frac{(-1)^j}{2^j\, j!}(1-x)^{-\alpha}(1+x)^{-\beta}\left(\frac{d}{dx}\right)^j\left((1-x)^{\alpha+j}(1+x)^{\beta+j}\right) \tag{4.1}$$

for $x \in (-1, 1)$. We recall, in particular, the symmetry relation

$$P_j^{(\alpha,\beta)}(-x) = (-1)^j P_j^{(\beta,\alpha)}(x), \tag{4.2}$$

for $j \in \mathbb{N}$, $\alpha, \beta > -1$ and $x \in \mathbb{R}$. Ultraspherical polynomials correspond to Jacobi polynomials with $\alpha = \beta$ [55,(4.7.1)].

4.1 Spectral theory of the Laplace–Beltrami operator

We first recall some well-known facts about the spectral theory of Δ_d (see, e.g., [54, Ch. 4] or [4, Ch. 5]). The operator Δ_d is essentially self-adjoint on $L^2(\mathbb{S}^d)$ and has discrete spectrum: its eigenvalues are given by

$$\lambda_\ell^d := (\ell + (d-1)/2)(\ell - (d-1)/2), \tag{4.3}$$

where $\ell \in \mathbb{N}_d$, and

$$\mathbb{N}_d = \mathbb{N} + (d-1)/2. \tag{4.4}$$

The corresponding eigenspaces, denoted by $\mathcal{H}^\ell(\mathbb{S}^d)$, consist of all spherical harmonics of degree $\ell' = \ell - (d-1)/2$, that is, of all restrictions to $\mathbb{S}^d$ of homogeneous harmonic polynomials on $\mathbb{R}^{1+d}$ of degree ℓ'; they are finite-dimensional spaces of dimension

$$\alpha_\ell(\mathbb{S}^d) = \binom{\ell'+d}{\ell'} - \binom{\ell'+d-2}{\ell'-2} = \frac{2\ell'+d-1}{d-1}\binom{\ell'+d-2}{d-2} \tag{4.5}$$

for $\ell \in \mathbb{N}_d$ (the last identity only makes sense when $d > 1$), and in particular

$$\alpha_\ell(\mathbb{S}^d) \simeq_d \ell^{d-1} \tag{4.6}$$

(this estimate is also valid when $d = 1$, provided we stipulate that $0^0 = 1$).

Since Δ_d is self-adjoint, its eigenspaces are mutually orthogonal, that is,

$$\mathcal{H}^\ell(\mathbb{S}^d) \perp \mathcal{H}^{\ell'}(\mathbb{S}^d)$$

for $\ell, \ell' \in \mathbb{N}_d$, $\ell \neq \ell'$. Moreover, if E_ℓ^d is an orthonormal basis of $\mathcal{H}^\ell(\mathbb{S}^d)$, then

$$\sum_{Z \in E_\ell^d} |Z(z)|^2 = \sigma(\mathbb{S}^d)^{-1}\,\alpha_\ell(\mathbb{S}^d) \tag{4.7}$$

for all $z \in \mathbb{S}^d$ [54, Ch. 4, Corollary 2.9].

4.2 Joint eigenfunctions of Δ_d and Δ_{d-1}

We start the construction of joint eigenfunctions with the case $k = d-1$, and look for eigenfunctions of Δ_d that are simultaneously eigenfunctions of Δ_{d-1}.

Following, for example, [56, §IX.5], one can use the expression (3.9) for Δ_d to solve the eigenfunction equation for Δ_d via separation of variables. More precisely, we look for functions W on $\mathbb{S}^d$ of the form $X \otimes Z$, that is,

$$W(\lfloor \omega, \psi \rceil) = X(\psi) Z(\omega)$$

in the coordinates (3.8), such that

$$\Delta_d W = \lambda_\ell^d W \qquad \text{and} \qquad \Delta_{d-1} Z = \lambda_m^k Z,$$

for some $\ell \in \mathbb{N}_d$, $m \in \mathbb{N}_{d-1}$. This leads to a differential equation for X that is solved in terms of ultraspherical polynomials. Namely, if $Z \in \mathcal{H}^m(\mathbb{S}^{d-1})$ is nonzero and $\ell \geq m$, then $W = X \otimes Z$ is in $\mathcal{H}^\ell(\mathbb{S}^d)$ if and only if X is a multiple of

$$X^d_{\ell,m}(\psi) = c_{\ell m} (\cos \psi)^{m-(d-2)/2} P^{(m,m)}_{\ell-m-1/2}(\sin \psi). \tag{4.8}$$

Here the normalization constant $c_{\ell m}$ is chosen so that

$$\int_{-\pi/2}^{\pi/2} |X^d_{\ell,m}(\psi)|^2 \cos^{d-1} \psi \, \mathrm{d}\psi = 1, \tag{4.9}$$

that is, by means of [55, (4.3.3)],

$$c_{\ell m} = \frac{\left[\ell\, \Gamma(\ell - m + 1/2)\, \Gamma(\ell + m + 1/2)\right]^{1/2}}{2^m\, \Gamma(\ell + 1/2)}. \tag{4.10}$$

We remark that, if $\ell \in \mathbb{N}_d$ and $m \in \mathbb{N}_{d-1}$, then 2ℓ and $2m$ have different parities (see (4.4)), so

$$\ell \geq m \iff \ell \geq m + 1/2; \tag{4.11}$$

this equivalence should be kept in mind in what follows.

Define

$$I_d = \{(\ell, m) \,:\, \ell \in \mathbb{N}_d,\ m \in \mathbb{N}_{d-1},\ \ell \geq m\}. \tag{4.12}$$

Then, for all $(\ell, m) \in I_d$, we obtain an injective linear map

$$\mathcal{H}^m(\mathbb{S}^{d-1}) \ni Z \mapsto X^d_{\ell,m} \otimes Z \in \mathcal{H}^\ell(\mathbb{S}^d),$$

which is an isometry with respect to the Hilbert space structures of $L^2(\mathbb{S}^{d-1})$ and $L^2(\mathbb{S}^d)$, and a decomposition

$$\mathcal{H}^\ell(\mathbb{S}^d) = \bigoplus_{\substack{m\in\mathbb{N}_{d-1} \\ m\leq\ell}} X^d_{\ell,m} \otimes \mathcal{H}^m(\mathbb{S}^{d-1}) \tag{4.13}$$

(cf. [56, page 466, eq. (1)]). The summands in the right-hand side of (4.13) are joint eigenspaces of Δ_d and Δ_{d-1} of eigenvalues λ^d_ℓ and λ^k_m respectively; hence they are pairwise orthogonal in $L^2(\mathbb{S}^d)$.

4.3 Joint eigenfunctions of $\Delta_d, \dots, \Delta_k$

We go back to the general case $1 \leq k < d$ and we look for a complete system of joint eigenfunctions of $\Delta_d, \dots, \Delta_k$.

It is natural to introduce the index set

$$J^{(k)}_d = \{(\ell_d, \ell_{d-1}, \dots, \ell_k) \in \mathbb{N}_d \times \mathbb{N}_{d-1} \times \cdots \times \mathbb{N}_k : \ell_d \geq \ell_{d-1} \geq \cdots \geq \ell_k\}.$$

For all $(\ell_d, \dots, \ell_k) \in J^{(k)}_d$, let us define $X^d_{\ell_d,\dots,\ell_k} : [-\pi/2, \pi/2]^{d-k} \to \mathbb{R}$ by

$$X^d_{\ell_d,\dots,\ell_k}(\psi) = X^d_{\ell_d,\ell_{d-1}}(\psi_d) \cdots X^{k+1}_{\ell_{k+1},\ell_k}(\psi_{k+1}),$$

where $\psi = (\psi_{k+1}, \dots, \psi_d)$, and the functions $X^r_{\ell_r,\ell_{r-1}}$ are defined in (4.8). Then, for all $(\ell_d, \dots, \ell_k) \in J^{(k)}_d$ and $Z \in \mathcal{H}^{\ell_k}(\mathbb{S}^k)$, the function $X^d_{\ell_d,\dots,\ell_k} \otimes Z$, defined, in the coordinates (3.10) on $\mathbb{S}^d$, by

$$X^d_{\ell_d,\dots,\ell_k} \otimes Z : \lfloor \omega, \psi \rceil \mapsto X^d_{\ell_d,\dots,\ell_k}(\psi) Z(\omega), \tag{4.14}$$

is an eigenfunction of $\Delta_d, \dots, \Delta_k$ of respective eigenvalues $\lambda^d_{\ell_d}, \dots, \lambda^k_{\ell_k}$. More precisely, iterating (4.13), we obtain the orthogonal direct sum decomposition

$$\mathcal{H}^\ell(\mathbb{S}^d) = \bigoplus_{\substack{(\ell_d,\dots,\ell_k)\in J^{(k)}_d \\ \ell_d=\ell}} X^d_{\ell_d,\dots,\ell_k} \otimes \mathcal{H}^{\ell_k}(\mathbb{S}^k).$$

As a consequence, each function $f \in L^2(\mathbb{S}^d)$ may be written as

$$f = \sum_{(\ell_d,\dots,\ell_k)\in J_d^{(k)}} \sum_{Z\in E_{\ell_k}^k} c_{\ell_d,\dots,\ell_k,Z} X_{\ell_d,\dots,\ell_k}^d \otimes Z, \tag{4.15}$$

where $E_{\ell_k}^k$ is an orthonormal basis of $\mathcal{H}^{\ell_k}(\mathbb{S}^k)$ and

$$c_{\ell_d,\dots,\ell_k,Z} = \langle f, X_{\ell_d,\dots,\ell_k}^d \otimes Z\rangle.$$

In particular, for all $(\ell_d,\dots,\ell_k) \in J_d^{(k)}$, the orthogonal projection $\pi_{\ell_d,\dots,\ell_k}^d$ of $L^2(\mathbb{S}^d)$ onto the joint eigenspace of $\Delta_d,\dots,\Delta_k$ of eigenvalues $\lambda_{\ell_d}^d,\dots,\lambda_{\ell_k}^k$ is given by

$$\pi_{\ell_d,\dots,\ell_k}^d : f \mapsto \sum_{Z\in E_{\ell_k}^k} \langle f, X_{\ell_d,\dots,\ell_k}^d \otimes Z\rangle X_{\ell_d,\dots,\ell_k}^d \otimes Z. \tag{4.16}$$

Consequently, the integral kernel $K_{\ell_d,\dots,\ell_k}^d$ of $\pi_{\ell_d,\dots,\ell_k}^d$ is given by

$$K_{\ell_d,\dots,\ell_k}^d(\lfloor \omega,\psi \rceil, \lfloor \omega',\psi' \rceil) = K_{\ell_k}^k(\omega,\omega') X_{\ell_d,\dots,\ell_k}^d(\psi)\, X_{\ell_d,\dots,\ell_k}^d(\psi'), \tag{4.17}$$

where

$$K_{\ell_k}^k(\omega,\omega') := \sum_{Z\in E_{\ell_k}^k} Z(\omega)\overline{Z(\omega')}$$

is the integral kernel of the orthogonal projection of $L^2(\mathbb{S}^k)$ onto $\mathcal{H}^{\ell_k}(\mathbb{S}^k)$.

For all bounded Borel functions $F : \mathbb{R}^{d-k+1} \to \mathbb{C}$, we can express the operator $F(\Delta_d,\dots,\Delta_k)$ in the joint functional calculus of $\Delta_d,\dots,\Delta_k$ on $L^2(\mathbb{S}^d)$ as

$$F(\Delta_d,\dots,\Delta_k) = \sum_{(\ell_d,\dots,\ell_k)\in J_d^{(k)}} F(\lambda_{\ell_d}^d,\dots,\lambda_{\ell_k}^k)\, \pi_{\ell_d,\dots,\ell_k}^d. \tag{4.18}$$

Correspondingly, the integral kernel $\mathcal{K}_{F(\Delta_d,\dots,\Delta_k)}$ of the operator $F(\Delta_d,\dots,\Delta_k)$ is given by

$$\mathcal{K}_{F(\Delta_d,\dots,\Delta_k)} = \sum_{(\ell_d,\dots,\ell_k)\in J_d^{(k)}} F(\lambda_{\ell_d}^d,\dots,\lambda_{\ell_k}^k)\, K_{\ell_d,\dots,\ell_k}^d, \tag{4.19}$$

and in particular, by (4.7) and (4.9), for all $z' = \lfloor \omega', \psi' \rceil \in \mathbb{S}^d$,

$$\|\mathcal{K}_{F(\Delta_d,\dots,\Delta_k)}(\cdot, z')\|^2_{L^2(\mathbb{S}^d)} = \frac{1}{\sigma_k(\mathbb{S}^k)} \sum_{(\ell_d,\dots,\ell_k)\in J_d^{(k)}} \alpha_{\ell_k}(\mathbb{S}^k)|F(\lambda^d_{\ell_d},\dots,\lambda^k_{\ell_k})|^2 |X^d_{\ell_d,\dots,\ell_k}(\psi')|^2, \quad (4.20)$$

where σ_k is the Lebesgue measure on $\mathbb{S}^k$, and $\alpha_{\ell_k}(\mathbb{S}^k)$ denotes the dimension of $\mathcal{H}^{\ell_k}(\mathbb{S}^k)$ as in (4.5).

We note that the operators of the form (4.18) include those in the functional calculus of the Grushin operator $\mathcal{L}_{d,k} = \Delta_d - \Delta_k$; namely,

$$F(\mathcal{L}_{d,k}) = \sum_{(\ell,m)\in I_d^{(k)}} F(\lambda^{d,k}_{\ell,m}) \sum_{\substack{(\ell_d,\dots,\ell_k)\in J_d^{(k)} \\ \ell_d=\ell,\ \ell_k=m}} \pi^d_{\ell_d,\dots,\ell_k},$$

where

$$\begin{aligned} I_d^{(k)} &= \{(\ell,m) \,:\, \ell \in \mathbb{N}_d,\ m \in \mathbb{N}_k,\ \ell \geq m + (d-k)/2\} \\ &= \{(\ell,m) \,:\, \exists(\ell_d,\dots,\ell_k) \in J_d^{(k)} \,:\, \ell_d = \ell,\ \ell_k = m\} \end{aligned}$$

(see (4.11) for the latter equality) and, for all $(\ell,m) \in I_d^{(k)}$,

$$\lambda^{d,k}_{\ell,m} = \lambda^d_\ell - \lambda^k_m. \quad (4.21)$$

In light of (4.21), from the positive semidefiniteness of $\mathcal{L}_{d,k}$ and (3.7) it follows that, for all $(\ell,m) \in I_d^{(k)}$,

$$\lambda^d_\ell \geq \lambda^k_m \quad (4.22)$$

and

$$\lambda^d_\ell = \lambda^k_m \quad \text{if and only if} \quad \lambda^d_\ell = 0. \quad (4.23)$$

4.4 Riesz-type bounds

In this section we prove certain weighted L^2 bounds involving the joint functional calculus of $\Delta_d,\dots,\Delta_k$, which, in combination with the weighted spectral cluster estimates in Section 4.5 below, play a fundamental role in satisfying the assumptions on the weight in the abstract theorem and proving our main result.

A somewhat similar estimate was obtained in [13, Lemma 2.5] in the case $d = 2$ and $k = 1$. Differently from [13], the estimate in Proposition 4.1 below is proved for arbitrarily large powers of the weight; this prevents us from using the elementary "quadratic form majorization" method exploited in the previous paper and requires a more careful analysis, based on the explicit eigenfunction expansion developed in the previous sections.

For later use, it is convenient to make a change of variable in the functions $X^d_{\ell,m}$ defined in (4.8): namely, we introduce the functions $\widetilde{X}^d_{\ell,m} : [-1,1] \to \mathbb{R}$ defined by

$$\widetilde{X}^d_{\ell,m}(x) = c_{\ell m}(1-x^2)^{m/2-(d-2)/4} P^{(m,m)}_{\ell-m-1/2}(x), \tag{4.24}$$

where $(\ell, m) \in I_d$ and $c_{\ell m}$ is given by (4.10).

Let $t_{d,d} : \mathbb{S}^d \to \mathbb{R}$ be defined, for all $(z_0, \dots, z_d) \in \mathbb{S}^d$, by

$$t_{d,d}(z_0, \dots, z_d) = z_d.$$

Moreover, for $k = 1, \dots, d-1$, let $t_{d,k} : \mathbb{S}^d \to \mathbb{R}$ be defined by

$$t_{d,k}(\lfloor \omega, \psi \rceil) = \begin{cases} t_{k,k}(\omega) & \text{if} \quad |\psi| < \pi/2, \\ 0 & \text{otherwise}, \end{cases} \tag{4.25}$$

for all $(\omega, \psi) \in \mathbb{S}^k \times [-\pi/2, \pi/2]^{d-k}$ (here $|\psi| = |\psi|_\infty$ as in (3.13)). Finally, we set, for $k = 1, \dots, d-1$,

$$\tau_{d,k} = \sum_{r=k+1}^{d} |t_{d,r}|. \tag{4.26}$$

Let us fix $k \in \{1, \dots, d-1\}$. From the above definitions it is readily seen that, for all $(\omega, \psi) \in \mathbb{S}^k \times (-\pi/2, \pi/2)^{d-k}$,

$$t_{d,r}(\lfloor \omega, \psi \rceil) = \sin \psi_r \qquad \text{for} \quad r = k+1, \dots, d$$

and consequently

$$\tau_{d,k}(\lfloor \omega, \psi \rceil) \simeq |\psi|. \tag{4.27}$$

Proposition 4.1. Let $1 \leq k < d$. For all $N \in [0,\infty)$ and all $f \in L^2(\mathbb{S}^d)$ such that $f \perp \ker \Delta_{k+1}$,

$$\|\tau_{d,k}^N f\|_{L^2(\mathbb{S}^d)} \lesssim_N \|(\mathcal{L}_{d,k}/\Delta_{k+1})^{N/2} f\|_{L^2(\mathbb{S}^d)}. \tag{4.28}$$

Proof. By interpolation, it is enough to prove the estimate in the case $N \in \mathbb{N}$.

Let us first prove the inequality in the case $k = d-1$. From (4.24) and known identities for Jacobi polynomials [22, §10.9, eqs. (4) and (13), pages 174–175], one easily deduces that

$$x\widetilde{X}^d_{\ell,m}(x) = \alpha_{\ell,m}\widetilde{X}^d_{\ell+1,m}(x) + \alpha_{\ell-1,m}\widetilde{X}^d_{\ell-1,m}(x) \tag{4.29}$$

for all $(\ell,m) \in I_d$, where

$$\alpha_{\ell,m} = \sqrt{\frac{(\ell-m+1/2)(\ell+m+1/2)}{4\ell(\ell+1)}}.$$

We remark that, in the case $(\ell-1,m) \notin I_d$, the condition $(\ell,m) \in I_d$ forces $\ell-m-1/2 = 0$ and $\alpha_{\ell-1,m} = 0$; in other words, the term with $\widetilde{X}_{\ell-1,m}$ in the right-hand side of (4.29) appears only when $(\ell-1,m) \in I_d$ too. On the other hand, if $(\ell,m) \in I_d$, then $\alpha_{\ell,m} \simeq \sqrt{(\ell-m)(\ell+m)}/\ell$. Consequently, by iterating (4.29), we easily obtain, for all $N \in \mathbb{N}$ and $(\ell,m) \in I_d$,

$$x^N\widetilde{X}^d_{\ell,m}(x) = \sum_{j=0}^{N} \alpha^{N,j}_{\ell,m}\widetilde{X}^d_{\ell-N+2j,m}(x), \tag{4.30}$$

where

$$\alpha^{N,j}_{\ell,m} \simeq_N \begin{cases} ((\ell-m)(\ell+m)/\ell^2)^{N/2} & \text{if} \quad (\ell-N+2j,m) \in I_d, \\ 0 & \text{otherwise.} \end{cases}$$

Let now $f \in L^2(\mathbb{S}^d)$; then we can write (see (4.15))

$$f = \sum_{(\ell,m)\in I_d} \sum_{Z \in E^{d-1}_m} a_{\ell,m,Z} X^d_{\ell,m} \otimes Z,$$

where, for all $m \in \mathbb{N}_{d-1}$, E_m^{d-1} is an orthonormal system of eigenfunctions of Δ_{d-1} on $\mathbb{S}^{d-1}$ of eigenvalue λ_m^{d-1}. Then from (4.30) we deduce

$$t_{d,d}^N f = \sum_{j=0}^{N} \sum_{(\ell,m)\in I_d} \sum_{Z\in E_m^{d-1}} a_{\ell,m,Z}\, \alpha_{\ell,m}^{N,j} X_{\ell-N+2j,m}^d \otimes Z$$

and consequently, by the orthogonality properties of the $X_{\ell,m} \otimes Z$ (see Section 4),

$$\|t_{d,d}^N f\|_{L^2(\mathbb{S}^d)}^2 \lesssim_N \sum_{(\ell,m)\in I_d} \left(\frac{(\ell-m)(\ell+m)}{\ell^2}\right)^N \sum_{Z\in E_m^{d-1}} a_{\ell,m,Z}^2. \tag{4.31}$$

Recall that $\Delta_d(X_{\ell,m}^d \otimes Z) = \lambda_\ell^d (X_{\ell,m}^d \otimes Z)$ and $\Delta_{d-1}(X_{\ell,m}^d \otimes Z) = \lambda_m^{d-1}(X_{\ell,m}^d \otimes Z)$, where

$$\lambda_\ell^d = \ell^2 - ((d-1)/2)^2 \quad \text{and} \quad \lambda_m^{d-1} = m^2 - ((d-2)/2)^2$$

by (4.3). From these formulas, together with (4.22) and (4.23), we deduce that, for all $(\ell, m) \in I_d$,

$$\lambda_\ell^d \simeq \ell^2, \qquad \lambda_\ell^d - \lambda_m^{d-1} \simeq \ell^2 - m^2 \qquad \text{whenever} \quad \lambda_\ell^d \neq 0. \tag{4.32}$$

If $f \perp \ker \Delta_d$, then the coefficients $a_{\ell,m,Z}$ in (4.31) vanish unless $\lambda_\ell^d \neq 0$, and from (4.32) we deduce

$$\|t_{d,d}^N f\|_{L^2(\mathbb{S}^d)} \lesssim_N \|((\Delta_d - \Delta_{d-1})/\Delta_d)^{N/2} f\|_{L^2(\mathbb{S}^d)}, \tag{4.33}$$

which is (4.28) in the case $k = d-1$.

Let now $2 \le r \le d$. By the discussion in Section 3, the parametrization $(\omega, \psi) \mapsto \lfloor \omega, \psi \rceil$ defined in (3.10) with $k = r$ allows us to identify, up to null sets, the sphere $\mathbb{S}^d$ with the product $\mathbb{S}^r \times [-\pi/2, \pi/2]^{d-r}$, where the measure σ on $\mathbb{S}^d$ corresponds to $\cos^{d-1}\psi_d \cdots \cos^r \psi_{r+1}\, d\psi\, d\sigma_r(\omega)$ on the product. Consequently the space $L^2(\mathbb{S}^d)$ is identified with the Hilbert tensor product of the Lebesgue spaces $L^2(\mathbb{S}^r)$ and $L^2([-\pi/2,\pi/2]^{d-r}, \cos^{d-1}\psi_d \cdots \cos^r \psi_{r+1}\, d\psi)$. Hence the inequality (4.33), applied with $d = r$, yields a corresponding inequality on the sphere $\mathbb{S}^d$, namely

$$\|t_{d,r}^N f\|_{L^2(\mathbb{S}^d)} \lesssim_N \|((\Delta_r - \Delta_{r-1})/\Delta_r)^{N/2} f\|_{L^2(\mathbb{S}^d)} \tag{4.34}$$

for all $f \perp \ker \Delta_r$ (here the relation between $t_{d,r}$ and $t_{r,r}$ in (4.25) was used).

Recall now that the operators Δ_r for $r = 1, \dots, d$ have a joint spectral decomposition (see Section 4) and that $\Delta_{r_1} \le \Delta_{r_2}$ spectrally whenever $1 \le r_1 \le r_2 \le d$ (see (4.22)). So (4.34) implies the inequality

$$\|t^N_{d,r} f\|_{L^2(\mathbb{S}^d)} \lesssim_N \|((\Delta_d - \Delta_k)/\Delta_{k+1})^{N/2} f\|_{L^2(\mathbb{S}^d)} \tag{4.35}$$

whenever $k < r \le d$ and $f \perp \ker \Delta_{k+1}$. The desired inequality (4.28) then follows by summing the inequalities (4.35) for $r = k+1, \dots, d$. ■

4.5 Weighted spectral cluster estimates

Let $d, k \in \mathbb{N}$ with $1 \le k < d$. For $(\ell, m) \in I^{(k)}_d$ and $x = (x_d, x_{d-1}, \dots, x_{k+1}) \in [-1,1]^{d-k}$, define

$$\mathscr{X}^{d,k}_{\ell,m}(x) = \sum_{\substack{(\ell_d,\dots,\ell_k) \in J^{(k)}_d \\ \ell_d = \ell,\ \ell_k = m}} \left|\widetilde{X}^d_{\ell_d,\ell_{d-1}}(x_d)\right|^2 \dots \left|\widetilde{X}^{k+1}_{\ell_{k+1},\ell_k}(x_{k+1})\right|^2, \tag{4.36}$$

where $\widetilde{X}^d_{r,s}$ has been defined in (4.24). Here we are interested in bounds for suitable weighted sums of the $\mathscr{X}^{d,k}_{\ell,m}$ for indices ℓ, m such that the eigenvalue $\sqrt{\lambda^{d,k}_{\ell,m}}$ of $\sqrt{\mathcal{L}_{d,k}}$ ranges in an interval of unit length (whence the name "spectral cluster"). The bounds that we obtain are different in nature according to whether $m \le \epsilon\ell$ or $m \ge \epsilon\ell$ for some fixed $\epsilon \in (0,1)$ and are presented as separate statements. We remark that, in the case $m \le \epsilon\ell$, the eigenvalue $\lambda^{d,k}_{\ell,m}$ of $\mathcal{L}_{d,k}$ is comparable with the eigenvalue λ^d_ℓ of Δ_d; consequently, the range $m \le \epsilon\ell$ will be referred to as the "elliptic regime," while the range $m \ge \epsilon\ell$ will be called the "subelliptic regime."

Proposition 4.2 (Subelliptic regime). Let $\epsilon \in (0,1)$ and $d \ge 2$. Fix $1 \le k \le d-1$. Then, for all $i \in \mathbb{N} \setminus \{0\}$, $\alpha \in [0, k/2)$, and $x \in [-1,1]^{d-k}$,

$$\sum_{\substack{(\ell,m) \in I^{(k)}_d \\ m \ge \epsilon\ell \\ \lambda^{d,k}_{\ell,m} \in [i^2,(i+1)^2]}} \alpha_m(\mathbb{S}^k)\ \mathscr{X}^{d,k}_{\ell,m}(x) \left[\sqrt{\lambda^{d,k}_{\ell,m}}/\ell\right]^{2\alpha} \lesssim_\epsilon i^{d-1} \min\{i, 1/|x|\}^{k-2\alpha}. \tag{4.37}$$

Proposition 4.3 (Elliptic regime). Let $\epsilon \in (0,1)$ and $d \geq 2$. Fix $1 \leq k \leq d-1$. Then, for all $i \in \mathbb{N} \setminus \{0\}$ and $x \in [-1,1]^{d-k}$,

$$\sum_{\substack{(\ell,m)\in I_d^{(k)} \\ m \leq \epsilon\ell \\ \lambda_{\ell,m}^{d,k} \in [i^2,(i+1)^2]}} \alpha_m(\mathbb{S}^k)\ \mathscr{X}_{\ell,m}^{d,k}(x) \lesssim_\epsilon i^{d-1}, \tag{4.38}$$

where $\mathscr{X}_{\ell,m}^{d,k}$ was defined in (4.36).

Analogous estimates are proved in [13, Section 4] in the case $d = 2$ and $k = 1$; in that case, each of the products in (4.36) reduces to a single factor. Treating the general case, with multiple factors, presents substantial additional difficulties. In order not to disrupt the presentation of the proof of our main theorem, the proofs of Propositions 4.2 and 4.3 are postponed to Section 6.

5 The Multiplier Theorem

Fix $d, k \in \mathbb{N}$ with $1 \leq k < d$. In this section we complete the proof of our main result, Theorem 1.1, for the spherical Grushin operator $\mathcal{L}_{d,k}$.

5.1 The weighted Plancherel-type estimate

By means of the estimates from Sections 4.4 and 4.5 we shall prove a "weighted Plancherel-type estimate" for the Grushin operator $\mathcal{L}_{d,k}$.

For all $r \in (0,\infty)$, we define the weight $\varpi_r : \mathbb{S}^d \times \mathbb{S}^d \to [0,\infty)$ by

$$\varpi_r(\lfloor \omega, \psi \rceil, \lfloor \omega', \psi' \rceil) = \frac{|\psi|}{\max\{r, |\psi'|\}} \tag{5.1}$$

for all $(\omega,\psi), (\omega',\psi') \in \mathbb{S}^k \times [-\pi/2,\pi/2]^{d-k}$; here $|\psi| = |\psi|_\infty$ as in (3.13).

Proposition 5.1. Let $\alpha \in [0,k/2)$ and $N \in \mathbb{N} \setminus \{0\}$. For all Borel functions $F : \mathbb{R} \to \mathbb{C}$ supported in $[0,N]$, and all $z' \in \mathbb{S}^d$,

$$\|(1+\varpi_{N^{-1}}(\cdot,z'))^\alpha\, \mathcal{K}_{F(\sqrt{\mathcal{L}_{d,k}})}(\cdot,z')\|_{L^2(\mathbb{S}^d)} \lesssim_\alpha V(z',N^{-1})^{-1/2} \|F(N\cdot)\|_{N,2}.$$

Proof. We shall prove the apparently weaker estimate

$$\|\varpi_{N^{-1}}(\cdot,z')^{\alpha}\,\mathcal{K}_{F(\sqrt{\mathcal{L}_{d,k}})}(\cdot,z')\|_{L^2(\mathbb{S}^d)} \lesssim_{\alpha} V(z',N^{-1})^{-1/2}\|F(N\cdot)\|_{N,2} \tag{5.2}$$

for all $z' \in \mathbb{S}^d$. Proposition 5.1 follows by combining the estimate (5.2) with the analogous one where $\alpha = 0$.

Recall that $\mathcal{L}_{d,k} = \Delta_d - \Delta_k$. Hence, by (4.19), we can write

$$\begin{aligned}
\mathcal{K}_{F(\sqrt{\mathcal{L}_{d,k}})} &= \sum_{(\ell_d,\dots,\ell_k)\in J_d^{(k)}} F\left(\sqrt{\lambda^{d,k}_{\ell_d,\ell_k}}\right) K^d_{\ell_d,\dots,\ell_k} \\
&= \sum_{\substack{(\ell_d,\dots,\ell_k)\in J_d^{(k)}\\ \ell_k\le\epsilon\ell_d}} + \sum_{\substack{(\ell_d,\dots,\ell_k)\in J_d^{(k)}\\ \ell_k>\epsilon\ell_d}} =: K_1 + K_2,
\end{aligned}$$

where $\epsilon = \max\{1/2,(k-1)/(d-1)\} \in (0,1)$ and $\lambda^{d,k}_{\ell_d,\ell_k}$ is given by (4.21). Consequently, for all $z' = \lfloor \omega',\psi' \rceil \in \mathbb{S}^d$,

$$\begin{aligned}
&\|\varpi_{N^{-1}}(\cdot,z')^{\alpha}\mathcal{K}_{F(\sqrt{\mathcal{L}_{d,k}})}(\cdot,z')\|_{L^2(\mathbb{S}^d)} \\
&\le \|\varpi_{N^{-1}}(\cdot,z')^{\alpha}K_1(\cdot,z')\|_{L^2(\mathbb{S}^d)} + \|\varpi_{N^{-1}}(\cdot,z')^{\alpha}K_2(\cdot,z')\|_{L^2(\mathbb{S}^d)} \\
&\lesssim_{\alpha} \min\{N,|\psi'|^{-1}\}^{\alpha}\left[\|K_1(\cdot,z')\|_{L^2(\mathbb{S}^d)} + \|\tau^{\alpha}_{d,k}K_2(\cdot,z')\|_{L^2(\mathbb{S}^d)}\right],
\end{aligned} \tag{5.3}$$

where $\tau_{d,k}$ is the function defined in (4.26), and the estimate (4.27) was used.

We note that, due to the choice of ϵ, for all $(\ell_d,\dots,\ell_k) \in J_d^{(k)}$ with $\ell_k > \epsilon\ell_d$,

$$\lambda^{k+1}_{\ell_{k+1}} \simeq \ell^2_{k+1} \simeq \ell^2_d \tag{5.4}$$

(see (4.3)). In particular, $K_2(\cdot,z') \perp \ker(\Delta_{k+1})$, and moreover

$$K_2(\cdot,z') = \mathcal{L}^{-\alpha/2}_{d,k}\Delta^{\alpha/2}_{k+1}K_{2,\alpha}(\cdot,z') \tag{5.5}$$

for all $z' \in \mathbb{S}^d$, where

$$K_{2,\alpha} = \sum_{\substack{(\ell_d,\dots,\ell_k)\in J_d^{(k)}\\ \ell_k>\epsilon\ell_d}} (\lambda^{d,k}_{\ell_d,\ell_k}/\lambda^{k+1}_{\ell_{k+1}})^{\alpha/2}F\left(\sqrt{\lambda^{d,k}_{\ell_d,\ell_k}}\right) K^d_{\ell_d,\dots,\ell_k};$$

indeed, recall that $K^d_{\ell_d,\dots,\ell_k}$ is the integral kernel of the orthogonal projection of the joint eigenspace of $\Delta_d,\dots,\Delta_k$ of eigenvalues $\lambda^d_{\ell_d},\dots,\lambda^k_{\ell_k}$ (see Section 4.3), so

$$\Delta_r K^d_{\ell_d,\dots,\ell_k}(\cdot,z') = \lambda^r_{\ell_r} K^d_{\ell_d,\dots,\ell_k}(\cdot,z')$$

for all $r = k,\dots,d$ and $z' \in \mathbb{S}^d$, and therefore (5.5) follows by comparing the definitions of K_2 and $K_{2,\alpha}$.

As a consequence, we can apply Proposition 4.1 with $f = K_2(\cdot,z')$, and from (5.3) we deduce that

$$\begin{aligned}&\|\varpi_{N^{-1}}(\cdot,z')^\alpha \mathcal{K}_{F(\sqrt{\mathcal{L}_{d,k}})}(\cdot,z')\|_{L^2(\mathbb{S}^d)}\\ &\lesssim_\alpha \min\{N,|\psi'|^{-1}\}^\alpha \left[\|K_1(\cdot,z')\|_{L^2(\mathbb{S}^d)} + \|K_{2,\alpha}(\cdot,z')\|_{L^2(\mathbb{S}^d)}\right]\end{aligned}$$

for all $z' = \lfloor \omega',\psi' \rceil \in \mathbb{S}^d$. In light of (3.18), the estimate (5.2) will follow from

$$\|K_1(\cdot,z')\|^2_{L^2(\mathbb{S}^2)} \lesssim_\alpha N^d \min\{N,|\psi'|^{-1}\}^{k-2\alpha} \|F(N\cdot)\|^2_{N,2}, \tag{5.6}$$

$$\|K_{2,\alpha}(\cdot,z')\|^2_{L^2(\mathbb{S}^d)} \lesssim_\alpha N^d \min\{N,|\psi'|^{-1}\}^{k-2\alpha} \|F(N\cdot)\|^2_{N,2}. \tag{5.7}$$

In fact, instead of (5.6), we shall prove the stronger estimate

$$\|K_1(\cdot,z')\|^2_{L^2(\mathbb{S}^d)} \lesssim N^d \|F(N\cdot)\|^2_{N,2}. \tag{5.8}$$

In view of (2.1), (20), and (5.4), we can rewrite (5.7) and (5.8) as

$$\begin{aligned}&\sum_{\substack{(\ell_d,\dots,\ell_k)\in J^{(k)}_d\\ \ell_k>\epsilon\ell_d}} \alpha_{\ell_k}(\mathbb{S}^k)(\lambda^{d,k}_{\ell_d,\ell_k}/\ell_d^2)^\alpha \left|F\left(\sqrt{\lambda^{d,k}_{\ell_d,\ell_k}}\right)\right|^2 |X^d_{\ell_d,\dots,\ell_k}(\psi')|^2\\ &\qquad\qquad\lesssim_\alpha N^{d-1}\min\{N,|\psi'|^{-1}\}^{k-2\alpha}\sum_{i=1}^N \sup_{\lambda\in[i-1,i]}|F(\lambda)|^2\end{aligned}$$

and

$$\sum_{\substack{(\ell_d,\dots,\ell_k)\in J^{(k)}_d\\ \ell_k\le\epsilon\ell_d}} \alpha_{\ell_k}(\mathbb{S}^k) \left|F\left(\sqrt{\lambda^{d,k}_{\ell_d,\ell_k}}\right)\right|^2 |X^d_{\ell_d,\dots,\ell_k}(\psi')|^2 \lesssim_\alpha N^{d-1}\sum_{i=1}^N \sup_{\lambda\in[i-1,i]}|F(\lambda)|^2.$$

So it is enough to prove that

$$\sum_{\substack{(\ell_d,\ldots,\ell_k)\in J_d^{(k)}\\ \ell_k>\epsilon\ell_d\\ \lambda^{d,k}_{\ell_d,\ell_k}\in[(i-1)^2,i^2]}} \alpha_{\ell_k}(\mathbb{S}^k)(\lambda^{d,k}_{\ell_d,\ell_k}/\ell_d^2)^\alpha |X^d_{\ell_d,\ldots,\ell_k}(\psi')|^2 \lesssim_\alpha N^{d-1}\min\{N,|\psi'|^{-1}\}^{k-2\alpha}$$

and

$$\sum_{\substack{(\ell_d,\ldots,\ell_k)\in J_d^{(k)}\\ \ell_k\le\epsilon\ell_d\\ \lambda^{d,k}_{\ell_d,\ell_k}\in[(i-1)^2,i^2]}} \alpha_{\ell_k}(\mathbb{S}^k)|X^d_{\ell_d,\ldots,\ell_k}(\psi')|^2 \lesssim N^{d-1}$$

for $i=1,\ldots,N$. For $i=1$ it is easy to verify the above estimates, since each of the sums contains at most two summands, with $(\ell_d-(d-1)/2,\ell_k-(k-1)/2)\in\{(0,0),(1,1)\}$, and the functions $X_{\ell_d,\ldots,\ell_k}$ are bounded. For $i=2,\ldots,N$, these estimates follow from Propositions 4.2 and 4.3, applied with $m=\ell_k$ and $\ell=\ell_d$. ■

5.2 Properties of the weight

We shall need some properties of the weights $\varpi_r:\mathbb{S}^d\times\mathbb{S}^d\to[0,\infty)$ defined in (5.1). The following lemma extends [13, Lemma 5.1], where only the case $d=2$, $k=1$ was treated. We refer to [40, Lemma 12] and [36, Lemma 4.1] for analogous results.

Lemma 5.2. For all $r>0$ and $\alpha,\beta\ge 0$ such that $\alpha+\beta>d+k$ and $\alpha<\min\{d-k,k\}$, and for all $z'\in\mathbb{S}^d$,

$$\int_{\mathbb{S}^d}(1+\varrho(z,z')/r)^{-\beta}(1+\varpi_r(z,z'))^{-\alpha}\,\mathrm{d}\sigma(z)\lesssim_{\alpha,\beta} V(z',r). \tag{5.9}$$

Moreover

$$1+\varpi_r(z,z')\lesssim(1+\varrho(z,z')/r) \tag{5.10}$$

for all $r>0$ and $z,z'\in\mathbb{S}^d$.

Proof. Due to the compactness of $\mathbb{S}^d$, both (5.9) and (5.10) are obvious for $r\ge 1$. In the following we assume therefore that $r<1$.

To prove (5.10), we observe that, for all $\lfloor \omega, \psi \rceil, \lfloor \omega', \psi' \rceil \in \mathbb{S}^d$,

$$1 + \frac{|\psi|}{\max\{r, |\psi'|\}} \simeq 1 + \frac{|\psi - \psi'|}{\max\{r, |\psi'|\}} \lesssim 1 + \varrho(\lfloor \omega, \psi \rceil, \lfloor \omega', \psi' \rceil)/r. \tag{5.11}$$

The last inequality follows immediately from (3.16) in the case $\max\{|\psi|, |\psi'|\} < \pi/4$, and it is trivial when $|\psi'| > \pi/8$ (since $|\psi|/\max\{r, |\psi'|\} \lesssim 1$ in that case); in the remaining case ($|\psi| \geq \pi/4$ and $|\psi'| \leq \pi/8$), the points $\lfloor \omega, \psi \rceil$ and $\lfloor \omega', \psi' \rceil$ belong to disjoint compact subsets of $\mathbb{S}^d$, whence

$$\varrho(\lfloor \omega, \psi \rceil, \lfloor \omega', \psi' \rceil) \simeq 1 \simeq |\psi - \psi'| \tag{5.12}$$

and the desired inequality follows.

In order to prove (5.9), we fix $z' = \lfloor \omega', \psi' \rceil \in \mathbb{S}^d$ and split the integral in the left-hand side of (5.9) into the sum $\sum_{j=0}^{3} \mathcal{I}_j$, where

$$\mathcal{I}_j = \int_{\mathcal{S}_j} (1 + \varrho(z, z')/r)^{-\beta} (1 + \varpi_r(z, z'))^{-\alpha} \, d\sigma(z)$$

and

$$\begin{aligned}
\mathcal{S}_0 &= \left\{ \lfloor \omega, \psi \rceil \in \mathbb{S}^d \,:\, \max\{|\psi|, |\psi'|\} \geq \pi/4 \right\}, \\
\mathcal{S}_1 &= \left\{ \lfloor \omega, \psi \rceil \in \mathbb{S}^d \setminus \mathcal{S}_0 \,:\, \rho_{R,\mathbb{S}^k}(\omega, \omega')^{1/2} \leq \frac{\rho_{R,\mathbb{S}^k}(\omega, \omega')}{\max\{|\psi|, |\psi'|\}} \right\}, \\
\mathcal{S}_2 &= \left\{ \lfloor \omega, \psi \rceil \in \mathbb{S}^d \setminus (\mathcal{S}_0 \cup \mathcal{S}_1) \,:\, |\psi'| \leq |\psi|/2 \right\}, \\
\mathcal{S}_3 &= \left\{ \lfloor \omega, \psi \rceil \in \mathbb{S}^d \setminus (\mathcal{S}_0 \cup \mathcal{S}_1) \,:\, |\psi|/2 < |\psi'| \right\}.
\end{aligned}$$

We first estimate $\mathcal{I}_0$. In the case $|\psi'| > \pi/8$, we use (3.17) to conclude that

$$\mathcal{I}_0 \lesssim \int_{\mathbb{S}^d} (1 + \varrho_R(z, z')/r)^{-\beta} \, \mathrm{d}\sigma(z) \lesssim r^d \simeq V(z', r),$$

since $r < 1$ and $\beta > d$ (cf. [20, Lemma 4.4]). In the case $|\psi'| \leq \pi/8$, instead, $\varrho(z, z') \simeq |\psi| \simeq 1$ by (5.12) for all $z \in \mathcal{S}_0$, and

$$\mathcal{I}_0 \simeq r^\beta \max\{r, |\psi'|\}^\alpha = r^d \max\{r, |\psi'|\}^k \frac{r^{\beta - d}}{\max\{r, |\psi|\}^{k - \alpha}} \lesssim V(z', r),$$

by (3.18), since $\beta - d > k - \alpha > 0$.

In order to estimate $\mathcal{I}_1$, we decompose $\beta = \beta_1 + \beta_2$, with $\beta_1 > d - k - \alpha$ and $\beta_2 > 2k$. Thus (3.16) and (5.11) imply

$$\begin{aligned}
\mathcal{I}_1 &\simeq \int_{\mathcal{S}_1} (1 + \varrho(z, z')/r)^{-\beta} \left(1 + \frac{|\psi - \psi'|}{\max\{r, |\psi'|\}}\right)^{-\alpha} \mathrm{d}\sigma(z) \\
&\le (\max\{r, |\psi'|\}/r)^{\alpha} \int_{\mathcal{S}_1} (1 + \varrho(z, z')/r)^{-\beta} \left(1 + |\psi - \psi'|/r\right)^{-\alpha} \mathrm{d}\sigma(z) \\
&\lesssim (\max\{r, |\psi'|\}/r)^{\alpha} \int_{\mathcal{S}_1} (1 + \varrho_{R,\mathbb{S}^k}(\omega, \omega')^{1/2}/r)^{-\beta_2} (1 + |\psi - \psi'|/r)^{-\alpha-\beta_1} \mathrm{d}\sigma(\lfloor \omega, \psi \rceil) \\
&\lesssim (\max\{r, |\psi'|\}/r)^{\alpha} \int_{\mathbb{S}^k} (1 + \varrho_{R,\mathbb{S}^k}(\omega, \omega')/r^2)^{-\beta_2/2} \mathrm{d}\sigma_k(\omega) \\
&\qquad \times \int_{[-\pi/4,\pi/4]^{d-k}} \left(1 + |\psi - \psi'|/r\right)^{-\alpha-\beta_1} \mathrm{d}\psi_d \dots \mathrm{d}\psi_{k+1} \\
&\lesssim (\max\{r, |\psi'|\}/r)^{\alpha} r^{2k} r^{d-k} = r^d \max\{r, |\psi'|\}^k (r/\max\{r, |\psi'|\})^{k-\alpha} \lesssim V(z', r),
\end{aligned}$$

since $\beta_2/2 > k$ and $\alpha < k$.

In order to estimate $\mathcal{I}_2$, instead, we write $\beta = \tilde{\beta}_1 + \tilde{\beta}_2$, with $\tilde{\beta}_1 > d - \alpha$ and $\tilde{\beta}_2 > k$, so, again by (3.16),

$$\begin{aligned}
\mathcal{I}_2 &\simeq \int_{\mathcal{S}_2} \left(1 + \frac{|\psi - \psi'|}{r} + \frac{\varrho_{R,\mathbb{S}^k}(\omega, \omega')}{r \max\{|\psi|, |\psi'|\}}\right)^{-\beta} \left(1 + \frac{|\psi|}{\max\{r, |\psi'|\}}\right)^{-\alpha} \mathrm{d}\sigma(\lfloor \omega, \psi \rceil) \\
&\lesssim \int_{2|\psi'| \le |\psi| \le \pi/4} \left(1 + \frac{|\psi|}{r}\right)^{-\tilde{\beta}_1} \left(1 + \frac{|\psi|}{\max\{r, |\psi'|\}}\right)^{-\alpha} \\
&\qquad \times \int_{\mathbb{S}^k} \left(1 + \frac{\varrho_{R,\mathbb{S}^k}(\omega, \omega')}{r|\psi|}\right)^{-\tilde{\beta}_2} \mathrm{d}\omega \, \mathrm{d}\psi_d \dots \mathrm{d}\psi_{k+1} \\
&\lesssim (\max\{r, |\psi'|\}/r)^{\alpha} \int_{[-\pi/4,\pi/4]^{d-k}} \left(1 + \frac{|\psi|}{r}\right)^{-\tilde{\beta}_1-\alpha} (r|\psi|)^k \, \mathrm{d}\psi_d \dots \mathrm{d}\psi_{k+1} \\
&\lesssim (\max\{r, |\psi'|\}/r)^{\alpha} r^{d+k} = r^d \max\{r, |\psi'|\}^k (r/\max\{r, |\psi'|\})^{k-\alpha} \lesssim V(z', r)
\end{aligned}$$

where we used the fact that $\max\{|\psi|, |\psi'|\} \simeq |\psi - \psi'| \simeq |\psi|$ on $\mathcal{S}_2$.

Finally, to estimate $\mathcal{I}_3$, we decompose $\beta = \tilde{\beta}_1 + \tilde{\beta}_2$ as above and get

$$\begin{aligned}
\mathcal{I}_3 &\lesssim \int_{\mathcal{S}_3} \left(1 + \frac{|\psi - \psi'|}{r}\right)^{-\tilde{\beta}_1} \left(1 + \frac{\rho_{R,\mathbb{S}^k}(\omega, \omega')}{r|\psi'|}\right)^{-\tilde{\beta}_2} \mathrm{d}\sigma(\lfloor \omega, \psi \rceil) \\
&\lesssim (r|\psi'|)^k \int_{[-\pi/4,\pi/4]^{d-k}} \left(1 + \frac{|\psi - \psi'|}{r}\right)^{-\tilde{\beta}_1} \mathrm{d}\psi \lesssim r^d |\psi'|^k \lesssim V(z', r),
\end{aligned}$$

where we used the fact that $\max\{|\psi|, |\psi'|\} \simeq |\psi'|$ on $\mathcal{S}_3$. ∎

5.3 Proof of the main result

The previous estimates finally allow us to verify the assumptions of the abstract theorem in Section 2 and prove our multiplier theorem for the Grushin operators $\mathcal{L}_{d,k}$.

Proof of Theorem 1.1. Let $\alpha \in [0, \min\{d-k, k\})$. We apply Theorem 2.1 with $(X, \varrho, \mu) = (\mathbb{S}^d, \varrho, \sigma)$, $\mathfrak{L} = \mathcal{L}_{d,k}$, $q = 2$, $\mathfrak{d} = d+k-\alpha$, $\pi_r = (1+\varpi_r)^\alpha$. Note that the assumptions (a) and (b) easily follow from [21]; as a matter of fact, (a) also follows from Proposition 3.1, and (b) could be derived from Proposition 5.1 via the results of [41, 51] (cf. the discussion in [13]). Moreover, the assumptions (c) and (d) are proved in Lemma 5.2, while the assumption (d) is proved in Proposition 5.1. By choosing α sufficiently close to $\min\{d-k, k\}$, we can make $\mathfrak{d} = d+k-\alpha$ arbitrarily close to $D = \max\{d, 2k\}$, and the desired results follow. ■

6 Proof of the Weighted Spectral Cluster Estimates

Here we discuss the proof of the estimates stated in Section 4.5. Specifically, the proofs of Propositions 4.2 and 4.3 are presented in Sections 6.3 and 6.4, respectively, while Sections 6.1 and 6.2 are devoted to the discussion of a number of preliminary results.

6.1 Estimates for ultraspherical polynomials

In this section we collect a number of estimates for the functions $X^d_{\ell,m}$ discussed in Section 4.2 (or rather, the $\widetilde{X}^d_{\ell,m}$ from (4.24)), which play a crucial role in the proof of the weighted spectral cluster estimates.

We first state some elementary uniform bounds that follow readily from the discussion in Section 4 (see especially (4.7) and (4.13)). In the statement below, we convene that $0^0 = 1$.

Proposition 6.1. Let $d \in \mathbb{N}$, $d \geq 2$.

(i) For all $\ell \in \mathbb{N}_d$ and $x \in [-1, 1]$,

$$\sum_{\substack{m \in \mathbb{N}_{d-1} \\ m \leq \ell}} m^{d-2} |\widetilde{X}^d_{\ell,m}(x)|^2 \lesssim_d \ell^{d-1}.$$

(ii) $\|\widetilde{X}^d_{\ell,m}\|_\infty \lesssim_d \ell^{(d-1)/2}/m^{(d-2)/2}$ for all $(\ell, m) \in I_d$.

More refined pointwise estimates can be derived from asymptotic approximations of ultraspherical polynomials in terms of Hermite polynomials and Bessel functions, obtained in works of Olver [47] and Boyd and Dunster [11] in the regimes $m \geq \epsilon\ell$ and $m \leq \epsilon\ell$, respectively, where $\epsilon \in (0,1)$.

Here and subsequently, for all $\ell, m \in \mathbb{R}$ with $\ell \neq 0$ and $0 \leq m \leq \ell$, $a_{\ell,m}$ and $b_{\ell,m}$ will denote the numbers in $[0,1]$ defined by

$$b_{\ell,m} = \frac{m}{\ell} \tag{6.1}$$

and

$$a_{\ell,m}^2 = 1 - b_{\ell,m}^2 = \frac{(\ell - m)(\ell + m)}{\ell^2}. \tag{6.2}$$

The points $\pm a_{\ell,m} \in [-1,1]$ play the role of "transition points" for the functions $\widetilde{X}_{\ell,m}^d$ in the estimates that follow.

Theorem 6.2. Let $d \in \mathbb{N}$, $d \geq 2$. Let $\epsilon \in (0,1)$. There exists $c \in (0,1)$ such that, for all $(\ell, m) \in I_d$, if $m \geq \epsilon\ell$ then

$$|\widetilde{X}_{\ell,m}^d(x)| \lesssim_{d,\epsilon} \begin{cases} (\ell^{-1} + |x^2 - a_{\ell,m}^2|)^{-1/4} & \text{for all} \quad x \in [-1,1], \\ |x|^{-1/2} \exp(-c\ell x^2) & \text{for} \quad |x| \geq 2a_{\ell,m}, \end{cases} \tag{6.3}$$

while, if $m \leq \epsilon\ell$, then

$$|\widetilde{X}_{\ell,m}^d(x)| \lesssim_{d,\epsilon} \begin{cases} y^{-(d-2)/2} \left(\frac{(1+m)^{4/3}}{\ell^2} + |y^2 - b_{\ell,m}^2| \right)^{-1/4} & \text{for all} \quad x \in [-1,1], \\ \ell^{(d-1)/2}\, 2^{-m} & \text{if} \quad y \leq b_{\ell,m}/(2e), \end{cases} \tag{6.4}$$

where $y = \sqrt{1 - x^2}$.

In the case $d = 2$, the derivation of the estimates in Theorem 6.2 from the asymptotic approximations in [11, 47] is presented in [13, Section 3]; a number of variations and new ideas are required in the general case $d \geq 2$, and we refer to [14] for a complete proof (indeed, in [14] a stronger decay is proved in the regime $m \geq \epsilon\ell$ for $|x| \geq 2a_{\ell,m}$ than the one given in (6.3)). Here we only remark that combining the above estimates yields the following bound.

Corollary 6.3. Let $d \in \mathbb{N}$, $d \geq 2$. There exists $c \in (0, \infty)$ such that, for all $(\ell, m) \in I_d$ and $x \in [-1, 1]$,

$$|\widetilde{X}^d_{\ell,m}(x)| \lesssim_d \begin{cases} y^{-(d-2)/2} \left(\frac{1+m}{\ell^2} + |y^2 - b^2_{\ell,m}| \right)^{-1/4} & \text{for all} \quad x \in [-1, 1], \\ \ell^{(d-1)/2} \exp(-cm) & \text{if} \quad y \leq b_{\ell,m}/(2e), \end{cases} \tag{6.5}$$

where $y = \sqrt{1 - x^2}$.

Proof. Let $\epsilon \in (0, 1)$ be a parameter to be fixed later. If $m \leq \epsilon\ell$, the desired estimates immediately follow from (6.4), by taking any $c \leq \log 2$ (indeed, note that $(1 + m)^{4/3} \geq 1 + m$).

On the other hand, for $m \geq \epsilon\ell$, we may apply the estimates (6.3). Note that $m \simeq \ell \gtrsim 1$ in this range, so $1/\ell \simeq (1 + m)/\ell^2$; moreover $|x^2 - a_{\ell,m}|^2 = |y^2 - b_{\ell,m}|^2$ and $y \leq 1$, so the 1st estimate in (6.5) immediately follows from the 1st estimate in (6.3).

Assume now that $y \leq b_{\ell,m}/(2e)$. Since $b_{\ell,m} \geq \epsilon$ in this range, $a^2_{\ell,m}/(1 - \epsilon^2) \leq 1$. Consequently

$$x^2 = 1 - y^2 \geq 1 - \frac{b^2_{\ell,m}}{4e^2} = \frac{(4e^2 - 1) + a^2_{\ell,m}}{4e^2} \geq \min\left\{ \frac{4e^2 - 1}{4e^2}, \frac{1 - (\epsilon/(2e))^2}{1 - \epsilon^2} a^2_{\ell,m} \right\}.$$

This shows that, on the one side, $|x| \gtrsim 1$; on the other side, if $\epsilon \in (0, 1)$ is chosen sufficiently large, then $|x| \geq 2a_{\ell,m}$. Therefore we can apply the 2nd estimate in (6.3) and obtain that

$$|\widetilde{X}^d_{\ell,m}(x)| \lesssim \exp(-c'\ell)$$

for a suitable constant $c' \in (0, \infty)$. Since $\ell \simeq m$ in this range, this clearly implies the 2nd estimate in (6.5) for an appropriate choice of c. ∎

6.2 Estimating sums with integrals

In the proofs of the weighted spectral cluster estimates, we will need multiple times to majorize a sum with the corresponding integral. For this purpose we will repeatedly invoke a couple of elementary lemmas, whose statements are reproduced below for the reader's convenience.

The following statement can be found in [18, Lemma 5.7].

Lemma 6.4. Let $\kappa \in [1,\infty)$. Let $\Omega \subseteq \mathbb{R}^n$ be open and convex and $\phi : \Omega \to (0,\infty)$ be locally Lipschitz and satisfying

$$|\nabla\phi(u)|_2 \leq \kappa\phi(u)$$

for almost all $u \in \Omega$. Let $P \subseteq \Omega$ be such that, for some $r \in (0,1]$,

$$\inf_{u\in P} \mathrm{Vol}(B_r(u) \cap \Omega) \geq \kappa^{-1}$$

(here $B_r(u)$ is the Euclidean ball centred at u of radius r, and Vol is the Lebesgue measure) and moreover we can decompose $P = P_1 \cup \cdots \cup P_N$ for some $N \leq \kappa$ so that

$$\inf_{j=1,\ldots,N} \inf_{\substack{u,u'\in P_j\\ u\neq u'}} |u-u'|_2 \geq 2r.$$

Then

$$\sum_{u\in P} \phi(u) \leq e\kappa^3 \int_\Omega \phi(u)\,\mathrm{d}x.$$

In the one-dimensional case, a simplified version of the above lemma can be found in [13, Lemma 4.1] and is stated below.

Lemma 6.5. Let $\kappa \in [1,\infty)$. Let $D \subseteq \mathbb{R}$ be open and $\phi : D \to \mathbb{R}$ be a nonnegative differentiable function satisfying

$$|\phi'(x)| \leq \kappa\phi(x)$$

for all $x \in D$. Let $R \subseteq \mathbb{R}$ be such that

$$\inf\{|x-x'| : x, x' \in R,\ x \neq x'\} \geq \kappa^{-1}.$$

Then, for all intervals $I \subseteq D$ with length $\mathrm{Vol}(I) \geq \kappa^{-1}$,

$$\sum_{x\in R\cap I} \phi(x) \leq C_\kappa \int_I \phi(x)\,\mathrm{d}x,$$

where the constant C_κ depends only on κ and not on I, R, ϕ.

Both Lemmas 6.5 and 6.4 require a control of the gradient of the integrand function in terms of the function itself. In order to verify this assumption in the applications below, the following lemma will be useful.

Lemma 6.6. For $a, t \in \mathbb{R}$, $s \in (0,\infty)$, define

$$\Xi(a,s,t) = (s + |a-t|)^{-1/2}. \tag{6.6}$$

Let $\kappa \in [1,\infty)$. Let $\Omega \subseteq \mathbb{R}^n$, and $\alpha_j : \Omega \to (0,\infty)$, $\beta_j : \Omega \to \mathbb{R}$ be such that

$$|\nabla \alpha_j|_2, |\nabla \beta_j|_2 \le \kappa \alpha_j$$

for $j = 1, \dots, N$. Define $\tilde{\Xi}(y,x) = \prod_{j=1}^N \Xi(y_j, \alpha_j(x), \beta_j(x))$ for $y \in \mathbb{R}^n$ and $x \in \Omega$. Then, for all $y \in \mathbb{R}^N$ and $x \in \Omega$,

$$|\nabla_x \tilde{\Xi}(y,x)|_2 \le N\kappa\, \tilde{\Xi}(y,x).$$

Proof. By the Leibniz rule, it is enough to consider the case $N = 1$. Set $\alpha = \alpha_1$, $\beta = \beta_1$. Define $X(a,s,t) = s + |a-t|$ and $\tilde{X}(y,x) = X(y,\alpha(x),\beta(x))$. Note now that

$$X(a,s,t) \ge s, \qquad |\partial_s X(a,s,t)|, |\partial_t X(a,s,t)| \le 1,$$

whence, by the chain rule,

$$|\nabla_x \tilde{X}(y,x)|_2 \le |\nabla_x \alpha(x)|_2 + |\nabla_x \beta(x)|_2 \le 2\kappa\alpha(x) \le 2\kappa \tilde{X}(y,x)$$

and

$$\frac{|\nabla_x \tilde{\Xi}(y,x)|_2}{\tilde{\Xi}(y,x)} = \frac{1}{2} \frac{|\nabla_x \tilde{X}(y,x)|_2}{\tilde{X}(y,x)} \le \kappa,$$

as desired. ∎

6.3 The subelliptic regime

Here we prove Proposition 4.2. To this aim, we first present a couple of lemmas that will allow us to perform a particularly useful change of variables in the proof.

Lemma 6.7. Let $w \in \mathbb{R}^n$ and define the matrix $M(w) = (m_{j,s}(w))_{j,s=1}^n$ by

$$m_{j,s}(w) = \begin{cases} 1 & \text{if} \quad j = s, \\ w_j & \text{if} \quad j > s, \\ -w_j & \text{if} \quad j < s. \end{cases}$$

Then

$$\det M(w) = \sum_{\substack{S \subseteq \{1,\dots,n\} \\ |S| \quad \text{even}}} \prod_{j \in S} w_j.$$

Proof. Observe that $m_{j,s}(w) = \delta_{j,s} + \rho_{j,s} w_j$, where $\rho_{j,s} = \operatorname{sgn}(j-s)$. Consequently, if $\mathfrak{S}_n$ denotes the group of permutations of the set $\{1,\dots,n\}$ and $\epsilon(\sigma)$ denotes the signature of the permutation σ, then

$$\begin{aligned} \det M(w) &= \sum_{\sigma \in \mathfrak{S}_n} \epsilon(\sigma) \prod_{j=1}^n m_{j,\sigma(j)}(w) \\ &= \sum_{\sigma \in \mathfrak{S}_n} \epsilon(\sigma) \prod_{j : \sigma(j) \neq j} \rho_{j,\sigma(j)} w_j \\ &= \sum_{S \subseteq \{1,\dots,n\}} \left(\prod_{j \in S} w_j \right) \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma|_{S^c} = \mathrm{id}}} \epsilon(\sigma) \prod_{j \in S} \rho_{j,\sigma(j)} \\ &= \sum_{S \subseteq \{1,\dots,n\}} \left(\prod_{j \in S} w_j \right) \det(\rho_{l,m})_{l,m=1}^{|S|}, \end{aligned}$$

where $S^{\mathrm{c}} = \{1,\dots,n\} \setminus S$. We note that $(\rho_{l,m})_{l,m=1}^{|S|}$ is a skewsymmetric matrix, so its determinant vanishes when $|S|$ is odd; if $|S|$ is even, instead, its determinant is the square of its pfaffian, and using the Laplace-type expansion for pfaffians (see, e.g., [3, §III.5, p. 142]) one can see inductively that the determinant is 1. ∎

Lemma 6.8. Let $\Omega = \{v \in \mathbb{R}^n \,:\, \hat{v}_j \neq -1 \quad \text{for all} \quad j = 1,\dots,n\}$, where

$$\hat{v}_j = \sum_{r=j+1}^n v_r - \sum_{r=1}^{j-1} v_r.$$

Let $v \mapsto w$ be the map from Ω to $\mathbb{R}^n$ defined by

$$w_j = \frac{v_j}{1+\hat{v}_j}$$

for $j = 1, \ldots, n$. Then

$$\det(\partial_{v_s} w_j)_{j,s=1}^n = \left(\prod_{j=1}^n \frac{1}{1+\hat{v}_j}\right) \sum_{\substack{S\subseteq\{1,\ldots,n\} \\ |S| \text{ even}}} \prod_{j\in S} \frac{v_j}{1+\hat{v}_j}.$$

Moreover, for all $\epsilon \in (0,1)$, the map $v \mapsto w$ is injective when restricted to

$$\Omega_\epsilon := \left\{ v \in \mathbb{R}^n \,:\, v_j \geq 0 \ \forall j = 1, \ldots, n, \ \sum_j v_j \leq \epsilon \right\}.$$

Proof. From the definition it is immediate that

$$\partial_{v_s} w_j = \frac{1}{1+\hat{v}_j} m_{j,s}(w),$$

where $M(w) = \{m_{j,s}(w)\}_{j,s=1}^n$ is the matrix defined in Lemma 6.7, so

$$\det(\partial_{v_s} w_j)_{j,s=1}^n = \left(\prod_{j=1}^n \frac{1}{1+\hat{v}_j}\right) \det M(w),$$

and the desired expression for the determinant follows from Lemma 6.7.

Note that, if $v \in \Omega_\epsilon$, then $0 \leq v_j, |\hat{v}_j| \leq \sum_j v_j \leq \epsilon < 1$, so $1 + \hat{v}_j > 0$ and $\Omega_\epsilon \subseteq \Omega$. In addition, the equations $w_j = v_j/(1+\hat{v}_j)$ are equivalent to $v_j - w_j \hat{v}_j = w_j$, that is,

$$M(w)v = w.$$

Since $w_j = v_j/(1+\hat{v}_j) \geq 0$, from Lemma 6.7 it follows that $\det M(w) \geq 1$, so the matrix $M(w)$ is invertible and the above equation is equivalent to $v = M(w)^{-1}w$; in other words, if $v \in \Omega_\epsilon$, then v is uniquely determined by its image w via the map $v \mapsto w$, that is, the map restricted to Ω_ϵ is injective. ∎

Proof of Proposition 4.2. We start by observing that, for all $(\ell, m) \in I_d^{(k)}$, if we assume $\epsilon\ell \leq m$, then, for all $(\ell_d, \dots, \ell_k) \in J_d^{(k)}$ with $\ell_d = \ell$ and $\ell_k = m$,

$$\epsilon\ell_{j+1} \leq \ell_j, \qquad j \in \{k, \dots, d-1\}; \tag{6.7}$$

in particular

$$\ell_j \simeq \ell \gtrsim 1, \quad \text{for all} \quad j \in \{k, \dots, d\} \tag{6.8}$$

and, by (4.6),

$$\alpha_m(\mathbb{S}^k) \simeq m^{k-1} \simeq_\epsilon \ell^{k-1}.$$

Thus, in view of (4.36), the estimate (4.37) can be equivalently rewritten as

$$\sum_{\substack{(\ell_d,\dots,\ell_k)\in J_d^{(k)} \\ \epsilon\ell_d\leq\ell_k \\ \lambda^{d,k}_{\ell_d,\ell_k}\in[i^2,(i+1)^2]}} \ell_d^{k-1-2\alpha} \left|\widetilde{X}^d_{\ell_d,\ell_{d-1}}(x_d)\right|^2 \dots \left|\widetilde{X}^{k+1}_{\ell_{k+1},\ell_k}(x_{k+1})\right|^2 \lesssim_\epsilon i^{d-1-2\alpha} \min\{i, |x|^{-1}\}^{k-2\alpha}. \tag{6.9}$$

We now note that, by (4.3) and (4.21),

$$\lambda^{d,k}_{\ell_d,\ell_k} + (d+k-2)(d-k)/4 = \ell_d^2 - \ell_k^2, \tag{6.10}$$

and therefore, for all $i \in \mathbb{N} \setminus \{0\}$,

$$\lambda^{d,k}_{\ell_d,\ell_k} \in [i^2, (i+1)^2] \quad \Longrightarrow \quad \ell_d^2 - \ell_k^2 \in [i^2, (i+h)^2],$$

where h is a positive integer depending only on d and k (one can take, e.g., $h = \lceil (d+k-2)(d-k)/4 \rceil$). Thus the estimate (6.9) will follow if we prove that

$$\sum_{\substack{(\ell_d,\dots,\ell_k)\in J_d^{(k)} \\ \epsilon\ell_d\leq\ell_k \\ \ell_d^2-\ell_k^2\in[i^2,(i+1)^2]}} \ell_d^{k-1-2\alpha} \left|\widetilde{X}^d_{\ell_d,\ell_{d-1}}(x_d)\right|^2 \dots \left|\widetilde{X}^{k+1}_{\ell_{k+1},\ell_k}(x_{k+1})\right|^2 \lesssim_\epsilon i^{d-1-2\alpha} \min\{i, |x|^{-1}\}^{k-2\alpha}, \tag{6.11}$$

for all $i \in \mathbb{N} \setminus \{0\}$ and $x \in [-1, 1]^{d-k}$; indeed, to deduce (6.9) it suffices to apply (6.11) h times, with i replaced by $i, i+1, \dots, i+h-1$, respectively.

Due to (4.2), we may restrict without loss of generality to $x \in [0,1]^{d-k}$. In addition, for each fixed i, the sum in the left-hand side of (6.11) is finite, since $\ell_d - \ell_k \gtrsim 1$ and therefore

$$\ell_d \le \ell_d + \ell_k \lesssim \ell_d^2 - \ell_k^2 \le (i+1)^2;$$

the boundedness of the functions $\widetilde{X}^{d-j+1}_{\ell_{d-j+1},\ell_{d-j}}$ (see Proposition 6.1(ii)) then shows that the estimate (6.11) is trivially true for each fixed i (with a constant depending on i), and therefore it is enough to prove it for i sufficiently large.

It is convenient to reindex the sum in (6.11). Let us set

$$p = \ell_d + \ell_k, \quad q_j = \ell_{d-j+1} - \ell_{d-j}, \qquad \text{for all} \quad j \in \{1, \dots, d-k\}, \tag{6.12}$$

and let us introduce the notation

$$Q := q_1 + \dots + q_{d-k}.$$

We need to determine how the conditions describing the summation range in (6.11) can be reinterpreted when using the indices $p, q_1, \dots, q_{d-k}$ instead of $\ell_d, \dots, \ell_k$. First, note that

$$\ell_d - \ell_k = Q,$$

so the condition $(\ell_d, \dots, \ell_k) \in J_d^{(k)}$ is equivalent to

$$q_1, \dots, q_{d-k} \in \mathbb{N} + 1/2, \qquad\qquad p \in \mathbb{N} + (d+k-2)/2, \tag{6.13}$$

$$p \ge Q + k - 1, \qquad\qquad p - Q \equiv k - 1 \pmod 2. \tag{6.14}$$

Moreover

$$\ell_d^2 - \ell_k^2 = pQ,$$

so the condition $\ell_d^2 - \ell_k^2 \in [i^2, (i+1)^2]$ is equivalent to

$$pQ \in [i^2, (i+1)^2]. \tag{6.15}$$

Furthermore

$$\frac{Q}{p} = \frac{1 - \ell_k/\ell_d}{1 + \ell_k/\ell_d}$$

and $t \mapsto \frac{1-t}{1+t}$ is strictly decreasing on $[0, \infty)$; consequently, the condition $\epsilon \ell_d \leq \ell_k$ is equivalent to

$$Q \leq \bar{\epsilon}^4 p, \tag{6.16}$$

where $\bar{\epsilon} = \left(\frac{1-\epsilon}{1+\epsilon}\right)^{1/4} \in (0, 1)$.

As previously discussed, it will be enough to prove the estimate (6.11) for i sufficiently large; in the following we will assume that

$$1 + 1/i \leq \bar{\epsilon}^{-1}.$$

Under this assumption on i, from (6.15) and (6.16) we deduce that

$$Q \leq \bar{\epsilon}^2 \sqrt{pQ} \leq \bar{\epsilon}^2 (i + 1) \leq \bar{\epsilon} i. \tag{6.17}$$

We also remark that, for $j = 1, \ldots, d - k$,

$$\ell_{d-j+1} + \ell_{d-j} = p + \hat{q}_j, \qquad \text{where} \quad \hat{q}_j := \sum_{r=j+1}^{d-k} q_r - \sum_{r=1}^{j-1} q_r; \tag{6.18}$$

in particular, by (6.2), (6.12), and (6.18),

$$a^2_{\ell_{d-j+1}, \ell_{d-j}} = 1 - \frac{\ell^2_{d-j}}{\ell^2_{d-j+1}} = \frac{4 q_j (p + \hat{q}_j)}{(p + \hat{q}_j + q_j)^2} \simeq \frac{q_j}{p}, \tag{6.19}$$

where the latter estimate follows from (6.13)–(6.14). Moreover, by (6.12) and (6.18),

$$\frac{q_j}{p + \hat{q}_j} = \frac{1 - \ell_{d-j}/\ell_{d-j+1}}{1 + \ell_{d-j}/\ell_{d-j+1}},$$

so (6.7) implies

$$q_j \leq \bar{\epsilon}^4 (p + \hat{q}_j) \tag{6.20}$$

for $j = 1, \ldots, d - k$.

To prove the estimate (6.11), we split the sum into two parts. Let us first consider the range

$$|x| \geq 2 \max_{j\in\{1,\dots,d-k\}} a_{\ell_{d-j+1},\ell_{d-j}}; \tag{6.21}$$

here, and in what follows,

$$|x| = |x|_\infty = \max_{j\in\{1,\dots,d-k\}} |x_{d-j+1}|.$$

In light of (6.3), the inequalities

$$|\widetilde{X}^{d-j+1}_{\ell_{d-j+1},\ell_{d-j}}(x_{d-j+1})| \lesssim_\epsilon \ell^{1/4}_{d-j+1} \simeq p^{1/4}$$

hold for all $j \in \{1,\dots,d-k\}$. Moreover, for one of the quantities

$$|\widetilde{X}^{d}_{\ell_d,\ell_{d-1}}(x_d)|, \; \dots, \; |\widetilde{X}^{k+1}_{\ell_{k+1},\ell_k}(x_{k+1})|$$

the better bound $|x|^{-1/2}\exp(-cp|x|^2)$ holds for some $c > 0$, thanks to the 2nd estimate in (6.3) and to (6.8). As a consequence, we obtain

$$\begin{aligned}\left|\widetilde{X}^{d}_{\ell_d,\ell_{v-1}}(x_d)\right|^2 \cdots \left|\widetilde{X}^{k+1}_{\ell_{k+1},\ell_k}(x_{k+1})\right|^2 &\lesssim_\epsilon p^{(d-k-1)/2}|x|^{-1}\exp(-2cp|x|^2)\\ &\lesssim_N |x|^{-(d-k)}(p|x|^2)^{-N}\end{aligned}$$

for arbitrarily large $N \in \mathbb{N}$. Note then that the condition (6.21), together with (6.19), implies that

$$|x|^2 \gtrsim Q/p,$$

which, together with $\ell_d^2 - \ell_k^2 = pQ \in [i^2,(i+1)^2]$, yields

$$i|x| \gtrsim Q \gtrsim 1.$$

Then

$$
\begin{aligned}
&\sum_{\substack{(\ell_d,\dots,\ell_k)\in J_d^{(k)} \\ \epsilon\ell_d\le\ell_k \\ \ell_d^2-\ell_k^2\in[i^2,(i+1)^2] \\ |x|\ge 2\max_j a_{\ell_{d-j+1},\ell_{d-j}}}} \ell_d^{k-1-2\alpha}\left|\widetilde{X}^d_{\ell_d,\ell_{d-1}}(x_d)\right|^2\cdots\left|\widetilde{X}^{k+1}_{\ell_{k+1},\ell_k}(x_{k+1})\right|^2 \\
&\lesssim_{\epsilon,N} |x|^{-(d-k)-2N} \sum_{\substack{Q\le\bar{\epsilon}^4 p \\ pQ\in[i^2,(i+1)^2] \\ |x|^2\gtrsim Q/p}} p^{k-1-2\alpha-N} \\
&\lesssim |x|^{-(d-k)-2N} \sum_{Q\lesssim i|x|} \sum_{p\in[i^2/Q,(i+1)^2/Q]} p^{k-1-2\alpha-N} \\
&\lesssim |x|^{-(d-k)-2N} \sum_{Q\lesssim i|x|} (i/Q)(i^2/Q)^{k-1-2\alpha-N} \\
&= i^{2k-1-4\alpha-2N}|x|^{-(d-k)-2N} \sum_{Q\lesssim i|x|} Q^{N-k+2\alpha} \\
&\lesssim i^{2k-1-4\alpha-2N}|x|^{-(d-k)-2N}(i|x|)^{N+d-2k+2\alpha} \\
&= i^{d-1-2\alpha}|x|^{-k+2\alpha}\,(i|x|)^{-N} \lesssim i^{d-1-2\alpha}\min\{i,|x|^{-1}\}^{k-2\alpha},
\end{aligned}
$$

since $i|x|\gtrsim 1$ and $k-2\alpha>0$, provided N is large enough. Note that, in estimating the sum in p, we used the fact that the interval $[i^2/Q,(i+1)^2/Q]$ has length $(2i+1)/Q\simeq i/Q\gtrsim 1$. This concludes the proof of (6.11) in the range (6.21).

Let us now discuss the range

$$|x|\le 2\max_{j\in\{1,\dots,d-k\}} a_{\ell_{d-j+1},\ell_{d-j}}. \tag{6.22}$$

We first note that (6.22) and (6.18) imply

$$|x|^2\lesssim Q/p,$$

which, combined with $pQ\in[i^2,(i+1)^2]$ and $Q\in\mathbb{N}+(d-k)/2$, implies

$$p\lesssim i/|x| \qquad \text{and} \qquad Q\gtrsim\max\{1,i|x|\}.$$

Note that, by (6.19), for all $j = 1, \dots, d-k$,

$$(a_{\ell_{d-j+1},\ell_{d-j}})^2 = \varphi(q_j/(p + \hat{q}_j)), \tag{6.23}$$

where $\varphi(w) = 4w/(1+w)^2$. Note that the map $\varphi : [0,1] \to [0,1]$ is an increasing bijection, such that $w \le \varphi(w) \le 4w$; its derivative is given by $\varphi'(w) = 4\frac{1-w}{(1+w)^3}$ and vanishes only at $w = 1$. As a consequence, setting $\bar{x}_j = \sqrt{\varphi^{-1}(x_j^2)}$, with $j \in \{1, \dots, d-k\}$, one has $\bar{x}_j \simeq |x_j|$; moreover, in light of (6.23) and (6.20),

$$\left|x_{d-j+1}^2 - (a_{\ell_{d-j+1},\ell_{d-j}})^2\right| \simeq_\epsilon \left|\bar{x}_{d-j+1}^2 - q_j/(p+\hat{q}_j)\right|,$$

uniformly for $x \in [0,1]^{d-k}$. In particular, in this range, by (6.3),

$$\left|\widetilde{X}^{d-j+1}_{\ell_{d-j+1},\ell_{d-j}}(x_{d-j+1})\right|^2 \lesssim_\epsilon \Xi(\bar{x}_{d-j+1}^2, 1/p, q_j/(p+\hat{q}_j))$$

for all $j = 1, \dots, d-k$, where Ξ is defined as in (6.6). Then

$$\begin{aligned}
&\sum_{\substack{(\ell_d,\dots,\ell_k)\in J_d^{(k)} \\ \epsilon\ell_d \le \ell_k \\ \ell_d^2-\ell_k^2 \in [i^2,(i+1)^2] \\ |x| \le 2\max_j a_{\ell_{d-j+1},\ell_{d-j}}}} \ell_d^{k-1-2\alpha} \left|\widetilde{X}^d_{\ell_d,\ell_{d-1}}(x_d)\right|^2 \dots \left|\widetilde{X}^{k+1}_{\ell_{k+1},\ell_k}(x_{k+1})\right|^2 \\
&\lesssim_\epsilon \sum_{\substack{Q \le \bar{\epsilon}^4 p \\ pQ \in [i^2,(i+1)^2] \\ |x|^2 \lesssim Q/p}} p^{k-1-2\alpha} \prod_{j=1}^{d-k} \Xi(\bar{x}_{d-j+1}^2, 1/p, q_j/(p+\hat{q}_j)) \\
&\lesssim \sum_{\max\{1, i|x|\} \lesssim Q \le \bar{\epsilon} i} \left(\frac{i^2}{Q}\right)^{k-1-2\alpha} \sum_{p \in [i^2/Q,(i+1)^2/Q]} \vec{\Xi}(\bar{x}, p, q),
\end{aligned}$$

where $\bar{x} = (\bar{x}_d, \dots, \bar{x}_{k+1})$, $q = (q_1, \dots, q_{d-k})$ and

$$\vec{\Xi}(\bar{x}, p, q) = \prod_{j=1}^{d-k} \Xi(\bar{x}_{d-j+1}^2, 1/p, q_j/(p+\hat{q}_j)).$$

We now want to bound the inner sum in p with the corresponding integral. To justify this, we first note that

$$|\partial_p(1/p)|, |\partial_p(q_j/(p+\hat{q}_j))| \lesssim_\epsilon 1/p$$

for all $j = 1, \ldots, d-k$, on the range of summation; here we are using the fact that the condition $Q \le \bar\epsilon i$ implies that $q_j + |\hat q_j| \le Q \le \bar\epsilon^2 i^2/Q \le \bar\epsilon^2 p$ and $\bar\epsilon < 1$, whence $p + \hat q_j \simeq_\epsilon p \gtrsim q_j \gtrsim 1$. Thus, by Lemma 6.6,

$$|\partial_p \vec\Xi(\bar x, p, q)| \lesssim_\epsilon \vec\Xi(\bar x, p, q).$$

Moreover the interval $[i^2/Q, (i+1)^2/Q]$ has length $(2i+1)/Q \simeq i/Q \gtrsim 1$. Hence, by Lemma 6.5,

$$\begin{aligned}
&\sum_{\max\{1, i|x|\} \lesssim Q \le \bar\epsilon i} \left(\frac{i^2}{Q}\right)^{k-1-2\alpha} \sum_{p \in [i^2/Q, (i+1)^2/Q]} \vec\Xi(\bar x, p, q) \\
&\lesssim \sum_{\max\{1, i|x|\} \lesssim Q \le \bar\epsilon i} \left(\frac{i^2}{Q}\right)^{k-1-2\alpha} \int_{i^2/Q}^{(i+1)^2/Q} \vec\Xi(\bar x, p, q)\, \mathrm{d}p \\
&\simeq i^{2k-1-4\alpha} \int_i^{i+1} \sum_{\max\{1, i|x|\} \lesssim Q \le \bar\epsilon i} \hat\Xi(\bar x, u, q)\, \mathrm{d}u,
\end{aligned}$$

where the change of variables $p = u^2/Q$ was used, and

$$\begin{aligned}
\hat\Xi(\bar x, u, q) &= Q^{2\alpha-k} \vec\Xi(\bar x, u^2/Q, q) \\
&= Q^{2\alpha-k} \prod_{j=1}^{d-k} \Xi(\bar x_{d-j+1}^2, Q/u^2, q_j Q/(u^2 + \hat q_j Q)).
\end{aligned}$$

At this point, we can also bound the remaining sum in $q_1, \ldots, q_{d-k}$ with the corresponding integral. Indeed, it is easily checked that

$$|\nabla_q (Q/u^2)|, |\nabla_q (q_j Q/(u^2 + \hat q_j Q))| \lesssim_\epsilon Q/u^2$$

for all $j = 1, \ldots, d-k$, on the range of summation; here we are using that $|\hat q_j| Q \le Q^2 \le \bar\epsilon^2 i^2 \le \bar\epsilon^2 u^2$ and $\bar\epsilon < 1$, so $u^2 + \hat q_j Q \simeq_\epsilon u^2$. Therefore, by Lemma 6.6 and the Leibniz rule,

$$|\nabla_q \hat\Xi(\bar x, u, q)| \lesssim_\epsilon \hat\Xi(\bar x, u, q).$$

Hence, by Lemma 6.4,

$$
\begin{aligned}
& i^{2k-1-4\alpha} \int_i^{i+1} \sum_{\max\{1, i|x|\} \lesssim Q \le \bar{\epsilon} i} \hat{\Xi}(\bar{x}, u, q)\, \mathrm{d}u \\
& \lesssim_\epsilon i^{2k-1-4\alpha} \int_i^{i+1} \int_{\max\{1, i|x|\} \lesssim Q \le \bar{\epsilon} i} \hat{\Xi}(\bar{x}, u, q)\, \mathrm{d}q\, \mathrm{d}u \\
& \simeq \iint_{\substack{\max\{1, i|x|\} \lesssim Q \le \bar{\epsilon} i \\ pQ \in [i^2, (i+1)^2]}} \vec{\Xi}(\bar{x}, p, q)\, p^{k-1-2\alpha}\, \mathrm{d}q\, \mathrm{d}p \\
& \lesssim \iint_{\substack{\max\{i^{-1}, |x|\}^2 \lesssim V \le \bar{\epsilon}^2 \\ p^2 V \in [i^2, (i+1)^2]}} \vec{\Xi}(\bar{x}, p, pv)\, p^{d-1-2\alpha}\, \mathrm{d}p\, \mathrm{d}v
\end{aligned}
$$

where the change of variables $q_j = pv_j$ was used, and $V := \sum_{j=1}^{d-k} v_j$ (note that $Q \le \bar{\epsilon} i$ and $pQ \ge i^2$ implies $V = Q^2/(pQ) \le \bar{\epsilon}^2$). Now,

$$
\begin{aligned}
& \iint_{\substack{\max\{i^{-1}, |x|\}^2 \lesssim V \le \bar{\epsilon}^2 \\ p^2 V \in [i^2, (i+1)^2]}} \vec{\Xi}(\bar{x}, p, pv)\, p^{d-1-2\alpha}\, \mathrm{d}p\, \mathrm{d}v \\
& \lesssim \iint_{\substack{\max\{i^{-1}, |x|\}^2 \lesssim V \le \bar{\epsilon}^2 \\ p^2 V \in [i^2, (i+1)^2]}} \left(\frac{i}{\sqrt{V}} \right)^{d-1-2\alpha} \prod_{j=1}^{d-k} \left| \bar{x}_{d-j+1}^2 - \frac{v_j}{1 + \hat{v}_j} \right|^{-1/2} \mathrm{d}p\, \mathrm{d}v \\
& \lesssim i^{d-1-2\alpha} \int_{\max\{i^{-1}, |x|\}^2 \lesssim V \le \bar{\epsilon}^2} V^{-(d-2\alpha)/2} \prod_{j=1}^{d-k} \left| \bar{x}_{d-j+1}^2 - \frac{v_j}{1 + \hat{v}_j} \right|^{-1/2} \mathrm{d}v,
\end{aligned}
$$

where $\hat{v}_j = \sum_{r=j+1}^{d-k} v_r - \sum_{r=1}^{j-1} v_j$, and the fact that the interval $[i/\sqrt{V}, (i+1)/\sqrt{V}]$ has length $V^{-1/2}$ was used. We can now use the change of variables

$$
w_j = \frac{v_j}{1 + \hat{v}_j}, \qquad j = 1, \dots, d-k;
$$

indeed, $v_j, |\hat{v}_j| \in [0, \bar{\epsilon}^2]$ for all $j \in \{1, \dots, d-k\}$ on the domain of integration, and moreover $\bar{\epsilon} < 1$, whence

$$
w_j \simeq_\epsilon v_j \qquad \text{for all} \quad j \in \{1, \dots, d-k\}
$$

and (see Lemma 6.8)

$$\det(\partial_{v_s} w_j)_{j,s=1,\dots,d-k} = \left(\prod_{j=1}^{d-k} \frac{1}{1+\hat{v}_j}\right) \sum_{\substack{S\subseteq\{1,\dots,d-k\} \\ |S| \text{ even}}} \prod_{j\in S} \frac{v_j}{1+\hat{v}_j} \simeq_\epsilon 1,$$

so the change of variable yields

$$\begin{aligned} & i^{d-1-2\alpha} \int_{\max\{i^{-1},|x|\}^2 \lesssim V \le \bar{\epsilon}^2} V^{-(d-2\alpha)/2} \prod_{j=1}^{d-k} \left| \bar{x}_{d-j+1}^2 - \frac{v_j}{1+\hat{v}_j} \right|^{-1/2} \mathrm{d}v \\ & \simeq_\epsilon i^{d-1-2\alpha} \int_{\max\{i^{-1},|x|\}^2 \lesssim |w|} |w|^{-(d-2\alpha)/2} \prod_{j=1}^{d-k} \left| \bar{x}_{d-j+1}^2 - w_j \right|^{-1/2} \mathrm{d}w. \end{aligned}$$

In order to conclude, it is enough to bound the last integral with a multiple of $\min\{i, |x|^{-1}\}^{k-2\alpha}$. To do this, it is convenient to split the domain of integration according to whether w_j is larger or smaller than $2\bar{x}_{d-j+1}^2$ for each $j = 1, \dots, d-k$, and according to which j corresponds to the maximum component w_j of w. In other words,

$$\begin{aligned} & \int_{\max\{i^{-1},|x|\}^2 \lesssim |w|} |w|^{-(d-2\alpha)/2} \prod_{j=1}^{d-k} \left| \bar{x}_{d-j+1}^2 - w_j \right|^{-1/2} \mathrm{d}w \\ & \qquad \le \sum_{J\subseteq\{1,\dots,d-k\}} \sum_{j_*=1}^{d-k} \int_{\substack{\max\{i^{-1},|x|\}^2 \lesssim |w| \\ w_{j_*} = \max_j w_j \\ w_j \ge 2\bar{x}_{d-j+1}^2 \ \forall j\in J \\ w_j \le 2\bar{x}_{d-j+1}^2 \ \forall j \in J^{\mathrm{c}}}} |w|^{-(d-2\alpha)/2} \prod_{j=1}^{d-k} \left| \bar{x}_{d-j+1}^2 - w_j \right|^{-1/2} \mathrm{d}w, \end{aligned}$$

where $J^{\mathrm{c}} = \{1, \dots, d-k\} \setminus J$. We estimate separately each summand, depending on the choice of $j_* \in \{1, \dots, d-k\}$ and $J \subseteq \{1, \dots, d-k\}$, noting that, in the respective domain of integration, $|\bar{x}_{d-j+1}^2 - w_j|^{-1/2} \simeq w_j^{-1/2}$ for all $j \in J$.

Suppose first that $j_* \in J$, and set $J' = J \setminus \{j_*\}$. Then

$$\begin{aligned}
&\int_{\substack{\max\{i^{-1},|x|\}^2 \lesssim |w| \\ w_{j_*}=\max_j w_j \\ w_j \geq 2\bar{x}^2_{d-j+1} \ \forall j \in J \\ w_j \leq 2\bar{x}^2_{d-j+1} \ \forall j \in J^{\mathrm{c}}}} |w|^{-(d-2\alpha)/2} \prod_{j=1}^{d-k} \left| \bar{x}^2_{d-j+1} - w_j \right|^{-1/2} \mathrm{d}w \\
&\lesssim \int_{\max\{i^{-1},|x|\}^2 \lesssim w_{j_*}} w_{j_*}^{-(d-2\alpha)/2-1/2} \left(\prod_{j \in J'} \int_{w_j \leq w_{j_*}} w_j^{-1/2} \, \mathrm{d}w_j \right) \mathrm{d}w_{j_*} \\
&\qquad \times \left(\prod_{j \in J^{\mathrm{c}}} \int_{w_j \leq 2\bar{x}^2_{d-j+1}} \left| \bar{x}^2_{d-j+1} - w_j \right|^{-1/2} \mathrm{d}w_j \right) \\
&\lesssim \left(\prod_{j \in J^{\mathrm{c}}} |x_{d-j+1}| \right) \int_{\max\{i^{-1},|x|\}^2 \lesssim w_{j_*}} w_{j_*}^{-(d-|J|-2\alpha)/2-1} \, \mathrm{d}w_{j_*} \\
&\lesssim |x|^{|J^{\mathrm{c}}|} \max\{i^{-1}, |x|\}^{|J|-d+2\alpha} \leq \max\{i^{-1}, |x|\}^{-k+2\alpha} = \min\{i, |x|^{-1}\}^{k-2\alpha},
\end{aligned}$$

which is the desired estimate. Here we used that $d - |J| - 2\alpha \geq k - 2\alpha > 0$.

Suppose instead that $j_* \notin J$. In this range, $|x|^2 \lesssim \max\{i^{-1}, |x|\}^2 \lesssim |w| \simeq w_{j_*} \lesssim x^2_{d-j_*+1} \leq |x|^2$, whence $w_{j_*} \simeq |w| \simeq \max\{i^{-1}, |x|\}^2$. So

$$\begin{aligned}
&\int_{\substack{\max\{i^{-1},|x|\}^2 \lesssim |w| \\ w_{j_*}=\max_j w_j \\ w_j \geq 2\bar{x}^2_{d-j+1} \ \forall j \in J \\ w_j \leq 2\bar{x}^2_{d-j+1} \ \forall j \in J^{\mathrm{c}}}} |w|^{-(d-2\alpha)/2} \prod_{j=1}^{d-k} \left| \bar{x}^2_{d-j+1} - w_j \right|^{-1/2} \mathrm{d}w \\
&\lesssim \max\{i^{-1}, |x|\}^{-(d-2\alpha)} \left(\prod_{j \in J^{\mathrm{c}}} \int_{w_j \leq 2\bar{x}^2_{d-j+1}} \left| \bar{x}^2_{d-j+1} - w_j \right|^{-1/2} \mathrm{d}w_j \right) \\
&\qquad \times \left(\prod_{j \in J} \int_{w_j \lesssim \max\{i^{-1},|x|\}^2} w_j^{-1/2} \, \mathrm{d}w_j \right) \\
&\lesssim \max\{i^{-1}, |x|\}^{-(d-2\alpha)+|J|} |x|^{|J^{\mathrm{c}}|} \leq \min\{i, |x|^{-1}\}^{k-2\alpha},
\end{aligned}$$

and we are done. ■

6.4 The elliptic regime

We now discuss the proof of Proposition 4.3. We first observe that a straightforward iteration of Proposition 6.1(i) yields the following estimate.

Lemma 6.9. Fix $d \in \mathbb{N}$, $d \geq 2$, and $s \in \mathbb{N}$, $1 \leq s \leq d-1$. For all $\ell_d \in \mathbb{N}_d$ and all $(x_d, \dots, x_{s+1}) \in [-1,1]^{d-s}$,

$$\sum_{\substack{(\ell_{d-1},\dots,\ell_s)\in J^{(s)}_{d-1}\\ \ell_{d-1}\leq \ell_d}} \ell_s^{s-1} \left|\widetilde{X}^d_{\ell_d,\ell_{d-1}}(x_d)\right|^2 \dots \left|\widetilde{X}^{s+1}_{\ell_{s+1},\ell_s}(x_{s+1})\right|^2 \lesssim \ell_d^{d-1}.$$

Proof of Proposition 4.3. Arguing as at the beginning of the proof of Proposition 4.2, we readily see that it suffices to prove the estimate

$$\sum_{\substack{(\ell_d,\dots,\ell_k)\in J^{(k)}_{d}\\ \ell_k\leq \epsilon\ell_d \\ \ell_d^2-\ell_k^2\in[i^2,(i+1)^2]}} \ell_k^{k-1} \left|\widetilde{X}^d_{\ell_d,\ell_{d-1}}(x_d)\right|^2 \dots \left|\widetilde{X}^{k+1}_{\ell_{k+1},\ell_k}(x_{k+1})\right|^2 \lesssim_\epsilon i^{d-1} \tag{6.24}$$

for all $x \in [0,1]^{d-k}$ and $i \in \mathbb{N}\setminus\{0\}$.

We preliminary remark that, since $\epsilon \in (0,1)$, the conditions $\ell_k \leq \epsilon\ell_d$ and $\ell_d^2-\ell_k^2 \in [i^2,(i+1)^2]$ imply that

$$\ell_k \leq \ell_d \simeq_\epsilon i. \tag{6.25}$$

We first deal with the terms in the sum with $\ell_k = 0$ (observe that this may happen only for $k=1$). The condition $\ell_d^2 \in [i^2,(i+1)^2]$ uniquely determines the value of ℓ_d. Using the estimate in Proposition 6.1(ii) to bound $\widetilde{X}^{k+1}_{\ell_{k+1},0}(x_{k+1})$ in the left-hand side of (6.24) and then applying Lemma 6.9, we obtain

$$\begin{aligned}
&\sum_{\substack{(\ell_d,\dots,\ell_{k+1})\in J^{(k+1)}_{d}\\ \ell_d^2\in[i^2,(i+1)^2]}} \left|\widetilde{X}^d_{\ell_d,\ell_{d-1}}(x_d)\right|^2 \dots \left|\widetilde{X}^{k+2}_{\ell_{k+2},\ell_{k+1}}(x_{k+2})\right|^2 \left|\widetilde{X}^{k+1}_{\ell_{k+1},0}(x_{k+1})\right|^2 \\
&\lesssim \sum_{\substack{\ell_d\in\mathbb{N}_d\\ \ell_d^2\in[i^2,(i+1)^2]}} \sum_{\substack{(\ell_{d-1},\dots,\ell_{k+1})\in J^{(k+1)}_{d-1}\\ \ell_{d-1}\leq\ell_d}} \ell_{k+1}^k \left|\widetilde{X}^d_{\ell_d,\ell_{d-1}}(x_d)\right|^2 \dots \left|\widetilde{X}^{k+2}_{\ell_{k+2},\ell_{k+1}}(x_{k+2})\right|^2 \\
&\lesssim \sum_{\substack{\ell_d\in\mathbb{N}_d\\ \ell_d^2\in[i^2,(i+1)^2]}} \ell_d^{d-1} \\
&\lesssim i^{d-1},
\end{aligned}$$

which is the desired bound. In what follows, we shall therefore assume that $\ell_k > 0$ in the range of summation.

Define $y_j := \sqrt{1-x_j^2}$ for $j = k+1, \dots, d$, and recall the notation (6.1). Fix $j_* \in \{k+1, \dots, d\}$, and let us consider the range of the sum in (6.24) where

$$y_{j_*} \leq b_{\ell_{j_*}, \ell_{j_*-1}}/(2e). \tag{6.26}$$

By (6.5), in this case,

$$|\widetilde{X}^{j_*}_{\ell_{j_*}, \ell_{j_*-1}}(x_{j_*})|^2 \lesssim \ell_{j_*}^{j_*-1}\, e^{-2c\ell_{j_*-1}},$$

for a suitable $c \in (0, \infty)$. Hence, by (6.25),

$$\begin{aligned}
&\sum_{\substack{(\ell_d, \dots, \ell_k) \in J_d^{(k)} \\ 0<\ell_k \leq \epsilon \ell_d \\ \ell_d^2 - \ell_k^2 \in [i^2, (i+1)^2] \\ y_k \leq b_{\ell_{j_*}, \ell_{j_*-1}}/(2e)}} \ell_k^{k-1} \left|\widetilde{X}^d_{\ell_d, \ell_{d-1}}(x_d)\right|^2 \dots \left|\widetilde{X}^{k+1}_{\ell_{k+1}, \ell_k}(x_{k+1})\right|^2 \\
&\lesssim \sum_{\ell_{j_*-1} \in \mathbb{N}_{j_*-1}} e^{-2c\ell_{j_*-1}} \\
&\quad \times \sum_{\substack{(\ell_{j_*-2}, \dots, \ell_k) \in J_{j_*-2}^{(k)} \\ \ell_{j_*-2} \leq \ell_{j_*-1} \\ \ell_k \lesssim_\epsilon i}} \ell_k^{k-1} \left|\widetilde{X}^{j_*-1}_{\ell_{j_*-1}, \ell_{j_*-2}}(x_{j_*-1})\right|^2 \dots \left|\widetilde{X}^{k+1}_{\ell_{k+1}, \ell_k}(x_{k+1})\right|^2 \\
&\quad \times \sum_{\substack{(\ell_d, \dots, \ell_{j_*}) \in J_d^{(j_*)} \\ \ell_{j_*} \geq \ell_{j_*-1} \\ \ell_d \in [\sqrt{\ell_k^2 + i^2}, \sqrt{\ell_k^2 + (i+1)^2}]}} \left|\widetilde{X}^d_{\ell_d, \ell_{d-1}}(x_d)\right|^2 \dots \left|\widetilde{X}^{J+1}_{\ell_{j_*+1}, \ell_{j_*}}(x_{j_*+1})\right|^2 \ell_{j_*}^{j_*-1}.
\end{aligned}$$

Now, for a fixed $\ell_k \lesssim_\epsilon i$, the interval $[\sqrt{\ell_k^2 + i^2}, \sqrt{\ell_k^2 + (i+1)^2}]$ has length $\simeq_\epsilon 1$; so the sum over ℓ_d essentially contains only one term, and moreover $\ell_d \simeq_\epsilon i$. Thus, by applying

Lemma 6.9 first to the sum over $\ell_{j_*}, \cdots, \ell_{d-1}$ and then to the sum over $\ell_{j_*-2}, \ldots, \ell_k$, we get

$$\begin{aligned}
&\sum_{\substack{(\ell_d,\ldots,\ell_k)\in J_d^{(k)}\\ 0<\ell_k\le\epsilon\ell_d\\ \ell_d^2-\ell_k^2\in[i^2,(i+1)^2]\\ y_k\le b_{\ell_{j_*},\ell_{j_*-1}}/(2e)}} \ell_k^{k-1}\left|\widetilde{X}^d_{\ell_d,\ell_{d-1}}(x_d)\right|^2\ldots\left|\widetilde{X}^{k+1}_{\ell_{k+1},\ell_k}(x_{k+1})\right|^2\\
&\lesssim_\epsilon i^{d-1}\sum_{\ell_{j_*-1}\in\mathbb{N}_{j_*-1}} e^{-2c\ell_{j_*-1}}\\
&\quad\times\sum_{\substack{(\ell_{j_*-2},\ldots,\ell_k)\in J_{j_*-2}^{(k)}\\ \ell_{j_*-2}\le\ell_{j_*-1}}} \ell_k^{k-1}\left|\widetilde{X}^{j_*-1}_{\ell_{j_*-1},\ell_{j_*-2}}(x_{j_*-1})\right|^2\ldots\left|\widetilde{X}^{k+1}_{\ell_{k+1},\ell_k}(x_{k+1})\right|^2\\
&\lesssim i^{d-1}\sum_{\ell_{j_*-1}\in\mathbb{N}_{j_*-1}} e^{-2c\ell_{j_*-1}}\ell_{j_*-1}\lesssim i^{d-1}.
\end{aligned}$$

This concludes the proof of the estimate (6.24) in each range (6.26) corresponding to any $j_*\in\{k+1,\ldots,d\}$.

It remains to consider the range of the sum where

$$y_j > b_{\ell_j,\ell_{j-1}}/(2e) \qquad \text{for all} \quad j\in\{k+1,\ldots,d\}.$$

Here we may assume $y_j>0$ for $j=k+1,\ldots,d$ (otherwise the range is empty). We split the range of summation further, according to the value of k_*, defined as the smallest index in $\{k,\ldots,d\}$ for which one has

$$b_{\ell_j,\ell_{j-1}}/(2e) < y_j \le 2b_{\ell_j,\ell_{j-1}} \qquad \text{for all} \quad j>k_*.$$

Note that the above inequality implies that

$$\ell_{j-1}\simeq\ell_j y_j \qquad \text{for all} \quad j>k_*, \tag{6.27}$$

and moreover, by Corollary 6.3,

$$\left|\widetilde{X}^j_{\ell_j,\ell_{j-1}}(x_j)\right|^2 \lesssim y_j^{-(j-2)}\ \Xi(y_j^2,\ell_{j-1}/\ell_j^2,\ell_{j-1}^2/\ell_j^2) \tag{6.28}$$

for $k_*<j\le d$, where Ξ was defined in (6.6).

Assume first that $k_* > k$. Then $y_{k_*} > 2b_{\ell_{k_*},\ell_{k_*-1}}$, that is,

$$\ell_{k_*-1} < \frac{1}{2} y_{k_*} \ell_{k_*}, \tag{6.29}$$

whence, by Corollary 6.3,

$$|\widetilde{X}^{k_*}_{\ell_{k_*},\ell_{k_*-1}}(x_{k_*})|^2 \lesssim y_{k_*}^{-(k_*-1)}.$$

Moreover, from (6.25), (6.27), and (6.29) we deduce that

$$\ell_{k_*-1} \lesssim_\epsilon i y_{k_*} y_{k_*+1} \cdots y_d.$$

Hence

$$\begin{aligned} &\sum_{\substack{(\ell_d,\dots,\ell_k)\in J_d^{(k)} \\ 0<\ell_k\le\epsilon\ell_d \\ \ell_d^2-\ell_k^2\in[i^2,(i+1)^2] \\ b_{\ell_j,\ell_{j-1}}/(2e)<y_j\le 2b_{\ell_j,\ell_{j-1}}\ \forall j>k_* \\ y_{k_*}>2b_{\ell_{k_*},\ell_{k_*-1}}}} \ell_k^{k-1} \left|\widetilde{X}^d_{\ell_d,\ell_{d-1}}(x_d)\right|^2 \dots \left|\widetilde{X}^{k+1}_{\ell_{k+1},\ell_k}(x_{k+1})\right|^2 \\ &\lesssim \sum_{\substack{(\ell_{k_*-1},\dots,\ell_k)\in J_{k_*-1}^{(k)} \\ \ell_{k_*-1}\lesssim_\epsilon i y_{k_*} y_{k_*+1}\cdots y_d}} \ell_k^{k-1} \left|\widetilde{X}^d_{\ell_{k_*-1},\ell_{k_*-2}}(x_{k_*-1})\right|^2 \dots \left|\widetilde{X}^{k+1}_{\ell_{k+1},\ell_k}(x_{k+1})\right|^2 \\ &\quad\times \sum_{\substack{(\ell_d,\dots,\ell_{k_*})\in J_d^{(k_*)} \\ \ell_{j-1}\simeq\ell_j y_j\ \forall j>k_* \\ \ell_d\in[\sqrt{i^2+\ell_k^2},\sqrt{(i+1)^2+\ell_k^2}]}} y_{k_*}^{-(k_*-1)} \prod_{j=k_*+1}^d y_j^{-(j-2)}\, \Xi(y_j^2,\ell_{j-1}/\ell_j^2,\ell_{j-1}^2/\ell_j^2). \end{aligned} \tag{6.30}$$

We now want to bound the inner sum with the corresponding integral. Note that, for $j = k_*+1,\dots,d$,

$$|\nabla_{(\ell_d,\dots,\ell_{k_*})}(\ell_{j-1}/\ell_j^2)|, |\nabla_{(\ell_d,\dots,\ell_{k_*})}(\ell_{j-1}^2/\ell_j^2)| \lesssim \ell_{j-1}/\ell_j^2$$

in the range of summation; moreover the interval $[\sqrt{i^2+\ell_k^2}, \sqrt{(i+1)^2+\ell_k^2}]$ has length $\simeq 1$ and its endpoints are $\simeq i$, because $\ell_k \lesssim i$. Hence, in view of Lemma 6.6, we can apply

Lemma 6.4 to the inner sum and obtain that

$$
\begin{aligned}
&\sum_{\substack{(\ell_d,\dots,\ell_{k_*})\in J_d^{(k_*)}\\ \ell_{j-1}\simeq\ell_j y_j\ \forall j>k_*\\ \ell_d\in[\sqrt{i^2+\ell_k^2},\sqrt{(i+1)^2+\ell_k^2}]}} y_{k_*}^{-(k_*-1)}\prod_{j=k_*+1}^{d} y_j^{-(j-2)}\,\Xi(y_j^2,\ell_{j-1}/\ell_j^2,\ell_{j-1}^2/\ell_j^2)\\
&\lesssim\left(y_{k_*}^{-1}\prod_{j=k_*}^{d} y_j^{-(j-2)}\right)\int_{\substack{\ell_d\in[\sqrt{i^2+\ell_k^2},\sqrt{(i+1)^2+\ell_k^2}]\\ \ell_{j-1}\simeq\ell_j y_j\ \forall j>k_*}}\prod_{j=k_*+1}^{d}\left|y_j^2-\frac{\ell_{j-1}^2}{\ell_j^2}\right|^{-1/2}\mathrm{d}\ell_{k_*}\cdots\mathrm{d}\ell_d.
\end{aligned}
\tag{6.31}
$$

The change of variables $t_{j-1}=\ell_{j-1}/(\ell_j y_j)$, $j=k_*+1,\dots,d$, then gives

$$
\begin{aligned}
&\int_{\substack{\ell_d\in[\sqrt{i^2+\ell_k^2},\sqrt{(i+1)^2+\ell_k^2}]\\ \ell_{j-1}\simeq\ell_j y_j\ \forall j>k_*}}\prod_{j=k_*+1}^{d}\left|y_j^2-\frac{\ell_{j-1}^2}{\ell_j^2}\right|^{-1/2}\mathrm{d}\ell_{k_*}\cdots\mathrm{d}\ell_d\\
&\lesssim\int_{\ell_d\in[\sqrt{i^2+\ell_k^2},\sqrt{(i+1)^2+\ell_k^2}]}\int_{t_{k_*},\dots,t_{d-1}\simeq 1}\prod_{j=k_*+1}^{d}\frac{y_{j+1}\cdots y_d\, i}{|1-t_{j-1}^2|^{1/2}}\,\mathrm{d}t_{k_*}\cdots\mathrm{d}t_{d-1}\,\mathrm{d}\ell_d\\
&\simeq\prod_{j=k_*+1}^{d}(y_{j+1}\cdots y_d\, i)=i^{d-k_*}\prod_{j=k_*+2}^{d} y_j^{j-k_*-1},
\end{aligned}
$$

whence, by (6.31),

$$
\sum_{\substack{(\ell_d,\dots,\ell_{k_*})\in J_d^{(k_*)}\\ \ell_{j-1}\simeq\ell_j y_j\ \forall j>k_*\\ \ell_d\in[\sqrt{i^2+\ell_k^2},\sqrt{(i+1)^2+\ell_k^2}]}} y_{k_*}^{-(k_*-1)}\prod_{j=k_*+1}^{d} y_j^{-(j-2)}\,\Xi(y_j^2,\ell_{j-1}/\ell_j^2,\ell_{j-1}^2/\ell_j^2)\lesssim i^{d-k_*}(y_{k_*}\cdots y_d)^{1-k_*}
$$

and, by (6.30),

$$
\begin{aligned}
&\sum_{\substack{(\ell_d,\dots,\ell_k)\in J_d^{(k)}\\ 0<\ell_k\le\epsilon\ell_d\\ \ell_d^2-\ell_k^2\in[i^2,(i+1)^2]\\ b_{\ell_j,\ell_{j-1}}/(2e)<y_j\le 2b_{\ell_j,\ell_{j-1}}\ \forall j>k_*\\ y_{k_*}>2b_{\ell_{k_*},\ell_{k_*-1}}}}\ell_k^{k-1}\left|\widetilde{X}^d_{\ell_d,\ell_{d-1}}(x_d)\right|^2\dots\left|\widetilde{X}^{k+1}_{\ell_{k+1},\ell_k}(x_{k+1})\right|^2\\
&\lesssim i^{d-k_*}(y_{k_*}\cdots y_d)^{1-k_*}\\
&\qquad\times\sum_{\substack{(\ell_{k_*-1},\dots,\ell_k)\in J_{k_*-1}^{(k)}\\ \ell_{k_*-1}\lesssim_\epsilon i y_{k_*}y_{k_*+1}\cdots y_d}}\ell_k^{k-1}\left|\widetilde{X}^{k_*-1}_{\ell_{k_*-1},\ell_{k_*-2}}(x_{k_*-1})\right|^2\dots\left|\widetilde{X}^{k+1}_{\ell_{k+1},\ell_k}(x_{k+1})\right|^2\\
&\lesssim_\epsilon i^{d-k_*}(y_{k_*}\cdots y_d)^{1-k_*}\sum_{\ell_{k_*-1}\lesssim_\epsilon i y_{k_*}y_{k_*+1}\cdots y_d}\ell_{k_*-1}^{k_*-2}\lesssim_\epsilon i^{d-1},
\end{aligned}
$$

where Lemma 6.9 was applied to the sum in $(\ell_{k_*}, \dots, \ell_k)$ and the fact that $k_* \geq k+1 \geq 2$ was used. This concludes the proof of (6.24) in this range.

We now consider the case where $k_* = k$. Here, by (6.25), (6.27), and (6.28),

$$\begin{aligned}
&\sum_{\substack{(\ell_d,\dots,\ell_k)\in J_d^{(k)} \\ 0<\ell_k\leq\epsilon\ell_d \\ \ell_d^2-\ell_k^2\in[i^2,(i+1)^2] \\ b_{\ell_j,\ell_{j-1}}/(2e)<y_j\leq 2b_{\ell_j,\ell_{j-1}}\,\forall j>k}} \ell_k^{k-1}\left|\widetilde{X}^d_{\ell_d,\ell_{d-1}}(x_d)\right|^2\dots\left|\widetilde{X}^{k+1}_{\ell_{k+1},\ell_k}(x_{k+1})\right|^2 \\
&\lesssim \sum_{\substack{(\ell_d,\dots,\ell_k)\in J_d^{(k)} \\ \ell_{j-1}\simeq\ell_j y_j\ \forall j>k \\ \ell_d\simeq_\epsilon i \\ \ell_d\in[\sqrt{i^2+\ell_k^2},\sqrt{(i+1)^2+\ell_k^2}]}} \ell_k^{k-1}\prod_{j=k+1}^{d} y_j^{-(j-2)}\,\Xi(y_j^2,\ell_{j-1}/\ell_j^2,\ell_{j-1}^2/\ell_j^2) \\
&\lesssim_\epsilon i^{k-1}\left(\prod_{j=k+1}^{d} y_j^{k+1-j}\right)\sum_{\substack{(\ell_{d-1},\dots,\ell_k)\in J_{d-1}^{(k)} \\ \ell_j\simeq_\epsilon y_{j+1}\cdots y_d i\ \forall k\leq j<d}}\prod_{j=k+1}^{d-1}\Xi(y_j^2,\ell_{j-1}/\ell_j^2,\ell_{j-1}^2/\ell_j^2) \\
&\quad\times\sum_{\substack{\ell_d\in\mathbb{N}_d \\ \ell_d\in[\sqrt{i^2+\ell_k^2},\sqrt{(i+1)^2+\ell_k^2}]}}\Xi(y_d^2,\ell_{d-1}/\ell_d^2,\ell_{d-1}^2/\ell_d^2) \\
&\lesssim_\epsilon i^{k-1}\left(\prod_{j=k+1}^{d} y_j^{k+1-j}\right)\sum_{\substack{(\ell_{d-1},\dots,\ell_k)\in J_{d-1}^{(k)} \\ \ell_j\simeq_\epsilon y_{j+1}\cdots y_d i\ \forall k\leq j<d}}\prod_{j=k+1}^{d-1}\Xi(y_j^2,\ell_{j-1}/\ell_j^2,\ell_{j-1}^2/\ell_j^2) \\
&\quad\times\int_{\ell_d\in[\sqrt{i^2+\ell_k^2},\sqrt{(i+1)^2+\ell_k^2}]}\Xi(y_d^2,\ell_{d-1}/\ell_d^2,\ell_{d-1}^2/\ell_d^2)\,\mathrm{d}\ell_d,
\end{aligned}$$

where the last inequality follows from Lemma 6.5 together with Lemma 6.6, the fact that

$$|\partial_{\ell_d}(\ell_{d-1}/\ell_d^2)|,|\partial_{\ell_d}(\ell_{d-1}^2/\ell_d^2)|\lesssim\ell_{d-1}/\ell_d^2$$

in the range of summation and the fact that (since $\ell_k \lesssim_\epsilon i$) the length of the interval $[\sqrt{\ell_k^2+i^2},\sqrt{\ell_k^2+(i+1)^2}]$ is $\simeq_\epsilon 1$.

The change of variables $u = \sqrt{\ell_d^2 - \ell_k^2}$ in the inner integral then gives

$$\begin{aligned}
&\sum_{\substack{(\ell_d,\dots,\ell_{k_*}) \in J_d^{(k)} \\ 0<\ell_k \le \epsilon \ell_d \\ \ell_d^2 - \ell_k^2 \in [i^2,(i+1)^2] \\ b_{\ell_j,\ell_{j-1}}/(2e) < y_j \le 2 b_{\ell_j,\ell_{j-1}} \forall j>k}} \ell_k^{k-1} \left|\widetilde{X}^d_{\ell_d,\ell_{d-1}}(x_d)\right|^2 \dots \left|\widetilde{X}^{k+1}_{\ell_{k+1},\ell_k}(x_{k+1})\right|^2 \\
&\lesssim_\epsilon i^{k-1} \left(\prod_{j=k+1}^d y_j^{k+1-j}\right) \sum_{\substack{(\ell_{d-1},\dots,\ell_k) \in J_{d-1}^{(k)} \\ \ell_j \simeq_\epsilon y_{j+1} \cdots y_d i \ \forall k \le j < d}} \prod_{j=k+1}^{d-1} \Xi(y_j^2, \ell_{j-1}/\ell_j^2, \ell_{j-1}^2/\ell_j^2) \\
&\quad \times \int_i^{i+1} \Xi(y_d^2, \ell_{d-1}/(u^2+\ell_k^2), \ell_{d-1}^2/(u^2+\ell_k^2)) \, \mathrm{d}u \\
&= i^{k-1} \left(\prod_{j=k+1}^d y_j^{k+1-j}\right) \int_i^{i+1} \sum_{\substack{(\ell_{d-1},\dots,\ell_k) \in J_{d-1}^{(k)} \\ \ell_j \simeq_\epsilon y_{j+1} \cdots y_d i \ \forall k \le j < d}} \tilde{\Xi}(u, \vec{y}, \vec{\ell}) \, \mathrm{d}u,
\end{aligned}$$

where $\vec{\ell} = (\ell_{d-1}, \dots, \ell_k)$, $\vec{y} = (y_d, \dots, y_{k+1})$, and

$$\tilde{\Xi}(u, \vec{y}, \vec{\ell}) = \Xi(y_d^2, \ell_{d-1}/(u^2+\ell_k^2), \ell_{d-1}^2/(u^2+\ell_k^2)) \prod_{j=k+1}^{d-1} \Xi(y_j^2, \ell_{j-1}/\ell_j^2, \ell_{j-1}^2/\ell_j^2).$$

We now want to bound the inner sum with the corresponding integral. Observe that, since $u \in [i, i+1]$,

$$|\nabla_{\vec{\ell}} (\ell_{d-1}/(u^2+\ell_k^2))|, |\nabla_{\vec{\ell}} (\ell_{d-1}^2/(u^2+\ell_k^2))| \lesssim \ell_{d-1}/(u^2+\ell_k^2)$$

and

$$|\nabla_{\vec{\ell}} (\ell_{j-1}/\ell_j^2)|, |\nabla_{\vec{\ell}} (\ell_{j-1}^2/\ell_j^2)| \lesssim \ell_{j-1}/\ell_j^2 \qquad \text{for} \quad j = k+1, \dots, d-1$$

on the range of summation. Thanks to Lemma 6.6, we can apply Lemma 6.4 and obtain that

$$
\begin{aligned}
&\sum_{\substack{(\ell_d,\dots,\ell_k)\in J_d^{(k)}\\ 0<\ell_k\le\epsilon\ell_d\\ \ell_d^2-\ell_k^2\in[i^2,(i+1)^2]\\ b_{\ell_j,\ell_{j-1}}/(2e)<y_j\le 2b_{\ell_j,\ell_{j-1}}\,\forall j>k}} \ell_k^{k-1}\left|\widetilde{X}^d_{\ell_d,\ell_{d-1}}(x_d)\right|^2\dots\left|\widetilde{X}^{k+1}_{\ell_{k+1},\ell_k}(x_{k+1})\right|^2\\
&\lesssim_\epsilon i^{k-1}\left(\prod_{j=k+1}^d y_j^{k+1-j}\right)\int_i^{i+1}\int_{\ell_j\simeq_\epsilon y_{j+1}\cdots y_d i\ \forall k\le j<d}\tilde{\Xi}(u,\vec{y},\vec{\ell})\,\mathrm{d}\ell_k\cdots\mathrm{d}\ell_{d-1}\,\mathrm{d}u\\
&\lesssim i^{k-1}\left(\prod_{j=k+1}^d y_j^{k+1-j}\right)\int_i^{i+1}\int_{\ell_j\simeq_\epsilon y_{j+1}\cdots y_d i\ \forall k\le j<d}\left|y_d^2-\frac{\ell_{d-1}^2}{u^2+\ell_k^2}\right|^{-1/2}\\
&\quad\times\prod_{j=k+1}^{d-1}\left|y_j^2-\frac{\ell_{j-1}^2}{\ell_j^2}\right|^{-1/2}\mathrm{d}\ell_k\cdots\mathrm{d}\ell_{d-1}\,\mathrm{d}u.
\end{aligned}
$$

The change of variables $\ell_j = uy_{j+1}\cdots y_d\tau_j$, $j = k,\dots,d-1$, then gives

$$
\begin{aligned}
&\sum_{\substack{(\ell_d,\dots,\ell_k)\in J_d^{(k)}\\ 0<\ell_k\le\epsilon\ell_d\\ \ell_d^2-\ell_k^2\in[i^2,(i+1)^2]\\ b_{\ell_j,\ell_{j-1}}/(2e)<y_j\le 2b_{\ell_j,\ell_{j-1}}\,\forall j>k}} \ell_k^{k-1}\left|\widetilde{X}^d_{\ell_d,\ell_{d-1}}(x_d)\right|^2\dots\left|\widetilde{X}^{k+1}_{\ell_{k+1},\ell_k}(x_{k+1})\right|^2\\
&\lesssim_\epsilon i^{d-1}\left(\prod_{j=k+1}^d y_j\right)\int_i^{i+1}\int_{\tau_k,\dots,\tau_{d-1}\simeq_\epsilon 1}\left|y_d^2-\frac{y_d^2\tau_{d-1}^2}{1+y_{k+1}^2\cdots y_d^2\tau_k^2}\right|^{-1/2}\\
&\quad\times\prod_{j=k+1}^{d-1}\left|y_j^2-\frac{y_j^2\tau_{j-1}^2}{\tau_j^2}\right|^{-1/2}\mathrm{d}\tau_k\cdots\mathrm{d}\tau_{d-1}\,\mathrm{d}u\\
&= i^{d-1}\int_{\tau_k,\dots,\tau_{d-1}\simeq_\epsilon 1}\left|1-\frac{\tau_{d-1}^2}{1+y_{k+1}^2\cdots y_d^2\tau_k^2}\right|^{-1/2}\\
&\quad\times\prod_{j=k+1}^{d-1}\left|1-\frac{\tau_{j-1}^2}{\tau_j^2}\right|^{-1/2}\mathrm{d}\tau_k\cdots\mathrm{d}\tau_{d-1}\\
&\simeq_\epsilon i^{d-1}\int_{\tau_k,\dots,\tau_{d-1}\simeq_\epsilon 1}\left|\frac{1}{\tau_{d-1}^2}+\frac{y_{k+1}^2\cdots y_d^2\tau_k^2}{\tau_{d-1}^2}-1\right|^{-1/2}\\
&\quad\times\prod_{j=k+1}^{d-1}\left|1-\frac{\tau_{j-1}^2}{\tau_j^2}\right|^{-1/2}\mathrm{d}\tau_k\cdots\mathrm{d}\tau_{d-1}.
\end{aligned}
$$

Finally, the change of variables

$$t_{d-1} = \frac{1}{\tau_{d-1}}, \qquad t_j = \frac{\tau_j}{\tau_{j+1}} \quad \text{for} \quad j = k, \ldots, d-2$$

yields

$$\begin{aligned}
&\sum_{\substack{(\ell_d,\ldots,\ell_k)\in J_d^{(k)} \\ 0<\ell_k\leq\epsilon\ell_d \\ \ell_d^2-\ell_k^2\in[i^2,(i+1)^2] \\ b_{\ell_j,\ell_{j-1}}/(2e)<y_j\leq 2b_{\ell_j,\ell_{j-1}} \forall j>k}} \ell_k^{k-1} \left|\widetilde{X}^d_{\ell_d,\ell_{d-1}}(x_d)\right|^2 \ldots \left|\widetilde{X}^{k+1}_{\ell_{k+1},\ell_k}(x_{k+1})\right|^2 \\
&\lesssim_\epsilon i^{d-1} \int_{t_k,\ldots,t_{d-1}\simeq_\epsilon 1} \left|t_{d-1}^2 + y_{k+1}^2 \cdots y_d^2 t_k^2 \cdots t_{d-2}^2 - 1\right|^{-1/2} \\
&\quad \times \prod_{j=k+1}^{d-1} \left|1 - t_{j-1}^2\right|^{-1/2} \mathrm{d}t_k \cdots \mathrm{d}t_{d-1} \\
&\lesssim_\epsilon i^{d-1} \int_{t_k,\ldots,t_{d-2}\simeq_\epsilon 1} \prod_{j=k+1}^{d-1} \left|1 - t_{j-1}^2\right|^{-1/2} \int_{|v|\lesssim_\epsilon 1} |v|^{-1/2} \,\mathrm{d}v \,\mathrm{d}t_k \cdots \mathrm{d}t_{d-2} \lesssim_\epsilon i^{d-1},
\end{aligned}$$

and we are done. ■

7 Proof of the Abstract Multiplier Theorem

Here we give an outline proof of the multiplier theorem stated in Section 2. The proof combines ideas from multiple works on the subject, including [13, 17, 20, 31, 36], to which we refer for additional details.

Proof of Theorem 2.1. Similarly as in [36], for all $r \in (0,\infty)$, $\beta \in [0,\infty)$, $p \in [1,\infty]$ and $K : X \times X \to \mathbb{C}$, we define the norm

$$|||K|||_{p,\beta,r} = \operatorname{ess\,sup}_{z'\in X} \mu(B(z',r))^{1/p'} \|(1+\varrho(\cdot,z')/r)^\beta \, K(\cdot,z')\|_{L^p(X)},$$

where $p' = p/(p-1)$ is the conjugate exponent to p; if $r \in (0,1]$, we also define the norm

$$|||K|||^*_{p,\beta,r} = \operatorname{ess\,sup}_{z'\in X} \mu(B(z',r))^{1/p'} \|(1+\varrho(\cdot,z')/r)^\beta \, \pi_r(\cdot,z')^{1/p} \, K(\cdot,z')\|_{L^p(X)}.$$

Let also Q denote the homogeneous dimension of (X, ϱ, μ), that is, a positive constant (whose existence is a consequence of the doubling condition) such that

$$\mu(B(z, \lambda r)) \lesssim \lambda^Q \mu(B(z, r))$$

for all $z \in X$, $r > 0$ and $\lambda \geq 1$.

Due to the doubling condition and the heat kernel bounds, we can apply [36, Theorem 6.1] to obtain that, for all $\epsilon > 0$, all $\beta \geq 0$, all $R \in (0, \infty)$ and all $F : \mathbb{R} \to \mathbb{C}$ supported in $[-R^2, R^2]$,

$$|||\mathcal{K}_{F(\mathfrak{L})}|||_{2,\beta,R^{-1}} \lesssim_{\beta,\epsilon} \|F(R^2\cdot)\|_{L^\infty_{\beta+\epsilon}}, \tag{7.1}$$

$$\|F(\mathfrak{L})\|_{L^1(X)\to L^1(X)} \lesssim_\epsilon \|F(R^2\cdot)\|_{L^\infty_{Q/2+\epsilon}}, \tag{7.2}$$

It is worth noting that, since $\pi_r \gtrsim 1$ by (2.2), the estimate (2.3) trivially holds for all $\beta > Q$, $r > 0$ and $y \in X$ [20, Lemma 4.4]; so it is not restrictive to assume in what follows that $\mathfrak{d} \leq Q$.

Set $A_t = \exp(-t^2\mathfrak{L})$ if $t \in [0, \infty)$ and $A_t = 0$ if $t = \infty$. From (7.1) we deduce that, for all $t \in [0, \infty]$, all $\epsilon > 0$, all $\beta \geq 0$, all $R \in (0, \infty)$ and all $F : \mathbb{R} \to \mathbb{C}$ supported in $[R/16, R]$,

$$|||\mathcal{K}_{F(\sqrt{\mathfrak{L}})(1-A_t)}|||_{2,\beta,R^{-1}} \lesssim_{\beta,\epsilon} \|F(R\cdot)\|_{L^\infty_{\beta+\epsilon}} \min\{1, (Rt)^2\}.$$

Let $\xi \in C_c((-1/16, 1/16))$ be nonnegative with

$$\int_R \xi(t)\,\mathrm{d}t = 1 \qquad \text{and} \qquad \int_{\mathbb{R}} t^k \xi(t)\,\mathrm{d}t = 0 \quad \text{for} \quad k = 1, \ldots, 2Q+2.$$

(cf. [36, eq. (18)]). Then by Young's inequality we obtain that, for all $t \in [0, \infty]$, all $\epsilon > 0$, all $\beta \geq 0$, all $R \in [1, \infty)$ and all $F : \mathbb{R} \to \mathbb{C}$ supported in $[R/8, 7R/8]$,

$$|||\mathcal{K}_{(\xi*F)(\sqrt{\mathfrak{L}})(1-A_t)}|||_{2,\beta,R^{-1}} \lesssim_{\beta,\epsilon} \|F(R\cdot)\|_{L^\infty_{\beta+\epsilon}} \min\{1, (Rt)^2\}.$$

In particular, by (2.2) and Sobolev's embedding, for all $t \in [0, \infty]$, all $\epsilon > 0$, all $\beta \geq 0$, all $N \in \mathbb{N} \setminus \{0\}$ and all $F : \mathbb{R} \to \mathbb{C}$ supported in $[N/8, 7N/8]$,

$$|||\mathcal{K}_{(\xi*F)(\sqrt{\mathfrak{L}})(1-A_t)}|||^*_{2,\beta,N^{-1}} \lesssim_{\beta,\epsilon} \|F(N\cdot)\|_{L^q_{\beta+M_0+1/q+\epsilon}} \min\{1, (Nt)^2\}. \tag{7.3}$$

On the other hand, by (2.4), for all $t \in [0,\infty]$, all $N \in \mathbb{N} \setminus \{0\}$ and all $F : \mathbb{R} \to \mathbb{C}$ supported in $[N/16, N]$,

$$|||\mathcal{K}_{F(\sqrt{\mathfrak{L}})(1-A_t)}|||^*_{2,0,N^{-1}} \lesssim \|F(N\cdot)\|_{N,q} \min\{1,(Nt)^2\}.$$

Hence, by [20, eq. (4.9)], for all $t \in [0,\infty]$, all $N \in \mathbb{N} \setminus \{0\}$ and all $F : \mathbb{R} \to \mathbb{C}$ supported in $[N/8, 7N/8]$,

$$|||\mathcal{K}_{(\xi*F)(\sqrt{\mathfrak{L}})(1-A_t)}|||^*_{2,0,N^{-1}} \lesssim \|F(N\cdot)\|_{L^q} \min\{1,(Nt)^2\}. \tag{7.4}$$

Interpolation of (7.3) and (7.4) gives that, for all $t \in [0,\infty]$, all $\epsilon > 0$, all $\beta \geq 0$, all $N \in \mathbb{N} \setminus \{0\}$ and all $F : \mathbb{R} \to \mathbb{C}$ supported in $[N/4, 3N/4]$,

$$|||\mathcal{K}_{(\xi*F)(\sqrt{\mathfrak{L}})(1-A_t)}|||^*_{2,\beta,N^{-1}} \lesssim_{\beta,\epsilon} \|F(N\cdot)\|_{L^q_{\beta+\epsilon}} \min\{1,(Nt)^2\}. \tag{7.5}$$

By (2.3) and (7.5), we then deduce that, for all $r \in [0,\infty)$, all $t \in [0,\infty]$, all $s > \mathfrak{d}/2$, all $\epsilon \in [0, s-\mathfrak{d}/2)$, all $N \in \mathbb{N} \setminus \{0\}$ and all $F : \mathbb{R} \to \mathbb{C}$ supported in $[N/4, 3N/4]$,

$$\begin{aligned} &\operatorname{ess\,sup}_{z' \in X} \int_{X \setminus B(z',r)} |\mathcal{K}_{(\xi*F)(\sqrt{\mathfrak{L}})(1-A_t)}(z,z')| \,\mathrm{d}\mu(z) \\ &\quad \leq (1+Nr)^{-\epsilon} |||\mathcal{K}_{(\xi*F)(\sqrt{\mathfrak{L}})(1-A_t)}|||_{1,\epsilon,N^{-1}} \\ &\quad \lesssim_{s,\epsilon} (1+Nr)^{-\epsilon} |||\mathcal{K}_{(\xi*F)(\sqrt{\mathfrak{L}})(1-A_t)}|||^*_{2,\gamma,N^{-1}} \\ &\quad \lesssim_{s,\epsilon} (1+Nr)^{-\epsilon} \|F(N\cdot)\|_{L^q_s} \min\{1,(Nt)^2\}, \end{aligned} \tag{7.6}$$

where γ is the midpoint of $(\mathfrak{d}/2+\epsilon, s)$; more specifically, the 2nd inequality follows from (2.3) (applied with $\beta = 2(\gamma-\epsilon)$) and the Cauchy–Schwarz inequality, while the 3rd inequality is just (7.5) with γ and s in place of β and $\beta+\epsilon$.

On the other hand, from (2.3) and (2.4) we deduce that, for all $s > \eth/2$, all $\epsilon \in [0, \min\{s - \eth/2, \eth/2\})$, all $N \in \mathbb{N} \setminus \{0\}$ and all $F : \mathbb{R} \to \mathbb{C}$ supported in $[N/4, 3N/4]$,

$$\begin{aligned}
\|(F - \xi * F)(\sqrt{\mathfrak{L}})\|_{1\to 1} &= |||\mathcal{K}_{(F-\xi*F)(\sqrt{\mathfrak{L}})}|||_{1,0,N^{-1}} \\
&\lesssim_{s,\epsilon} |||\mathcal{K}_{(F-\xi*F)(\sqrt{\mathfrak{L}})}|||^*_{2,\gamma,N^{-1}} \\
&\leq (1+ND)^{\gamma} |||\mathcal{K}_{(F-\xi*F)(\sqrt{\mathfrak{L}})}|||^*_{2,0,N^{-1}} \\
&\lesssim_{s,\epsilon} N^{\gamma} \|(F - \xi * F)(N\cdot)\|_{N,q} \\
&\lesssim_{s,\epsilon} N^{-\epsilon} \|F(N\cdot)\|_{L^q_{\epsilon+\gamma}} \\
&\lesssim_{s,\epsilon} N^{-\epsilon} \|F(N\cdot)\|_{L^q_s},
\end{aligned} \tag{7.7}$$

where D is the ϱ-diameter of X and γ is the midpoint of $(\eth/2, \min\{\eth, s - \epsilon\})$; more specifically, the 2nd inequality follows from (2.3) (applied with $\beta = 2\gamma$) and the Cauchy–Schwarz inequality, the 4th inequality is (2.4), and the 5th follows from [20, Proposition 4.6].

Finally, from by (2.3) and (2.4) we deduce that, if $\operatorname{supp} F \subseteq [0, 1]$, then

$$\begin{aligned}
\|F(\sqrt{\mathfrak{L}})\|_{1\to 1} &= |||\mathcal{K}_{F(\sqrt{\mathfrak{L}})}|||_{1,0,1} \\
&\lesssim |||\mathcal{K}_{F(\sqrt{\mathfrak{L}})}|||^*_{2,\eth,1} \\
&\leq (1+D)^{\eth} |||\mathcal{K}_{F(\sqrt{\mathfrak{L}})}|||^*_{2,0,1} \\
&\lesssim \|F\|_{1,q} \\
&\leq \|F\|_{\infty};
\end{aligned} \tag{7.8}$$

more specifically, the 2nd inequality follows from (2.3) (applied with $\beta = \eth$) and the Cauchy–Schwarz inequality, while the 4th inequality is (2.4) applied with $r = N = 1$.

Combining (7.6) (applied with $t = \infty$, and $\epsilon = r = 0$) and (7.7) (applied with $\epsilon = 0$) gives in particular that, for all $s > \eth/2$, all $N \in \mathbb{N} \setminus \{0\}$ and all $F : \mathbb{R} \to \mathbb{C}$ supported in $[N/4, 3N/4]$,

$$\|F(\sqrt{\mathfrak{L}})\|_{1\to 1} \lesssim_s \|F(N\cdot)\|_{L^q_s}. \tag{7.9}$$

This estimate, combined with (7.8), easily gives a weak version of part (i): namely, for all $s > \eth/2$ and $F : \mathbb{R} \to \mathbb{C}$ supported in $[1/2, 1]$,

$$\sup_{t>0} \|F(t\sqrt{\mathfrak{L}})\|_{1\to 1} \lesssim_s \|F\|_{L^q_s}. \tag{7.10}$$

We now prove the full version of part (i). Fix an even cutoff function $\chi \in C_c^\infty(\mathbb{R})$ with $\chi(0) = 1$ and $\mathrm{supp}\chi \subseteq [-1, 1]$. Let $F : \mathbb{R} \to \mathbb{C}$ be supported in $[-1, 1]$ and set $\tilde{F} = F - F(0)\chi$. Note that, for all $k \in \mathbb{N}$,

$$\|F(0)\chi(\sqrt{\cdot})\|_{C^k} \lesssim_k \|F(0)\chi\|_{C^{2k}} \lesssim_k |F(0)| \lesssim_s \|F\|_{L^q_s},$$

by Sobolev's embedding, provided $s > 1/q$. In particular, from (7.2) it follows that

$$\sup_{t>0} \|F(0)\chi(t\sqrt{\mathfrak{L}})\|_{1\to 1} \lesssim_s \|F\|_{L^q_s} \tag{7.11}$$

for all $s > 1/q$, and moreover

$$\|\tilde{F}\|_{L^q_s} \lesssim_s \|F\|_{L^q_s}.$$

Let now $\xi \in C_c^\infty(\mathbb{R})$ be such that $\mathrm{supp}\xi \subseteq (1/2, 2)$ and $\sum_{k\in\mathbb{Z}} \xi(2^k\cdot) = 1$ on $(0, \infty)$. Decompose $\tilde{F} = \sum_{k\in\mathbb{N}} \tilde{F}_k(2^k\cdot)$ on $[0, \infty)$, where $\tilde{F}_k = \tilde{F}(2^{-k}\cdot)\xi$; since $\mathrm{supp}\tilde{F}_k \subseteq (1/2, 2)$, from (7.10) we deduce that

$$\sup_{t>0} \|\tilde{F}_k(t\sqrt{\mathfrak{L}})\|_{1\to 1} \lesssim_\beta \|\tilde{F}_k\|_{L^q_\beta}$$

provided $\beta > \eth/2$. On the other hand, arguing as in the proof of [35, Lemma 4.8], one deduces that, for all $\beta \geq 0$ and $s > \max\{\beta, 1/q\}$, there exists $\epsilon > 0$ such that

$$\|\tilde{F}_k\|_{L^q_\beta} \lesssim_{\beta,s} \|\tilde{F}_k\|_\infty + 2^{-k\epsilon}\|\tilde{F}\|_{L^q_s} \lesssim_s 2^{-k\epsilon}\|\tilde{F}\|_{L^q_s};$$

the latter estimate is due to the fact that $\tilde{F}(0) = 0$ and, by Sobolev's embedding, if $\|\tilde{F}\|_{L^q_s} < \infty$ for some $s > 1/q$, then $\tilde{F}$ is Hölder continuous. In conclusion, for all $t > 0$ and $s > \eth/2$,

$$\|\tilde{F}(t\sqrt{\mathfrak{L}})\|_{1\to 1} \leq \sum_{k\in\mathbb{N}} \|\tilde{F}_k(2^k t\sqrt{\mathfrak{L}})\|_{1\to 1} \lesssim_s \sum_{k\in\mathbb{N}} 2^{-k\epsilon}\|\tilde{F}\|_{L^q_s} \lesssim_s \|F\|_{L^q_s}; \tag{7.12}$$

combining the estimates (7.11) and (7.12) gives part (i).

As for part (ii), since the right-hand side of (2.5) is essentially independent of the cut-off function η, we may assume that $\operatorname{supp}\eta \subseteq (1/4, 1)$ and $\sum_{k\in\mathbb{Z}} \eta(2^k \cdot) = 1$ on $(0, \infty)$.

We now argue as in [20, proof of Theorems 3.1 and 3.2]. For a given $F : \mathbb{R} \to \mathbb{C}$ supported in $[1/2, \infty)$, we decompose dyadically $F = \sum_{k\in\mathbb{N}} \eta(2^{-k}\cdot)F$. By applying (7.6) to each dyadic piece and summing the corresponding estimate, we obtain, for all $s > \mathfrak{d}/2$ and $r > 0$,

$$\operatorname{ess\,sup}_{z'\in X} \int_{X\setminus B(z',r)} |\mathcal{K}_{(\xi*F)(\sqrt{\mathfrak{L}})(1-A_r)}(z, z')| \,\mathrm{d}\mu(z) \lesssim_s \sup_{k\in\mathbb{N}} \|F(2^k\cdot)\eta\|_{L^q_s}.$$

An application of [19, Theorem 1] then gives, for all $s > \mathfrak{d}/2$,

$$\|(\xi * F)(\sqrt{\mathfrak{L}})\|_{L^1\to L^{1,\infty}} \lesssim_s \sup_{k\in\mathbb{N}} \|\eta\, F(2^k\cdot)\|_{L^q_s}. \tag{7.13}$$

On the other hand, by applying (7.7) to each dyadic piece of F and summing the corresponding estimates, we obtain, for all $s > \mathfrak{d}/2$,

$$\|(F - \xi * F)(\sqrt{\mathfrak{L}})\|_{L^1\to L^1} \lesssim_s \sup_{k\in\mathbb{N}} \|\eta\, F(2^k\cdot)\|_{L^q_s}. \tag{7.14}$$

Combining the estimates (7.13) and (7.14) yields, for all $F : \mathbb{R} \to \mathbb{C}$ supported in $[1/2, \infty)$ and all $s > \mathfrak{d}/2$,

$$\|F(\sqrt{\mathfrak{L}})\|_{L^1\to L^{1,\infty}} \lesssim_s \sup_{k\in\mathbb{N}} \|\eta\, F(2^k\cdot)\|_{L^q_s}. \tag{7.15}$$

Via a partition of unity subordinated to $\{(1/2, \infty), (-\infty, 1)\}$, we can now combine (7.15) and (7.8) and obtain part (ii). ■

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References

[1] Agrachev, A. A., D. Barilari, and U. Boscain. *A Comprehensive Introduction to Sub-Riemannian Geometry*. Cambridge: Cambridge University Press, 2019.

[2] Ahrens, J., M. G. Cowling, A. Martini, and D. Müller. "Quaternionic spherical harmonics and a sharp multiplier theorem on quaternionic spheres." *Math. Z.* 294 (2020): 1659–86.

[3] Artin, E. *Geometric Algebra*. New York–London: Interscience Publishers, 1957.

[4] Axler, S., P. Bourdon, and W. Ramey. "Harmonic Function Theory." *Graduate Text in Mathematics*, vol. 137. New York: Springer, 2001.

[5] Baouendi, M. S. "Sur une classe d'opérateurs elliptiques dégénérés." *Bull. Soc. Math. France* 95 (1967): 45–87.

[6] Bauer, W., K. Furutani, and C. Iwasaki. "Spectral Analysis and Geometry of Sub-Laplacian and Related Grushin-Type Operators." *Partial Differential Equations and Spectral Theory, Oper. Theory Adv. Appl.*, vol. 211. 183–290. Basel: Birkhäuser/Springer Basel AG, 2011.

[7] Bauer, W., K. Furutani, and C. Iwasaki. "Fundamental solution of a higher step Grushin type operator." *Adv. Math.* 271 (2015): 188–234.

[8] Bellaïche, A. and J.-J. Risler. "Sub-Riemannian Geometry." *Progress in Mathematics*, vol. 144. Birkhäuser: Basel, 1996.

[9] Boscain, U. and C. Laurent. "The Laplace–Beltrami operator in almost-Riemannian geometry." *Univ. Grenoble. Ann. Institut Fourier. Univ. Grenoble I* 63 (2013): 1739–70.

[10] Boscain, U., D. Prandi, and M. Seri. "Spectral analysis and the Aharonov–Bohm effect on certain almost-Riemannian manifolds." *Comm. Partial Differential Equations* 41 (2016): 32–50.

[11] Boyd, W. G. C. and T. M. Dunster. "Uniform asymptotic solutions of a class of second-order linear differential equations having a turning point and a regular singularity, with an application to Legendre functions." *SIAM J. Math. Anal.* 17 (1986): 422–50.

[12] Calin, O. and D.-C. Chang. *Sub-Riemannian Geometry, Encyclopedia of Mathematics and Its Applications*, vol. 126. Cambridge: Cambridge University Press, 2009.

[13] Casarino, V., P. Ciatti, and A. Martini. "From refined estimates for spherical harmonics to a sharp multiplier theorem on the Grushin sphere." *Adv. Math.* 350 (2019): 816–59.

[14] Casarino, V., P. Ciatti, and A. Martini. "Uniform pointwise estimates for ultraspherical polynomials." (2020): preprint arXiv:2009.02323.

[15] Casarino, V., M. G. Cowling, A. Martini, and A. Sikora. "Spectral multipliers for the Kohn Laplacian on forms on the sphere in $\mathbb{C}^n$." *J. Geom. Anal.* 27 (2017): 3302–38.

[16] Cowling, M. G., O. Klima, and A. Sikora. “Spectral multipliers for the Kohn sublaplacian on the sphere in C^n.” *Trans. Amer. Math. Soc.* 363 (2011): 611–31.

[17] Cowling, M. G. and A. Sikora. “A spectral multiplier theorem for a sublaplacian on SU(2).” *Math. Z.* 238 (2001): 1–36.

[18] Dall’Ara, G. and A. Martini. “A robust approach to sharp multiplier theorems for Grushin operators.” *Trans. Amer. Math. Soc.* 373 (2020): 7533–74.

[19] Duong, X. T. and A. McIntosh. “Singular integral operators with non-smooth kernels on irregular domains.” *Rev. Mat. Iberoam.* 15 (1999): 233–65.

[20] Duong, X. T., E. M. Ouhabaz, and A. Sikora. “Plancherel-type estimates and sharp spectral multipliers.” *J. Funct. Anal.* 196 (2002): 443–85.

[21] Dziubański, J. and A. Sikora. “Lie group approach to Grushin operators.” *J. Lie Theory* 31 (2021): 1–14.

[22] Erdélyi, A., W. Magnus, F. Oberhettinger, and F. G. Tricomi. *Higher Transcendental Functions*, vol. II. Melbourne, Fla.: Robert E. Krieger Publishing Co. Inc., 1981.

[23] Franchi, B., C. E. Gutiérrez, and R. L. Wheeden. “Weighted Sobolev–Poincaré inequalities for Grushin type operators.” *Comm. Partial Differential Equations* 19 (1994): 523–604.

[24] Frank, R. L. and J. Sabin. “Spectral cluster bounds for orthonormal systems and oscillatory integral operators in Schatten spaces.” *Adv. Math.* 317 (2017): 157–92.

[25] Gallone, M., A. Michelangeli, and E. Pozzoli. “On geometric quantum confinement in Grushin-type manifolds.” *Z. Angew. Math. Phys.* 70 (2019).

[26] Gallone, M., A. Michelangeli, and E. Pozzoli. “Geometric confinement and dynamical transmission of a quantum particle in Grushin cylinder.” (2020): preprint arXiv:2003.07128.

[27] Geller, D. “The Laplacian and the Kohn Laplacian for the sphere.” *J. Differential Geom.* 15 (1980): 417–35.

[28] Grushin, V. V. “On a class of hypoelliptic operators.” *Math. USSR Sb.* 12 (1970): 458–76.

[29] Grushin, V. V. “A certain class of elliptic pseudodifferential operators that are degenerate on a submanifold.” *Mat. Sb. (N. S.)* 84 (1971): 163–95.

[30] Hebisch, W. “Multiplier theorem on generalized Heisenberg groups.” *Coll. Math.* 65 (1993): 231–9.

[31] Hebisch, W. “*Functional calculus for slowly decaying kernels*.” (1995): preprint http://www.math.uni.wroc.pl/~hebisch/.

[32] Hörmander, L. “Hypoelliptic second order differential equations.” *Acta Math.* 119 (1967): 147–71.

[33] Hörmander, L. “The spectral function of an elliptic operator.” *Acta Math.* 121 (1968): 193–218.

[34] Kenig, C. E., R. J. Stanton, and P. A. Tomas. “Divergence of eigenfunction expansions.” *J. Funct. Anal.* 46 (1982): 28–44.

[35] Martini, A. “Analysis of joint spectral multipliers on Lie groups of polynomial growth.” *Univ. Grenoble. Ann. Institut Fourier. Univ. Grenoble I* 62 (2012): 1215–63.

[36] Martini, A. “Joint functional calculi and a sharp multiplier theorem for the Kohn Laplacian on spheres.” *Math. Z.* 286 (2017): 1539–74.

[37] Martini, A. and D. Müller. "A sharp multiplier theorem for Grushin operators in arbitrary dimensions." *Rev. Mat. Iberoam.* 30 (2014): 1265–80.

[38] Martini, A. and D. Müller. "Spectral multipliers on 2-step groups: topological versus homogeneous dimension." *Geom. Funct. Anal.* 26 (2016): 680–702.

[39] Martini, A., D. Müller, and S. Nicolussi Golo. "Spectral multipliers and wave equation for sub-Laplacians: lower regularity bounds of Euclidean type." (2018): preprint arXiv:1812.02671.

[40] Martini, A. and A. Sikora. "Weighted Plancherel estimates and sharp spectral multipliers for the Grushin operators." *Math. Res. Lett.* 19 (2012): 1075–88.

[41] Melrose, R. B. "Propagation for the Wave Group of a Positive Subelliptic Second-Order Differential Operator." *Hyperbolic Equations and Related Topics (Katata/Kyoto, 1984)*. 181–92. Boston, MA: Academic Press, 1986.

[42] Mitjagin, B. S. "Divergenz von Spektralentwicklungen in L^p-Räumen." *Linear Operators and Approximation, II (Proc. Conf., Oberwolfach Math. Res. Inst., Oberwolfach, 1974)*. Internat. Ser. Numer. Math, vol. 25. 521–30. Berlin: Birkhäuser, 1974.

[43] Montgomery, R. "A Tour of Subriemannian Geometries." *Their Geodesics and Applications, Mathematical Surveys and Monographs*, vol. 91. Providence, RI: American Mathematical Society, 2002.

[44] Müller, D. and E. M. Stein. "On spectral multipliers for Heisenberg and related groups." *J. Math. Pures Appl.* 73 (1994): 413–40.

[45] Olver, F. W. J. "Second order linear differential equations with two turning points." *Philos. Trans. Roy. Soc. London Ser. A* 278 (1975): 137–74.

[46] Olver, F. W. J. *Asymptotics and Special Functions*. New York: Academic Press, 1974.

[47] Olver, F. W. J. "Legendre functions with both parameters large." *Philos. Trans. Roy. Soc. London Ser. A* 278 (1975): 175–85.

[48] Pesenson, I. "Parseval Space-Frequency Localized Frames on Sub-Riemannian Compact Homogeneous Manifolds." *Frames and Other Bases in Abstract and Function Spaces. Novel Methods in Harmonic Analysis, Volume 1*, Applied and and Numerical Harmonic Analysis, 413–33. Basel: Birkhäuser, 2017.

[49] Robinson, D. W. and A. Sikora. "Analysis of degenerate elliptic operators of Grušin type." *Math. Z.* 260 (2008): 475–508.

[50] Seeger, A. and C. D. Sogge. "On the boundedness of functions of (pseudo-) differential operators on compact manifolds." *Duke Math. J.* 59 (1989): 709–36.

[51] Sikora, A. "Riesz transform, Gaussian bounds and the method of wave equation." *Math. Z.* 247 (2004): 643–62.

[52] Sogge, C. D. "On the convergence of Riesz means on compact manifolds." *Ann. Math. (2)* 126 (1987): 439–47.

[53] Sogge, C. D. "Concerning the L^p norm of spectral clusters for second-order elliptic operators on compact manifolds." *J. Funct. Anal.* 77 (1988): 123–38.

[54] Stein, E. M. and G. L. Weiss. *Introduction to Fourier Analysis on Euclidean Spaces*. Princeton Mathematical Series, vol. 32. Princeton, NJ: Princeton University Press, 1971.

[55] Szegő, G. *Orthogonal Polynomials*. Amer. Math. Soc. Colloq. Publ, vol. 23. Providence, RI: Amererican Mathematical Society, 1974.

[56] Vilenkin, N. Ja. *Special Functions and the Theory of Group Representations*. Amer. Math. Soc. Colloq. Publ, vol. 22. Providence, RI: American Mathematical Society, 1968.