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Spectral multipliers on Heisenberg-Reiter and related groups

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Abstract Let L be a homogeneous sublaplacian on a 2-step stratified Lie group G of topological dimension d and homogeneous dimension Q . By a theorem due to Christ and to Mauceri and Meda, an operator of the form $F(L)$ is bounded on L^p for $1 < p < \infty$ if F satisfies a scale-invariant smoothness condition of order $s > Q/2$. Under suitable assumptions on G and L , here we show that a smoothness condition of order $s > d/2$ is sufficient. This extends to a larger class of 2-step groups the results for the Heisenberg and related groups by Müller and Stein and by Hebisch, and for the free group $N_{3,2}$ by Müller and the author.

Keywords nilpotent Lie groups · Heisenberg-Reiter groups · spectral multipliers · sublaplacians · Mihlin–Hörmander multipliers · singular integral operators

Mathematics Subject Classification (2000) 43A22 · 42B15

1 Introduction

Let L be a homogeneous sublaplacian on a stratified Lie group G of homogeneous dimension Q . Since L is a positive selfadjoint operator on $L^2(G)$, a functional calculus for L is defined via the spectral theorem and, for all Borel functions $F : \mathbb{R} \rightarrow \mathbb{C}$, the operator $F(L)$ is bounded on $L^2(G)$ whenever the “spectral multiplier” F is bounded. As for the L^p -boundedness for $p \neq 2$ of $F(L)$, a sufficient condition in terms of smoothness properties of the multiplier F is given by a theorem of Mihlin–Hörmander type due to Christ [4] and Mauceri and Meda [21]: the operator $F(L)$ is of weak type $(1, 1)$ and bounded on $L^p(G)$ for all $p \in]1, \infty[$ whenever

$$\|F\|_{MW_2^s} := \sup_{t>0} \|F(t \cdot) \eta\|_{W_2^s} < \infty$$

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for some $s > Q/2$, where $W_2^s(\mathbb{R})$ is the L^2 Sobolev space of fractional order s and $\eta \in C_c^\infty(]0, \infty[)$ is a nontrivial auxiliary function.

A natural question that arises is if the smoothness condition $s > Q/2$ is sharp. This is clearly true when G is abelian, so Q coincides with the topological dimension d of G , and L is essentially the Laplace operator on \mathbb{R}^d . Take however the smallest nonabelian example of a stratified group, that is, the Heisenberg group H_1 , which is defined by endowing $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ with the group law

$$(x, y, u) \cdot (x', y', u') = (x + x', y + y', u + u' + (xy' - x'y)/2) \quad (1)$$

and with the automorphic dilations

$$\delta_t(x, y, u) = (tx, ty, t^2u). \quad (2)$$

H_1 is a 2-step stratified group, and the homogeneous dimension of H_1 is 4. Nevertheless, a result by Müller and Stein [24] and Hebisch [12] shows that, for a homogeneous sublaplacian on H_1 , the smoothness condition on the multiplier can be pushed down to $s > d/2$, where $d = 3$ is the topological dimension of H_1 (in [24] it is also proved that the condition $s > d/2$ is sharp). Such an improvement of the Christ–Mauceri–Meda theorem holds not only for H_1 , but for the larger class of Métivier groups (and for direct products of Métivier and abelian groups), and also for differential operators other than sublaplacians (see, e.g., [14, 17]); moreover, as shown subsequently by Cowling and Sikora [5] (see also [6]), the sharp result on H_1 can be obtained by transplantation from an analogous result for a distinguished sublaplacian on the (non-stratified) group SU_2 (which in turn improves, in the case of SU_2 , an extension of the Christ–Mauceri–Meda theorem to spaces of homogeneous type [1, 13, 8]). However it is still an open question whether, for a general stratified Lie group (or even for a general 2-step stratified group), the homogeneous dimension in the smoothness condition can be replaced by the topological dimension.

The aim of this paper is to extend the class of the 2-step stratified groups and sublaplacians for which the smoothness condition in the multiplier theorem can be pushed down to half the topological dimension.

Take for instance the Heisenberg-Reiter group H_{d_1, d_2} (cf. [27]), defined by endowing $\mathbb{R}^{d_2 \times d_1} \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ with the group law (1) and the automorphic dilations (2); here however $\mathbb{R}^{d_2 \times d_1}$ is the set of the real $d_2 \times d_1$ matrices, and the products $xy', x'y$ in (1) are interpreted in the sense of matrix multiplication. H_{d_1, d_2} is a 2-step stratified group of homogeneous dimension $Q = d_1 d_2 + d_1 + 2d_2$ and topological dimension $d = d_1 d_2 + d_1 + d_2$. Despite the formal similarity with H_1 , the group H_{d_1, d_2} does not fall into the class of Métivier groups, unless $d_2 = 1$ (in fact, $H_{d_1, 1}$ is the $(2d_1 + 1)$ -dimensional Heisenberg group H_{d_1}). Nevertheless, the technique presented here allows one to handle the case $d_2 > 1$ too.

Namely, let $X_{1,1}, \dots, X_{d_2, d_1}, Y_1, \dots, Y_{d_1}, U_1, \dots, U_{d_2}$ be the left-invariant vector fields on H_{d_1, d_2} extending the standard basis of $\mathbb{R}^{d_2 \times d_1} \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ at the identity, and define the homogeneous sublaplacian L by

$$L = - \sum_{j=1}^{d_1} \sum_{k=1}^{d_2} X_{k,j}^2 - \sum_{j=1}^{d_2} Y_j^2.$$

Then a particular instance of our main result reads as follows.

Theorem 1. *Suppose that a function $F : \mathbb{R} \rightarrow \mathbb{C}$ satisfies*

$$\|F\|_{MW_2^s} < \infty$$

for some $s > d/2$. Then the operator $F(L)$ is of weak type $(1, 1)$ and bounded on $L^p(\mathbb{H}_{d_1, d_2})$ for all $p \in]1, \infty[$.

To the best of our knowledge, this result is new, at least in the case $d_2 > d_1$. In fact, in the case $d_2 \leq d_1$, the extension described in [17] of the technique of [12, 14] would give the same result. However the technique presented here is different, and yields the result irrespective of the parameters d_1, d_2 .

The left quotient of \mathbb{H}_{d_1, d_2} by the subgroup $\mathbb{R}^{d_2 \times d_1} \times \{0\} \times \{0\}$ gives a homogeneous space diffeomorphic to $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, and the sublaplacian L corresponds in the quotient to a Grushin operator. In recent joint works with Sikora [20] and Müller [18], we proved for these Grushin operators on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ a sharp spectral multiplier theorem of Mihlin–Hörmander type, where the smoothness requirement is again half the topological dimension of the ambient space.

The proofs in [20, 18] rely heavily on properties of the eigenfunction expansions for the Hermite operators. Since a homogeneous sublaplacian on a 2-step stratified group reduces to a Hermite operator in almost all irreducible unitary representations of the group, it is conceivable that an adaptation of the methods of [20, 18] may give an improvement to the multiplier theorem for 2-step stratified groups, even outside of the Métivier setting. A first result in this direction is shown in [19], where the free 2-step nilpotent Lie group $N_{3,2}$ on three generators is considered, and properties of Laguerre polynomials are exploited (somehow in the spirit of [7, 24, 23]). The argument presented here refines and extends the one in [19].

Theorem 1 above is just a particular case of the result presented here, and we refer the reader to the next section for a precise statement. We remark that the analogue of Theorem 1 holds on \mathbb{H}_{d_1, d_2} when the sublaplacian L has the more general form

$$L = - \sum_{j=1}^{d_1} \sum_{k, k'=0}^{d_2} a_{k, k'}^j X_{k, j} X_{k', j} \quad (3)$$

where $X_{0, j} = Y_j$ and $(a_{k, k'}^j)_{k, k'=0, \dots, d_2}$ is a positive-definite symmetric matrix for all $j \in \{1, \dots, d_1\}$. Other groups can be considered too, e.g., the complexification of a Heisenberg-Reiter group, or the quotient of the direct product of $\mathbb{H}_{1,3}$ and $N_{3,2}$ given by identifying the respective centers.

2 The general setting

Let G be a connected, simply connected nilpotent Lie group of step 2. Recall that, via exponential coordinates, G may be identified with its Lie algebra \mathfrak{g} , that is, the tangent space of G at the identity. In turn, \mathfrak{g} may be identified with the Lie algebra of left-invariant vector fields on G . We refer to [11] for the basic definitions and further details.

Let \mathfrak{g} be decomposed as $\mathfrak{v} \oplus \mathfrak{z}$, where \mathfrak{z} is the center of \mathfrak{g} , and let $\langle \cdot, \cdot \rangle$ be an inner product on \mathfrak{v} . The sublaplacian L associated with the inner product is defined by $L = - \sum_j X_j^2$, where $\{X_j\}_j$ is any orthonormal basis of \mathfrak{v} . Note that, vice versa, by the Poincaré-Birkhoff-Witt theorem, any second-order operator L

of the form $-\sum_j X_j^2$ for some basis $\{X_j\}_j$ of \mathfrak{g} modulo \mathfrak{z} determines uniquely a linear complement $\mathfrak{v} = \text{span}\{X_j\}_j$ of \mathfrak{z} and an inner product on \mathfrak{v} such that $\{X_j\}_j$ is orthonormal.

Let \mathfrak{z}^* be the dual of \mathfrak{z} and, for all $\eta \in \mathfrak{z}^*$, define J_η as the linear endomorphism of \mathfrak{v} such that $\eta([z, z']) = \langle J_\eta z, z' \rangle$ for all $z, z' \in \mathfrak{v}$. Clearly J_η is skewadjoint with respect to the inner product, hence J_η^2 is selfadjoint and negative semidefinite, with even rank, for all $\eta \in \mathfrak{z}^*$. Set moreover $\mathfrak{j} = \mathfrak{z}^* \setminus \{0\}$.

Assumption (A). *There exist integers $r_1, \dots, r_{d_1} > 0$ and an orthogonal decomposition $\mathfrak{v} = \mathfrak{v}_1 \oplus \dots \oplus \mathfrak{v}_{d_1}$ such that, if P_1, \dots, P_{d_1} are the corresponding orthogonal projections, then $J_\eta P_j = P_j J_\eta$ and $J_\eta^2 P_j$ has rank $2r_j$ and a unique nonzero eigenvalue for all $\eta \in \mathfrak{j}$ and all $j \in \{1, \dots, d_1\}$.*

Note that from Assumption (A) it follows that $J_\eta \neq 0$ for all $\eta \in \mathfrak{j}$. Therefore $[\mathfrak{v}, \mathfrak{v}] = \mathfrak{z}$, that is, the decomposition $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$ is a stratification of \mathfrak{g} , and the sublaplacian L is hypoelliptic.

In fact J_η has constant rank $2(r_1 + \dots + r_k)$ for all $\eta \in \mathfrak{j}$. If J_η is invertible for all $\eta \in \mathfrak{j}$, then G is a Métivier group, and if in particular $J_\eta^2 = -|\eta|^2 \text{id}_{\mathfrak{v}}$ for some inner product norm $|\cdot|$ on \mathfrak{z}^* , then G is an H-type group. The main novelty of our Assumption (A) is that it allows J_η to have a nonzero kernel when $\eta \in \mathfrak{j}$, although the dimension of the kernel must be constant.

The fact that J_η has constant rank for $\eta \in \mathfrak{j}$ depends only on the algebraic structure of G . What depends on the inner product, that is, on the sublaplacian L , are the values and multiplicities of the eigenvalues of the J_η . The above Assumption (A) asks for a sort of simultaneous diagonalizability of the J_η .

Under our Assumption (A) on the group G and the sublaplacian L , we are able to prove the following multiplier theorem.

Theorem 2. *Suppose that a function $F : \mathbb{R} \rightarrow \mathbb{C}$ satisfies*

$$\|F\|_{MW_2^s} < \infty$$

for some $s > (\dim G)/2$. Then the operator $F(L)$ is of weak type $(1, 1)$ and bounded on $L^p(G)$ for all $p \in]1, \infty[$.

The previously mentioned Heisenberg-Reiter groups H_{d_1, d_2} satisfy Assumption (A), where the inner product is determined by the sublaplacian (3), and the orthogonal decomposition of the first layer is given by the natural isomorphism $\mathbb{R}^{d_2 \times d_1} \times \mathbb{R}^{d_1} \cong (\mathbb{R}^{d_2} \times \mathbb{R})^{d_1}$. Other examples are the free 2-step nilpotent Lie group $N_{3,2}$ on 3 generators, considered in [19], and its complexification $N_{3,2}^{\mathbb{C}}$. Moreover, if G_1 and G_2 satisfy Assumption (A), and their centers have the same dimension, then the quotient of $G_1 \times G_2$ given by any linear identification of the centers satisfy Assumption (A). Note that the direct product $G_1 \times G_2$ itself does not satisfy Assumption (A), but an adaptation of the argument presented here allows one to consider that case too. We postpone to the end of this paper a more detailed discussion of these remarks.

From now on, unless otherwise specified, we assume that G and L are a 2-step stratified group and a homogeneous sublaplacian on G satisfying Assumption (A). Since L is a left-invariant operator, so is any operator of the form $F(L)$. Let $\mathcal{K}_{F(L)}$ denote the convolution kernel of $F(L)$. As shown, e.g., by [17, Theorem 4.6], the previous theorem is a consequence of the following estimate.

Proposition 3. *For all $s > (\dim G)/2$, there exists a weight $w_s : G \rightarrow [1, \infty[$ such that $w_s^{-1} \in L^2(G)$ and, for all compact sets $K \subseteq \mathbb{R}$ and for all functions $F : \mathbb{R} \rightarrow \mathbb{C}$ with $\text{supp } F \subseteq K$,*

$$\|w_s \mathcal{K}_{F(L)}\|_2 \leq C_{K,s} \|F\|_{W_2^s}; \quad (4)$$

in particular

$$\|\mathcal{K}_{F(L)}\|_1 \leq C_{K,s} \|F\|_{W_2^s}. \quad (5)$$

The rest of the paper, except for the last section, is devoted to the proof of this estimate.

3 The joint functional calculus

Let $d_2 = \dim \mathfrak{z}$, and let U_1, \dots, U_{d_2} be any basis of the center \mathfrak{z} . Let moreover the ‘‘partial sublaplacian’’ L_j be defined as $L_j = -\sum_l X_{j,l}^2$, where $\{X_{j,l}\}_l$ is any orthonormal basis of \mathfrak{v}_j , for all $j \in \{1, \dots, d_1\}$; in particular $L = L_1 + \dots + L_{d_1}$. Then the left-invariant differential operators

$$L_1, \dots, L_{d_1}, -iU_1, \dots, -iU_{d_2} \quad (6)$$

are essentially selfadjoint and commute strongly, hence they admit a joint functional calculus (see, e.g., [16]). Therefore, if \mathbf{L} and \mathbf{U} denote the ‘‘vectors of operators’’ (L_1, \dots, L_{d_1}) and $(-iU_1, \dots, -iU_{d_2})$, and if we identify \mathfrak{z}^* with \mathbb{R}^{d_2} via the dual basis of U_1, \dots, U_{d_2} , then, for all bounded Borel functions $H : \mathbb{R}^{d_1} \times \mathfrak{z}^* \rightarrow \mathbb{C}$, the operator $H(\mathbf{L}, \mathbf{U})$ is defined and bounded on $L^2(G)$. Moreover $H(\mathbf{L}, \mathbf{U})$ is left-invariant, and we can express its convolution kernel $\mathcal{K}_{H(\mathbf{L}, \mathbf{U})}$ in terms of Laguerre functions.

Namely, for all $n, k \in \mathbb{N}$, let

$$L_n^{(k)}(t) = \frac{t^{-k} e^t}{n!} \left(\frac{d}{dt} \right)^n (t^{k+n} e^{-t})$$

be the n -th Laguerre polynomial of type k , and define

$$\mathcal{L}_n^{(k)}(t) = (-1)^n e^{-t} L_n^{(k)}(2t).$$

Note that, by Assumption (A), for all $\eta \in \mathfrak{j}$ and $j \in \{1, \dots, d_1\}$,

$$J_\eta^2 P_j = -(b_j^\eta)^2 P_j^\eta$$

for some orthogonal projection P_j^η of rank $2r_j$ and some $b_j^\eta > 0$. Set moreover

$$\bar{P}_j^\eta = P_j - P_j^\eta.$$

Modulo reordering the \mathfrak{v}_j in the decomposition of \mathfrak{v}_2 we may suppose that there exists $\tilde{d}_1 \in \{0, \dots, d_1\}$ such that $\dim \mathfrak{v}_j > 2r_j$ if $j \leq \tilde{d}_1$, and $\dim \mathfrak{v}_j = 2r_j$ if $j > \tilde{d}_1$. In particular, $\bar{P}_j^\eta = 0$ and $P_j^\eta = P_j$ for all $j > \tilde{d}_1$ and $\eta \in \mathfrak{j}$. We will also use the abbreviations $r = (r_1, \dots, r_{d_1})$, $\mathbb{R}^r = \mathbb{R}^{r_1} \times \dots \times \mathbb{R}^{r_{d_1}}$, $\mathbb{N}^r = \mathbb{N}^{r_1} \times \dots \times \mathbb{N}^{r_{d_1}}$, $|r| = r_1 + \dots + r_{d_1}$. Moreover $\langle \cdot, \cdot \rangle$ will also denote the duality pairing $\mathfrak{z}^* \times \mathfrak{z} \rightarrow \mathbb{R}$.

Proposition 4. *Let $H : \mathbb{R}^{d_1} \times \mathfrak{z}^* \rightarrow \mathbb{C}$ be in the Schwartz class, and set*

$$m(n, \mu, \eta) = H((2n_1 + r_1)b_1^\eta + \mu_1, \dots, (2n_{\bar{d}_1} + r_{\bar{d}_1})b_{\bar{d}_1}^\eta + \mu_{\bar{d}_1}, \\ (2n_{\bar{d}_1+1} + r_{\bar{d}_1+1})b_{\bar{d}_1+1}^\eta, \dots, (2n_{d_1} + r_{d_1})b_{d_1}^\eta, \eta) \quad (7)$$

for all $n \in \mathbb{N}^{d_1}$, $\mu \in \mathbb{R}^{\bar{d}_1}$, $\eta \in \mathfrak{z}$. Then, for all $(z, u) \in G$,

$$\mathcal{K}_{H(\mathbf{L}, \mathbf{U})}(z, u) = \frac{2^{|r|}}{(2\pi)^{\dim G}} \int_{\mathfrak{z}} \int_{\mathfrak{v}} \sum_{n \in \mathbb{N}^{d_1}} m(n, (|\bar{P}_1^\eta \xi|^2, \dots, |\bar{P}_{\bar{d}_1}^\eta \xi|^2), \eta) \\ \times \left[\prod_{j=1}^{d_1} \mathcal{L}_{n_j}^{(r_j-1)}(|P_j^\eta \xi|^2 / b_j^\eta) \right] e^{i\langle \xi, z \rangle} e^{i\langle \eta, u \rangle} d\xi d\eta. \quad (8)$$

Proof. For all $\eta \in \mathfrak{z}$ and $j \in \{1, \dots, d_1\}$, let $E_{j,1}^\eta, \bar{E}_{j,1}^\eta, \dots, E_{j,r_j}^\eta, \bar{E}_{j,r_j}^\eta$ be an orthonormal basis of the range of P_j^η such that

$$J_\eta E_{j,l}^\eta = b_j^\eta \bar{E}_{j,l}^\eta, \quad J_\eta \bar{E}_{j,l}^\eta = -b_j^\eta E_{j,l}^\eta, \quad \text{for } l = 1, \dots, r_j.$$

Hence, for all $z \in \mathfrak{v}$, $\eta \in \mathfrak{z}$, and $j \in \{1, \dots, d_1\}$, we can write

$$P_j^\eta z = \sum_{l=1}^{r_j} (z_{j,l}^\eta E_{j,l}^\eta + \bar{z}_{j,l}^\eta \bar{E}_{j,l}^\eta)$$

for some uniquely determined $z_{j,l}^\eta, \bar{z}_{j,l}^\eta \in \mathbb{R}$; set then $z_j^\eta = (z_{j,1}^\eta, \dots, z_{j,r_j}^\eta)$, $\bar{z}_j^\eta = (\bar{z}_{j,1}^\eta, \dots, \bar{z}_{j,r_j}^\eta)$, and moreover $z^\eta = (z_1^\eta, \dots, z_{d_1}^\eta)$ and $\bar{z}^\eta = (\bar{z}_1^\eta, \dots, \bar{z}_{d_1}^\eta)$.

For all $\eta \in \mathfrak{z}$ and all $\rho \in \ker J_\eta$, an irreducible unitary representation $\pi_{\eta,\rho}$ of G on $L^2(\mathbb{R}^r)$ is defined by

$$\pi_{\eta,\rho}(z, u)\phi(v) = e^{i\langle \eta, u \rangle} e^{i\langle \rho, \bar{P}^\eta z \rangle} e^{i \sum_{j=1}^{d_1} b_j^\eta \langle v_j + z_j^\eta / 2, \bar{z}_j^\eta \rangle} \phi(z^\eta + v)$$

for all $(z, u) \in G$, $v \in \mathbb{R}^r$, $\phi \in L^2(\mathbb{R}^r)$, where $\bar{P}^\eta = \bar{P}_1^\eta + \dots + \bar{P}_{\bar{d}_1}^\eta$ is the orthogonal projection onto $\ker J_\eta$. Following, e.g., [2, §2], one can see that these representations are sufficient to write the Plancherel formula for the group Fourier transform of G , and the corresponding Fourier inversion formula:

$$f(z, u) = (2\pi)^{|r| - \dim G} \int_{\mathfrak{z}} \int_{\ker J_\eta} \text{tr}(\pi_{\eta,\rho}(z, u) \pi_{\eta,\rho}(f)) \prod_{j=1}^{d_1} (b_j^\eta)^{r_j} d\rho d\eta \quad (9)$$

for all $f : G \rightarrow \mathbb{C}$ in the Schwartz class and all $(z, u) \in G$, where $\pi_{\eta,\rho}(f) = \int_G f(g) \pi_{\eta,\rho}(g^{-1}) dg$.

Fix $\eta \in \mathfrak{z}$ and $\rho \in \ker J_\eta$. The operators (6) are represented in $\pi_{\eta,\rho}$ as

$$d\pi_{\eta,\rho}(L_j) = -\Delta_{v_j}^2 + (b_j^\eta)^2 |v_j|^2 + |P_j \rho|^2, \quad d\pi_{\eta,\rho}(-iU_k) = \eta_k, \quad (10)$$

for all $j \in \{1, \dots, d_1\}$ and $k \in \{1, \dots, d_2\}$, where $v_j \in \mathbb{R}^{r_j}$ denotes the j -th component of $v \in \mathbb{R}^r$, and Δ_{v_j} denotes the corresponding partial Laplacian. Let h_ℓ denote the ℓ -th Hermite function, that is,

$$h_\ell(t) = (-1)^\ell (\ell! 2^\ell \sqrt{\pi})^{-1/2} e^{t^2/2} \left(\frac{d}{dt} \right)^\ell e^{-t^2},$$

and, for all $\omega \in \mathbb{N}^r$, define $\tilde{h}_{\eta,\omega} : \mathbb{R}^r \rightarrow \mathbb{R}$ by

$$\tilde{h}_{\eta,\omega} = \tilde{h}_{\eta,\omega,1} \otimes \cdots \otimes \tilde{h}_{\eta,\omega,d_1}, \quad \tilde{h}_{\eta,\omega,j}(v_j) = (b_j^\eta)^{r_j/4} \prod_{l=1}^{r_j} h_{\omega_{j,l}}((b_j^\eta)^{1/2} v_{j,l}),$$

for all $j \in \{1, \dots, d_1\}$, where $\omega_{j,l}$ and $v_{j,l}$ denote the l -th components of $\omega_j \in \mathbb{N}^{r_j}$ and $v_j \in \mathbb{R}^{r_j}$. Then $\{\tilde{h}_{\eta,\omega}\}_{\omega \in \mathbb{N}^r}$ is a complete orthonormal system for $L^2(\mathbb{R}^r)$, made of joint eigenfunctions of the operators (10). In fact,

$$\begin{aligned} d\pi_{\eta,\rho}(L_j)\tilde{h}_{\eta,\omega} &= ((2|\omega_j| + r_j)b_j^\eta + |P_j\rho|^2)\tilde{h}_{\eta,\omega}, \\ d\pi_{\eta,\rho}(-iU_k)\tilde{h}_{\eta,\omega} &= \eta_k \tilde{h}_{\eta,\omega}, \end{aligned} \quad (11)$$

where $|\omega_j| = \omega_{j,1} + \cdots + \omega_{j,r_j}$; it should be observed that $P_j\rho = 0$ if $j > \tilde{d}_1$.

Since $H : \mathbb{R}^{d_1} \times \mathfrak{z}^* \rightarrow \mathbb{C}$ is in the Schwartz class, $\mathcal{K}_{H(\mathbf{L}, \mathbf{U})} : G \rightarrow \mathbb{C}$ is in the Schwartz class too (see [3, Theorem 5.2] or [15, §4.2]). Moreover

$$\pi_{\eta,\rho}(\mathcal{K}_{H(\mathbf{L}, \mathbf{U})})\tilde{h}_{\eta,\omega} = m((|\omega_1|, \dots, |\omega_{d_1}|), (|P_1\rho|^2, \dots, |P_{\tilde{d}_1}\rho|^2), \eta)\tilde{h}_{\eta,\omega}$$

by (11) and [22, Proposition 1.1]; hence, if $\varphi_{\eta,\rho,\omega}(z, u) = \langle \pi_{\eta,\rho}(z, u)\tilde{h}_{\eta,\omega}, \tilde{h}_{\eta,\omega} \rangle$ is the corresponding diagonal matrix coefficient of $\pi_{\eta,\rho}$, then

$$\langle \pi_{\eta,\rho}(z, u)\pi_{\eta,\rho}(\mathcal{K}_{H(\mathbf{L}, \mathbf{U})})\tilde{h}_{\eta,\omega}, \tilde{h}_{\eta,\omega} \rangle = m((|\omega_j|)_{j \leq d_1}, (|P_j\rho|^2)_{j \leq \tilde{d}_1}, \eta)\varphi_{\eta,\rho,\omega}(z, u).$$

Therefore (9) gives that

$$\begin{aligned} &\mathcal{K}_{H(\mathbf{L}, \mathbf{U})}(z, u) \\ &= (2\pi)^{|r| - \dim G} \int_{\mathfrak{z}} \int_{\ker J_\eta} \sum_{n \in \mathbb{N}^{d_1}} m(n, (|P_j\rho|^2)_{j \leq \tilde{d}_1}, \eta) \psi_{\eta,\rho,n}(z, u) \prod_{j=1}^{d_1} (b_j^\eta)^{r_j} d\rho d\eta, \end{aligned} \quad (12)$$

where

$$\psi_{\eta,\rho,n}(z, u) = \sum_{\substack{\omega \in \mathbb{N}^r \\ |\omega_1| = n_1, \dots, |\omega_{d_1}| = n_{d_1}}} \varphi_{\eta,\rho,\omega}(z, u).$$

On the other hand,

$$\begin{aligned} \varphi_{\eta,\rho,\omega}(z, u) &= e^{i\langle \eta, u \rangle} e^{i\langle \rho, \bar{P}^\eta z \rangle} \prod_{j=1}^{d_1} \prod_{l=1}^{r_j} \left[(b_j^\eta)^{1/2} \right. \\ &\quad \left. \times \int_{\mathbb{R}} e^{ib_j^\eta s \bar{z}_{j,l}^\eta} h_{\omega_{j,l}}((b_j^\eta)^{1/2}(s + z_{j,l}^\eta/2)) h_{\omega_{j,l}}((b_j^\eta)^{1/2}(s - z_{j,l}^\eta/2)) ds \right]. \end{aligned}$$

The last integral is essentially the Fourier-Wigner transform of a pair of Hermite functions, whose bidimensional Fourier transform is a Fourier-Wigner transform too [10, formula (1.90)]. The parity properties of the Hermite functions then yield

$$\begin{aligned} \varphi_{\eta,\rho,\omega}(z, u) &= e^{i\langle \eta, u \rangle} e^{i\langle \rho, \bar{P}^\eta z \rangle} \prod_{j=1}^{d_1} \prod_{l=1}^{r_j} \left[\frac{(-1)^{\omega_{j,l}}}{\pi b_j^\eta} \int_{\mathbb{R} \times \mathbb{R}} e^{i\theta_1 z_{j,l}^\eta} e^{i\theta_2 \bar{z}_{j,l}^\eta} \right. \\ &\quad \left. \times \int_{\mathbb{R}} e^{it(2\theta_1/(b_j^\eta)^{1/2})} h_{\omega_{j,l}}(t + \theta_2/(b_j^\eta)^{1/2}) h_{\omega_{j,l}}(t - \theta_2/(b_j^\eta)^{1/2}) dt d\theta_1 d\theta_2 \right]. \end{aligned}$$

Since the Fourier-Wigner transform of a pair of Hermite functions can be expressed in terms of Laguerre polynomials (see [10, Theorem 1.104] or [26, Theorem 1.3.4]), we obtain that

$$\begin{aligned} \varphi_{\eta,\rho,\omega}(z,u) &= \frac{e^{i\langle \eta, u \rangle} e^{i\langle \rho, \bar{P}^\eta z \rangle}}{\pi^{|\mathbb{R}^r|}} \int_{\mathbb{R}^r \times \mathbb{R}^r} e^{i\langle \zeta_1, z^\eta \rangle} e^{i\langle \zeta_2, \bar{z}^\eta \rangle} \\ &\quad \times \prod_{j=1}^{d_1} \left[(b_j^\eta)^{-r_j} \prod_{l=1}^{r_j} \mathcal{L}_{\omega_{j,l}}^{(0)}((\zeta_{1,j,l}^2 + \zeta_{2,j,l}^2)/b_j^\eta) \right] d\zeta_1 d\zeta_2 \end{aligned}$$

Consequently, for all $n \in \mathbb{N}^{d_1}$,

$$\begin{aligned} \psi_{\eta,\rho,n}(z,u) &= \frac{e^{i\langle \eta, u \rangle} e^{i\langle \rho, \bar{P}^\eta z \rangle}}{\pi^{|\mathbb{R}^r|}} \int_{\mathbb{R}^r \times \mathbb{R}^r} e^{i\langle \zeta_1, z^\eta \rangle} e^{i\langle \zeta_2, \bar{z}^\eta \rangle} \\ &\quad \times \prod_{j=1}^{d_1} \left[(b_j^\eta)^{-r_j} \mathcal{L}_{n_j}^{(r_j-1)}(|\zeta_{1,j}|^2 + |\zeta_{2,j}|^2)/b_j^\eta \right] d\zeta_1 d\zeta_2 \quad (13) \end{aligned}$$

[9, §10.12, formula (41)]. The conclusion then follows by plugging (13) into (12) and performing a change of variable by rotation in the inner integrals. \square

4 A weighted Plancherel estimate

Proposition 4 expresses the convolution kernel $\mathcal{K}_{H(\mathbf{L}, \mathbf{U})}$ as the inverse Fourier transform of a function of the multiplier H . Due to the properties of the Fourier transform, it is not unreasonable to think that multiplying the kernel by a polynomial weight might correspond to taking derivatives of the multiplier. As a matter of fact, the presence of the Laguerre expansion leads us to consider both “discrete” and “continuous” derivatives of the reparametrization $m : \mathbb{N}^{d_1} \times \mathbb{R}^{d_1} \times \mathfrak{j} \rightarrow \mathbb{C}$ of the multiplier H given by (7).

For convenience, set $\mathcal{L}_n^{(k)} = 0$ for all $n < 0$. From the properties of Laguerre polynomials (see, e.g., [9, §10.12]) one can easily derive the following identities.

Lemma 5. *For all $k, n, m \in \mathbb{N}$ and $t \in \mathbb{R}$,*

$$\mathcal{L}_n^{(k)}(t) = \mathcal{L}_{n-1}^{(k+1)}(t) + \mathcal{L}_n^{(k+1)}(t), \quad (14)$$

$$\frac{d}{dt} \mathcal{L}_n^{(k)}(t) = \mathcal{L}_{n-1}^{(k+1)}(t) - \mathcal{L}_n^{(k+1)}(t), \quad (15)$$

$$\int_0^\infty \mathcal{L}_n^{(k)}(t) \mathcal{L}_m^{(k)}(t) t^k dt = \begin{cases} \frac{(n+k)!}{2^{k+1} n!} & \text{if } n = m, \\ 0 & \text{otherwise.} \end{cases} \quad (16)$$

Let e_1, \dots, e_{d_1} denote the standard basis of \mathbb{R}^{d_1} . We introduce some operators on functions $f : \mathbb{N}^{d_1} \times \mathbb{R}^{d_1} \times \mathfrak{j} \rightarrow \mathbb{C}$:

$$\begin{aligned} \tau_j f(n, \mu, \eta) &= f(n + e_j, \mu, \eta), \\ \delta_j f(n, \mu, \eta) &= f(n + e_j, \mu, \eta) - f(n, \mu, \eta), \\ \partial_{\mu_l} f(n, \mu, \eta) &= \frac{\partial}{\partial \mu_l} f(n, \mu, \eta), \\ \partial_{\eta_k} f(n, \mu, \eta) &= \frac{\partial}{\partial \eta_k} f(n, \mu, \eta) \end{aligned}$$

for all $j \in \{1, \dots, d_1\}$, $l \in \{1, \dots, \tilde{d}_1\}$, $k \in \{1, \dots, d_2\}$.

For all $h \in \mathbb{N}$ and all multiindices $\alpha \in \mathbb{N}^h$, we denote by $|\alpha|$ the length $\alpha_1 + \dots + \alpha_h$ of α . Inequalities between multiindices, such as $\alpha \leq \alpha'$, shall be interpreted componentwise. Set moreover $(\alpha)_+ = ((\alpha_1)_+, \dots, (\alpha_h)_+)$, where $(\ell)_+ = \max\{\ell, 0\}$.

A function $\Psi : \mathfrak{z} \times \mathfrak{v} \rightarrow \mathbb{C}$ will be called *multihomogeneous* if there exist $h_0, h_1, \dots, h_{d_1} \in \mathbb{R}$ such that

$$\Psi \left(\lambda_0 \eta, \sum_{j=1}^{d_1} \lambda_j P_j \xi \right) = \lambda_0^{h_0} \lambda_1^{h_1} \dots \lambda_{d_1}^{h_{d_1}} \Psi(\eta, \xi)$$

for all $\eta \in \mathfrak{z}$, $\xi \in \mathfrak{v}$, $\lambda_0, \lambda_1, \dots, \lambda_{d_1} \in]0, \infty[$; the homogeneity degrees h_0, h_1, \dots, h_{d_1} of Ψ will also be denoted as $\deg_{\mathfrak{z}} \Psi, \deg_{\mathfrak{v}_1} \Psi, \dots, \deg_{\mathfrak{v}_{d_1}} \Psi$. Note that, if Ψ is multihomogeneous and continuous, then $\deg_{\mathfrak{v}_j} \Psi \geq 0$ for all $j \in \{1, \dots, d_1\}$.

Proposition 6. *Let $H : \mathbb{R}^{d_1} \times \mathfrak{z}^* \rightarrow \mathbb{C}$ be smooth and compactly supported in $\mathbb{R}^{d_1} \times \mathfrak{z}$, and let $m(n, \mu, \eta)$ be defined by (7). For all $\alpha \in \mathbb{N}^{d_2}$,*

$$\begin{aligned} u^\alpha \mathcal{K}_{H(\mathbf{L}, \mathbf{U})}(z, u) &= \sum_{\iota \in I_\alpha} \int_{\mathfrak{z}} \int_{\mathfrak{v}} \sum_{n \in \mathbb{N}^{d_1}} \partial_\eta^{\gamma^\iota} \partial_\mu^{\theta^\iota} \delta^{\beta^\iota} m(n, (|\bar{P}_j^\eta \xi|^2)_{j \leq \tilde{d}_1}, \eta) \\ &\quad \times \Psi_\iota(\eta, \xi) \left[\prod_{j=1}^{d_1} \mathcal{L}_{n_j}^{(r_j-1+\beta_j^\iota)} (|P_j^\eta \xi|^2 / b_j^\eta) \right] e^{i\langle \xi, z \rangle} e^{i\langle \eta, u \rangle} d\xi d\eta, \end{aligned}$$

for almost all $(z, u) \in G$, where I_α is a finite set and, for all $\iota \in I_\alpha$,

- $\gamma^\iota \in \mathbb{N}^{d_2}$, $\theta^\iota \in \mathbb{N}^{\tilde{d}_1}$, $\beta^\iota \in \mathbb{N}^{d_1}$, $\gamma^\iota \leq \alpha$,
- $\Psi_\iota = \Psi_{\iota,0} \Psi_{\iota,1} \dots \Psi_{\iota,d_1}$, where $\Psi_{\iota,j} : \mathfrak{z} \times \mathfrak{v} \rightarrow \mathbb{C}$ is smooth and multihomogeneous for all $j \in \{0, \dots, d_1\}$,
- $\deg_{\mathfrak{z}} \Psi_\iota = |\gamma^\iota| - |\alpha| - |\beta^\iota|$ and $\deg_{\mathfrak{v}_j} \Psi_\iota = 2\beta_j^\iota + 2\theta_j^\iota$ for all $j \in \{1, \dots, d_1\}$,
- for all $j \in \{1, \dots, d_1\}$, $\Psi_{\iota,j}(\eta, \xi)$ is a product of factors of the form $|P_j^\eta \xi|^2$ or $\partial_{\eta_k} |P_j^\eta \xi|^2$ for $k \in \{1, \dots, d_2\}$,
- $|\gamma^\iota| + |\theta^\iota| + |\beta^\iota| + \sum_{j=1}^{d_1} (\beta_j^\iota - (\deg_{\mathfrak{v}_j} \Psi_{\iota,j})/2)_+ \leq |\alpha|$.

Proof. By Proposition 4 and the properties of the Fourier transform, we are reduced to proving that, for all $\alpha \in \mathbb{N}^{d_2}$, $\eta \in \mathfrak{z}$, $\xi \in \mathfrak{v}$,

$$\begin{aligned} &\left(\frac{\partial}{\partial \eta} \right)^\alpha \sum_{n \in \mathbb{N}^{d_1}} m(n, (|\bar{P}_j^\eta \xi|^2)_{j \leq \tilde{d}_1}, \eta) \prod_{j=1}^{d_1} \mathcal{L}_{n_j}^{(r_j-1)} (|P_j^\eta \xi|^2 / b_j^\eta) \\ &= \sum_{\iota \in I_\alpha} \sum_{n \in \mathbb{N}^{d_1}} \partial_\eta^{\gamma^\iota} \partial_\mu^{\theta^\iota} \delta^{\beta^\iota} m(n, (|\bar{P}_j^\eta \xi|^2)_{j \leq \tilde{d}_1}, \eta) \Psi_\iota(\eta, \xi) \prod_{j=1}^{d_1} \mathcal{L}_{n_j}^{(r_j-1+\beta_j^\iota)} (|P_j^\eta \xi|^2 / b_j^\eta), \end{aligned}$$

where $I_\alpha, \gamma^\iota, \theta^\iota, \beta^\iota, \Psi_\iota$ are as in the above statement.

This is easily proved by induction on $|\alpha|$. For $|\alpha| = 0$ it is trivially verified. For the inductive step, one applies Leibniz' rule, and exploits the following observations:

- when a derivative ∂_{η_k} hits a Laguerre function, by the identity (15) and summation by parts, the type of the Laguerre function is increased by 1, as well as the corresponding component of β^t ;
- for all $j \in \{1, \dots, d_1\}$, $b_j^\eta = \sqrt{\text{tr}(-J_\eta^2 P_j)/(2r_j)}$ is a smooth function of $\eta \in \mathfrak{z}$, homogeneous of degree 1;
- for all $j \in \{1, \dots, d_1\}$, $P_j^\eta = -J_\eta^2 P_j / (b_j^\eta)^2$ is a smooth function of $\eta \in \mathfrak{z}$, homogeneous of degree 0, and in fact it is constant if $j > \tilde{d}_1$;
- for all $j \in \{1, \dots, \tilde{d}_1\}$, $|P_j^\eta \xi|^2 = \langle P_j^\eta P_j \xi, P_j \xi \rangle$ is a smooth bihomogeneous function of $(\eta, P_j \xi) \in \mathfrak{z} \times \mathfrak{v}_j$ of bidegree $(0, 2)$, and moreover

$$\begin{aligned} |\bar{P}_j^\eta \xi|^2 &= |P_j \xi|^2 - |P_j^\eta \xi|^2, & \partial_{\eta_k} |\bar{P}_j^\eta \xi|^2 &= -\partial_{\eta_k} |P_j^\eta \xi|^2, \\ \partial_{\eta_k} (|P_j^\eta \xi|^2 / b_j^\eta) &= |P_j^\eta \xi|^2 \partial_{\eta_k} (1/b_j^\eta) + (\partial_{\eta_k} |P_j^\eta \xi|^2) / b_j^\eta \end{aligned}$$

for all $k \in \{1, \dots, d_2\}$.

The conclusion follows. \square

Note that, for all $j \in \{1, \dots, d_1\}$, $\mu \in \mathbb{R}^{\tilde{d}_1}$, $\eta \in \mathfrak{z}$, the quantities $\tau_j f(\cdot, \mu, \eta)$, $\delta_j f(\cdot, \mu, \eta)$ depend only on $f(\cdot, \mu, \eta)$; in other words, τ_j and δ_j can be considered as operators on functions $\mathbb{N}^{d_1} \rightarrow \mathbb{C}$.

The following lemma exploits the orthogonality properties (16) of the Laguerre functions, together with (14), and shows that a mismatch between the type of the Laguerre function and the exponent of the weight attached to the measure may be turned in some cases into discrete differentiation.

Lemma 7. *For all $h, k \in \mathbb{N}^{d_1}$ and all compactly supported $f : \mathbb{N}^{d_1} \rightarrow \mathbb{C}$,*

$$\begin{aligned} \int_{]0, \infty[^{d_1}} \left| \sum_{n \in \mathbb{N}^{d_1}} f(n) \prod_{j=1}^{d_1} \mathcal{L}_{n_j}^{(k_j)}(t_j) \right|^2 t^h dt \\ \leq C_{h,k} \sum_{n \in \mathbb{N}^{d_1}} |\delta^{(k-h)_+} f(n)|^2 \prod_{j=1}^{d_1} (1+n_j)^{h_j + 2(k_j - h_j)_+}. \end{aligned}$$

Proof. Via an inductive argument, we may reduce to the case $d_1 = 1$.

Note that, if f is compactly supported, then $\tau^l f$ is null for all sufficiently large $l \in \mathbb{N}$. Hence the operator $1 + \tau$, when restricted to the set of compactly supported functions, is invertible, with inverse given by

$$(1 + \tau)^{-1} f = \sum_{l \in \mathbb{N}} (-1)^l \tau^l f.$$

Then by (14) we deduce that, for all $k \in \mathbb{N}$,

$$\begin{aligned} \sum_{n \in \mathbb{N}} f(n) \mathcal{L}_n^{(k)}(t) &= \sum_{n \in \mathbb{N}} (1 + \tau) f(n) \mathcal{L}_n^{(k+1)}(t), \\ \sum_{n \in \mathbb{N}} f(n) \mathcal{L}_n^{(k+1)}(t) &= \sum_{n \in \mathbb{N}} (1 + \tau)^{-1} f(n) \mathcal{L}_n^{(k)}(t), \end{aligned}$$

and consequently, for all $h, k \in \mathbb{N}$,

$$\sum_{n \in \mathbb{N}} f(n) \mathcal{L}_n^{(k)}(t) = \sum_{n \in \mathbb{N}} (1 + \tau)^{h-k} f(n) \mathcal{L}_n^{(h)}(t)$$

Thus the orthogonality properties (16) of the Laguerre functions give us that

$$\int_0^\infty \left| \sum_{n \in \mathbb{N}} f(n) \mathcal{L}_n^{(k)}(t) \right|^2 t^h dt \leq C_{h,k} \sum_{n \in \mathbb{N}} |(1+\tau)^{h-k} f(n)|^2 \langle n \rangle^h,$$

where $\langle n \rangle = 1 + n$.

In the case $h \geq k$, $(1+\tau)^{h-k}$ is given by the finite sum

$$(1+\tau)^{h-k} = \sum_{\ell=0}^{h-k} \binom{h-k}{\ell} \tau^\ell,$$

and the conclusion follows immediately by the triangular inequality.

In the case $h < k$, instead, since $\delta = \tau - 1$, from the identity $1 - \tau^2 = (1 - \tau)(1 + \tau)$ we deduce that

$$(1+\tau)^{h-k} = (-\delta)^{k-h} (1 - \tau^2)^{h-k} = (-1)^{k-h} \sum_{\ell \geq 0} \binom{\ell + k - h - 1}{\ell} \delta^{k-h} \tau^{2\ell},$$

hence

$$\begin{aligned} \sum_{n \in \mathbb{N}} |(1+\tau)^{h-k} f(n)|^2 \langle n \rangle^h &= \sum_{n \in \mathbb{N}} \left| \sum_{\ell \geq 0} \binom{\ell + k - h - 1}{\ell} \delta^{k-h} f(n+2\ell) \right|^2 \langle n \rangle^h \\ &\leq C_{h,k} \sum_{n \in \mathbb{N}} \left| \sum_{\ell \geq n} \langle \ell \rangle^{k-h-1} \delta^{k-h} f(\ell) \right|^2 \langle n \rangle^h \\ &\leq C_{h,k} \sum_{n \in \mathbb{N}} \langle n \rangle^{-1/2} \sum_{\ell \geq n} |\langle \ell \rangle^{k-h-1/4} \delta^{k-h} f(\ell)|^2 \langle n \rangle^h \\ &\leq C_{h,k} \sum_{\ell \in \mathbb{N}} \langle \ell \rangle^{2k-2h-1/2} |\delta^{k-h} f(\ell)|^2 \sum_{n=0}^{\ell} \langle n \rangle^{h-1/2} \\ &\leq C_{h,k} \sum_{\ell \in \mathbb{N}} \langle \ell \rangle^{2k-h} |\delta^{k-h} f(\ell)|^2, \end{aligned}$$

by the Cauchy-Schwarz inequality, and we are done. \square

Let $|\cdot|$ denote any Euclidean norm on \mathfrak{z}^* . The previous lemma, together with Plancherel's formula for the Fourier transform, yields the following L^2 -estimate.

Proposition 8. *Under the hypotheses of Proposition 6, for all $\alpha \in \mathbb{N}^{d_2}$,*

$$\begin{aligned} \int_G |u^\alpha \mathcal{K}_{H(\mathbf{L}, \mathbf{U})}(z, u)|^2 dz du &\leq C_\alpha \sum_{\iota \in \tilde{I}_\alpha} \int_{\mathfrak{z}} \int_{[0, \infty[^{d_1}} \sum_{n \in \mathbb{N}^{d_1}} |\partial_\eta^{\gamma^\iota} \partial_\mu^{\theta^\iota} \delta^{\beta^\iota} m(n, \mu, \eta)|^2 \\ &\quad \times |\eta|^{2|\gamma^\iota| - 2|\alpha| - 2|\beta^\iota| + |\alpha^\iota| + d_1} (1+n_1)^{a_1^\iota} \dots (1+n_{d_1})^{a_{d_1}^\iota} d\sigma_\iota(\mu) d\eta, \quad (17) \end{aligned}$$

where \tilde{I}_α is a finite set and, for all $\iota \in \tilde{I}_\alpha$,

- $\gamma^\iota \in \mathbb{N}^{d_2}$, $\theta^\iota \in \mathbb{N}^{d_1}$, $\alpha^\iota, \beta^\iota \in \mathbb{N}^{d_1}$,
- $\gamma^\iota \leq \alpha$, $|\gamma^\iota| + |\theta^\iota| + |\beta^\iota| \leq |\alpha|$,
- σ_ι is a regular Borel measure on $[0, \infty[^{d_1}$.

Proof. Note that, for all $j \in \{1, \dots, d_1\}$,

$$\partial_{\eta_k} (|P_j^\eta \xi|^2) = 2 \langle \partial_{\eta_k} P_j^\eta, P_j^\eta \xi, P_j^\eta \xi \rangle \leq C |\eta|^{-1} |P_j^\eta \xi| |P_j \xi|;$$

consequently, if $\Psi_\iota, \Psi_{\iota,j}, \gamma^\iota, \theta^\iota, \beta^\iota$ are as in the statement of Proposition 6, then

$$|\Psi_{\iota,j}(\eta, \xi)|^2 \leq C_\iota |\eta|^{2 \deg_{\mathfrak{s}} \Psi_{\iota,j}} |P_j^\eta \xi|^{\deg_{\mathfrak{v}_j} \Psi_{\iota,j}} |P_j \xi|^{\deg_{\mathfrak{v}_j} \Psi_{\iota,j}}$$

for all $j \in \{1, \dots, d_1\}$, hence

$$\begin{aligned} |\Psi_\iota(\eta, \xi)|^2 &\leq C_\iota |\eta|^{2 \deg_{\mathfrak{s}} \Psi_\iota} \prod_{j=1}^{d_1} |P_j^\eta \xi|^{\deg_{\mathfrak{v}_j} \Psi_{\iota,j}} |P_j \xi|^{2 \deg_{\mathfrak{v}_j} \Psi_{\iota,0} + \deg_{\mathfrak{v}_j} \Psi_{\iota,j}} \\ &\leq C_\iota |\eta|^{2|\gamma^\iota| - 2|\alpha| - 2|\beta^\iota|} \prod_{j=1}^{d_1} \sum_{h_j = (\deg_{\mathfrak{v}_j} \Psi_{\iota,j})/2}^{2\theta_j^\iota + 2\beta_j^\iota} |P_j^\eta \xi|^{2h_j} |\bar{P}_j^\eta \xi|^{4\theta_j^\iota + 4\beta_j^\iota - 2h_j}, \end{aligned}$$

and moreover, for all $h \in \mathbb{N}^{d_1}$, if $h_j \geq (\deg_{\mathfrak{v}_j} \Psi_{\iota,j})/2$ for all $j \in \{1, \dots, d_1\}$, then

$$|\gamma^\iota| + |\theta^\iota| + |\beta^\iota| + \sum_{j=1}^{\tilde{d}_1} (\beta_j^\iota - h_j)_+ \leq |\alpha|.$$

By Proposition 6, Plancherel's formula and the triangular inequality, we then obtain that the left-hand side of (17) is majorized by a finite sum of terms of the form

$$\begin{aligned} \int_{\mathfrak{z}} \int_{\mathfrak{v}} \left| \sum_{n \in \mathbb{N}^{d_1}} \partial_{\eta_j}^\gamma \partial_{\mu_j}^\theta \delta^\beta m(n, (|\bar{P}_j^\eta \xi|^2)_{j \leq \tilde{d}_1}, \eta) \prod_{j=1}^{d_1} \mathcal{L}_{n_j}^{(r_j - 1 + \beta_j)} (|P_j^\eta \xi|^2 / b_j^\eta) \right|^2 \\ \times |\eta|^{2|\gamma| - 2|\alpha| - 2|\beta|} \prod_{j=1}^{d_1} |P_j^\eta \xi|^{2h_j} \prod_{j=1}^{\tilde{d}_1} |\bar{P}_j^\eta \xi|^{2k_j} d\xi d\eta, \quad (18) \end{aligned}$$

where $\gamma \in \mathbb{N}^{d_2}$, $\theta, k \in \mathbb{N}^{\tilde{d}_1}$, $\beta, h \in \mathbb{N}^{d_1}$ and $|\gamma| + |\theta| + |\beta| + (\beta - h)_+ \leq |\alpha|$. Simple changes of variables (rotation, polar coordinates and rescaling) allow one to rewrite (18) as a constant times

$$\begin{aligned} \int_{\mathfrak{z}} \int_{]0, \infty[^{\tilde{d}_1}} \int_{]0, \infty[^{d_1}} \left| \sum_{n \in \mathbb{N}^{d_1}} \partial_{\eta_j}^\gamma \partial_{\mu_j}^\theta \delta^\beta m(n, \mu, \eta) \prod_{j=1}^{d_1} \mathcal{L}_{n_j}^{(r_j - 1 + \beta_j)}(t_j) \right|^2 \prod_{j=1}^{d_1} t_j^{r_j - 1 + h_j} dt \\ \times |\eta|^{2|\gamma| - 2|\alpha| - 2|\beta|} \prod_{j=1}^{d_1} (b_j^\eta)^{h_j + r_j} \prod_{j=1}^{\tilde{d}_1} \mu_j^{k_j + (\dim \mathfrak{v}_j - 2r_j)/2} \frac{d\mu}{\mu_1 \cdots \mu_{\tilde{d}_1}} d\eta. \end{aligned}$$

By exploiting the fact that the b_j^η are smooth functions of $\eta \in \mathfrak{z}$, homogeneous of degree 1 (see the proof of Proposition 6), and applying Lemma 7 to the inner integral, the last quantity is majorized by

$$\begin{aligned} C \int_{\mathfrak{z}} \int_{]0, \infty[^{\tilde{d}_1}} \sum_{n \in \mathbb{N}^{d_1}} |\partial_{\eta_j}^\gamma \partial_{\mu_j}^\theta \delta^{\beta + (\beta - h)_+} m(n, \mu, \eta)|^2 \prod_{j=1}^{d_1} (1 + n_j)^{r_j - 1 + h_j + 2(\beta_j - h_j)_+} \\ \times |\eta|^{2|\gamma| - 2|\alpha| - 2|\beta| + |h| + |r|} \prod_{j=1}^{\tilde{d}_1} \mu_j^{k_j + (\dim \mathfrak{v}_j - 2r_j)/2} \frac{d\mu}{\mu_1 \cdots \mu_{\tilde{d}_1}} d\eta, \end{aligned}$$

and since the exponents $k_j + (\dim \mathfrak{v}_j - 2r_j)/2$ are strictly positive, while

$$-2|\beta| + |h| + |r| = -2|\beta| + (\beta - h)_+ + \sum_{j=1}^{d_1} (r_j - 1 + h_j + 2(\beta_j - h_j)_+) + d_1$$

and $|\gamma| + |\theta| + |\beta| + (\beta - h)_+ \leq |\alpha|$, the conclusion follows by suitably renaming the multiindices. \square

5 From discrete to continuous

Via the fundamental theorem of integral calculus, finite differences can be estimated by continuous derivatives. The next lemma is a multivariate analogue of [19, Lemma 6], and we omit the proof (see also [18, Lemma 7]).

Lemma 9. *Let $f : \mathbb{N}^{d_1} \rightarrow \mathbb{C}$ have a smooth extension $\tilde{f} : [0, \infty[^{d_1} \rightarrow \mathbb{C}$, and let $\beta \in \mathbb{N}^{d_1}$. Then*

$$\delta^\beta f(n) = \int_{J_\beta} \partial^\beta \tilde{f}(n+s) d\nu_\beta(s)$$

for all $n \in \mathbb{N}$, where $J_\beta = \prod_{j=1}^{d_1} [0, \beta_j]$ and ν_β is a Borel probability measure on J_β . In particular

$$|\delta^\beta f(n)|^2 \leq \int_{J_\beta} |\partial^\beta \tilde{f}(n+s)|^2 d\nu_\beta(s)$$

for all $n \in \mathbb{N}^{d_1}$.

We give now a simplified version of the right-hand side of (17), in the case we restrict to the functional calculus of L alone. In order to avoid issues of divergent series, it is however convenient at first to truncate the multiplier along the spectrum of \mathbf{U} .

Lemma 10. *Let $\chi \in C_c^\infty(\mathbb{R})$ be supported in $[1/2, 2]$, $K \subseteq \mathbb{R}$ be compact and $M \in]0, \infty[$. If $F : \mathbb{R} \rightarrow \mathbb{C}$ is smooth and supported in K , and $F_M : \mathbb{R} \times \mathfrak{z}^* \rightarrow \mathbb{C}$ is given by*

$$F_M(\lambda, \eta) = F(\lambda) \chi(|\eta|/M),$$

then, for all $r \in [0, \infty[$,

$$\int_G \| |u|^r \mathcal{K}_{F_M(L, \mathbf{U})}(z, u) \|^2 dz du \leq C_{K, \chi, r} M^{d_2 - 2r} \|F\|_{W_2^r}^2.$$

Proof. We may restrict to the case $r \in \mathbb{N}$, the other cases being recovered a posteriori by interpolation. Hence we need to prove that

$$\int_G |u^\alpha \mathcal{K}_{F_M(L, \mathbf{U})}(z, u)|^2 dz du \leq C_{K, \chi, \alpha} M^{d_2 - 2|\alpha|} \|F\|_{W_2^{|\alpha|}}^2 \quad (19)$$

for all $\alpha \in \mathbb{N}^{d_1}$. On the other hand, if m is defined by

$$m(n, \mu, \eta) = F\left(\sum_{j=1}^{d_1} b_j^\eta \langle n_j \rangle_j + |\mu|_\Sigma\right) \chi(|\eta|/M), \quad (20)$$

where $\langle \ell \rangle_j = 2\ell + r_j$ and $|\mu|_\Sigma = \sum_{j=1}^{\tilde{d}_1} \mu_j$, then the left-hand side of (19) is majorized by the right-hand side of (17), and we are reduced to proving that

$$\sum_{n \in \mathbb{N}^{d_1}} \int_{\mathfrak{J}} \int_{[0, \infty[^{\tilde{d}_1}} |\partial_\eta^{\gamma^\iota} \partial_\mu^{\theta^\iota} \delta^{\beta^\iota} m(n, \mu, \eta)|^2 |\eta|^{2|\gamma^\iota| - 2|\alpha| - 2|\beta^\iota| + |\alpha^\iota| + d_1} \times (1 + n_1)^{\alpha_1} \dots (1 + n_{d_1})^{\alpha_{d_1}} d\sigma_\iota(\mu) d\eta \leq C_{K, \chi, \alpha} M^{d_2 - 2|\alpha|} \|F\|_{W_2^{|\alpha|}}^2 \quad (21)$$

for all $\iota \in \tilde{I}_\alpha$, where \tilde{I}_α , γ^ι , θ^ι , β^ι , α^ι , σ_ι are as in Proposition 8.

Note that the right-hand side of (20) makes sense for all $n \in \mathbb{R}^{d_1}$, and defines a smooth extension of m , which we still denote by m by a slight abuse of notation. Hence, by Lemma 9,

$$|\partial_\eta^{\gamma^\iota} \partial_\mu^{\theta^\iota} \delta^{\beta^\iota} m(n, \mu, \eta)|^2 \leq \int_{J_\iota} |\partial_\eta^{\gamma^\iota} \partial_\mu^{\theta^\iota} \partial_n^{\beta^\iota} m(n + s, \mu, \eta)|^2 d\nu_\iota(s), \quad (22)$$

where $J_\iota = \prod_{j=1}^{d_1} [0, \beta_j^\iota]$ and ν_ι is a suitable probability measure on J_ι . Moreover the measure σ_ι in (21) is finite on compacta, and the right-hand side of (22) vanishes when $|\mu|_\Sigma > \max K$, because $\text{supp } F \subseteq K$. Consequently (21) will be proved if we show that

$$\sum_{n \in \mathbb{N}^{d_1}} \int_{\mathfrak{J}} |\partial_\eta^{\gamma^\iota} \partial_\mu^{\theta^\iota} \partial_n^{\beta^\iota} m(n + s, \mu, \eta)|^2 |\eta|^{2|\gamma^\iota| - 2|\alpha| - 2|\beta^\iota| + |\alpha^\iota| + d_1} \times (1 + n_1)^{\alpha_1} \dots (1 + n_{d_1})^{\alpha_{d_1}} d\eta \leq C_{K, \chi, \alpha} M^{d_2 - 2|\alpha|} \|F\|_{W_2^{|\alpha|}}^2 \quad (23)$$

for all $s \in J_\iota$ and $\mu \in [0, \max K]^{\tilde{d}_1}$, uniformly in s and μ .

As observed in the proof of Proposition 6, the b_j^η are positive, smooth functions of $\eta \in \mathfrak{J}$, homogeneous of degree 1; therefore, for all $n \in \mathbb{N}^{d_1}$, $j \in \{1, \dots, d_1\}$, $\eta \in \mathfrak{J}$, $s \in [0, \infty[^{d_1}$, $\mu \in [0, \infty[^{\tilde{d}_1}$,

$$|\eta|(1 + n_j) \sim b_j^\eta \langle n_j \rangle_j \leq \sum_{l=1}^{d_1} b_l^\eta \langle n_l + s_l \rangle_l + |\mu|_\Sigma, \quad (24)$$

and the last quantity is bounded by the constant $\max K$ whenever $(n + s, \mu, \eta) \in \text{supp } m$, because $\text{supp } F \subseteq K$. Hence the factors $|\eta|(1 + n_j)$ in the left-hand side of (23) can be discarded, that is, we are reduced to proving (23) in the case $\alpha^\iota = 0$.

From (20) it follows immediately that

$$\partial_\mu^{\theta^\iota} \partial_n^{\beta^\iota} m(n, \mu, \eta) = F^{(|\theta^\iota| + |\beta^\iota|)} \left(\sum_{j=1}^{d_1} b_j^\eta \langle n_j \rangle_j + |\mu|_\Sigma \right) \chi(|\eta|/M) \prod_{j=1}^{d_1} (2b_j^\eta)^{\beta_j^\iota}$$

and then it is easily proved inductively that

$$\begin{aligned} \partial_\eta^{\gamma^\iota} \partial_\mu^{\theta^\iota} \partial_n^{\beta^\iota} m(n, \mu, \eta) &= \sum_{\substack{v \in \mathbb{N}^{d_1} \\ |v| \leq |\gamma^\iota|}} \sum_{q=0}^{|\gamma^\iota| - |v|} F^{(|\theta^\iota| + |\beta^\iota| + |v|)} \left(\sum_{j=1}^{d_1} b_j^\eta \langle n_j \rangle_j + |\mu|_\Sigma \right) \\ &\quad \times \Psi_{\iota, v, q}(\eta) M^{-q} \chi^{(q)}(|\eta|/M) \prod_{j=1}^{d_1} \langle n_j \rangle_j^{v_j} \end{aligned}$$

where $\Psi_{\ell, v, q} : \mathfrak{J} \rightarrow \mathbb{R}$ is smooth and homogeneous of degree $|\beta^\ell| + |v| + q - |\gamma^\ell|$. By exploiting again (24) and the fact that $\text{supp } F \subseteq K$, we can majorize the factors $\langle n_j \rangle_j$ in the right-hand side by $|\eta|^{-1} \sim M^{-1}$ and obtain that

$$|\partial_\eta^{\gamma^\ell} \partial_\mu^{\theta^\ell} \partial_n^{\beta^\ell} m(n, \mu, \eta)|^2 \leq C_{K, \chi, \alpha} M^{2|\beta^\ell| - 2|\gamma^\ell|} \tilde{\chi}(|\eta|/M) \\ \times \sum_{v=0}^{|\gamma^\ell|} \left| F^{(|\beta^\ell| + |\theta^\ell| + v)} \left(\sum_{j=1}^{d_1} b_j^\eta \langle n_j \rangle_j + |\mu|_\Sigma \right) \right|^2,$$

where $\tilde{\chi}$ is the characteristic function of $[1/2, 2]$. Hence the left-hand side of (23), when $a^\ell = 0$, is majorized by

$$C_{K, \chi, \alpha} M^{d_1 - 2|\alpha|} \\ \times \sum_{v=0}^{|\gamma^\ell|} \int_{\mathfrak{J}} \sum_{n \in \mathbb{N}^{d_1}} \left| F^{(|\beta^\ell| + |\theta^\ell| + v)} \left(\sum_{j=1}^{d_1} b_j^\eta \langle n_j + s_j \rangle_j + |\mu|_\Sigma \right) \right|^2 \tilde{\chi}(|\eta|/M) d\eta.$$

Let S denote the unit sphere in \mathfrak{J}^* . By passing to polar coordinates and exploiting the homogeneity of the b_j^η , the integral in the above formula is majorized by

$$C \int_S \int_0^\infty \sum_{n \in \mathbb{N}^{d_1}} \left| F^{(|\beta^\ell| + |\theta^\ell| + v)} \left(\rho \sum_{j=1}^{d_1} b_j^\omega \langle n_j + s_j \rangle_j + |\mu|_\Sigma \right) \right|^2 \tilde{\chi}(\rho/M) \rho^{d_2} \frac{d\rho}{\rho} d\omega \\ \leq C M^{d_2} \int_0^\infty |F^{(|\beta^\ell| + |\theta^\ell| + v)}(\rho + |\mu|_\Sigma)|^2 \int_S \sum_{n \in \mathbb{N}^{d_1}} \tilde{\chi}(\rho/(M \langle n \rangle_{\omega, s})) d\omega \frac{d\rho}{\rho} \quad (25)$$

where $\langle n \rangle_{\omega, s} = \sum_{j=1}^{d_1} b_j^\omega \langle n_j + s_j \rangle_j \sim 1 + |n|$ uniformly in $\omega \in S$ and $s \in J_\ell$. Since $\tilde{\chi}(\rho/(M \langle n \rangle_{\omega, s}))$ vanishes unless $\langle n \rangle_{\omega, s} \sim \rho/M$, the sum in the right-hand side of (25) has at most $C_\ell (\rho/M)^{d_1}$ nonvanishing summands, and the integral on S is majorized by $C_\ell (\rho/M)^{d_1}$. In conclusion, the left-hand side of (23) is majorized by

$$C_{K, \chi, \alpha} M^{d_2 - 2|\alpha|} \sum_{v=0}^{|\gamma^\ell|} \int_0^\infty |F^{(|\beta^\ell| + |\theta^\ell| + v)}(\rho + |\mu|_\Sigma)|^2 \rho^{d_1 - 1} d\rho \\ \leq C_{K, \chi, \alpha} M^{d_2 - 2|\alpha|} \|F\|_{W_2^{|\alpha|}}^2,$$

because $d_1 \geq 1$, $\text{supp } F \subseteq K$ and $|\beta^\ell| + |\theta^\ell| + |\gamma^\ell| \leq |\alpha|$, and we are done. \square

Proposition 11. *Let $F : \mathbb{R} \rightarrow \mathbb{C}$ be smooth and such that $\text{supp } F \subseteq K$ for some compact set $K \subseteq \mathbb{R}$. For all $r \in [0, d_2/2[$,*

$$\int_G |(1 + |u|)^r \mathcal{K}_{F(L)}(z, u)|^2 dz du \leq C_{K, r} \|F\|_{W_2^r}^2.$$

Proof. Take $\chi \in C_c^\infty(]0, \infty[)$ such that $\text{supp } \chi \subseteq [1/2, 2]$ and $\sum_{k \in \mathbb{Z}} \chi(2^{-k} t) = 1$ for all $t \in]0, \infty[$. If F_M is defined for all $M \in]0, \infty[$ as in Lemma 10, then $\mathcal{K}_{F_M(L, \mathbf{U})}$ is given by the right-hand side of (8), where m is defined by (20), and moreover

$$\sum_{j=1}^{d_1} b_j^\eta \langle n_j \rangle_j + |\mu|_\Sigma \geq C^{-1} |\eta|$$

for all $\eta \in \mathfrak{z}$, $\mu \in [0, \infty[^{\tilde{d}_1}$ and $n \in \mathbb{N}^{d_1}$, therefore $F_M(L, \mathbf{U}) = 0$ whenever $M > 2C \max K$. Hence, if $k_K \in \mathbb{Z}$ is sufficiently large so that $2^{k_K} > 2C \max K$, then

$$F(L) = \sum_{k \in \mathbb{Z}, k \leq k_K} F_{2^k}(L, \mathbf{U})$$

(with convergence in the strong sense). Consequently an estimate for $\mathcal{K}_{F(L)}$ can be obtained, via Minkowski's inequality, by summing the corresponding estimates for $\mathcal{K}_{F_{2^k}}(L, \mathbf{U})$ given by Lemma 10. If $r < d/2$, then the series $\sum_{k \leq k_K} (2^k)^{d_2/2-r}$ converges, thus

$$\int_G ||u|^r \mathcal{K}_{F(L)}(z, u)|^2 dz du \leq C_{K,r} \|F\|_{W_2^r}^2.$$

The conclusion follows by combining the last inequality with the corresponding one for $r = 0$. \square

Let $|\cdot|_\delta$ be a δ_t -homogeneous norm on G ; take, e.g., $|(z, u)|_\delta = |z| + |u|^{1/2}$. Interpolation then allows us to improve the standard weighted estimate for a homogeneous sublaplacian on a stratified group.

Proposition 12. *Let $F : \mathbb{R} \rightarrow \mathbb{C}$ be smooth and such that $\text{supp } F \subseteq K$ for some compact set $K \subseteq \mathbb{R}$. For all $r \in [0, d_2/2[$, $\alpha \geq 0$ and $\beta > \alpha + r$,*

$$\int_G |(1 + |(z, u)|_\delta)^\alpha (1 + |u|)^r \mathcal{K}_{F(L)}(z, u)|^2 dz du \leq C_{K,\alpha,\beta,r} \|F\|_{W_2^\beta}^2. \quad (26)$$

Proof. Note that $1 + |u| \leq C(1 + |(z, u)|_\delta)^2$. Hence, in the case $\alpha \geq 0$, $\beta > \alpha + 2r$, the inequality (26) follows by the mentioned standard estimate (see [21, Lemma 1.2] or [17, Theorem 2.7]). On the other hand, if $\alpha = 0$ and $\beta \geq r$, then (26) is given by Proposition 11. The full range of α and β is then obtained by interpolation. \square

We can finally prove the crucial estimate.

Proof of Proposition 3. Take $r \in](\dim G)/2 + d_2/2 - s, d_2/2[$. Then

$$s - r > (\dim G)/2 + d_2/2 - 2r = (\dim \mathfrak{v})/2 + d_2 - 2r,$$

hence we can find $\alpha_1 > (\dim \mathfrak{v})/2$ and $\alpha_2 > d_2 - 2r$ such that $s - r > \alpha_1 + \alpha_2$. Set $w_s(z, u) = (1 + |(z, u)|_\delta)^\alpha (1 + |u|)^r$. The L^2 -estimate (4) then follows from Proposition 12. On the other hand, for all $(z, u) \in G$,

$$w_s^{-2}(z, u) \leq C_s (1 + |z|)^{-2\alpha_1} (1 + |u|)^{-\alpha_2 - 2r},$$

and the right-hand side is integrable over $G \cong \mathfrak{v} \times \mathfrak{z}$ since $2\alpha_1 > \dim \mathfrak{v}$ and $\alpha_2 + 2r > d_2 = \dim \mathfrak{z}$. Therefore $w_s^{-1} \in L^2(G)$, and the L^1 -estimate (5) follows from (4) and Hölder's inequality. \square

6 Remarks on the validity of the assumption and direct products

In this section we do no longer suppose that G and L are a 2-step stratified Lie group and a sublaplacian satisfying Assumption (A).

As observed in §2, a necessary condition for the validity of Assumption (A) is that the skewadjoint endomorphism J_η of the first layer \mathfrak{v} has constant rank for η ranging in $\mathfrak{j} = \mathfrak{z}^* \setminus \{0\}$. Here we show that this condition is also sufficient when the rank is minimal.

Proposition 13. *Let G be a 2-step nilpotent Lie group, with Lie algebra $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$, and let $\langle \cdot, \cdot \rangle$ be an inner product on \mathfrak{v} . Suppose that the skewadjoint endomorphism J_η of \mathfrak{v} has rank 2 for all $\eta \in \mathfrak{j}$. Then G satisfies Assumption (A) with the sublaplacian L associated to the given inner product, and also with any other sublaplacian associated to an inner product on a complement of \mathfrak{z} .*

Let moreover $G_{\mathbb{C}}$ be the complexification of G , considered as a real 2-step group, with Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{v}_{\mathbb{C}} \oplus \mathfrak{z}_{\mathbb{C}}$, and let $\mathfrak{v}_{\mathbb{C}}$ be endowed with the real inner product induced by the inner product on \mathfrak{v} . Then $G_{\mathbb{C}}$, with the sublaplacian associated to the given inner product, satisfies Assumption (A).

Proof. From the normal form for skewadjoint endomorphisms, it follows immediately that, if J_η has rank 2, then J_η^2 has exactly one nonzero eigenvalue, and Assumption (A) is trivially verified. Moreover, if \mathfrak{v} is identified with $\mathfrak{g}/\mathfrak{z}$, then $\ker J_\eta$ corresponds to the subspace

$$N_\eta = \{x + \mathfrak{z} : x \in \mathfrak{g} \text{ and } \eta([x, x']) = 0 \text{ for all } x' \in \mathfrak{g}\}$$

of $\mathfrak{g}/\mathfrak{z}$; hence the rank condition on J_η can be rephrased by saying that N_η has codimension 2 for all $\eta \in \mathfrak{j}$, and this condition does not depend on the sublaplacian L chosen on G .

Let $R(J_\eta)$ denote the range of J_η . We show now that, for all $\eta, \eta' \in \mathfrak{j}$, the intersection $R(J_\eta) \cap R(J_{\eta'})$ is nontrivial. If it were trivial, since $J_{\eta+\eta'} = J_\eta + J_{\eta'}$, we would have $\ker J_{\eta+\eta'} = \ker J_\eta \cap \ker J_{\eta'}$, hence

$$R(J_{\eta+\eta'}) = (\ker J_{\eta+\eta'})^\perp = R(J_\eta) \oplus R(J_{\eta'}),$$

thus $J_{\eta+\eta'}$ would have rank 4, contradiction.

Consider now the complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$. Via the linear identifications $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \times \mathfrak{g}$, $\mathfrak{z}_{\mathbb{C}}^* = \mathfrak{z}^* \times \mathfrak{z}^*$, $\mathfrak{v}_{\mathbb{C}} = \mathfrak{v} \times \mathfrak{v}$, the skewsymmetric endomorphism \tilde{J}_η of the first layer $\mathfrak{v}_{\mathbb{C}}$ corresponding to the element $\eta = (\eta_R, \eta_I) \in \mathfrak{z}_{\mathbb{C}}^*$ is given by

$$\tilde{J}_\eta(x_R, x_I) = (J_{\eta_R}x_R + J_{\eta_I}x_I, J_{\eta_I}x_R - J_{\eta_R}x_I). \quad (27)$$

Take now $\eta = (\eta_R, \eta_I) \in \mathfrak{z}_{\mathbb{C}}^*$; we want to show that \tilde{J}_η^2 has rank 4 and a unique nonzero eigenvalue. We distinguish several cases.

If $\eta_I = 0$, then $\tilde{J}_\eta = J_{\eta_R} \times (-J_{\eta_R})$, hence $\tilde{J}_\eta^2 = J_{\eta_R}^2 \times J_{\eta_R}^2$ satisfies the condition. The same argument gives the conclusion in the case $\eta_R = 0$.

If both $\eta_R, \eta_I \in \mathfrak{j}$, then $R(J_{\eta_R}) \cap R(J_{\eta_I}) \neq 0$, hence $\dim(R(J_{\eta_R}) \cap R(J_{\eta_I}))$ is either 2 or 1. In the first case, $R(J_{\eta_R}) = R(J_{\eta_I})$, so J_{η_R} and J_{η_I} commute and (27) implies that

$$\tilde{J}_\eta^2 = (J_{\eta_R}^2 + J_{\eta_I}^2) \times (J_{\eta_R}^2 + J_{\eta_I}^2);$$

since $J_{\eta_R}^2$ and $J_{\eta_I}^2$ are negative multiples of the same orthogonal projection, the conclusion follows.

Suppose now that $R(J_{\eta_R}) \cap R(J_{\eta_I}) = \mathbb{R}x$ for some unit vector $x \in \mathfrak{v}$, and set $y_R = J_{\eta_R}x$, $y_I = J_{\eta_I}x$, $b_R = |y_R|$, $b_I = |y_I|$; in particular $J_{\eta_R}^2x = -b_R^2x$ and $J_{\eta_I}^2x = -b_I^2x$. Since J_{η_R} and J_{η_I} are skewadjoint and of rank 2, necessarily $J_{\eta_R}x, J_{\eta_I}x \in x^\perp$ and $J_{\eta_R}(x^\perp) = J_{\eta_I}(x^\perp) = \mathbb{R}x$, therefore $J_{\eta_R}J_{\eta_I}x$ and $J_{\eta_I}J_{\eta_R}x$ are both multiples of x ; on the other hand,

$$\langle J_{\eta_R}J_{\eta_I}x, x \rangle = -\langle J_{\eta_I}x, J_{\eta_R}x \rangle = \langle x, J_{\eta_I}J_{\eta_R}x \rangle,$$

hence $J_{\eta_R}J_{\eta_I}x = J_{\eta_I}J_{\eta_R}x$. This identity, together with (27), allows us easily to show that

$$\begin{aligned} \tilde{J}_\eta(x, 0) &= (y_R, y_I), & \tilde{J}_\eta(y_R, y_I) &= -(b_R^2 + b_I^2)(x, 0), \\ \tilde{J}_\eta(0, x) &= (y_I, -y_R), & \tilde{J}_\eta(y_I, -y_R) &= -(b_R^2 + b_I^2)(0, x). \end{aligned}$$

Note that $b_R^2 + b_I^2$ is the squared norm of both (y_R, y_I) and $(y_I, -y_R)$. Hence we would be done if we knew that $R(\tilde{J}_\mu)$ coincides with the linear span W of $(x, 0)$, $(0, x)$, (y_R, y_I) , $(y_I, -y_R)$.

In fact, we just need to show that $R(\tilde{J}_\eta)$ is contained in W , or equivalently, that W^\perp is contained in $\ker \tilde{J}_\eta$. On the other hand, if $v = (v_R, v_I) \in W^\perp$, then $v_R, v_I \in x^\perp$ and moreover

$$\langle v_R, y_R \rangle + \langle v_I, y_I \rangle = 0, \quad \langle v_R, y_I \rangle - \langle v_I, y_R \rangle = 0,$$

hence $J_{\eta_R}v_R, J_{\eta_R}v_I, J_{\eta_I}v_R, J_{\eta_I}v_I \in \mathbb{R}x$, and

$$\begin{aligned} \langle J_{\eta_R}v_R, x \rangle &= -\langle v_R, y_R \rangle = \langle v_I, y_I \rangle = -\langle J_{\eta_I}v_I, x \rangle, \\ \langle J_{\eta_I}v_R, x \rangle &= -\langle v_R, y_I \rangle = -\langle v_I, y_R \rangle = \langle J_{\eta_R}v_I, x \rangle, \end{aligned}$$

therefore $J_{\eta_R}v_R = -J_{\eta_I}v_I$ and $J_{\eta_I}v_R = J_{\eta_R}v_I$, from which it follows immediately that $\tilde{J}_\eta(v_R, v_I) = 0$. \square

The next proposition shows how groups and sublaplacians satisfying Assumption (A) may be “glued together”, so to give a higher-dimensional group and a sublaplacian that satisfy Assumption (A) too.

Proposition 14. *Suppose that, for $j = 1, 2$, the sublaplacian L_j on the 2-step stratified Lie group G_j satisfies Assumption (A). Suppose further that the centers of G_1 and G_2 have the same dimension. Let G be the quotient of $G_1 \times G_2$ given by any linear identification of the respective centers, and let $L = L_1^\# + L_2^\#$, where $L_j^\#$ is the pushforward of L_j to G . Then the sublaplacian L on the group G satisfies Assumption (A).*

Proof. Let \mathfrak{g}_j be the Lie algebra of G_j , and let \mathfrak{v}_j and $\langle \cdot, \cdot \rangle_j$ be the linear complement of the center \mathfrak{z}_j and the inner product on \mathfrak{v}_j determined by the sublaplacian L_j ; denote moreover by $J_{j,\eta}$ the skewadjoint endomorphism of \mathfrak{v}_j determined by $\eta \in \mathfrak{z}_j^*$.

The linear identification of the centers of G_1 and G_2 corresponds to a linear isomorphism $\phi : \mathfrak{z}_1 \rightarrow \mathfrak{z}_2$, and the Lie algebra \mathfrak{g} of the quotient G can be identified with $\mathfrak{v}_1 \times \mathfrak{v}_2 \times \mathfrak{z}_2$, with Lie bracket

$$[(v_1, v_2, z), (v'_1, v'_2, z')] = (0, 0, \phi([v_1, v'_1]) + [v_2, v'_2]).$$

Then the sublaplacian L on G corresponds to the inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{v}_1 \times \mathfrak{v}_2$ defined by

$$\langle (v_1, v_2), (v'_1, v'_2) \rangle = \langle v_1, v'_1 \rangle_1 + \langle v_2, v'_2 \rangle_2.$$

In particular, if $\phi^* : \mathfrak{z}_2^* \rightarrow \mathfrak{z}_1^*$ denotes the adjoint map of $\phi : \mathfrak{z}_1 \rightarrow \mathfrak{z}_2$, then it is easily checked that the skewadjoint endomorphism of the first layer $\mathfrak{v}_1 \times \mathfrak{v}_2$ of \mathfrak{g} corresponding to an element η of the dual \mathfrak{z}_2^* of the center of \mathfrak{g} is given by $J_\eta = J_{1, \phi^* \eta} \times J_{2, \eta}$. Hence the orthogonal decomposition of $\mathfrak{v}_1 \times \mathfrak{v}_2$ giving the “simultaneous diagonalization” of the J_η for all $\eta \in \mathfrak{z}_2^*$ (in the sense of §2) is simply obtained by juxtaposing the corresponding orthogonal decompositions of \mathfrak{v}_1 and \mathfrak{v}_2 . \square

Note that the direct product $G_1 \times G_2$ itself need not satisfy Assumption (A), even if the factors G_1 and G_2 do. However a functional-analytic argument, as in [24, §4], can be used to deal with that case.

The key step in our proof of Theorem 2 is the weighted L^2 -estimate (4) of Proposition 3. Let us now turn the conclusion of Proposition 3 into an assumption on a homogeneous sublaplacian L on a stratified group G .

Assumption (B_t). *For all $s > t$ there exist a weight $w_s : G \rightarrow [1, \infty[$ such that $w_s^{-1} \in L^2(G)$ and, for all compact sets $K \subseteq \mathbb{R}$ and all Borel functions $F : \mathbb{R} \rightarrow \mathbb{C}$ with $\text{supp } F \subseteq K$,*

$$\|w_s \mathcal{K}_{F(L)}\|_{L^2(G)} \leq C_{K,s} \|F\|_{W_2^s(\mathbb{R})}. \quad (28)$$

Our Proposition 3 can then be rephrased by saying that Assumption (A) implies Assumption (B_t) for $t = (\dim G)/2$. Note, on the other hand, that Assumption (B_t) makes sense for homogeneous sublaplacians on stratified groups G of step other than 2. In fact, every homogeneous sublaplacian on a stratified group of homogeneous dimension Q satisfies Assumption (B_t) for $t = Q/2$, by [21, Lemma 1.2] (suitably extended so to admit multipliers that do not vanish in a neighborhood of the origin of \mathbb{R} ; see, e.g., [24, Lemma 3.1] for the 1-dimensional case, and [17, Theorem 2.7] for the higher-dimensional case).

Differently from Assumption (A), the new Assumption (B_t) “behaves well” under direct products.

Proposition 15. *For $j = 1, \dots, n$, let L_j be a homogeneous sublaplacian on a stratified Lie group G_j satisfying Assumption (B_{t_j}) for some $t_j > 0$. Let $G = G_1 \times \dots \times G_n$ and $L = L_1^\sharp + \dots + L_n^\sharp$, where L_j^\sharp is the pushforward to G of the operator L_j . Then the sublaplacian L on G satisfies Assumption (B_t), where $t = t_1 + \dots + t_n$.*

Proof. Take $s > t$. Then we can choose s_1, \dots, s_n such that $s_1 > t_1, \dots, s_n > t_n$ and $s = s_1 + \dots + s_n$. Let then $w_{j,s_j} : G_j \rightarrow [1, \infty[$ be the weight corresponding to s_j given by Assumption (B_{t_j}) on G_j and L_j , for $j = 1, \dots, n$. In particular $w_{j,s_j}^{-1} \in L^2(G_j)$ and, for all $\phi \in C_c^\infty(\mathbb{R})$, the map $F \mapsto \mathcal{K}_{(\phi F)(L_j)}$ is a bounded linear map of Hilbert spaces $W_2^{s_j}(\mathbb{R}) \rightarrow L^2(G_j, w_{j,s_j}^2(x_j) dx_j)$, where dx_j denotes the Haar measure on G_j .

The operators $L_1^\sharp, \dots, L_n^\sharp$ are essentially selfadjoint and commute strongly, that is, they admit a joint spectral resolution and a joint functional calculus on $L^2(G)$, and moreover, for all bounded Borel functions $F_1, \dots, F_n : \mathbb{R} \rightarrow \mathbb{C}$,

$$\mathcal{K}_{(F_1 \otimes \dots \otimes F_n)(L_1^\sharp, \dots, L_n^\sharp)} = \mathcal{K}_{F_1(L_1)} \otimes \dots \otimes \mathcal{K}_{F_n(L_n)}$$

[16, Corollary 5.5]. Hence, for all $\phi_1, \dots, \phi_n \in C_c^\infty(\mathbb{R})$, if $\phi = \phi_1 \otimes \dots \otimes \phi_n$, then the map $H \mapsto \mathcal{K}_{(\phi H)(L_1^\sharp, \dots, L_n^\sharp)}$ is the tensor product of the maps $F_j \mapsto \mathcal{K}_{(\phi_j F_j)(L_j)}$. Since these maps are bounded $W_2^{s_j}(\mathbb{R}) \rightarrow L^2(G_j, w_{j,s_j}^2(x_j) dx_j)$, the map $H \mapsto \mathcal{K}_{(\phi H)(L_1^\sharp, \dots, L_n^\sharp)}$ is bounded $S_2^{(s_1, \dots, s_n)} W(\mathbb{R}^n) \rightarrow L^2(G, w_s^2(x) dx)$, where $S_2^{(s_1, \dots, s_n)} W(\mathbb{R}^n) = W_2^{s_1}(\mathbb{R}) \otimes \dots \otimes W_2^{s_n}(\mathbb{R})$ is the L^2 Sobolev space with dominating mixed smoothness [25] of order (s_1, \dots, s_n) , and $w_s = w_{1,s_1} \otimes \dots \otimes w_{n,s_n}$ is the product weight on G . In particular, for all compact sets $K \subseteq \mathbb{R}$, if we choose the cutoffs $\phi_j \in C_c^\infty(\mathbb{R})$ so that $\phi_j|_K = 1$, then we deduce that, for all $H : \mathbb{R}^n \rightarrow \mathbb{C}$ with $\text{supp } H \subseteq K^n$,

$$\|w_s \mathcal{K}_{H(L_1^\sharp, \dots, L_n^\sharp)}\|_{L^2(G)} \leq C_{K,s} \|H\|_{S_2^{(s_1, \dots, s_n)} W(\mathbb{R}^n)}.$$

(cf. [17, Proposition 5.2]). Since

$$\begin{aligned} \|f\|_{S_2^{(s_1, \dots, s_n)} W(\mathbb{R}^n)}^2 &\sim \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1 + |\xi_1|)^{2s_1} \dots (1 + |\xi_n|)^{2s_n} d\xi \\ &\leq \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1 + |\xi|)^{2s_1 + \dots + 2s_n} d\xi \sim \|f\|_{W_2^s(\mathbb{R}^n)}^2, \end{aligned}$$

where \hat{f} denotes the Euclidean Fourier transform of f , we see immediately that the estimate

$$\|w_s \mathcal{K}_{H(L_1^\sharp, \dots, L_n^\sharp)}\|_{L^2(G)} \leq C_{K,s_1, \dots, s_n} \|H\|_{W_2^s(\mathbb{R}^n)}, \quad (29)$$

holds true whenever $K \subseteq \mathbb{R}$ is compact and $H : \mathbb{R}^n \rightarrow \mathbb{C}$ is supported in K^n .

Take now a compact set $K \subseteq \mathbb{R}$ and choose a smooth cutoff $\eta_K \in C_c^\infty(\mathbb{R})$ such that $\eta_K|_{[0, \max K]} = 1$. Let $F : \mathbb{R} \rightarrow \mathbb{C}$ be such that $\text{supp } F \subseteq K$, and define $H : \mathbb{R}^n \rightarrow \mathbb{C}$ by

$$H(\lambda_1, \dots, \lambda_n) = F(\lambda_1 + \dots + \lambda_n) \eta_K(\lambda_1) \dots \eta_K(\lambda_n)$$

for all $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$. Then $\text{supp } H \subseteq (\text{supp } \eta_K)^n$, and

$$F(\lambda_1 + \dots + \lambda_n) = H(\lambda_1, \dots, \lambda_n)$$

for all $(\lambda_1, \dots, \lambda_n) \in [0, \infty]^n$. Since the operators L_1, \dots, L_n are nonnegative, the joint spectrum of $L_1^\sharp, \dots, L_n^\sharp$ is contained in $[0, \infty]^n$, hence

$$F(L) = F(L_1^\sharp + \dots + L_n^\sharp) = H(L_1^\sharp, \dots, L_n^\sharp).$$

Consequently, by (29) and the smoothness of the map $(\lambda_1, \dots, \lambda_n) \mapsto \lambda_1 + \dots + \lambda_n$ we obtain that

$$\|w_s \mathcal{K}_{F(L)}\|_{L^2(G)} \leq C_{K,s} \|H\|_{W_2^s(\mathbb{R}^n)} \leq C_{K,s} \|F\|_{W_2^s(\mathbb{R})}.$$

Since clearly $w_s^{-1} = w_{1,s_1}^{-1} \otimes \dots \otimes w_{n,s_n}^{-1} \in L^2(G)$, we are done. \square

The previous results, together with the known weighted estimates for abelian [24, Lemma 3.1] and Métivier [12, 14, 17] groups, then yield the following extension of Theorem 2.

Theorem 16. For $j = 1, \dots, n$, suppose that L_j is a homogeneous sublaplacian on a stratified Lie group G_j . Suppose further that, for each $j \in \{1, \dots, n\}$, at least one of the following conditions holds:

- G_j and L_j satisfy Assumption (A);
- G_j is a Métivier group;
- G_j is abelian.

Let $G = G_1 \times \dots \times G_n$ and $L = L_1^\sharp + \dots + L_n^\sharp$, as in Proposition 15. If $F : \mathbb{R} \rightarrow \mathbb{C}$ satisfies

$$\|F\|_{MW_2^s} < \infty$$

for some $s > (\dim G)/2$, then $F(L)$ is of weak type $(1, 1)$ and bounded on $L^p(G)$ for all $p \in]1, \infty[$.

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