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Abstract: In this paper, we review the so-called Myrzakulov Gravity models (MG-N, with N $=$ I, II, ..., VIII) and derive their respective metric-affine generalizations (MAMG-N), discussing also their particular sub-cases. The field equations of the theories are obtained by regarding the metric tensor and the general affine connection as independent variables. We then focus on the case in which the function characterizing the aforementioned metric-affine models is linear and consider a Friedmann-Lemaître–Robertson–Walker background to study cosmological aspects and applications. Historical motivation for this research is thoroughly reviewed and specific physical motivations are provided for the aforementioned family of alternative theories of gravity.

Keywords: modified gravity; metric-affine gravity; gravitation and cosmology

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1. Introduction

In the 20th century, physics experienced extraordinary progress with the formulation of General Relativity (GR), Einstein's well-celebrated theory of gravity. However, despite its great success and solid predictive power, GR is not devoid of limitations, which manifest as shortages at both very small and large scales [1,2], along with contradictions between theory and observations. In particular, the flaws of GR at large scales are grievous given that gravity appears to be the force that rules cosmic evolution. In this context, GR is indeed unable to explain the observed late-time accelerated expansion of the universe. Another problematic point is the inability of GR to explicate the rotational curves of galaxies without the need for dark matter. Further open issues which highly involve gravity at cosmological scales and are related to each other include the horizon and so-called flatness problem, cosmic inflation and early universe, size, origin and future of the latter, the abundant and mysterious dark energy, the cosmological constant and coincidence problems.

Consequently, many physicists will agree that gravity, even though related to phenomena that we experience in everyday life, still stands as the most enigmatic of the fundamental interactions. The lack of a clear understanding of gravity has led over the years to the formulation of disparate alternative theoretical frameworks, which collectively go by the name of modified gravity [3]. In fact, the terminologies "alternative theory of gravity" and "modified gravity" have become standard for gravitational theories differing from the most conventional one, where the latter is considered to be GR (whose rigorous mathematical formulations resides in Riemannian geometry). The literature on the subject is huge. Let us mention, for instance, (Palatini and metric-affine) $f(R)$ gravity [4–8], teleparallel $f(T)$ gravity theories [9,10], symmetric teleparallel $f(Q)$ gravity [11,12], and scalar-tensor theories [13,14]. Here, let us mention that teleparallel theories of gravity are typically "gauge" theories of gravity. (We put the word "gauge" in inverted commas as, commonly, gravitational theories are invariant under diffeomorphisms by construction, but they are not invariant under spacetime translations. Thus, they are not true gauge theory of the associated group considered. However, we shall adopt the terminology "gauge theory of gravity" since it is widely used in the literature, keeping in mind that in fact, in that case we just have diffeomorphisms invariance rather than invariance under spacetime translations.) On one hand, one can fix a priori symmetries and form of the connection in such a way to determine the explicit form of torsion (cf., e.g., [15]) or nonmetricity in terms of what turns out to be the dynamical field (i.e., in the tetrads formalism, the vielbein). On the other hand, one can start with no a priori assumption on the symmetries of the connection which, however, is related to non-Riemannian quantities by the definition of the field-strengths associated with the gauge group at hand. By contrast, in this review we will consider gravitational theories out of this gauge realm, as we will further discuss in the following.

In the context of modified gravity, a wide part of the physics scientific community claims that the understanding and solution to open issues regarding the gravitational interaction may need generalizations and extensions of Riemannian geometry. It is well-known that one way to go beyond Riemannian geometry is to release the Riemannian assumptions of metric compatibility and torsionlessness of the connection and therefore allow, as we have already anticipated above, for non-vanishing torsion and nonmetricity (along with curvature). This is the framework of non-Riemannian geometry (Regarding non-Riemannian geometry, we refer the reader to [16,17]. Moreover, we highlight [18] for a recent review of Einstein manifolds with torsion and nonmetricity and applications in physics (for further interesting applications cf., for instance, [19] and references therein).) and, in particular, the "geometric arena" where Metric-Affine Gravity (MAG) theories are developed [20–23]. Let us stress that even if the metric-affine approach has been widely used to interpret gravity as a gauge theory, there is no conceptual or physical problem in studying metric-affine theories outside this realm. This is in fact what we are going do in the present paper, where we will not deal with gauge theories of gravity. Specifically, we will work in the first order formalism, considering the metric and the connection a priori as independent, without assuming any symmetry or constraint on the connection from the very beginning. In this setting, the final form of the connection in terms of non-Riemannian objects is then obtained from the study of the field equations of the theory.

MAGs in the first order formalism are gravitational theories alternative to GR exhibiting a very general setup, with the potential of properly describing various physical scenarios, where the metric and the general affine connection (i.e., involving, in principle, torsion and nonmetricity) are considered, a priori, as independent. In particular, no symmetry is imposed a priori on the connection. An additional motivation for studying MAG theories emerges when one considers coupling with matter, as the matter Lagrangian depends on the connection as well. Therefore, in MAG there is a new physical object that comes into play when varying the matter part of the action with respect to the connection, which is the so-called hypermomentum tensor [24–26], which encompasses the microscopic characteristics of matter. In this setup, the energy-momentum tensor sources spacetime curvature by means of the metric field equations, while the hypermomentum is source of spacetime torsion and nonmetricity through the connection field equations.

Following the line of thought based on the idea that considering alternative geometric frameworks one can effectively gain better insights towards a deeper and more complete understanding of gravity than the one provided by GR, in this paper we collect and review a rather general class of gravity theories, the so-called Myrzakulov Gravity (MG) models, in the literature also referred to as MG-N, $\mathrm{N} = \mathrm{I}, \mathrm{II}, \dots, \mathrm{VIII}$ [27], and derive their respective metric-affine generalizations, which will go by the name of Metric-Affine Myrzakulov Gravity (MAMG) models. The action of MG theories is characterized by a generic function F of non-Riemannian scalars (the scalar curvature of the general affine connection, the torsion scalar, the nonmetricity scalar, and, moreover, the energy-momentum trace), which takes different form depending on the specific model. Moreover, in the metric-affine framework one may generalize the theories by including a dependence on the divergence of the dilation current, the latter being a trace of the hypermomentum tensor, in F. We will consider four spacetime dimensions and work in the first order (Palatini) formalism, where the metric tensor $g_{\mu\nu}$ and the general affine connection $\Gamma^\lambda{}_{\mu\nu}$ are treated, a priori, as independent variables, following the lines of [28]. Subsequently, we will focus on the case in which the function characterizing the aforementioned metric-affine models is linear and consider a Friedmann-Lemaître–Robertson–Walker (FLRW) background to study cosmological aspects.

The paper is organized as follows: In Sections 2–9 we review the MG-N models and present their respective metric-affine generalizations (MAMG-N), together with their particular sub-cases, while Section 10 is devoted to the study of a non-Riemannian cosmological setup in which, in particular, we derive the modified Friedmann equations for the linear MAMG-N theories and discuss cosmological applications of the results (another application is also presented in Appendix C). In Section 11 we make some final remark and discuss possible future developments. In Appendix A we collect notation, conventions, definitions, and useful MAG formulas, while Appendix B gathers key expressions in the context of cosmology with torsion and nonmetricity in a homogeneous, non-Riemannian FLRW spacetime.

2. MG-I and MAMG-I

We first describe the MG-I model. Its action reads [27]

$$\mathcal{S}^{(\mathrm{I})}[g,\Gamma,\varphi] = \frac{1}{2\kappa}\int \sqrt{-g}d^4x[F(R,T)+2\kappa\mathcal{L}_{\mathrm{m}}]\,, \tag{1}$$

where $\kappa = 8\pi G$ is the gravitational constant, R is the curvature scalar of the general affine connection $\Gamma^\lambda{}_{\mu\nu}$ involving torsion and nonmetricity and T is the torsion scalar (see Appendix A). In Equation (1), $F = F(R,T)$ is a generic function of R and T. In fact, the MG-I model represents an extension of both the $F(R)$ and $F(T)$ gravity theories. The action (1) depends on the metric field $g_{\mu\nu}$, the affine connection $\Gamma^\lambda{}_{\mu\nu}$, and the matter fields, collectively denoted by φ, appearing in the matter Lagrangian $\mathcal{L}_{\mathrm{m}}$.

The variation of (1) with respect to the metric field yields

$$-\frac{1}{2}g_{\mu\nu}F + F'_R R_{(\mu\nu)} + F'_T\left(2S_{\nu\alpha\beta}S_\mu{}^{\alpha\beta} - S_{\alpha\beta\mu}S^{\alpha\beta}{}_\nu + 2S_{\nu\alpha\beta}S_\mu{}^{\beta\alpha} - 4S_\mu S_\nu\right) = \kappa T_{\mu\nu}\,, \tag{2}$$

where $R_{(\mu\nu)}$ is the symmetric part of the Ricci tensor of Γ, $S_{\mu\nu}{}^\lambda$ is the torsion tensor, S_μ is the torsion trace, $T_{\mu\nu}$ is the energy-momentum tensor (cf. the respective definitions in Appendix A), and $F'_R := \frac{\partial F}{\partial R}$, $F'_T := \frac{\partial F}{\partial T}$. (Here and in the following we adopt the notation $F'_X := \frac{\partial F}{\partial X}$ to denote the derivative of F with respect to any scalar X of which F is function.)

On the other hand, the connection field equations are

$$P_\lambda{}^{\mu\nu}(F'_R) + 2F'_T\left(S^{\mu\nu}{}_\lambda - 2S_\lambda{}^{[\mu\nu]} - 4S^{[\mu}\delta^{\nu]}_\lambda\right) = 0\,, \tag{3}$$

where $P_\lambda{}^{\mu\nu}(F'_R)$ is the modified Palatini tensor,

$$P_\lambda{}^{\mu\nu}(F'_R) := -\frac{\nabla_\lambda(\sqrt{-g}F'_R g^{\mu\nu})}{\sqrt{-g}} + \frac{\nabla_\alpha\left(\sqrt{-g}F'_R g^{\mu\alpha}\delta^\nu_\lambda\right)}{\sqrt{-g}} + 2F'_R(S_\lambda g^{\mu\nu} - S^\mu\delta^\nu_\lambda - S_\lambda{}^{\mu\nu})\,, \tag{4}$$

being ∇ the covariant derivative associated with the general affine connection Γ.

For some cosmological implications of the MG-I model we refer the reader to [29,30], while observational constraints on the theory were studied in [31]. We will come back to these points in Section 10.

2.1. *Metric-Affine Generalizations of the MG-I Model*

As we have already mentioned in the introduction, in the metric-affine setup the matter Lagrangian depends on the connection as well. In this framework, the theory is assumed to have, in principle, a non-vanishing hypermomentum tensor, $\Delta_\lambda{}^{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_{\mathrm{m}})}{\delta\Gamma^\lambda{}_{\mu\nu}}$, which is one of the sources of MAG theories (along with the energy-momentum tensor, cf.

Appendix A). The hypermomentum has a direct physical interpretation when split into its irreducible pieces of spin, dilation, and shear [24–26]. Moreover, as observed in [28], in the metric-affine setup one may also consider the function F appearing in the various MG models to depend on the contribution

$$\mathcal{D} := \frac{1}{\sqrt{-g}}\partial_\nu\left(\sqrt{-g}\Delta^\nu\right), \tag{5}$$

where

$$\Delta^\nu := \Delta_\mu{}^{\mu\nu} \tag{6}$$

is the dilation current. (In particular, the energy-momentum trace $\mathcal{T} := g^{\mu\nu}T_{\mu\nu}$ and $\mathcal{D}$ can be placed on an equal footing (see [28] for details). The first, in fact, will appear in the MG-IV, MG-VI, MG-VII, and MG-VIII theories and in their metric-affine generalization, in which case we will also include the $\mathcal{D}$ contribution.)

Taking all of this into account, the MAMG-I action, which is the metric-affine generalization of (1), reads as follows:

$$\mathcal{S}^{(\mathrm{I})}_{\mathrm{MAMG}} = \frac{1}{2\kappa}\int \sqrt{-g}d^4x[F(R,T,\mathcal{D}) + 2\kappa\mathcal{L}_{\mathrm{m}}], \tag{7}$$

where we have also introduced a dependence on $\mathcal{D}$ in the function F.

The metric field equations of the theory are

$$-\frac{1}{2}g_{\mu\nu}F + F'_R R_{(\mu\nu)} + F'_T\left(2S_{\nu\alpha\beta}S_\mu{}^{\alpha\beta} - S_{\alpha\beta\mu}S^{\alpha\beta}{}_\nu + 2S_{\nu\alpha\beta}S_\mu{}^{\beta\alpha} - 4S_\mu S_\nu\right) + F'_{\mathcal{D}}M_{\mu\nu} = \kappa T_{\mu\nu}, \tag{8}$$

where

$$M_{\mu\nu} := \frac{\delta\mathcal{D}}{\delta g^{\mu\nu}}, \tag{9}$$

while the variation of (7) with respect to the general affine connection $\Gamma^\lambda{}_{\mu\nu}$ yields

$$P_\lambda{}^{\mu\nu}(F'_R) + 2F'_T\left(S^{\mu\nu}{}_\lambda - 2S_\lambda{}^{[\mu\nu]} - 4S^{[\mu}\delta^{\nu]}_\lambda\right) - M_\lambda{}^{\mu\nu\rho}\partial_\rho F'_{\mathcal{D}} = \kappa\Delta_\lambda{}^{\mu\nu}, \tag{10}$$

where we have defined

$$M_\lambda{}^{\mu\nu\rho} := \frac{\delta\Delta^\rho}{\delta\Gamma^\lambda{}_{\mu\nu}}. \tag{11}$$

Let us mention, here, that one could also consider a "minimal" metric-affine generalization of the MG-I theory, obtained without including a dependence on $\mathcal{D}$ in the function F. In this case, the metric field equations would coincide with those of the MG-I model, namely (2), while the connection field equations would be

$$P_\lambda{}^{\mu\nu}(F'_R) + 2F'_T\left(S^{\mu\nu}{}_\lambda - 2S_\lambda{}^{[\mu\nu]} - 4S^{[\mu}\delta^{\nu]}_\lambda\right) = \kappa\Delta_\lambda{}^{\mu\nu}, \tag{12}$$

as we still have the hypermomentum contribution obtained by varying the matter Lagrangian $\mathcal{L}_{\mathrm{m}}$ with respect to the connection. The same result is obtained for specific matter such that $\Delta^\nu = 0$. (In fact, for specific matter one has an explicit, specific expression for the whole $\Delta_\lambda{}^{\mu\nu}$).

2.2. Particular Sub-Cases of the MAMG-I Theory

In this subsection we collect some particular sub-cases of the MAMG-I model. In each sub-case, if we remove the dependence of the matter Lagrangian on the general affine connection, we are left with the respective sub-case of the MG-I model. We have already mentioned above one sub-case, corresponding to the minimal MAMG-I theory (minimal metric-affine generalization of MG-I), that is the metric-affine $F(R,T)$ theory. Let us discuss other sub-cases in the following.

2.2.1. Metric-Affine $F(R)$ Theory

First, we restrict ourselves to $F(R,T,\mathcal{D}) \to F(R)$, i.e., we assume that F is independent of the torsion scalar T and we also remove the $\mathcal{D}$ dependence in F. Then, the MAMG-I model reduces to the well-known metric-affine $F(R)$ gravity. The action of the theory has the form

$$\mathcal{S}_{F(R)} = \frac{1}{2\kappa}\int \sqrt{-g}d^4x[F(R) + 2\kappa\mathcal{L}_{\rm m}]\,, \tag{13}$$

where R is the curvature scalar of the general affine connection. The variation of the action (13) with respect to the metric and the affine connection gives the following set of field equations:

$$\begin{aligned} &-\frac{1}{2}g_{\mu\nu}F + F'_R R_{(\mu\nu)} = \kappa T_{\mu\nu}\,, \\ &P_\lambda{}^{\mu\nu}(F'_R) = \kappa\Delta_\lambda{}^{\mu\nu}\,, \end{aligned} \tag{14}$$

where $P_\lambda{}^{\mu\nu}(F'_R)$ is the modified Palatini tensor defined in (4).

2.2.2. Metric-Affine $F(T)$ Theory

Now we assume that $F(R,T,\mathcal{D}) \to F(T)$. This corresponds to the metric-affine $F(T)$ theory, which is another sub-case of the MAMG-I model. The action of the theory reads

$$\mathcal{S}_{F(T)} = \frac{1}{2\kappa}\int \sqrt{-g}d^4x[F(T) + 2\kappa\mathcal{L}_{\rm m}]\,, \tag{15}$$

where T is the torsion scalar. Varying the action (15) with respect to the metric field and the general affine connection, we obtain

$$\begin{aligned} &-\frac{1}{2}g_{\mu\nu}F + F'_T\left(2S_{\nu\alpha\beta}S_\mu{}^{\alpha\beta} - S_{\alpha\beta\mu}S^{\alpha\beta}{}_\nu + 2S_{\nu\alpha\beta}S_\mu{}^{\beta\alpha} - 4S_\mu S_\nu\right) = \kappa T_{\mu\nu}\,, \\ &2F'_T\left(S^{\mu\nu}{}_\lambda - 2S_\lambda{}^{[\mu\nu]} - 4S^{[\mu}\delta^{\nu]}_\lambda\right) = \kappa\Delta_\lambda{}^{\mu\nu}\,, \end{aligned} \tag{16}$$

respectively. Here observe that defining a specific form for the connection from the very beginning, as done in [15], where the Weitzenböck connection [32] was considered, in $F(T)$ gravity one could then derive the Einstein equations for the metric field by exploiting the tetrads formalism, i.e., introducing the vielbein vector with the use of its component form $\mathbf{V}_a = V^\mu_a\partial_\mu$, with $a = 0,1,2,3$, and expressing the torsion in terms of the latter. (For a discussion on the degrees of freedom in $F(T)$ gravity we refer the reader to [15].) Nevertheless, recall that here we are not assuming a specific form for the connection a priori and we are not dealing with a gauge theory of gravity. In our formalism, the final form of the connection arises when one studies in detail the field equations of the theory for given matter field contents, i.e., once the explicit expression of $\Delta_\lambda{}^{\mu\nu}$ is consequently found. Hence, torsion and, in particular, hypermomentum variables (i.e., sources) encode the full dynamics of the theory. Notice that analogous arguments apply to the $F(Q)$ (sub-)case we will briefly review in Section 3.

2.2.3. Metric-Affine $F(R,\mathcal{D})$ Theory

Let us now restrict ourselves to $F(R,T,\mathcal{D}) \to F(R,\mathcal{D})$, i.e., we assume that F is independent of the torsion scalar T. In this case, the MAMG-I model boils down to the metric-affine $F(R,\mathcal{D})$ theory, which is an extension of the $F(R)$ gravity involving a dependence on the divergence of the dilation current $\mathcal{D}$ in F. The action of the model at hand has the form

$$\mathcal{S}_{F(R,\mathcal{D})} = \frac{1}{2\kappa}\int \sqrt{-g}d^4x[F(R,\mathcal{D}) + 2\kappa\mathcal{L}_{\rm m}]\,. \tag{17}$$

The variation of the action (17) with respect to the metric and the general affine connection gives the following set of field equations:

$$
\begin{aligned}
& -\frac{1}{2}g_{\mu\nu}F + F'_R R_{(\mu\nu)} + F'_{\mathcal{D}} M_{\mu\nu} = \kappa T_{\mu\nu}\,, \\
& P_\lambda{}^{\mu\nu}(F'_R) - M_\lambda{}^{\mu\nu\rho}\partial_\rho F'_{\mathcal{D}} = \kappa \Delta_\lambda{}^{\mu\nu}\,,
\end{aligned}
\tag{18}
$$

with $P_\lambda{}^{\mu\nu}(F'_R)$ defined in (4), while $M_{\mu\nu}$ and $M_\lambda{}^{\mu\nu\rho}$ are given by (9) and (11), respectively.

2.2.4. Metric-Affine $F(T,\mathcal{D})$ Theory

Considering $F(R,T,\mathcal{D}) \to F(T,\mathcal{D})$, the MAMG-I model reduces to the metric-affine $F(T,\mathcal{D})$ theory, which, in turn, is an extension of the $F(T)$ theory involving a dependence on the divergence of the dilation current in F. The action of the theory reads

$$
\mathcal{S}_{F(T,\mathcal{D})} = \frac{1}{2\kappa}\int \sqrt{-g}d^4x[F(T,\mathcal{D}) + 2\kappa\mathcal{L}_{\mathrm{m}}]\,,
\tag{19}
$$

where T is the torsion scalar and $\mathcal{D}$ the divergence of the dilation current. Varying the action (19) with respect to the metric and the general affine connection, we find

$$
\begin{aligned}
& -\frac{1}{2}g_{\mu\nu}F + F'_T\left(2S_{\nu\alpha\beta}S_\mu{}^{\alpha\beta} - S_{\alpha\beta\mu}S^{\alpha\beta}{}_\nu + 2S_{\nu\alpha\beta}S_\mu{}^{\beta\alpha} - 4S_\mu S_\nu\right) + F'_{\mathcal{D}}M_{\mu\nu} = \kappa T_{\mu\nu}\,, \\
& 2F'_T\left(S^{\mu\nu}{}_\lambda - 2S_\lambda{}^{[\mu\nu]} - 4S^{[\mu}\delta^{\nu]}_\lambda\right) - M_\lambda{}^{\mu\nu\rho}\partial_\rho F'_{\mathcal{D}} = \kappa\Delta_\lambda{}^{\mu\nu}\,,
\end{aligned}
\tag{20}
$$

respectively.

3. MG-II and MAMG-II

In this section, we start from the description of the MG-II theory and subsequently generalize it by considering a metric-affine setup. The action of the MG-II model is [27]

$$
\mathcal{S}^{(\mathrm{II})}[g,\Gamma,\varphi] = \frac{1}{2\kappa}\int \sqrt{-g}d^4x[F(R,Q) + 2\kappa\mathcal{L}_{\mathrm{m}}]\,.
\tag{21}
$$

It is an extension of both the $F(R)$ and $F(Q)$ theories. Indeed, the function $F = F(R,Q)$ in (21) is a generic function of the scalar curvature R (of the general affine connection Γ) and of Q, where Q is the nonmetricity scalar (cf. Appendix A).

The metric field equations of the theory read as follows:

$$
-\frac{1}{2}g_{\mu\nu}F + F'_R R_{(\mu\nu)} + F'_Q L_{(\mu\nu)} + \hat{\nabla}_\lambda\left(F'_Q J^\lambda{}_{(\mu\nu)}\right) + g_{\mu\nu}\hat{\nabla}_\lambda\left(F'_Q\zeta^\lambda\right) = \kappa T_{\mu\nu}\,,
\tag{22}
$$

having defined

$$
\hat{\nabla}_\lambda := \frac{1}{\sqrt{-g}}(2S_\lambda - \nabla_\lambda)
\tag{23}
$$

and

$$
\begin{aligned}
L_{\mu\nu} &:= \frac{1}{4}\left[(Q_{\mu\alpha\beta} - 2Q_{\alpha\beta\mu})Q_\nu{}^{\alpha\beta} + (Q_\mu + 2q_\mu)Q_\nu + (2Q_{\mu\nu\alpha} - Q_{\alpha\mu\nu})Q^\alpha\right] \\
&\quad - \Xi^{\alpha\beta}{}_\nu Q_{\alpha\beta\mu} - \Xi_{\alpha\mu\beta}Q^{\alpha\beta}{}_\nu\,, \\
J^\lambda{}_{\mu\nu} &:= \sqrt{-g}\left(\frac{1}{4}Q^\lambda{}_{\mu\nu} - \frac{1}{2}Q_{\mu\nu}{}^\lambda + \Xi^\lambda{}_{\mu\nu}\right), \\
\zeta^\lambda &:= \sqrt{-g}\left(-\frac{1}{4}Q^\lambda + \frac{1}{2}q^\lambda\right),
\end{aligned}
\tag{24}
$$

where $Q_{\lambda\mu\nu}$ is the nonmetricity tensor, Q_λ and q_λ are its trace parts (see Appendix A), and $\Xi_{\lambda\mu\nu}$ is the so-called (nonmetricity) "superpotential" defined in the first line of (A15).

The connection field equations are

$$P_\lambda{}^{\mu\nu}(F'_R) + F'_Q\left[2Q^{[\nu\mu]}{}_\lambda - Q_\lambda{}^{\mu\nu} + (q^\nu - Q^\nu)\delta^\mu_\lambda + Q_\lambda g^{\mu\nu} + \frac{1}{2}Q^\mu\delta^\nu_\lambda\right] = 0\,, \tag{25}$$

where $P_\lambda{}^{\mu\nu}(F'_R)$ is given by (4).

3.1. Metric-Affine Generalizations of the MG-II Model

We can now move on to the metric-affine generalization of the MG-II theory. Hence, taking into account the discussion in Section 2, one may write the MAMG-II action as follows:

$$\mathcal{S}^{(\mathrm{II})}_{\mathrm{MAMG}} = \frac{1}{2\kappa}\int\sqrt{-g}d^4x[F(R,Q,\mathcal{D}) + 2\kappa\mathcal{L}_{\mathrm{m}}]\,, \tag{26}$$

where, in particular, we have introduced in F a dependence on $\mathcal{D}$, the latter being the divergence of the dilation current, defined in Equation (5).

The variation of (26) with respect to the metric yields

$$-\frac{1}{2}g_{\mu\nu}F + F'_R R_{(\mu\nu)} + F'_Q L_{(\mu\nu)} + \hat{\nabla}_\lambda\left(F'_Q J^\lambda{}_{(\mu\nu)}\right) + g_{\mu\nu}\hat{\nabla}_\lambda\left(F'_Q\zeta^\lambda\right) + F'_{\mathcal{D}}M_{\mu\nu} = \kappa T_{\mu\nu}\,, \tag{27}$$

where $M_{\mu\nu}$ has been defined in (9).

On the other hand, the connection field equations of the MAMG-II theory read

$$P_\lambda{}^{\mu\nu}(F'_R) + F'_Q\left[2Q^{[\nu\mu]}{}_\lambda - Q_\lambda{}^{\mu\nu} + (q^\nu - Q^\nu)\delta^\mu_\lambda + Q_\lambda g^{\mu\nu} + \frac{1}{2}Q^\mu\delta^\nu_\lambda\right] - M_\lambda{}^{\mu\nu\rho}\partial_\rho F'_{\mathcal{D}} = \kappa\Delta_\lambda{}^{\mu\nu}\,, \tag{28}$$

where $M_\lambda{}^{\mu\nu\rho}$ is given by (11). Analogously to what we have mentioned in Section 2 for the $\mathrm{N} = \mathrm{I}$ case, one might also consider a minimal metric-affine generalization of the MG-II theory, excluding the $\mathcal{D}$ dependence in the function F. Then, the metric field equations would coincide with (22), while the connection field equations would be

$$P_\lambda{}^{\mu\nu}(F'_R) + F'_Q\left[2Q^{[\nu\mu]}{}_\lambda - Q_\lambda{}^{\mu\nu} + (q^\nu - Q^\nu)\delta^\mu_\lambda + Q_\lambda g^{\mu\nu} + \frac{1}{2}Q^\mu\delta^\nu_\lambda\right] = \kappa\Delta_\lambda{}^{\mu\nu}\,. \tag{29}$$

The same result can be obtained for specific matter couplings fulfilling $\Delta^\nu = 0$.

3.2. Particular Sub-cases of the MAMG-II Theory

Here we collect some particular sub-cases of the MAMG-II theory. In each sub-case, if we remove the dependence of the matter Lagrangian on the general affine connection, we are left with the respective sub-case of the MG-II model. We have already mentioned above one sub-case, corresponding to the minimal MAMG-II model (minimal metric-affine generalization of MG-II), i.e., the metric-affine $F(R,Q)$ theory. The other two sub-cases correspond to the metric-affine $F(R)$ and $F(R,\mathcal{D})$ theories previously discussed (cf. (13) and (17), respectively). Let us report in the following also other two sub-cases.

3.2.1. Metric-Affine $F(Q)$ Theory

We first restrict ourselves to $F(R,Q,\mathcal{D}) \to F(Q)$, i.e., we assume that F is independent of the scalar curvature R and we also remove the $\mathcal{D}$ dependence in F. With this assumption the MAMG-II model reduces to the metric-affine $F(Q)$ theory, whose action is

$$\mathcal{S}_{F(Q)} = \frac{1}{2\kappa}\int\sqrt{-g}d^4x[F(Q) + 2\kappa\mathcal{L}_{\mathrm{m}}]\,, \tag{30}$$

where Q is the nonmetricity scalar. The variation of the action (30) with respect to the metric and the affine connection gives the following set of field equations:

$$\begin{aligned} &-\frac{1}{2}g_{\mu\nu}F + F'_Q L_{(\mu\nu)} + \hat{\nabla}_\lambda\left(F'_Q J^\lambda{}_{(\mu\nu)}\right) + g_{\mu\nu}\hat{\nabla}_\lambda\left(F'_Q\zeta^\lambda\right) = \kappa T_{\mu\nu}\,, \\ &F'_Q\left[2Q^{[\nu\mu]}{}_\lambda - Q_\lambda{}^{\mu\nu} + (q^\nu - Q^\nu)\delta^\mu_\lambda + Q_\lambda g^{\mu\nu} + \frac{1}{2}Q^\mu\delta^\nu_\lambda\right] = \kappa\Delta_\lambda{}^{\mu\nu}\,, \end{aligned} \tag{31}$$

with $L_{\mu\nu}$, $J^\lambda{}_{\mu\nu}$, and ζ^λ given by (24). Here one could then apply arguments analogous to those made in the case of the $F(T)$ theory.

3.2.2. Metric-Affine $F(Q,\mathcal{D})$ Theory

On the other hand, considering $F(R,Q,\mathcal{D}) \to F(Q,\mathcal{D})$, the MAMG-II model boils down to the metric-affine $F(Q,\mathcal{D})$ theory, which, in turn, is an extension of the $F(Q)$ theory above involving a dependence on the divergence of the dilation current in F. The action of the metric-affine $F(Q,\mathcal{D})$ model has the following form:

$$\mathcal{S}_{F(Q,\mathcal{D})} = \frac{1}{2\kappa}\int\sqrt{-g}d^4x[F(Q,\mathcal{D}) + 2\kappa\mathcal{L}_\mathrm{m}]\,, \tag{32}$$

where Q is the nonmetricity scalar and $\mathcal{D}$ the divergence of the dilation current. Varying the action (32) with respect to the metric and the general affine connection, we obtain the set of field equations

$$\begin{aligned} &-\frac{1}{2}g_{\mu\nu}F + F'_Q L_{(\mu\nu)} + \hat{\nabla}_\lambda\left(F'_Q J^\lambda{}_{(\mu\nu)}\right) + g_{\mu\nu}\hat{\nabla}_\lambda\left(F'_Q\zeta^\lambda\right) + F'_\mathcal{D}M_{\mu\nu} = \kappa T_{\mu\nu}\,, \\ &F'_Q\left[2Q^{[\nu\mu]}{}_\lambda - Q_\lambda{}^{\mu\nu} + (q^\nu - Q^\nu)\delta^\mu_\lambda + Q_\lambda g^{\mu\nu} + \frac{1}{2}Q^\mu\delta^\nu_\lambda\right] - M_\lambda{}^{\mu\nu\rho}\partial_\rho F'_\mathcal{D} = \kappa\Delta_\lambda{}^{\mu\nu}\,, \end{aligned} \tag{33}$$

with $M_{\mu\nu}$ and $M_\lambda{}^{\mu\nu\rho}$ given by (9) and (11), respectively.

4. MG-III and MAMG-III

Let us start by describing the MG-III theory, whose action is [27]

$$\mathcal{S}^{(\mathrm{III})}[g,\Gamma,\varphi] = \frac{1}{2\kappa}\int\sqrt{-g}d^4x[F(T,Q) + 2\kappa\mathcal{L}_\mathrm{m}]\,. \tag{34}$$

In this model, the function $F = F(T,Q)$ is a generic function of the torsion scalar T and the nonmetricity scalar Q. The action (34) is an extension of both the $F(T)$ and $F(Q)$ theories.

Varying (34) with respect to the metric field we obtain

$$\begin{aligned} &-\frac{1}{2}g_{\mu\nu}F + F'_T\left(2S_{\nu\alpha\beta}S_\mu{}^{\alpha\beta} - S_{\alpha\beta\mu}S^{\alpha\beta}{}_\nu + 2S_{\nu\alpha\beta}S_\mu{}^{\beta\alpha} - 4S_\mu S_\nu\right) + F'_Q L_{(\mu\nu)} \\ &+ \hat{\nabla}_\lambda\left(F'_Q J^\lambda{}_{(\mu\nu)}\right) + g_{\mu\nu}\hat{\nabla}_\lambda\left(F'_Q\zeta^\lambda\right) = \kappa T_{\mu\nu}\,, \end{aligned} \tag{35}$$

where $\hat{\nabla}$ is given by (23), while the tensor $L_{\mu\nu}$ and the tensor and vector densities $J^\lambda{}_{\mu\nu}$ and ζ^λ, respectively, are defined in (24).

On the other hand, from the variation of (34) with respect to the general affine connection $\Gamma^\lambda{}_{\mu\nu}$ we obtain

$$2F'_T\left(S^{\mu\nu}{}_\lambda - 2S_\lambda{}^{[\mu\nu]} - 4S^{[\mu}\delta^{\nu]}_\lambda\right) + F'_Q\left[2Q^{[\nu\mu]}{}_\lambda - Q_\lambda{}^{\mu\nu} + (q^\nu - Q^\nu)\delta^\mu_\lambda + Q_\lambda g^{\mu\nu} + \frac{1}{2}Q^\mu\delta^\nu_\lambda\right] = 0\,. \tag{36}$$

We can now move on to the metric-affine generalization of the theory.

Metric-Affine Generalizations of the MG-III Model

The action of the MAMG-III model is

$$\mathcal{S}^{(\mathrm{III})}_{\mathrm{MAMG}} = \frac{1}{2\kappa} \int \sqrt{-g} d^4x [F(T, Q, \mathcal{D}) + 2\kappa \mathcal{L}_{\mathrm{m}}] \,, \tag{37}$$

where $\mathcal{D}$ is defined in (5).

The metric field equations of the theory read

$$\begin{aligned} &-\frac{1}{2} g_{\mu\nu} F + F'_T \left(2S_{\nu\alpha\beta} S_{\mu}{}^{\alpha\beta} - S_{\alpha\beta\mu} S^{\alpha\beta}{}_{\nu} + 2S_{\nu\alpha\beta} S_{\mu}{}^{\beta\alpha} - 4S_{\mu} S_{\nu} \right) + F'_Q L_{(\mu\nu)} \\ &+ \hat{\nabla}_{\lambda} \left(F'_Q J^{\lambda}{}_{(\mu\nu)} \right) + g_{\mu\nu} \hat{\nabla}_{\lambda} \left(F'_Q \zeta^{\lambda} \right) + F'_{\mathcal{D}} M_{\mu\nu} = \kappa T_{\mu\nu} \,, \end{aligned} \tag{38}$$

where $M_{\mu\nu}$ is given by (9).

The connection field equations are

$$\begin{aligned} &2F'_T \left(S^{\mu\nu}{}_{\lambda} - 2S_{\lambda}{}^{[\mu\nu]} - 4S^{[\mu} \delta^{\nu]}_{\lambda} \right) + F'_Q \left[2Q^{[\nu\mu]}{}_{\lambda} - Q_{\lambda}{}^{\mu\nu} + (q^{\nu} - Q^{\nu}) \delta^{\mu}_{\lambda} + Q_{\lambda} g^{\mu\nu} + \frac{1}{2} Q^{\mu} \delta^{\nu}_{\lambda} \right] \\ &- M_{\lambda}{}^{\mu\nu\rho} \partial_{\rho} F'_{\mathcal{D}} = \kappa \Delta_{\lambda}{}^{\mu\nu} \,, \end{aligned} \tag{39}$$

where $M_{\lambda}{}^{\mu\nu\rho}$ is defined in (11). There also exists a minimal metric-affine generalization of the MG-III theory, which excludes the $\mathcal{D}$ dependence in the function F. In this minimal case, the metric field equations coincide with (35), while the connection field equations are

$$\begin{aligned} &2F'_T \left(S^{\mu\nu}{}_{\lambda} - 2S_{\lambda}{}^{[\mu\nu]} - 4S^{[\mu} \delta^{\nu]}_{\lambda} \right) + F'_Q \left[2Q^{[\nu\mu]}{}_{\lambda} - Q_{\lambda}{}^{\mu\nu} + (q^{\nu} - Q^{\nu}) \delta^{\mu}_{\lambda} + Q_{\lambda} g^{\mu\nu} + \frac{1}{2} Q^{\mu} \delta^{\nu}_{\lambda} \right] \\ &= \kappa \Delta_{\lambda}{}^{\mu\nu} \,. \end{aligned} \tag{40}$$

The same equations can be obtained considering some specific matter couplings, always in the metric-affine framework, such that $\Delta^{\nu} = 0$.

Let us conclude this section by mentioning that there are other particular sub-cases, besides the metric-affine $F(T, Q)$ one above, of the MAMG-III model, corresponding to the metric-affine $F(T)$, $F(T, \mathcal{D})$, $F(Q)$, and $F(Q, \mathcal{D})$ theories previously discussed (cf., respectively, (15), (19), (30), and (32)). (In each sub-case, removing the dependence of the matter Lagrangian on the general affine connection, one is left with the respective sub-case of the MG-III model).

5. MG-IV and MAMG-IV

We first describe the MG-IV theory, whose action is [27]

$$\mathcal{S}^{(\mathrm{IV})}[g, \Gamma, \varphi] = \frac{1}{2\kappa} \int \sqrt{-g} d^4x [F(R, T, \mathcal{T}) + 2\kappa \mathcal{L}_{\mathrm{m}}] \,, \tag{41}$$

where $F = F(R, T, \mathcal{T})$ is a generic function of the scalar curvature R, the torsion scalar T, and the energy-momentum trace $\mathcal{T} := g^{\mu\nu} T_{\mu\nu}$. The MG-IV model extends the $F(R, \mathcal{T})$ (cf. [33]), $F(T)$, $F(R)$, and $F(T, \mathcal{T})$ (cf. [34]) theories. In particular, (41) represents an extension of the MG-I theory, as it also involves a dependence on $\mathcal{T}$ in F.

The metric field equations of (41) are

$$\begin{aligned} &-\frac{1}{2} g_{\mu\nu} F + F'_R R_{(\mu\nu)} + F'_T \left(2S_{\nu\alpha\beta} S_{\mu}{}^{\alpha\beta} - S_{\alpha\beta\mu} S^{\alpha\beta}{}_{\nu} + 2S_{\nu\alpha\beta} S_{\mu}{}^{\beta\alpha} - 4S_{\mu} S_{\nu} \right) \\ &+ F'_{\mathcal{T}} (\Theta_{\mu\nu} + T_{\mu\nu}) = \kappa T_{\mu\nu} \,, \end{aligned} \tag{42}$$

where $\hat{\nabla}$ is defined in (23), $L_{\mu\nu}$, $J^{\lambda}{}_{\mu\nu}$ and ζ^{λ} are given in (24), and

$$\Theta_{\mu\nu} := g^{\alpha\beta}\frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}} \,. \tag{43}$$

The connection field equations read as follows:

$$P_{\lambda}{}^{\mu\nu}(F'_R) + 2F'_T\left(S^{\mu\nu}{}_{\lambda} - 2S_{\lambda}{}^{[\mu\nu]} - 4S^{[\mu}\delta^{\nu]}_{\lambda}\right) = 0 \,, \tag{44}$$

where $P_{\lambda}{}^{\mu\nu}(F'_R)$ is defined in (4). Here, observe that (44) coincides with the connection field Equation (3) of the MG-I theory. This is so due to the fact that in the MG theories matter does not couple to the connection and, therefore, the dependence on $\mathcal{T}$ in F does not affect the connection field equations. Accordingly, we have no contribution to the connection field equations from the matter Lagrangian $\mathcal{L}_{\text{m}}$.

5.1. Metric-Affine Generalizations of the MG-IV Model

Following the discussion in Section 2, we write the MAMG-IV action as follows:

$$\mathcal{S}^{(\text{IV})}_{\text{MAMG}} = \frac{1}{2\kappa}\int \sqrt{-g}d^4x[F(R,T,\mathcal{T},\mathcal{D}) + 2\kappa\mathcal{L}_{\text{m}}] \,, \tag{45}$$

where $\mathcal{D}$ is the hypermomentum trace given by (5). In particular, here the energy-momentum trace $\mathcal{T}$ and the divergence of the dilation current $\mathcal{D}$ are placed on an equal footing (cf. [28]).

Varying the action (45) with respect to the metric field we obtain

$$\begin{aligned} &-\frac{1}{2}g_{\mu\nu}F + F'_R R_{(\mu\nu)} + F'_T\left(2S_{\nu\alpha\beta}S_{\mu}{}^{\alpha\beta} - S_{\alpha\beta\mu}S^{\alpha\beta}{}_{\nu} + 2S_{\nu\alpha\beta}S_{\mu}{}^{\beta\alpha} - 4S_{\mu}S_{\nu}\right) \\ &+ F'_{\mathcal{T}}(\Theta_{\mu\nu} + T_{\mu\nu}) + F'_{\mathcal{D}}M_{\mu\nu} = \kappa T_{\mu\nu} \,, \end{aligned} \tag{46}$$

with $M_{\mu\nu}$ and $\Theta_{\mu\nu}$ defined in (9) and (43), respectively.

On the other hand, the variation of (45) with respect to the general affine connection $\Gamma^{\lambda}{}_{\mu\nu}$ yields

$$P_{\lambda}{}^{\mu\nu}(F'_R) + 2F'_T\left(S^{\mu\nu}{}_{\lambda} - 2S_{\lambda}{}^{[\mu\nu]} - 4S^{[\mu}\delta^{\nu]}_{\lambda}\right) - M_{\lambda}{}^{\mu\nu\rho}\partial_{\rho}F'_{\mathcal{D}} = F'_{\mathcal{T}}\Theta_{\lambda}{}^{\mu\nu} + \kappa\Delta_{\lambda}{}^{\mu\nu} \,, \tag{47}$$

where $M_{\lambda}{}^{\mu\nu\rho}$ is given by (11) and

$$\Theta_{\lambda}{}^{\mu\nu} := -\frac{\delta\mathcal{T}}{\delta\Gamma^{\lambda}{}_{\mu\nu}} \,. \tag{48}$$

The latter appears in (47) since, in the metric-affine setup, the energy-momentum trace $\mathcal{T}$ may have, in principle, a non-trivial dependence on the hypermomentum tensor. Please note that if matter does not couple to the connection (which is the case, for instance, of a classical perfect fluid with no inner structure) one has $\Theta_{\lambda}{}^{\mu\nu} = 0$, $\Delta_{\lambda}{}^{\mu\nu} = 0$, and $M_{\lambda}{}^{\mu\nu\rho} = 0$ (together with $\mathcal{D} = 0$, i.e., no hypermomentum contribution also in the metric field equations). Let us mention that one can also consider a minimal metric-affine generalization of the MG-IV theory, excluding the $\mathcal{D}$ dependence in the function F. The metric field equations, in this case, would coincide with (42), while the connection field equations would be

$$P_{\lambda}{}^{\mu\nu}(F'_R) + 2F'_T\left(S^{\mu\nu}{}_{\lambda} - 2S_{\lambda}{}^{[\mu\nu]} - 4S^{[\mu}\delta^{\nu]}_{\lambda}\right) = F'_{\mathcal{T}}\Theta_{\lambda}{}^{\mu\nu} + \kappa\Delta_{\lambda}{}^{\mu\nu} \,. \tag{49}$$

The same result would follow from some specific matter fulfilling $\Delta^{\nu} = 0$. Observe that if $\Theta_{\lambda}{}^{\mu\nu} = 0$ the connection field equations (47) reduce to (10), which are those of the MAMG-I theory. Analogously, if $\Theta_\lambda{}^{\mu\nu} = 0$, then (49) boil down to (12), namely to the connection field equations of the minimal MAMG-I model.

5.2. Particular Sub-cases of the MAMG-IV Theory

Regarding the MAMG-IV model, here we discuss its sub-cases. Two of them are the minimal MAMG-IV model, minimal metric-affine generalization of MG-IV, i.e., the metric-affine $F(R, T, \mathcal{T})$ theory, and, of course, the minimal MAMG-I model, minimal metric-affine generalization of MG-I, i.e., the metric-affine $F(R, T)$ theory. The other sub-cases consist of the $F(R)$, $F(T)$, $F(R, \mathcal{D})$, and $F(T, \mathcal{D})$ theories we described earlier (cf. (13), (15), (17), and (19), respectively). Let us briefly sketch other four sub-cases in the following. In each of the MAMG-IV sub-cases, if we remove the dependence of the matter Lagrangian on the general affine connection, we are left with the respective sub-cases of the MG-IV model.

5.2.1. Metric-Affine $F(R, \mathcal{T})$ Theory

Let us consider $F(R, T, \mathcal{T}, \mathcal{D}) \to F(R, \mathcal{T})$. Then, the MAMG-IV model reduces to the metric-affine $F(R, \mathcal{T})$ theory [33], which is also an extension of the metric-affine $F(R)$ gravity including a dependence on the energy-momentum trace $\mathcal{T}$ in F. The action of the theory has the form

$$\mathcal{S}_{F(R,\mathcal{T})} = \frac{1}{2\kappa} \int \sqrt{-g} d^4x [F(R, \mathcal{T}) + 2\kappa \mathcal{L}_{\mathrm{m}}] \,, \tag{50}$$

where R is the curvature scalar of the general affine connection. The variation of the action (50) with respect to the metric and the affine connection gives the following set of field equations:

$$\begin{aligned} & -\frac{1}{2} g_{\mu\nu} F + F'_R R_{(\mu\nu)} + F'_{\mathcal{T}} (\Theta_{\mu\nu} + T_{\mu\nu}) = \kappa T_{\mu\nu} \,, \\ & P_\lambda{}^{\mu\nu}(F'_R) = F'_{\mathcal{T}} \Theta_\lambda{}^{\mu\nu} + \kappa \Delta_\lambda{}^{\mu\nu} \,, \end{aligned} \tag{51}$$

where $P_\lambda{}^{\mu\nu}(F'_R)$ is the modified Palatini tensor defined in (4), while $\Theta_{\mu\nu}$ and $\Theta_\lambda{}^{\mu\nu}$ are given by Equations (43) and (48), respectively.

5.2.2. Metric-Affine $F(T, \mathcal{T})$ Theory

On the other hand, considering $F(R, T, \mathcal{T}, \mathcal{D}) \to F(T, \mathcal{T})$, the MAMG-IV model boils down to the metric-affine $F(T, \mathcal{T})$ theory [34], which is also an extension of the metric-affine $F(T)$ model as it includes a dependence on the energy-momentum trace $\mathcal{T}$ in F as well. The action of the metric-affine $F(T, \mathcal{T})$ theory reads

$$\mathcal{S}_{F(T,\mathcal{T})} = \frac{1}{2\kappa} \int \sqrt{-g} d^4x [F(T, \mathcal{T}) + 2\kappa \mathcal{L}_{\mathrm{m}}] \,, \tag{52}$$

where T is the torsion scalar. From the variation of the action (52) with respect to the metric and the general affine connection we obtain

$$\begin{aligned} & -\frac{1}{2} g_{\mu\nu} F + F'_T \left(2S_{\nu\alpha\beta} S_\mu{}^{\alpha\beta} - S_{\alpha\beta\mu} S^{\alpha\beta}{}_\nu + 2S_{\nu\alpha\beta} S_\mu{}^{\beta\alpha} - 4S_\mu S_\nu \right) + F'_{\mathcal{T}} (\Theta_{\mu\nu} + T_{\mu\nu}) = \kappa T_{\mu\nu} \,, \\ & 2F'_T \left(S^{\mu\nu}{}_\lambda - 2S_\lambda{}^{[\mu\nu]} - 4S^{[\mu} \delta^{\nu]}_\lambda \right) = F'_{\mathcal{T}} \Theta_\lambda{}^{\mu\nu} + \kappa \Delta_\lambda{}^{\mu\nu} \,, \end{aligned} \tag{53}$$

respectively.

5.2.3. Metric-Affine $F(R, \mathcal{T}, \mathcal{D})$ Theory

Now, we consider the sub-case in which $F(R, T, \mathcal{T}, \mathcal{D}) \to F(R, \mathcal{T}, \mathcal{D})$, i.e., we exclude only the T dependence in F. In this case, the MAMG-IV theory reduces to the metric-affine $F(R, \mathcal{T}, \mathcal{D})$ theory, which is an extension of the metric-affine $F(R, \mathcal{T})$ model (50) with the inclusion of a dependence on the divergence of the dilation current in F. The action of the metric-affine $F(R,\mathcal{T},\mathcal{D})$ theory is given by

$$\mathcal{S}_{F(R,\mathcal{T},\mathcal{D})} = \frac{1}{2\kappa}\int \sqrt{-g}d^4x[F(R,\mathcal{T},\mathcal{D}) + 2\kappa\mathcal{L}_{\mathrm{m}}] \,. \tag{54}$$

The variation of (54) with respect to the metric and the affine connection gives the following set of field equations:

$$\begin{aligned} &-\frac{1}{2}g_{\mu\nu}F + F'_R R_{(\mu\nu)} + F'_{\mathcal{T}}(\Theta_{\mu\nu} + T_{\mu\nu}) + F'_{\mathcal{D}}M_{\mu\nu} = \kappa T_{\mu\nu}\,, \\ &P_\lambda{}^{\mu\nu}(F'_R) - M_\lambda{}^{\mu\nu\rho}\partial_\rho F'_{\mathcal{D}} = F'_{\mathcal{T}}\Theta_\lambda{}^{\mu\nu} + \kappa\Delta_\lambda{}^{\mu\nu}\,, \end{aligned} \tag{55}$$

with $P_\lambda{}^{\mu\nu}(F'_R)$ defined in (4), while $M_{\mu\nu}$, $M_\lambda{}^{\mu\nu\rho}$, $\Theta_{\mu\nu}$, and $\Theta_\lambda{}^{\mu\nu}$ are given in Equations (9), (11), (43), and (48), respectively.

5.2.4. Metric-Affine $F(T,\mathcal{T},\mathcal{D})$ Theory

Finally, taking $F(R,T,\mathcal{T},\mathcal{D}) \to F(T,\mathcal{T},\mathcal{D})$, the MAMG-IV theory reduces to the metric-affine $F(T,\mathcal{T},\mathcal{D})$ model, extension of the metric-affine $F(T,\mathcal{T})$ theory (52) with the inclusion of a dependence on the divergence of the dilation current in F. The action of the metric-affine $F(T,\mathcal{T},\mathcal{D})$ theory reads

$$\mathcal{S}_{F(T,\mathcal{T},\mathcal{D})} = \frac{1}{2\kappa}\int \sqrt{-g}d^4x[F(T,\mathcal{T},\mathcal{D}) + 2\kappa\mathcal{L}_{\mathrm{m}}] \,. \tag{56}$$

The variation of (56) with respect to the metric and the general affine connection yields

$$\begin{aligned} &-\frac{1}{2}g_{\mu\nu}F + F'_T\Big(2S_{\nu\alpha\beta}S_\mu{}^{\alpha\beta} - S_{\alpha\beta\mu}S^{\alpha\beta}{}_\nu + 2S_{\nu\alpha\beta}S_\mu{}^{\beta\alpha} - 4S_\mu S_\nu\Big) + F'_{\mathcal{T}}(\Theta_{\mu\nu} + T_{\mu\nu}) + F'_{\mathcal{D}}M_{\mu\nu} \\ &= \kappa T_{\mu\nu}\,, \\ &2F'_T\Big(S^{\mu\nu}{}_\lambda - 2S_\lambda{}^{[\mu\nu]} - 4S^{[\mu}\delta^{\nu]}_\lambda\Big) - M_\lambda{}^{\mu\nu\rho}\partial_\rho F'_{\mathcal{D}} = F'_{\mathcal{T}}\Theta_\lambda{}^{\mu\nu} + \kappa\Delta_\lambda{}^{\mu\nu}\,, \end{aligned} \tag{57}$$

respectively.

6. MG-V and MAMG-V

In this section, we start from the description of the MG-V model and then generalize the latter in the metric-affine framework. The MG-V action is given by [27]

$$\mathcal{S}^{(\mathrm{V})}[g,\Gamma,\varphi] = \frac{1}{2\kappa}\int \sqrt{-g}d^4x[F(R,T,Q) + 2\kappa\mathcal{L}_{\mathrm{m}}] \,. \tag{58}$$

The MG-V model is an extension of the $F(R)$, $F(T)$, and $F(Q)$ theories. In particular, it represents an extension of the MG-I model also involving a dependence on Q in F, and an extension of the MG-II theory as well, involving a dependence on T in F. In fact, $F = F(R,T,Q)$ is a generic function of the scalar curvature R of the general affine connection $\Gamma^\lambda{}_{\mu\nu}$, the torsion scalar T, and the nonmetricity scalar Q.

The variation of (58) with respect to the metric field yields

$$\begin{aligned} &-\frac{1}{2}g_{\mu\nu}F + F'_R R_{(\mu\nu)} + F'_T\Big(2S_{\nu\alpha\beta}S_\mu{}^{\alpha\beta} - S_{\alpha\beta\mu}S^{\alpha\beta}{}_\nu + 2S_{\nu\alpha\beta}S_\mu{}^{\beta\alpha} - 4S_\mu S_\nu\Big) + F'_Q L_{(\mu\nu)} \\ &+ \hat{\nabla}_\lambda\Big(F'_Q J^\lambda{}_{(\mu\nu)}\Big) + g_{\mu\nu}\hat{\nabla}_\lambda\Big(F'_Q\zeta^\lambda\Big) = \kappa T_{\mu\nu}\,, \end{aligned} \tag{59}$$

where $\hat{\nabla}$ is defined in (23), while the tensor $L_{\mu\nu}$ and the densities $J^\lambda{}_{\mu\nu}$ and ζ^λ are given by (24).

The equations obtained by varying the action (58) with respect to the general affine connection $\Gamma^{\lambda}{}_{\mu\nu}$ read as follows:

$$
\begin{aligned}
& P_{\lambda}{}^{\mu\nu}(F'_R) + 2F'_T\left(S^{\mu\nu}{}_{\lambda} - 2S_{\lambda}{}^{[\mu\nu]} - 4S^{[\mu}\delta^{\nu]}_{\lambda}\right) \\
& + F'_Q\left[2Q^{[\nu\mu]}{}_{\lambda} - Q_{\lambda}{}^{\mu\nu} + (q^{\nu} - Q^{\nu})\delta^{\mu}_{\lambda} + Q_{\lambda}g^{\mu\nu} + \frac{1}{2}Q^{\mu}\delta^{\nu}_{\lambda}\right] = 0\,,
\end{aligned}
\tag{60}
$$

where $P_{\lambda}{}^{\mu\nu}(F'_R)$ is the modified Palatini tensor defined in (4).

Metric-Affine Generalizations of the MG-V Model

We can now move on to the metric-affine generalization of the MG-V theory. Hence, in accordance with the discussion in Section 2, let us write the MAMG-V action as

$$
\mathcal{S}^{(\mathrm{V})}_{\mathrm{MAMG}} = \frac{1}{2\kappa}\int \sqrt{-g}d^4x[F(R,T,Q,\mathcal{D}) + 2\kappa\mathcal{L}_{\mathrm{m}}]\,,
\tag{61}
$$

where $\mathcal{D}$ is defined in (5).

The variation of (61) with respect to the metric yields

$$
\begin{aligned}
& -\frac{1}{2}g_{\mu\nu}F + F'_R R_{(\mu\nu)} + F'_T\left(2S_{\nu\alpha\beta}S_{\mu}{}^{\alpha\beta} - S_{\alpha\beta\mu}S^{\alpha\beta}{}_{\nu} + 2S_{\nu\alpha\beta}S_{\mu}{}^{\beta\alpha} - 4S_{\mu}S_{\nu}\right) + F'_Q L_{(\mu\nu)} \\
& + \hat{\nabla}_{\lambda}\left(F'_Q J^{\lambda}{}_{(\mu\nu)}\right) + g_{\mu\nu}\hat{\nabla}_{\lambda}\left(F'_Q\zeta^{\lambda}\right) + F'_{\mathcal{D}}M_{\mu\nu} = \kappa T_{\mu\nu}\,,
\end{aligned}
\tag{62}
$$

where $M_{\mu\nu}$ is given by (9).

The connection field equations of the theory read

$$
\begin{aligned}
& P_{\lambda}{}^{\mu\nu}(F'_R) + 2F'_T\left(S^{\mu\nu}{}_{\lambda} - 2S_{\lambda}{}^{[\mu\nu]} - 4S^{[\mu}\delta^{\nu]}_{\lambda}\right) \\
& + F'_Q\left[2Q^{[\nu\mu]}{}_{\lambda} - Q_{\lambda}{}^{\mu\nu} + (q^{\nu} - Q^{\nu})\delta^{\mu}_{\lambda} + Q_{\lambda}g^{\mu\nu} + \frac{1}{2}Q^{\mu}\delta^{\nu}_{\lambda}\right] - M_{\lambda}{}^{\mu\nu\rho}\partial_{\rho}F'_{\mathcal{D}} = \kappa\Delta_{\lambda}{}^{\mu\nu}\,,
\end{aligned}
\tag{63}
$$

with $M_{\lambda}{}^{\mu\nu\rho}$ defined in (11). Analogously to what we have seen for the previous models, here we mention that a minimal metric-affine generalization of the MG-V theory might also be considered, in which the function F does not exhibits a dependence on the divergence of the dilation current $\mathcal{D}$. In the minimal case the metric field equations boil down to (59) and the connection field equations become

$$
\begin{aligned}
& P_{\lambda}{}^{\mu\nu}(F'_R) + 2F'_T\left(S^{\mu\nu}{}_{\lambda} - 2S_{\lambda}{}^{[\mu\nu]} - 4S^{[\mu}\delta^{\nu]}_{\lambda}\right) \\
& + F'_Q\left[2Q^{[\nu\mu]}{}_{\lambda} - Q_{\lambda}{}^{\mu\nu} + (q^{\nu} - Q^{\nu})\delta^{\mu}_{\lambda} + Q_{\lambda}g^{\mu\nu} + \frac{1}{2}Q^{\mu}\delta^{\nu}_{\lambda}\right] = \kappa\Delta_{\lambda}{}^{\mu\nu}\,,
\end{aligned}
\tag{64}
$$

One can end up with the same equations by considering some specific matter coupling, always in the metric-affine setup, satisfying $\Delta^{\nu} = 0$.

Always concerning the MAMG-V theory, besides the sub-case we have already mentioned above, corresponding to the minimal MAMG-V model (minimal metric-affine generalization of MG-V, which is the metric-affine $F(R,T,Q)$ theory), we also have the following sub-cases: the metric-affine $F(R)$, $F(T)$, $F(Q)$ and $F(R,\mathcal{D})$, $F(T,\mathcal{D})$, $F(Q,\mathcal{D})$ theories, the metric-affine $F(R,T)$ model (minimal MAMG-I), the metric-affine $F(R,Q)$ one (minimal MAMG-II), the metric-affine $F(T,Q)$ theory (minimal MAMG-III), and the metric-affine $F(R,T,\mathcal{D})$, $F(R,Q,\mathcal{D})$, $F(T,Q,\mathcal{D})$ models (corresponding, respectively, to MAMG-I, MAMG-II, and MAMG-III). In each sub-case, if we remove the dependence of the matter Lagrangian on the general affine connection, we are left with the respective sub-case of the MG-V theory.

7. MG-VI and MAMG-VI

Here we start with the action of the MG-VI model, which reads [27]

$$\mathcal{S}^{(\mathrm{VI})}[g,\Gamma,\varphi]=\frac{1}{2\kappa}\int\sqrt{-g}d^4x[F(R,Q,\mathcal{T})+2\kappa\mathcal{L}_{\mathrm{m}}]\,, \tag{65}$$

where $F=F(R,Q,\mathcal{T})$ is a generic function of the scalar curvature R of the general affine connection Γ, Q is the nonmetricity scalar, and $\mathcal{T}$ is the energy-momentum trace. The action (65) represents and extension of the $F(R,\mathcal{T})$, $F(Q)$, $F(R)$, and $F(Q,\mathcal{T})$ (cf. [35]) theories. In particular, (65) is an extension of the action (21) of the MG-II model, as it includes a dependence on $\mathcal{T}$ in F.

The metric field equations of the theory are given by

$$-\frac{1}{2}g_{\mu\nu}F+F'_R R_{(\mu\nu)}+F'_Q L_{(\mu\nu)}+\hat{\nabla}_\lambda\left(F'_Q J^\lambda{}_{(\mu\nu)}\right)+g_{\mu\nu}\hat{\nabla}_\lambda\left(F'_Q\zeta^\lambda\right)+F'_{\mathcal{T}}(\Theta_{\mu\nu}+T_{\mu\nu})=\kappa T_{\mu\nu}\,, \tag{66}$$

where $\hat{\nabla}$ is given by (23), $L_{\mu\nu}$, $J^\lambda{}_{\mu\nu}$, ζ^λ are defined in (24), while $\Theta_{\mu\nu}$ is defined in (43).

The connection field equations read

$$P_\lambda{}^{\mu\nu}(F'_R)+F'_Q\left[2Q^{[\nu\mu]}{}_\lambda-Q_\lambda{}^{\mu\nu}+(q^\nu-Q^\nu)\delta^\mu_\lambda+Q_\lambda g^{\mu\nu}+\frac{1}{2}Q^\mu\delta^\nu_\lambda\right]=0\,, \tag{67}$$

where $P_\lambda{}^{\mu\nu}(F'_R)$ is defined in (4). Please note that (67) coincides with the connection field equations (25) of the MG-II model, which is a consequence of the fact that in the MG theories matter does not couple to the connection Γ.

7.1. Metric-Affine Generalizations of the MG-VI Model

Now, we construct the metric-affine generalization of the MG-VI theory, i.e., we allow matter to couple to the general affine connection $\Gamma^\lambda{}_{\mu\nu}$. On the same lines of what we have done in the previous sections, we write the MAMG-VI action as

$$\mathcal{S}^{(\mathrm{VI})}_{\mathrm{MAMG}}=\frac{1}{2\kappa}\int\sqrt{-g}d^4x[F(R,Q,\mathcal{T},\mathcal{D})+2\kappa\mathcal{L}_{\mathrm{m}}]\,, \tag{68}$$

where as usual, $\mathcal{D}$ is the divergence of the dilation current defined in (5). In (68) the energy-momentum trace $\mathcal{T}$ and $\mathcal{D}$ are placed on an equal footing.

Varying the action (68) with respect to the metric field we obtain

$$\begin{aligned}&-\frac{1}{2}g_{\mu\nu}F+F'_R R_{(\mu\nu)}+F'_Q L_{(\mu\nu)}+\hat{\nabla}_\lambda\left(F'_Q J^\lambda{}_{(\mu\nu)}\right)+g_{\mu\nu}\hat{\nabla}_\lambda\left(F'_Q\zeta^\lambda\right)+F'_{\mathcal{T}}(\Theta_{\mu\nu}+T_{\mu\nu})+F'_{\mathcal{D}}M_{\mu\nu}\\&=\kappa T_{\mu\nu}\,,\end{aligned} \tag{69}$$

where we recall that $M_{\mu\nu}$ is given by (9).

On the other hand, the variation of (68) with respect to $\Gamma^\lambda{}_{\mu\nu}$ yields

$$\begin{aligned}&P_\lambda{}^{\mu\nu}(F'_R)+F'_Q\left[2Q^{[\nu\mu]}{}_\lambda-Q_\lambda{}^{\mu\nu}+(q^\nu-Q^\nu)\delta^\mu_\lambda+Q_\lambda g^{\mu\nu}+\frac{1}{2}Q^\mu\delta^\nu_\lambda\right]-M_\lambda{}^{\mu\nu\rho}\partial_\rho F'_{\mathcal{D}}\\&=F'_{\mathcal{T}}\Theta_\lambda{}^{\mu\nu}+\kappa\Delta_\lambda{}^{\mu\nu}\,,\end{aligned} \tag{70}$$

where $M_\lambda{}^{\mu\nu\rho}$ and $\Theta_\lambda{}^{\mu\nu}$ are defined in (11) and (48), respectively. One can also consider a minimal metric-affine generalization of the MG-VI theory, excluding the $\mathcal{D}$ dependence in the function F. In this case, the metric field equations would coincide with (66), while the connection field equations would be

$$P_\lambda{}^{\mu\nu}(F'_R)+F'_Q\left[2Q^{[\nu\mu]}{}_\lambda-Q_\lambda{}^{\mu\nu}+(q^\nu-Q^\nu)\delta^\mu_\lambda+Q_\lambda g^{\mu\nu}+\frac{1}{2}Q^\mu\delta^\nu_\lambda\right]=F'_{\mathcal{T}}\Theta_\lambda{}^{\mu\nu}+\kappa\Delta_\lambda{}^{\mu\nu}\,. \tag{71}$$

The same result would follow by considering some specific matter fulfilling $\Delta^{\nu}=0$. Let us also notice that if $\Theta_{\lambda}{}^{\mu\nu}=0$ the connection field equations (70) reduce to those of the MAMG-II theory, namely (28). Analogously, if $\Theta_{\lambda}{}^{\mu\nu}=0$ then (71) boil down to (29), i.e., to the connection field equations of the minimal MAMG-II model.

7.2. Particular Sub-Cases of the MAMG-VI Theory

Regarding the MAMG-VI theory, besides the sub-case we have already mentioned above, corresponding to the minimal MAMG-VI model (minimal metric-affine generalization of MG-VI, which is the metric-affine $F(R,Q,\mathcal{T})$ theory), we also have the following sub-cases: the metric-affine $F(R)$, $F(Q)$ and $F(R,\mathcal{D})$, $F(Q,\mathcal{D})$ theories, the metric-affine $F(R,Q)$ model (minimal MAMG-II), the metric-affine $F(R,\mathcal{T})$ and $F(R,\mathcal{T},\mathcal{D})$ theories (cf. (50) and (54), respectively), and the metric-affine $F(R,Q,\mathcal{D})$ model (i.e., MAMG-II). For the sake of completeness, let us finally report in the following other two sub-cases, namely the metric-affine $F(Q,\mathcal{T})$ and $F(Q,\mathcal{T},\mathcal{D})$ theories. In each of these sub-cases, removing the dependence of the matter Lagrangian on the general affine connection, one is left with the respective sub-cases of the MG-VI theory.

7.2.1. Metric-Affine $F(Q,\mathcal{T})$ Theory

Restricting ourselves to $F(R,Q,\mathcal{T},\mathcal{D})\to F(Q,\mathcal{T})$, the MAMG-VII model boils down to the metric-affine $F(Q,\mathcal{T})$ theory, whose action reads

$$\mathcal{S}_{F(Q,\mathcal{T})}=\frac{1}{2\kappa}\int\sqrt{-g}d^4x[F(Q,\mathcal{T})+2\kappa\mathcal{L}_{\rm m}]\,, \tag{72}$$

where Q is the nonmetricity scalar and $\mathcal{T}$ the energy-momentum trace. The variation of the action (72) with respect to the metric and the affine connection gives the following set of field equations:

$$\begin{aligned}
&-\frac{1}{2}g_{\mu\nu}F+F'_QL_{(\mu\nu)}+\hat{\nabla}_{\lambda}\left(F'_QJ^{\lambda}{}_{(\mu\nu)}\right)+g_{\mu\nu}\hat{\nabla}_{\lambda}\left(F'_Q\zeta^{\lambda}\right)+F'_{\mathcal{T}}(\Theta_{\mu\nu}+T_{\mu\nu})=\kappa T_{\mu\nu}\,,\\
&F'_Q\left[2Q^{[\nu\mu]}{}_{\lambda}-Q_{\lambda}{}^{\mu\nu}+(q^{\nu}-Q^{\nu})\delta^{\mu}_{\lambda}+Q_{\lambda}g^{\mu\nu}+\frac{1}{2}Q^{\mu}\delta^{\nu}_{\lambda}\right]=F'_{\mathcal{T}}\Theta_{\lambda}{}^{\mu\nu}+\kappa\Delta_{\lambda}{}^{\mu\nu}\,,
\end{aligned} \tag{73}$$

where $L_{\mu\nu}$, $J^{\lambda}{}_{\mu\nu}$, and ζ^{λ} have been defined in (24), while $\Theta_{\mu nu}$ and $\Theta_{\lambda}{}^{\mu\nu}$ are given by Equations (43) and (48), respectively.

7.2.2. Metric-Affine $F(Q,\mathcal{T},\mathcal{D})$ Theory

On the other hand, considering $F(R,Q,\mathcal{T},\mathcal{D})\to F(Q,\mathcal{T},\mathcal{D})$, namely excluding the dependence on the scalar curvature R in the MAMG-VI model, the latter reduces to the metric-affine $F(Q,\mathcal{T},\mathcal{D})$ theory, which, in turn, is an extension of the $F(Q,\mathcal{T})$ theory above involving a dependence on the divergence of the dilation current in F as well. The action of the metric-affine $F(Q,\mathcal{T},\mathcal{D})$ model has the following form:

$$\mathcal{S}_{F(Q,\mathcal{T},\mathcal{D})}=\frac{1}{2\kappa}\int\sqrt{-g}d^4x[F(Q,\mathcal{T},\mathcal{D})+2\kappa\mathcal{L}_{\rm m}]\,, \tag{74}$$

where Q is the nonmetricity scalar, $\mathcal{T}$ the energy-momentum trace, and $\mathcal{D}$ the divergence of the dilation current. Varying the action (74) with respect to the metric and the general affine connection, we obtain the set of field equations

$$\begin{aligned}
&-\frac{1}{2}g_{\mu\nu}F+F'_QL_{(\mu\nu)}+\hat{\nabla}_{\lambda}\left(F'_QJ^{\lambda}{}_{(\mu\nu)}\right)+g_{\mu\nu}\hat{\nabla}_{\lambda}\left(F'_Q\zeta^{\lambda}\right)+F'_{\mathcal{T}}(\Theta_{\mu\nu}+T_{\mu\nu})+F'_{\mathcal{D}}M_{\mu\nu}=\kappa T_{\mu\nu}\,,\\
&F'_Q\left[2Q^{[\nu\mu]}{}_{\lambda}-Q_{\lambda}{}^{\mu\nu}+(q^{\nu}-Q^{\nu})\delta^{\mu}_{\lambda}+Q_{\lambda}g^{\mu\nu}+\frac{1}{2}Q^{\mu}\delta^{\nu}_{\lambda}\right]-M_{\lambda}{}^{\mu\nu\rho}\partial_{\rho}F'_{\mathcal{D}}=F'_{\mathcal{T}}\Theta_{\lambda}{}^{\mu\nu}+\kappa\Delta_{\lambda}{}^{\mu\nu}\,,
\end{aligned} \tag{75}$$

with $M_{\mu\nu}$ and $M_{\lambda}{}^{\mu\nu\rho}$ given by (9) and (11), respectively.

8. MG-VII and MAMG-VII

In this section, we first consider the MG-VII theory and subsequently generalize it by considering a metric-affine setup. The MG-VII action [27] reads as follows:

$$\mathcal{S}^{(\mathrm{VII})}[g,\Gamma,\varphi]=\frac{1}{2\kappa}\int\sqrt{-g}d^4x[F(T,Q,\mathcal{T})+2\kappa\mathcal{L}_{\mathrm{m}}]\,, \tag{76}$$

where $F=F(T,Q,\mathcal{T})$ is a generic function of the torsion and nonmetricity scalars (T and Q, respectively) and of the trace, $\mathcal{T}$, of the energy-momentum tensor. In fact, (76) is an extension of the action (34) of the MG-III model as it also includes a dependence on $\mathcal{T}$ in F.

By varying the action (76) with respect to the metric we obtain

$$\begin{aligned}&-\frac{1}{2}g_{\mu\nu}F+F'_T\left(2S_{\nu\alpha\beta}S_{\mu}{}^{\alpha\beta}-S_{\alpha\beta\mu}S^{\alpha\beta}{}_{\nu}+2S_{\nu\alpha\beta}S_{\mu}{}^{\beta\alpha}-4S_{\mu}S_{\nu}\right)+F'_QL_{(\mu\nu)}\\&+\hat{\nabla}_{\lambda}\left(F'_QJ^{\lambda}{}_{(\mu\nu)}\right)+g_{\mu\nu}\hat{\nabla}_{\lambda}\left(F'_Q\zeta^{\lambda}\right)+F'_{\mathcal{T}}(\Theta_{\mu\nu}+T_{\mu\nu})=\kappa T_{\mu\nu}\,,\end{aligned} \tag{77}$$

where $\hat{\nabla}$ is given in (23), $L_{\mu\nu}$, $J^{\lambda}{}_{\mu\nu}$ and ζ^{λ} are defined in (24), while $\Theta_{\mu\nu}$ is given by (43).

The variation of (76) with respect to the general affine connection gives

$$2F'_T\left(S^{\mu\nu}{}_{\lambda}-2S_{\lambda}{}^{[\mu\nu]}-4S^{[\mu}\delta^{\nu]}_{\lambda}\right)+F'_Q\left[2Q^{[\nu\mu]}{}_{\lambda}-Q_{\lambda}{}^{\mu\nu}+(q^{\nu}-Q^{\nu})\delta^{\mu}_{\lambda}+Q_{\lambda}g^{\mu\nu}+\frac{1}{2}Q^{\mu}\delta^{\nu}_{\lambda}\right]=0\,. \tag{78}$$

Let us observe that (78) coincides with the connection field equations (36) of the MG-III theory as, in the MG models, matter does not couple to the connection. We can now move on to the metric-affine generalization of the MG-VII model.

Metric-Affine Generalizations of the MG-VII Model

On the same lines of what we have done in the previous sections, let us write the MAMG-VII action as

$$\mathcal{S}^{(\mathrm{VII})}_{\mathrm{MAMG}}=\frac{1}{2\kappa}\int\sqrt{-g}d^4x[F(T,Q,\mathcal{T},\mathcal{D})+2\kappa\mathcal{L}_{\mathrm{m}}]\,, \tag{79}$$

where $\mathcal{D}$ is given by (5). Again, $\mathcal{T}$ and $\mathcal{D}$ are placed on the same footing.

The metric field equations of the theory are

$$\begin{aligned}&-\frac{1}{2}g_{\mu\nu}F+F'_T\left(2S_{\nu\alpha\beta}S_{\mu}{}^{\alpha\beta}-S_{\alpha\beta\mu}S^{\alpha\beta}{}_{\nu}+2S_{\nu\alpha\beta}S_{\mu}{}^{\beta\alpha}-4S_{\mu}S_{\nu}\right)+F'_QL_{(\mu\nu)}\\&+\hat{\nabla}_{\lambda}\left(F'_QJ^{\lambda}{}_{(\mu\nu)}\right)+g_{\mu\nu}\hat{\nabla}_{\lambda}\left(F'_Q\zeta^{\lambda}\right)+F'_{\mathcal{T}}(\Theta_{\mu\nu}+T_{\mu\nu})+F'_{\mathcal{D}}M_{\mu\nu}=\kappa T_{\mu\nu}\,,\end{aligned} \tag{80}$$

where we recall that $M_{\mu\nu}$ is defined in (9).

The connection field equations read as follows:

$$\begin{aligned}&2F'_T\left(S^{\mu\nu}{}_{\lambda}-2S_{\lambda}{}^{[\mu\nu]}-4S^{[\mu}\delta^{\nu]}_{\lambda}\right)+F'_Q\left[2Q^{[\nu\mu]}{}_{\lambda}-Q_{\lambda}{}^{\mu\nu}+(q^{\nu}-Q^{\nu})\delta^{\mu}_{\lambda}+Q_{\lambda}g^{\mu\nu}+\frac{1}{2}Q^{\mu}\delta^{\nu}_{\lambda}\right]\\&-M_{\lambda}{}^{\mu\nu\rho}\partial_{\rho}F'_{\mathcal{D}}=F'_{\mathcal{T}}\Theta_{\lambda}{}^{\mu\nu}+\kappa\Delta_{\lambda}{}^{\mu\nu}\,,\end{aligned} \tag{81}$$

where $M_{\lambda}{}^{\mu\nu\rho}$ and $\Theta_{\lambda}{}^{\mu\nu}$ are respectively defined in Equations (11) and (48). We can also consider a minimal metric-affine generalization of the MG-VII theory by excluding the $\mathcal{D}$ dependence in the function F. In this minimal case, the metric field equations coincide with (77), while the connection field equations are

$$\begin{aligned}&2F'_T\left(S^{\mu\nu}{}_{\lambda}-2S_{\lambda}{}^{[\mu\nu]}-4S^{[\mu}\delta^{\nu]}_{\lambda}\right)+F'_Q\left[2Q^{[\nu\mu]}{}_{\lambda}-Q_{\lambda}{}^{\mu\nu}+(q^{\nu}-Q^{\nu})\delta^{\mu}_{\lambda}+Q_{\lambda}g^{\mu\nu}+\frac{1}{2}Q^{\mu}\delta^{\nu}_{\lambda}\right]\\&=F'_{\mathcal{T}}\Theta_{\lambda}{}^{\mu\nu}+\kappa\Delta_{\lambda}{}^{\mu\nu}\,.\end{aligned} \tag{82}$$

Analogously to what we have seen in the previous sections, the same result would follow by considering some specific matter fulfilling $\Delta^{\nu} = 0$. Observe that if $\Theta_{\lambda}{}^{\mu\nu} = 0$ then the connection field equations (81) reduce to (39), which are those of the MAMG-III theory. Similarly, if $\Theta_{\lambda}{}^{\mu\nu} = 0$ then (82) boil down to (82), namely to the connection field equations of the minimal MAMG-III theory.

We conclude this section by mentioning that regarding MAMG-VII, besides the sub-case we have already mentioned above, corresponding to the minimal MAMG-VII model (minimal metric-affine generalization of MG-VII, which is the metric-affine $F(T,Q,\mathcal{T})$ theory), we also have the following sub-cases: the metric-affine $F(T)$, $F(Q)$ and $F(T,\mathcal{D})$, $F(Q,\mathcal{D})$ theories, the metric-affine $F(T,Q)$ model (minimal MAMG-III), the metric-affine $F(T,\mathcal{T})$ and $F(T,\mathcal{T},\mathcal{D})$ theories (cf. (52) and (56), respectively), the metric-affine $F(Q,\mathcal{T})$ and $F(Q,\mathcal{T},\mathcal{D})$ models (see (72) and (74), respectively), and the metric-affine $F(T,Q,\mathcal{D})$ theory (i.e., MAMG-III). In each sub-case, by removing the dependence of the matter Lagrangian on the general affine connection one is left with the respective sub-case of the MG-VII theory.

9. MG-VIII and MAMG-VIII

In this section, we review the metric-affine generalization of the MG-VIII model, introduced in [28]. Before moving on to the MAG generalization, let us introduce the MG-VIII model, whose action is given by [27,36]

$$\mathcal{S}^{(\mathrm{VIII})}[g,\Gamma,\varphi] = \frac{1}{2\kappa}\int \sqrt{-g}d^4x[F(R,T,Q,\mathcal{T}) + 2\kappa\mathcal{L}_{\mathrm{m}}]\,, \tag{83}$$

extending the $F(R)$, $F(T)$, $F(Q)$, $F(R,\mathcal{T})$ (cf. [33]), $F(T,\mathcal{T})$ (cf. [34]), and $F(Q,\mathcal{T})$ (cf. [35]) theories. (In particular, assuming flatness (namely $R^{\lambda}{}_{\mu\nu\rho} = 0$) and vanishing nonmetricity, (83) boils down to the torsionful theories of [10,37], while demanding flatness and vanishing torsion one is left with the models of [11,12]. On the other hand, imposing only teleparallelism in (83) we are left with the generalized theories of [38,39].) The MG-VIII model is the most general of the MG theories, the function $F = F(R,T,Q,\mathcal{T})$ in (83) being a generic function of the scalar curvature R of the general affine connection $\Gamma^{\lambda}{}_{\mu\nu}$, the torsion scalar T, the nonmetricity scalar Q, and the energy-momentum trace $\mathcal{T}$.

The metric field equations of the theory read

$$\begin{aligned}
&-\frac{1}{2}g_{\mu\nu}F + F'_R R_{(\mu\nu)} + F'_T\left(2S_{\nu\alpha\beta}S_{\mu}{}^{\alpha\beta} - S_{\alpha\beta\mu}S^{\alpha\beta}{}_{\nu} + 2S_{\nu\alpha\beta}S_{\mu}{}^{\beta\alpha} - 4S_{\mu}S_{\nu}\right) + F'_Q L_{(\mu\nu)} \\
&+ \hat{\nabla}_{\lambda}\left(F'_Q J^{\lambda}{}_{(\mu\nu)}\right) + g_{\mu\nu}\hat{\nabla}_{\lambda}\left(F'_Q\zeta^{\lambda}\right) + F'_{\mathcal{T}}\left(\Theta_{\mu\nu} + T_{\mu\nu}\right) = \kappa T_{\mu\nu}\,,
\end{aligned} \tag{84}$$

where $\hat{\nabla}$ is defined in (23), while $L_{\mu\nu}$ and the densities $J^{\lambda}{}_{\mu\nu}$ and ζ^{λ} have been defined in (24), and $\Theta_{\mu\nu}$ is given by (43).

On the other hand, the connection field equations are

$$\begin{aligned}
&P_{\lambda}{}^{\mu\nu}(F'_R) + 2F'_T\left(S^{\mu\nu}{}_{\lambda} - 2S_{\lambda}{}^{[\mu\nu]} - 4S^{[\mu}\delta^{\nu]}_{\lambda}\right) \\
&+ F'_Q\left[2Q^{[\nu\mu]}{}_{\lambda} - Q_{\lambda}{}^{\mu\nu} + (q^{\nu} - Q^{\nu})\delta^{\mu}_{\lambda} + Q_{\lambda}g^{\mu\nu} + \frac{1}{2}Q^{\mu}\delta^{\nu}_{\lambda}\right] = 0\,,
\end{aligned} \tag{85}$$

where $P_{\lambda}{}^{\mu\nu}(F'_R)$ is the modified Palatini tensor defined in (4). Equation (85) coincides with (60) obtained for the MG-V model, due to the fact that in the MG theories matter does not couple to the connection.

Metric-Affine Generalizations of the MG-VIII Model

The MAMG-VIII action, generalizing to the metric-affine case (83), is [28]

$$\mathcal{S}^{(\mathrm{VIII})}_{\mathrm{MAMG}} = \frac{1}{2\kappa}\int \sqrt{-g}d^4x[F(R,T,Q,\mathcal{T},\mathcal{D}) + 2\kappa\mathcal{L}_{\mathrm{m}}]\,, \tag{86}$$

where the divergence of the dilation current $\mathcal{D}$ is given by (5) and in (86) it is placed on the same footing of $\mathcal{T}$.

The variation of the action (86) with respect to the metric yields

$$
\begin{aligned}
&-\frac{1}{2}g_{\mu\nu}F + F'_R R_{(\mu\nu)} + F'_T\Big(2S_{\nu\alpha\beta}S_\mu{}^{\alpha\beta} - S_{\alpha\beta\mu}S^{\alpha\beta}{}_\nu + 2S_{\nu\alpha\beta}S_\mu{}^{\beta\alpha} - 4S_\mu S_\nu\Big) + F'_Q L_{(\mu\nu)} \\
&+\hat{\nabla}_\lambda\Big(F'_Q J^\lambda{}_{(\mu\nu)}\Big) + g_{\mu\nu}\hat{\nabla}_\lambda\Big(F'_Q\zeta^\lambda\Big) + F'_{\mathcal{T}}\left(\Theta_{\mu\nu} + T_{\mu\nu}\right) + F'_{\mathcal{D}}M_{\mu\nu} = \kappa T_{\mu\nu}\,,
\end{aligned}
\tag{87}
$$

where in particular, $M_{\mu\nu}$ is given by (9).

On the other hand, from the variation of the action (86) with respect to the general affine connection $\Gamma^\lambda{}_{\mu\nu}$ we find

$$
\begin{aligned}
&P_\lambda{}^{\mu\nu}(F'_R) + 2F'_T\Big(S^{\mu\nu}{}_\lambda - 2S_\lambda{}^{[\mu\nu]} - 4S^{[\mu}\delta^{\nu]}_\lambda\Big) \\
&+F'_Q\left[2Q^{[\nu\mu]}{}_\lambda - Q_\lambda{}^{\mu\nu} + (q^\nu - Q^\nu)\delta^\mu_\lambda + Q_\lambda g^{\mu\nu} + \frac{1}{2}Q^\mu\delta^\nu_\lambda\right] - M_\lambda{}^{\mu\nu\rho}\partial_\rho F'_{\mathcal{D}} = F'_{\mathcal{T}}\Theta_\lambda{}^{\mu\nu} + \kappa\Delta_\lambda{}^{\mu\nu}\,,
\end{aligned}
\tag{88}
$$

where $M_\lambda{}^{\mu\nu\rho}$ and $\Theta_\lambda{}^{\mu\nu}$ have been defined in (11) and (48), respectively. On the same lines of what we have seen for all the theories previously discussed, also here one might consider a minimal metric-affine generalization of the MG-VIII model by excluding the $\mathcal{D}$ dependence in the function F. The metric field equations in this minimal case coincide with (84), while the connection field equations read

$$
\begin{aligned}
&P_\lambda{}^{\mu\nu}(F'_R) + 2F'_T\Big(S^{\mu\nu}{}_\lambda - 2S_\lambda{}^{[\mu\nu]} - 4S^{[\mu}\delta^{\nu]}_\lambda\Big) \\
&+F'_Q\left[2Q^{[\nu\mu]}{}_\lambda - Q_\lambda{}^{\mu\nu} + (q^\nu - Q^\nu)\delta^\mu_\lambda + Q_\lambda g^{\mu\nu} + \frac{1}{2}Q^\mu\delta^\nu_\lambda\right] = F'_{\mathcal{T}}\Theta_\lambda{}^{\mu\nu} + \kappa\Delta_\lambda{}^{\mu\nu}\,.
\end{aligned}
\tag{89}
$$

The same result would follow by considering some specific matter fulfilling $\Delta^\nu = 0$. Finally, observe that if $\Theta_\lambda{}^{\mu\nu} = 0$ the connection field equations (88) reduce to (63), namely to those of the MAMG-V theory. Analogously, if $\Theta_\lambda{}^{\mu\nu} = 0$ Equation (89) boil down to (64), i.e., to the connection field equations of the minimal MAMG-V model.

Let us conclude this section by mentioning that concerning the MAMG-VIII theory, besides the sub-case we have already mentioned above, corresponding to minimal MAMG-VIII (minimal metric-affine generalization of MG-VIII, which is the metric-affine $F(R,T,Q,\mathcal{T})$ theory), we also have the following sub-cases: the metric-affine $F(R)$, $F(T)$, $F(Q)$, $F(R,\mathcal{D})$, $F(T,\mathcal{D})$, $F(Q,\mathcal{D})$, $F(R,\mathcal{T})$, $F(T,\mathcal{T})$, $F(Q,\mathcal{T})$, $F(R,\mathcal{T},\mathcal{D})$, $F(T,\mathcal{T},\mathcal{D})$, $F(Q,\mathcal{T},\mathcal{D})$ theories, the metric-affine $F(R,T)$, $F(R,Q)$, $F(T,Q)$ models (namely the minimal MAMG-I, minimal MAMG-II, minimal MAMG-III theories, respectively), the metric-affine $F(R,T,\mathcal{D})$, $F(R,Q,\mathcal{D})$, $F(T,Q,\mathcal{D})$ models (MAMG-I, MAMG-II, MAMG-III, respectively), the metric-affine $F(R,T,\mathcal{T})$, $F(R,T,Q)$, $F(R,Q,\mathcal{T})$, $F(T,Q,\mathcal{T})$ theories (namely minimal MAMG-IV, minimal MAMG-V, minimal MAMG-VI, minimal MAMG-VII, respectively), and the metric-affine $F(R,T,\mathcal{T},\mathcal{D})$, $F(R,T,Q,\mathcal{D})$, $F(R,Q,\mathcal{T},\mathcal{D})$, and $F(T,Q,\mathcal{T},\mathcal{D})$ models (i.e., respectively, MAMG-IV, MAMG-V, MAMG-VI, and MAMG-VII).

10. Cosmological Aspects of MAMG Theories

Before proceeding, let us highlight some cosmological features of various modified gravity models previously reviewed, mentioning in particular the way in which they offer solutions to diverse issues in the cosmological context.

First of all, it is well-known that any extra source term such as Einstein's cosmological constant could be added into the energy-momentum tensor and serve as a candidate for dark energy. The most favored candidate to play such role is the cosmological constant Λ, and the Lambda Cold Dark Matter (ΛCDM) model has an optimal fit with many observational data. However, there exist many dark energy models able to explain the accelerated expansion of the universe such as, e.g., quintessence, K-essence, phantom, etc.; moreover, another way to explain late-time acceleration consists of considering modified gravity. Indeed, modified gravity theories, such as teleparallel gravity, symmetric teleparallel gravity, $F(R)$ gravity and further generalizations previously discussed, turned out to be rather prominent setups to study modern cosmology especially to explore the elusive nature of dark energy and the reason behind late-time cosmic acceleration.

As we have already mentioned, cosmological implications of $F(R,T)$ gravity were discussed in [29,30]. In particular, in [29], considering a FLRW background, it was found that the model can describe the accelerated expansion of the universe. In [30], even for the case in which the action is linear in R and T, it was proved that the Friedmann equations contain new terms of geometrical origin and, applying the theory at late-times, it was found that in the model dark energy can be quintessence-like or phantom-like, or behave as a cosmological constant, reproducing ΛCDM cosmology. On the other hand, considering early-time the de Sitter solution was recovered, along with an inflationary scenario with the desired scale factor evolution. Observational constraints on $F(R,T)$ gravity have been recently introduced in [31], using data from Supernovae (SNIa) Pantheon sample, Baryonic Acoustic Oscillations (BAO), and Cosmic Chronometers measurements of the Hubble parameter (CC), alongside arguments from Big Bang Nucleosynthesis (BBN), using a specific connection and considering two specific cosmological models in the $F(R,T)$ setup (the aforementioned models differ in two scalars that quantify the effect of the specific imposed connection). Both models lead to $\sim 1\sigma$ compatibility in all cases. Moreover, reconstructing the Hubble function and the dark energy equation-of-state parameter as a function of redshift, it was shown that the first model is very close to the ΛCDM scenario, while for the second model at earlier times deviations are allowed. Even though the second model does not contain ΛCDM cosmology as a limit, both models present a very efficient fitting behavior and are at least statistically equivalent to ΛCDM cosmology.

For a review of cosmological aspects of $F(T)$ teleparallel gravity, instead, we refer the reader to, e.g., [15]. Moreover, in [40] diverse interacting dark energy cosmological models in $F(T)$ gravity were discussed, finding a good agreement with observational data and obtaining, from the study of the behavior of the deceleration parameter, a phase transition from a decelerated expanding universe to the present accelerated expanding one. Furthermore, a complete cosmological scenario of FLRW universe in the context of teleparallel gravity has been recently proposed in [41], where cosmology is much simplified since teleparallel gravity is a second order theory. More precisely, the authors have considered a $F(T)$ theory with $F(T) = T + \beta(-T)^{\alpha}$, where α and β are parameters, deriving the profiles of energy density, pressure, equation-of-state parameter, analyzing deviation from the ΛCDM model (in particular, at late-time the findings are consistent with standard cosmology), discussing the nature of dark energy, and studying energy conditions. The strong energy condition is violated in support of the acceleration of the universe, in accordance with current observations. $F(T)$ gravity was also considered in the framework of so-called fractal spacetime in [42], assuming that only time has a fractal profile, finding consistency with cosmological observations.

As for $F(T)$ gravity, also the $F(Q)$ theory features in second order field equations. The expansion history of the universe in $F(Q)$ gravity has received lot of attention, and the theory is successful in explaining the accelerated expansion of the universe at least to the same level of statistic precision of most renowned modified gravities (cf., e.g., [43–51]). Moreover, observational constraints on the theory have been put forward in [52,53], and first demonstrates that $F(Q)$ gravity can challenge the ΛCDM model, placing itself as a good alternative candidate to describe the cosmological aspects of the universe, have been disclosed in [54,55]. Moreover, spherically symmetric configurations in $F(Q)$ gravity have been recently studied in [56].

In recent years, many investigations have been carried out to describe the present cosmic acceleration also in the setup of $F(R,\mathcal{T})$ gravity [57–63]. Moreover, several homogeneous isotropic and anisotropic cosmological models have been constructed in this context [64–71]. In particular, in [68] a $\Lambda(t)$ cosmological model capable of explaining the recent astronomical observations of accelerating expansion of the universe with a decelerating phase of evolution in the past was obtained by a simple parametrization of the Hubble parameter in a flat FLRW spacetime in $F(R,\mathcal{T})$ gravity. The field equations were derived by taking the functional form of $F(R,\mathcal{T}) = F(R) + F(\mathcal{T})$ into consideration, which leads to general relativistic field equations with a trace $\mathcal{T}$ dependent term that is the cosmological constant $\Lambda(\mathcal{T})$ (see also [72]). Moreover, in [73] it was reconstructed a cosmological model in $F(R,\mathcal{T})$ gravity able to discuss the expansion history of the model in GR by dark matter as well as by holographic dark energy. The compatibility of $F(R,\mathcal{T})$ models with the accelerated expansion of the universe has also been proved and discussed in [74]. Furthermore, in [75] it was considered non-minimally coupled $F(R,\mathcal{T})$ gravity admitting minimal coupling with scalar field models in a generalized spacetime which corresponds to different anisotropic and homogeneous universe models. For dust and perfect fluids exact solutions have been derived and, in particular, it has been shown that for a perfect fluid with dominating potential energy over kinetic energy the current cosmic expansion is recovered for both phantom as well as quintessence models. Moreover, following the idea that a varying cosmological constant may solve some of the standard ΛCDM model problems such as the fine-tuning and cosmic coincidence, a stable flat universe with variable cosmological constant in $F(R,\mathcal{T})$ gravity was studied in [76]. Subsequently, in [77] a specific $F(R,T)$ gravity reconstruction of the evolution for cyclic models in which the Hubble parameter oscillates and keeps positive was considered, obtaining a singularity-free cyclic universe with negative varying cosmological constant, which supports the role suggested for a negative Λ in stopping the eternal acceleration. The cosmological solutions were obtained for the case of a flat universe, and it was found that the cosmic pressure grows without singular values, it is positive during the early-time decelerated expansion and negative during the late-time accelerating epoch. For every single cycle, the universe accelerates after an epoch of deceleration which agrees with observations.

Another fundamental question in modern cosmology is the so-called (initial) singularity problem. In this context, one of the attractive possible alternatives to the well-celebrated inflationary model to solve such problem resides in bouncing models of the universe, according to which the latter may have emerged from a prior contracting phase capable of expanding without singularity or experiencing a bouncing process. Bouncing models in the $F(R,\mathcal{T})$ setup have been proposed and analyzed in, e.g., [78,79]. In particular, the model of [79] is a non-singular bouncing cosmological model in $F(R,\mathcal{T})$ gravity within a flat FLRW background metric with a specific parametrization of the Hubble parameter, able to describe an expanding universe from the prior period of contraction with specific constraints on the parameters. Quintessence-like and phantom-like scalar fields have also been discussed in the same model for the specific parametrization of the Hubble parameter.

Cosmological issues regarding the future fate of the universe, i.e., whether the universe expands forever or ends with a Big Rip, in the context of $F(R,\mathcal{T})$ gravity have been explored in [80], where it was observed that considering a quadratic variation of the deceleration parameter as a function of cosmic time which describes a smooth transition from the decelerating phase of the universe to an accelerating one, the outcome is in favor of Big Rip.

In addition, locally rotationally symmetric Bianchi type-I viscous and non-viscous cosmological models have been explored and compared in GR and $F(R,\mathcal{T})$ gravity in [81], where it was shown that the metric potentials remain the same in both GR and $F(R,\mathcal{T})$ gravity, while in general, in $F(R,\mathcal{T})$ gravity the effect of bulk viscosity diminishes. On the other hand, anisotropic Bianchi type-III perfect fluid cosmological models in $F(R,\mathcal{T})$ gravity were studied in [82]. These could be physically significant to discuss early stages of the evolution of the universe. Moreover, a new holographic dark energy model with bulk viscosity in $F(R,\mathcal{T})$ gravity has been recently proposed in [83] (the model considered there approaches the ΛCDM one in the late-time evolution of the universe). Furthermore, in [84] a bulk viscous cosmological model in $F(R,\mathcal{T})$ gravity was considered. A cosmological solution was obtained for the special case $F(R,\mathcal{T}) = R + 2\lambda\mathcal{T}$, being λ a constant, and the findings turned out to be compatible with recent observational data. The cosmic evolution of non-minimally coupled $F(R,\mathcal{T})$ gravity in the presence of matter fluids consisting of collisional self-interacting dark matter and radiation was studied in [85], comparing the results with those corresponding to non-collisional matter and to the ΛCDM model. In particular, a flat FLRW universe was considered, focusing on the late-time dynamical evolution of the model. The results are well accommodated to recent observational data based on physical parameters. Always in the context of $F(R,\mathcal{T})$ gravity, a locally rotationally symmetric Bianchi type-I magnetized strange quark matter (SQM) cosmological model was studied in [86], where the exact solutions of the field equations were derived with linearly time varying deceleration parameter, which is consistent with observational data (from SNIa, BAO, and CMB) of standard cosmology. The model begins with Big Bang and ends with a Big Rip. The transition of the deceleration parameter from a decelerating phase to an accelerating one with respect to redshift obtained in the model of [86] fits with the observational data of [87].

Regarding, instead, $F(Q,\mathcal{T})$ gravity, cosmological aspects were analyzed in a FLRW background in [88]. In particular, in the quoted paper the non-linear model $F(Q,\mathcal{T}) = -\alpha Q - \beta\mathcal{T}^2$, where $\alpha > 0$ and $\beta > 0$ are constants, were considered, exploring the evolution of the universe by examining the energy conditions and determining the numerical solutions of the Hubble and deceleration parameters, the apparent magnitude, and the luminosity distance (Supernova data have been used to obtain consistent results of apparent magnitude and luminosity distance). The theoretical results on the Hubble parameter, given in terms redshift, have been compared with those of the ΛCDM model. For lower redshift the values are closer to those of the ΛCDM with respect to the case of higher redshift values. Moreover, in [35] diverse $F(Q,\mathcal{T})$ models have been considered. In all of them, the universe experiences an accelerating expansion, ending with a de Sitter type evolution and representing a valid alternative to ΛCDM cosmology. The late-time de Sitter phase is induced by the coupling between nonmetricity and matter. Cosmological implications of Weyl-type $F(Q,\mathcal{T})$ gravity have been investigated in [89] by constraining the model parameters using recent Hubble and Pantheon Supernovae data. In the same paper, statefinder analysis was implemented to study various dark energy models to address the current cosmic acceleration. It has been found that the solution which mimics the power-law fits with the Pantheon data better than with the Hubble data. Moreover, observational constraints in $F(Q,\mathcal{T})$ gravity have been put forward in [90], revealing that the theory can be promising in addressing the current cosmic acceleration, possibly offering another suitable alternative to the dark energy problem. Finally, in [91] non-singular matter bounce has been explored in $F(Q,\mathcal{T})$ gravity models, validating the latter by means of cosmographic tests and stability analysis.

On the other hand, more general cases remain poorly explored, in particular under the cosmological point of view. In this direction, we now move on to the of study cosmological aspects of all the MAMG theories previously introduced, which also represent generalizations of the prospective sub-cases we have just mentioned above. We focus on the case in which the function characterizing MAMG is linear and consider a homogeneous FLRW background in the presence of torsion and nonmetricity (see Appendix B for a collection of useful formulas in this context). In particular, we derive the Friedmann equations for the aforementioned theories in this cosmological framework.

Let us start by recalling the second Friedmann equation (i.e., the acceleration equation, also known as Raychaudhuri equation) for general non-Riemannian cosmological setups, which was obtained in [92] and reads as follows:

$$\frac{\ddot{a}}{a} = -\frac{1}{3}R_{\mu\nu}u^\mu u^\nu + 2\left(\frac{\dot{a}}{a}\right)\Phi + 2\dot{\Phi} + \left(\frac{\dot{a}}{a}\right)\left(A + \frac{C}{2}\right) + \frac{\dot{A}}{2} - \frac{A^2}{2} - \frac{1}{2}AC - 2A\Phi - 2C\Phi\,, \tag{90}$$

where $a = a(t)$ is the scale factor of the universe, u^μ is the normalized four-velocity, and $\Phi = \Phi(t)$ and $A = A(t)$, $C = C(t)$ are functions appearing, respectively, in the expressions of torsion and nonmetricity given by (A30), obtained, in fact, in the highly symmetric spacetime we are considering. We now derive (variants of) the first Friedmann equation for the MAMG theories, focusing on the linear case. Before moving on to the analysis of the various models, let us observe that since $\sqrt{-g}\mathcal{D}$ is a total divergence, at the linear level the dilation current dependence in the function F characterizing the theories does not contribute to the field equations. Thus, we will directly focus on the linear, minimal MAMG models, namely on those excluding the $\mathcal{D}$ dependence in F, as at the linear level the on-shell result would actually coincide with the one of the non-minimal cases.

Cosmology in linear MAMG-I

We consider the linear MAMG-I case in which

$$F = R + \beta T \,, \tag{91}$$

where β is, in principle, a free parameter. (In fact, in this linear case one can write $F = \alpha R + \beta T$ and fix the normalization of the theory choosing $\alpha = 1$.) The metric field equations (8) (actually, (2), as we have safely excluded the $\mathcal{D}$ dependence in F) now take the form

$$-\frac{1}{2}g_{\mu\nu}F + R_{(\mu\nu)} + \beta\left(2S_{\nu\alpha\beta}S_{\mu}{}^{\alpha\beta} - S_{\alpha\beta\mu}S^{\alpha\beta}{}_{\nu} + 2S_{\nu\alpha\beta}S_{\mu}{}^{\beta\alpha} - 4S_{\mu}S_{\nu}\right) = \kappa T_{\mu\nu} \,. \tag{92}$$

Taking the trace of (92), let us mention here that one might also contract (92) with $u^\mu u^\nu$ and exploit the second Friedmann equation (90) in order to express everything in terms of the scale factor and the torsion and nonmetricity parameters. Using the post-Riemannian expansion of the scalar curvature R given in (A37) and the expressions of torsion and nonmetricity in (A30), we obtain the following variant of the modified first Friedmann equation:

$$\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + (1+\beta)\left(4\Phi^2 - P^2\right) + \frac{1}{8}\Big[2A^2 + B(C-A)\Big] + \Phi(2A-B) + \dot{f}_1 + 3Hf_1 = -\frac{\kappa}{6}\mathcal{T} \,, \tag{93}$$

where $\Phi = \Phi(t)$, $P = P(t)$ and $A = A(t)$, $B = B(t)$, $C = C(t)$ are the functions appearing in (A30), and we have also defined

$$f_1 := \frac{1}{2}\left(\frac{B}{2} - A - 4\Phi\right) . \tag{94}$$

In [28] it was provided a cosmological application of these results, considering the case in which the matter Lagrangian is the one for a scalar field ϕ coupled to torsion, restricting the theory to the torsionful case with vanishing nonmetricity. We will review this application and further elaborate on it in Section 10.1.

On the other hand, in this linear case the connection field equations of the model read

$$P_{\lambda}{}^{\mu\nu} + 2\beta\left(S^{\mu\nu}{}_{\lambda} - 2S_{\lambda}{}^{[\mu\nu]} - 4S^{[\mu}\delta^{\nu]}_{\lambda}\right) = \kappa\Delta_{\lambda}{}^{\mu\nu} \,, \tag{95}$$

where $P_{\lambda}{}^{\mu\nu}$ is the Palatini tensor defined in Equation (A7). Let us also mention, here, that in the cosmological setup we are considering, upon use of Equations (A30) (first line) and (A35), Equation (95) takes the following form:

$$\begin{aligned}&\left(\frac{1}{2}A + 4\Phi - \frac{C}{2}\right)u_{\lambda}h^{\mu\nu} + \left(B - \frac{3}{2}A - 4\Phi - \frac{C}{2}\right)u^{\mu}h_{\lambda}{}^{\nu} - \frac{B}{2}u^{\nu}h^{\mu}{}_{\lambda} - \frac{3}{2}Bu_{\lambda}u^{\mu}u^{\nu} - 2\varepsilon_{\lambda}{}^{\mu\nu}{}_{\rho}u^{\rho}P \\ &+ 2\beta\Big[4\left(\delta^{[\nu}_{\lambda}h^{\mu]}{}_{\rho}u^{\rho} + 3\delta^{[\mu}_{\lambda}u^{\nu]} + h_{\lambda}{}^{[\nu}u^{\mu]}\right)\Phi - \varepsilon_{\lambda}{}^{\mu\nu}{}_{\rho}u^{\rho}P\Big] = \kappa\Delta_{\lambda}{}^{\mu\nu} \,,\end{aligned} \tag{96}$$

where, in particular, the explicit expressions of $\Delta_{\lambda}{}^{\mu\nu}$ depend on the specific matter one might then consider (see, e.g., the explicit expression of the hypermomentum tensor in the presence of a cosmological hyperfluid given in Ref. [93]).

Cosmology in linear MAMG-II

We now consider the linear MAMG-II case in which

$$F = R + \gamma Q \,, \tag{97}$$

where γ is a free parameter. Equation (27) now takes the following form:

$$-\frac{1}{2} g_{\mu\nu} F + R_{(\mu\nu)} + \gamma \Big(L_{(\mu\nu)} + \hat{\nabla}_\lambda J^\lambda{}_{(\mu\nu)} + g_{\mu\nu} \hat{\nabla}_\lambda \zeta^\lambda \Big) = \kappa T_{\mu\nu} \,. \tag{98}$$

Taking the trace of (98) and following the procedure described in the previous paragraph, one ends up with

$$\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + 4\Phi^2 - P^2 + \frac{1}{8}\Big[2A^2 + B(C-A)\Big] + \Phi(2A - B) + \dot{f}_2 + 3H f_2 = -\frac{\kappa}{6}\mathcal{T} \,, \tag{99}$$

where

$$f_2 := \frac{1}{2}\left[(1-\gamma)\left(\frac{B}{2} - A\right) - 4\Phi\right] . \tag{100}$$

Equation (99) is a variant of the modified first Friedmann equation for the linear MAMG-II theory.

Moreover, the connection field equations of the model now read

$$P_\lambda{}^{\mu\nu} + \gamma \left[2Q^{[\nu\mu]}{}_\lambda - Q_\lambda{}^{\mu\nu} + (q^\nu - Q^\nu)\delta^\mu_\lambda + Q_\lambda g^{\mu\nu} + \frac{1}{2} Q^\mu \delta^\nu_\lambda \right] = \kappa \Delta_\lambda{}^{\mu\nu} \,. \tag{101}$$

The latter, upon use of Equations (A30) (second line) and (A35), take the form

$$\begin{aligned}
&\left(\frac{1}{2}A + 4\Phi - \frac{C}{2}\right) u_\lambda h^{\mu\nu} + \left(B - \frac{3}{2}A - 4\Phi - \frac{C}{2}\right) u^\mu h_\lambda{}^\nu - \frac{B}{2} u^\nu h^\mu{}_\lambda - \frac{3}{2} B u_\lambda u^\mu u^\nu - 2\varepsilon_\lambda{}^{\mu\nu}{}_\rho u^\rho P \\
&+ \gamma \left[\frac{B}{2} \delta^\nu_\lambda h^\mu{}_\rho u^\rho + B h_{\lambda\rho} h^{\mu\nu} u^\rho + \left(A - \frac{1}{2}B\right) \delta^\mu_\lambda h^\nu{}_\rho u^\rho + (2A - C) h^{\mu\nu} u_\lambda + \frac{1}{2}(3A - C)\delta^\nu_\lambda u^\mu \right. \\
&\left. - A h_\lambda{}^\nu u^\mu + 3\left(\frac{1}{2}B - A\right)\delta^\mu_\lambda u^\nu + (A - B) h_\lambda{}^\mu u^\nu - B h_{\lambda\rho} u^\rho u^\mu u^\nu - 3A u_\lambda u^\mu u^\nu \right] = \kappa \Delta_\lambda{}^{\mu\nu}
\end{aligned} \tag{102}$$

in the cosmological setup we are considering.

Cosmology in linear MAMG-III

Here we consider the linear MAMG-III case in which

$$F = T + \gamma Q \,, \tag{103}$$

where γ is a free parameter. (In fact, in the linear case at hand one may write $F = \beta T + \gamma Q$ and fix the normalization of the theory choosing $\beta = 1$.) The metric field equations of the theory now read

$$\begin{aligned}
&-\frac{1}{2} g_{\mu\nu} F + 2 S_{\nu\alpha\beta} S_\mu{}^{\alpha\beta} - S_{\alpha\beta\mu} S^{\alpha\beta}{}_\nu + 2 S_{\nu\alpha\beta} S_\mu{}^{\beta\alpha} - 4 S_\mu S_\nu \\
&+ \gamma \Big(L_{(\mu\nu)} + \hat{\nabla}_\lambda J^\lambda{}_{(\mu\nu)} + g_{\mu\nu} \hat{\nabla}_\lambda \zeta^\lambda \Big) = \kappa T_{\mu\nu} \,.
\end{aligned} \tag{104}$$

Taking the trace of the latter and using the expressions of torsion and nonmetricity in (A30) we find

$$4\Phi^2 - P^2 + \dot{f}_3 + 3H f_3 = -\frac{\kappa}{6}\mathcal{T} \,, \tag{105}$$

where we have defined

$$f_3 := -\frac{\gamma}{2}\left(\frac{B}{2} - A\right). \tag{106}$$

Notice that the scale factor $a(t)$ appears in (105) only through the Hubble parameter $H := \frac{\dot{a}}{a}$, due to the fact that the function F in (103) does not depend on the scalar curvature R. Therefore, Equation (105) gives the expression of H in terms of the other parameters of the linear MAMG-III theory and of $\mathcal{T}$. In particular, this must be so since we are not in the realm of a gauge theory of gravity. In the case of a gauge theory of gravity, in fact, in the tetrads formalism one can fix the form of the affine connection from the very beginning in terms of the dynamical vielbein and find the consequent effective Friedmann equations.

On the other hand, let us also mention that the connection field equations of the model now become

$$\begin{aligned}&2\beta\left(S^{\mu\nu}{}_{\lambda} - 2S_{\lambda}{}^{[\mu\nu]} - 4S^{[\mu}\delta^{\nu]}_{\lambda}\right) + \gamma\left[2Q^{[\nu\mu]}{}_{\lambda} - Q_{\lambda}{}^{\mu\nu} + (q^{\nu} - Q^{\nu})\delta^{\mu}_{\lambda} + Q_{\lambda}g^{\mu\nu} + \frac{1}{2}Q^{\mu}\delta^{\nu}_{\lambda}\right]\\ &= \kappa\Delta_{\lambda}{}^{\mu\nu}.\end{aligned} \tag{107}$$

In the cosmological setup we are considering, Equation (107) takes the form

$$\begin{aligned}&2\beta\left[4\left(\delta^{[\nu}_{\lambda}h^{\mu]}{}_{\rho}u^{\rho} + 3\delta^{[\mu}_{\lambda}u^{\nu]} + h_{\lambda}{}^{[\nu}u^{\mu]}\right)\Phi - \varepsilon_{\lambda}{}^{\mu\nu}{}_{\rho}u^{\rho}P\right] + \gamma\left[\frac{B}{2}\delta^{\nu}_{\lambda}h^{\mu}{}_{\rho}u^{\rho} + Bh_{\lambda\rho}h^{\mu\nu}u^{\rho}\right.\\ &+\left(A - \frac{1}{2}B\right)\delta^{\mu}_{\lambda}h^{\nu}{}_{\rho}u^{\rho} + (2A - C)h^{\mu\nu}u_{\lambda} + \frac{1}{2}(3A - C)\delta^{\nu}_{\lambda}u^{\mu} - Ah_{\lambda}{}^{\nu}u^{\mu} + 3\left(\frac{1}{2}B - A\right)\delta^{\mu}_{\lambda}u^{\nu}\\ &\left.+ (A - B)h_{\lambda}{}^{\mu}u^{\nu} - Bh_{\lambda\rho}u^{\rho}u^{\mu}u^{\nu} - 3Au_{\lambda}u^{\mu}u^{\nu}\right] = \kappa\Delta_{\lambda}{}^{\mu\nu},\end{aligned} \tag{108}$$

upon use of Equations (A30) and (A35).

Cosmology in linear MAMG-IV

Let us now move on to the linear MAMG-IV case, in which

$$F = R + \beta T + \mu\mathcal{T}, \tag{109}$$

where β and μ are free parameters. The metric field equations of the theory take the following form:

$$-\frac{1}{2}g_{\mu\nu}F + R_{(\mu\nu)} + \beta\left(2S_{\nu\alpha\beta}S_{\mu}{}^{\alpha\beta} - S_{\alpha\beta\mu}S^{\alpha\beta}{}_{\nu} + 2S_{\nu\alpha\beta}S_{\mu}{}^{\beta\alpha} - 4S_{\mu}S_{\nu}\right) + \mu(\Theta_{\mu\nu} + T_{\mu\nu}) = \kappa T_{\mu\nu}. \tag{110}$$

Taking the trace of (110), using the post-Riemannian expansion of R, given by Equation (A37), along with (A30), we obtain a variant of the modified first Friedmann equation of the linear MAMG-IV theory, i.e.,

$$\begin{aligned}&\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + (1+\beta)\left(4\Phi^2 - P^2\right) + \frac{1}{8}\left[2A^2 + B(C - A)\right] + \Phi(2A - B) + \dot{f}_1 + 3Hf_1\\ &= \frac{\mu}{6}(\Theta - \mathcal{T}) - \frac{\kappa}{6}\mathcal{T},\end{aligned} \tag{111}$$

where f_1 is defined in (94) and

$$\Theta := \Theta_{\mu\nu}g^{\mu\nu}. \tag{112}$$

The first term on the right-hand side of (111) is consequence of the (linear) $\mathcal{T}$ dependence in (109), while all the other terms in (111) coincide with those appearing in Equation (93).

Furthermore, in this linear case the connection field equations of the theory read as follows:

$$P_\lambda{}^{\mu\nu} + 2\beta\left(S^{\mu\nu}{}_\lambda - 2S_\lambda{}^{[\mu\nu]} - 4S^{[\mu}\delta^{\nu]}_\lambda\right) = \mu\Theta_\lambda{}^{\mu\nu} + \kappa\Delta_\lambda{}^{\mu\nu} \,. \tag{113}$$

The latter, in the cosmological scenario we are considering, upon use of Equations (A30) (first line) and (A35), become

$$\begin{aligned}
&\left(\frac{1}{2}A + 4\Phi - \frac{C}{2}\right)u_\lambda h^{\mu\nu} + \left(B - \frac{3}{2}A - 4\Phi - \frac{C}{2}\right)u^\mu h_\lambda{}^\nu - \frac{B}{2}u^\nu h^\mu{}_\lambda - \frac{3}{2}Bu_\lambda u^\mu u^\nu - 2\varepsilon_\lambda{}^{\mu\nu}{}_\rho u^\rho P \\
&+ 2\beta\left[4\left(\delta^{[\nu}_\lambda h^{\mu]}{}_\rho u^\rho + 3\delta^{[\mu}_\lambda u^{\nu]} + h_\lambda{}^{[\nu}u^{\mu]}\right)\Phi - \varepsilon_\lambda{}^{\mu\nu}{}_\rho u^\rho P\right] = \mu\Theta_\lambda{}^{\mu\nu} + \kappa\Delta_\lambda{}^{\mu\nu} \,,
\end{aligned} \tag{114}$$

where, in particular, the explicit expressions of $\Theta_\lambda{}^{\mu\nu}$ and $\Delta_\lambda{}^{\mu\nu}$ depend on the specific matter one might then consider.

Cosmology in linear MAMG-V

In this paragraph we consider the linear MAMG-V case in which

$$F = R + \beta T + \gamma Q \,, \tag{115}$$

where β and γ are free parameters. The metric field equations of the theory read as follows:

$$\begin{aligned}
&-\frac{1}{2}g_{\mu\nu}F + R_{(\mu\nu)} + \beta\left(2S_{\nu\alpha\beta}S_\mu{}^{\alpha\beta} - S_{\alpha\beta\mu}S^{\alpha\beta}{}_\nu + 2S_{\nu\alpha\beta}S_\mu{}^{\beta\alpha} - 4S_\mu S_\nu\right) \\
&+ \gamma\left(L_{(\mu\nu)} + \hat{\nabla}_\lambda J^\lambda{}_{(\mu\nu)} + g_{\mu\nu}\hat{\nabla}_\lambda\zeta^\lambda\right) = \kappa T_{\mu\nu} \,.
\end{aligned} \tag{116}$$

Taking the trace of the latter and following what we have done in the previous paragraph, we are left with a variant of the modified first Friedmann equation of the linear MAMG-V theory, i.e.,

$$\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + (1+\beta)\left(4\Phi^2 - P^2\right) + \frac{1}{8}\left[2A^2 + B(C-A)\right] + \Phi(2A-B) + \dot{f}_2 + 3Hf_2 = -\frac{\kappa}{6}\mathcal{T} \,, \tag{117}$$

where f_2 is defined in (100).

Moreover, in the linear case (115) the connection field equations of the theory become

$$\begin{aligned}
&P_\lambda{}^{\mu\nu} + 2\beta\left(S^{\mu\nu}{}_\lambda - 2S_\lambda{}^{[\mu\nu]} - 4S^{[\mu}\delta^{\nu]}_\lambda\right) \\
&+ \gamma\left[2Q^{[\nu\mu]}{}_\lambda - Q_\lambda{}^{\mu\nu} + (q^\nu - Q^\nu)\delta^\mu_\lambda + Q_\lambda g^{\mu\nu} + \frac{1}{2}Q^\mu\delta^\nu_\lambda\right] = \kappa\Delta_\lambda{}^{\mu\nu} \,.
\end{aligned} \tag{118}$$

The latter, in the cosmological setup we are considering, take the form

$$\begin{aligned}
&\left(\frac{1}{2}A + 4\Phi - \frac{C}{2}\right)u_\lambda h^{\mu\nu} + \left(B - \frac{3}{2}A - 4\Phi - \frac{C}{2}\right)u^\mu h_\lambda{}^\nu - \frac{B}{2}u^\nu h^\mu{}_\lambda - \frac{3}{2}Bu_\lambda u^\mu u^\nu - 2\varepsilon_\lambda{}^{\mu\nu}{}_\rho u^\rho P \\
&+ 2\beta\left[4\left(\delta^{[\nu}_\lambda h^{\mu]}{}_\rho u^\rho + 3\delta^{[\mu}_\lambda u^{\nu]} + h_\lambda{}^{[\nu}u^{\mu]}\right)\Phi - \varepsilon_\lambda{}^{\mu\nu}{}_\rho u^\rho P\right] + \gamma\Bigg[\frac{B}{2}\delta^\nu_\lambda h^\mu{}_\rho u^\rho + Bh_{\lambda\rho}h^{\mu\nu}u^\rho \\
&+ \left(A - \frac{1}{2}B\right)\delta^\mu_\lambda h^\nu{}_\rho u^\rho + (2A-C)h^{\mu\nu}u_\lambda + \frac{1}{2}(3A-C)\delta^\nu_\lambda u^\mu - Ah_\lambda{}^\nu u^\mu + 3\left(\frac{1}{2}B - A\right)\delta^\mu_\lambda u^\nu \\
&+ (A-B)h_\lambda{}^\mu u^\nu - Bh_{\lambda\rho}u^\rho u^\mu u^\nu - 3Au_\lambda u^\mu u^\nu\Bigg] = \kappa\Delta_\lambda{}^{\mu\nu}
\end{aligned} \tag{119}$$

upon use of Equations (A30) and (A35).

Cosmology in linear MAMG-VI

We now consider the linear MAMG-VI theory in which

$$F = R + \gamma Q + \mu \mathcal{T} \,, \tag{120}$$

where γ and μ are free parameters. The metric field equations of the model at hand read

$$-\frac{1}{2} g_{\mu\nu} F + R_{(\mu\nu)} + \gamma \left(L_{(\mu\nu)} + \hat{\nabla}_\lambda J^\lambda{}_{(\mu\nu)} + g_{\mu\nu} \hat{\nabla}_\lambda \zeta^\lambda \right) + \mu (\Theta_{\mu\nu} + T_{\mu\nu}) = \kappa T_{\mu\nu} \,. \tag{121}$$

Taking the trace of (121) and using (A37) and (A30), we arrive at

$$\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + 4\Phi^2 - P^2 + \frac{1}{8}\left[2A^2 + B(C - A)\right] + \Phi(2A - B) + \dot{f}_2 + 3H f_2 = \frac{\mu}{6}(\Theta - \mathcal{T}) - \frac{\kappa}{6}\mathcal{T} \,, \tag{122}$$

where f_2 and Θ are defined, respectively, in (100) and (112). Equation (122) is a variant of the modified first Friedmann equation of the linear MAMG-VI model.

Moreover, in the linear case at hand the connection field equations of the model read

$$P_\lambda{}^{\mu\nu} + \gamma \left[2Q^{[\nu\mu]}{}_\lambda - Q_\lambda{}^{\mu\nu} + (q^\nu - Q^\nu)\delta^\mu_\lambda + Q_\lambda g^{\mu\nu} + \frac{1}{2} Q^\mu \delta^\nu_\lambda \right] = \mu \Theta_\lambda{}^{\mu\nu} + \kappa \Delta_\lambda{}^{\mu\nu} \,, \tag{123}$$

which, in the cosmological setup we are considering, take the form

$$\begin{aligned}
&\left(\frac{1}{2}A + 4\Phi - \frac{C}{2}\right) u_\lambda h^{\mu\nu} + \left(B - \frac{3}{2}A - 4\Phi - \frac{C}{2}\right) u^\mu h_\lambda{}^\nu - \frac{B}{2} u^\nu h^\mu{}_\lambda - \frac{3}{2} B u_\lambda u^\mu u^\nu - 2\varepsilon_\lambda{}^{\mu\nu}{}_\rho u^\rho P \\
&+ \gamma \left[\frac{B}{2} \delta^\nu_\lambda h^\mu{}_\rho u^\rho + B h_{\lambda\rho} h^{\mu\nu} u^\rho + \left(A - \frac{1}{2}B\right) \delta^\mu_\lambda h^\nu{}_\rho u^\rho + (2A - C) h^{\mu\nu} u_\lambda + \frac{1}{2}(3A - C)\delta^\nu_\lambda u^\mu \right. \\
&\left. - A h_\lambda{}^\nu u^\mu + 3\left(\frac{1}{2}B - A\right)\delta^\mu_\lambda u^\nu + (A - B) h_\lambda{}^\mu u^\nu - B h_{\lambda\rho} u^\rho u^\mu u^\nu - 3A u_\lambda u^\mu u^\nu \right] \\
&= \mu \Theta_\lambda{}^{\mu\nu} + \kappa \Delta_\lambda{}^{\mu\nu} \,,
\end{aligned} \tag{124}$$

where we have used Equations (A30) (second line) and (A35).

Cosmology in linear MAMG-VII

Here we focus on the linear MAMG-VII case in which

$$F = T + \gamma Q + \mu \mathcal{T} \,, \tag{125}$$

where γ and μ are free parameters. The metric field equations of the theory become

$$\begin{aligned}
&-\frac{1}{2} g_{\mu\nu} F + 2 S_{\nu\alpha\beta} S_\mu{}^{\alpha\beta} - S_{\alpha\beta\mu} S^{\alpha\beta}{}_\nu + 2 S_{\nu\alpha\beta} S_\mu{}^{\beta\alpha} - 4 S_\mu S_\nu \\
&+ \gamma \left(L_{(\mu\nu)} + \hat{\nabla}_\lambda J^\lambda{}_{(\mu\nu)} + g_{\mu\nu} \hat{\nabla}_\lambda \zeta^\lambda \right) + \mu (\Theta_{\mu\nu} + T_{\mu\nu}) = \kappa T_{\mu\nu} \,.
\end{aligned} \tag{126}$$

Taking the trace of Equation (126) and using the expressions of torsion and nonmetricity in (A30), we obtain

$$4\Phi^2 - P^2 + \dot{f}_3 + 3H f_3 = \frac{\mu}{6}(\Theta - \mathcal{T}) - \frac{\kappa}{6}\mathcal{T} \,, \tag{127}$$

where f_3 is given by (106), while Θ is defined in (112). Observe that in this case, analogously to what happens in the linear MAMG-III theory, the scale factor $a(t)$ appears in (127) only through the Hubble parameter H, which is due to the fact that the function F in (125) does not depend on the scalar curvature R of the general affine connection Γ. One could then apply arguments analogous to those made in the case of the MAMG-III cosmology.

Moreover, in the linear case given by Equation (125), the connection field equations of the theory take the following form:

$$
\begin{aligned}
& 2\beta\left(S^{\mu\nu}{}_{\lambda} - 2S_{\lambda}{}^{[\mu\nu]} - 4S^{[\mu}\delta^{\nu]}_{\lambda}\right) + \gamma\left[2Q^{[\nu\mu]}{}_{\lambda} - Q_{\lambda}{}^{\mu\nu} + (q^{\nu} - Q^{\nu})\delta^{\mu}_{\lambda} + Q_{\lambda}g^{\mu\nu} + \frac{1}{2}Q^{\mu}\delta^{\nu}_{\lambda}\right] \\
& = \mu\Theta_{\lambda}{}^{\mu\nu} + \kappa\Delta_{\lambda}{}^{\mu\nu} \,,
\end{aligned}
\tag{128}
$$

which exploiting once again Equations (A30) and (A35), become

$$
\begin{aligned}
& + 2\beta\left[4\left(\delta^{[\nu}_{\lambda}h^{\mu]}{}_{\rho}u^{\rho} + 3\delta^{[\mu}_{\lambda}u^{\nu]} + h_{\lambda}{}^{[\nu}u^{\mu]}\right)\Phi - \varepsilon_{\lambda}{}^{\mu\nu}{}_{\rho}u^{\rho}P\right] + \gamma\left[\frac{B}{2}\delta^{\nu}_{\lambda}h^{\mu}{}_{\rho}u^{\rho} + Bh_{\lambda\rho}h^{\mu\nu}u^{\rho}\right. \\
& + \left(A - \frac{1}{2}B\right)\delta^{\mu}_{\lambda}h^{\nu}{}_{\rho}u^{\rho} + (2A - C)h^{\mu\nu}u_{\lambda} + \frac{1}{2}(3A - C)\delta^{\nu}_{\lambda}u^{\mu} - Ah_{\lambda}{}^{\nu}u^{\mu} + 3\left(\frac{1}{2}B - A\right)\delta^{\mu}_{\lambda}u^{\nu} \\
& \left. + (A - B)h_{\lambda}{}^{\mu}u^{\nu} - Bh_{\lambda\rho}u^{\rho}u^{\mu}u^{\nu} - 3Au_{\lambda}u^{\mu}u^{\nu}\right] = \mu\Theta_{\lambda}{}^{\mu\nu} + \kappa\Delta_{\lambda}{}^{\mu\nu}
\end{aligned}
\tag{129}
$$

in the cosmological scenario we are considering.

Cosmology in linear MAMG-VIII

Finally, let us report in the following the cosmological aspects of the linear MAMG-VIII theory. In this case, we have

$$
F = R + \beta T + \gamma Q + \mu \mathcal{T} \,,
\tag{130}
$$

β, γ, and μ being free parameters, and the metric field equations of the theory take the form

$$
\begin{aligned}
& -\frac{1}{2}g_{\mu\nu}F + R_{(\mu\nu)} + \beta\left(2S_{\nu\alpha\beta}S_{\mu}{}^{\alpha\beta} - S_{\alpha\beta\mu}S^{\alpha\beta}{}_{\nu} + 2S_{\nu\alpha\beta}S_{\mu}{}^{\beta\alpha} - 4S_{\mu}S_{\nu}\right) \\
& + \gamma\left(L_{(\mu\nu)} + \hat{\nabla}_{\lambda}J^{\lambda}{}_{(\mu\nu)} + g_{\mu\nu}\hat{\nabla}_{\lambda}\zeta^{\lambda}\right) + \mu\left(\Theta_{\mu\nu} + T_{\mu\nu}\right) = \kappa T_{\mu\nu} \,.
\end{aligned}
\tag{131}
$$

Taking the trace of (131) and using (A37) and (A30), we arrive at

$$
\begin{aligned}
& \frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + (1+\beta)\left(4\Phi^2 - P^2\right) + \frac{1}{8}\left[2A^2 + B(C - A)\right] + \Phi(2A - B) + \dot{f}_2 + 3Hf_2 \\
& = \frac{\mu}{6}(\Theta - \mathcal{T}) - \frac{\kappa}{6}\mathcal{T} \,,
\end{aligned}
\tag{132}
$$

where f_2 and Θ are defined in (100) and (112), respectively. Equation (132) is a variant of the modified first Friedmann equation which together with the acceleration Equation (90), rules the cosmology of the linear MAMG-VIII model.

Moreover, in this linear case the connection field equations of the theory read

$$
\begin{aligned}
& P_{\lambda}{}^{\mu\nu} + 2\beta\left(S^{\mu\nu}{}_{\lambda} - 2S_{\lambda}{}^{[\mu\nu]} - 4S^{[\mu}\delta^{\nu]}_{\lambda}\right) \\
& + \gamma\left[2Q^{[\nu\mu]}{}_{\lambda} - Q_{\lambda}{}^{\mu\nu} + (q^{\nu} - Q^{\nu})\delta^{\mu}_{\lambda} + Q_{\lambda}g^{\mu\nu} + \frac{1}{2}Q^{\mu}\delta^{\nu}_{\lambda}\right] = \mu\Theta_{\lambda}{}^{\mu\nu} + \kappa\Delta_{\lambda}{}^{\mu\nu} \,.
\end{aligned}
\tag{133}
$$

The latter, in the cosmological setup we are considering, take the form

$$
\begin{aligned}
&\left(\frac{1}{2}A+4\Phi-\frac{C}{2}\right)u_\lambda h^{\mu\nu}+\left(B-\frac{3}{2}A-4\Phi-\frac{C}{2}\right)u^\mu h_\lambda{}^\nu-\frac{B}{2}u^\nu h^\mu{}_\lambda-\frac{3}{2}Bu_\lambda u^\mu u^\nu-2\varepsilon_\lambda{}^{\mu\nu}{}_\rho u^\rho P\\
&+2\beta\left[4\left(\delta^{[\nu}_\lambda h^{\mu]}{}_\rho u^\rho+3\delta^{[\mu}_\lambda u^{\nu]}+h_\lambda{}^{[\nu}u^{\mu]}\right)\Phi-\varepsilon_\lambda{}^{\mu\nu}{}_\rho u^\rho P\right]+\gamma\left[\frac{B}{2}\delta^\nu_\lambda h^\mu{}_\rho u^\rho+Bh_{\lambda\rho}h^{\mu\nu}u^\rho\right.\\
&+\left(A-\frac{1}{2}B\right)\delta^\mu_\lambda h^\nu{}_\rho u^\rho+(2A-C)h^{\mu\nu}u_\lambda+\frac{1}{2}(3A-C)\delta^\nu_\lambda u^\mu-Ah_\lambda{}^\nu u^\mu+3\left(\frac{1}{2}B-A\right)\delta^\mu_\lambda u^\nu\\
&\left.+(A-B)h_\lambda{}^\mu u^\nu-Bh_{\lambda\rho}u^\rho u^\mu u^\nu-3Au_\lambda u^\mu u^\nu\right]=\mu\Theta_\lambda{}^{\mu\nu}+\kappa\Delta_\lambda{}^{\mu\nu}\,,
\end{aligned}
\tag{134}
$$

where, as in the other models previously studied, we have used Equations (A30) and (A35).

10.1. On the Case of a Scalar Field Coupled to Torsion

We shall now review and rediscuss the cosmological application in [28] to the case in which the matter Lagrangian is given by the one for a scalar field ϕ coupled to torsion (in particular, by means of the torsion vector S^μ), where the nonmetricity has been set to zero. The matter Lagrangian reads

$$
\mathcal{L}_{\mathrm{m}}=-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi-V(\phi)+\lambda_0 S^\mu\nabla_\mu\phi\,, \tag{135}
$$

where λ_0 is a constant parameter and we recall that $\nabla_\mu\phi=\partial_\mu\phi$. In particular, the authors of [28] considered the linear MAMG-I case (91). Hence, the full action of the theory is

$$
\mathcal{S}=\frac{1}{2\kappa}\int\sqrt{-g}d^4x\left[R+\beta T+2\kappa\left(-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi-V(\phi)+\lambda_0 S^\mu\nabla_\mu\phi\right)\right]. \tag{136}
$$

The variation of (136) with respect to the scalar field ϕ yields

$$
\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}(\partial^\mu\phi-\lambda_0 S^\mu)\right]=\frac{\partial V}{\partial\phi}\,. \tag{137}
$$

Moreover, the hypermomentum in this case reads (It can be computed using Equation (A18) (more precisely, the variation of the torsion vector with respect to the general affine connection)).

$$
\Delta_\lambda{}^{\mu\nu}=2\lambda_0\delta^{[\mu}_\lambda\nabla^{\nu]}\phi\,. \tag{138}
$$

Varying the action (136) with respect to the general affine connection $\Gamma^\lambda{}_{\mu\nu}$ one finds

$$
P_\lambda{}^{\mu\nu}+2\beta\left(S^{\mu\nu}{}_\lambda-2S_\lambda{}^{[\mu\nu]}-4S^{[\mu}\delta^{\nu]}_\lambda\right)=2\kappa\lambda_0\delta^{[\mu}_\lambda\nabla^{\nu]}\phi\,, \tag{139}
$$

where $P_\lambda{}^{\mu\nu}$ is the Palatini tensor defined in Equation (A7). In particular, taking different contractions of (139), we obtain

$$
\begin{aligned}
&S_\lambda=0\,,\\
&S^\mu=\frac{3\kappa\lambda_0}{8\beta}\partial^\mu\phi\,,\\
&(1+\beta)t_\lambda=0\,,
\end{aligned}
\tag{140}
$$

where t_λ is the torsion pseudo-vector (cf. Appendix A). The equations in (140) indicate that the torsion vector vanishes and, therefore, to have non-trivial dynamics for the scalar field, one should set $\lambda_0 = 0$, namely remove the coupling to torsion in (135). Moreover, the last of (140) implies that either $\beta = -1$ or $t_\lambda = 0$. Fixing

$$\lambda_0 = 0\,, \quad \beta = -1\,, \tag{141}$$

and plugging (140) back into (139), the latter boils down to

$$Z_\lambda{}^{\mu\nu} = 0\,, \tag{142}$$

namely the traceless part of the torsion vanishes, while the only remaining torsion components are the four of the non-vanishing torsion pseudo-vector t_λ. In particular, one can prove that the choice (141) is the only one we can perform at this point in such a way to have non-trivial dynamics for ϕ in the presence of a non-vanishing (totally antisymmetric) torsion. Moreover, Equation (137) reduces to

$$\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}\partial^\mu\phi\right] = \frac{\partial V}{\partial\phi}\,, \tag{143}$$

which, in the simple case of a free scalar (i.e., $V(\phi) = 0$), becomes

$$\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}\partial^\mu\phi\right] = 0\,. \tag{144}$$

Following the same lines of [28], let us restrict ourselves to this case from now on.

Regarding the torsion, we are left with

$$S_{\lambda\mu}{}^{\nu} = \frac{1}{6}\varepsilon_{\lambda\mu\kappa\rho}g^{\kappa\nu}t^\rho\,. \tag{145}$$

Up to this point, the above considerations were general for the model at hand. Let us now focus on the homogeneous FLRW cosmology of this theory (for the cosmological setup we will consider in the following we refer the reader to Appendix B). From the first equation of (A30) we see that, in this case, $\Phi(t) = 0$. Hence, we find that the full torsion tensor is given by

$$S_{\mu\nu\alpha} = \varepsilon_{\mu\nu\alpha\rho}u^\rho P(t) \tag{146}$$

in a homogeneous, non-Riemannian (metric but torsionful) FLRW cosmological setup. Moreover, plugging back the above results into the action (136), the latter becomes

$$\mathcal{S} = \frac{1}{2\kappa}\int \sqrt{-g}d^4x\left[R - T - \kappa g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\right]. \tag{147}$$

Now, Equation (144) implies

$$\dot{\phi} = \frac{c_0}{a^3}\,, \tag{148}$$

where c_0 is an arbitrary constant and $a = a(t)$ is the scale factor of the universe.

On the other hand, the metric field equations of the theory (147) read

$$-\frac{1}{2}g_{\mu\nu}(R - T) + R_{(\mu\nu)} - \left(2S_{\nu\alpha\beta}S_\mu{}^{\alpha\beta} - S_{\alpha\beta\mu}S^{\alpha\beta}{}_\nu + 2S_{\nu\alpha\beta}S_\mu{}^{\beta\alpha}\right) = \kappa T_{\mu\nu}\,. \tag{149}$$

Here, let us introduce the following form of the energy-momentum tensor:

$$T_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu}\,, \tag{150}$$

where ρ and p are, respectively, the density and the pressure associated with the scalar field Lagrangian, while u^μ is the normalized four-velocity and $h_{\mu\nu}$ is the projection tensor projecting objects on the space orthogonal to u^μ (cf. the definition in Equation (A25)). Therefore, we have

$$\mathcal{T} = -\rho + 3p = \dot{\phi}^2 \,. \tag{151}$$

Taking the trace of (149) and using (146) and (A37), we obtain

$$\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = -\frac{\kappa}{6}\dot{\phi}^2 \,. \tag{152}$$

Let us now derive the acceleration equation for the case at hand. Contracting the field equations (149) with $u^\mu u^\nu$, using the useful formulas collected in Appendix B, we obtain

$$R_{\mu\nu}u^\mu u^\nu = \frac{\kappa}{2}(\rho + 3p) \,. \tag{153}$$

The latter, when substituted in Equation (90) in the case of vanishing nonmetricity and $\Phi = 0$, yields

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{6}(\rho + 3p) \,, \tag{154}$$

which is the acceleration equation of the theory. It can be rewritten as

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{3}\dot{\phi}^2 \,. \tag{155}$$

Notice that the non-Riemannian degrees of freedom of the torsion pseudo-vector do not affect the cosmological evolution in this case, as all the contributions in $P(t)$ cancel out in the above equations. Finally, observe that using (155) to eliminate the double temporal derivative of the scale factor in (152), the latter becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa}{6}\dot{\phi}^2 \,, \tag{156}$$

which is the first Friedmann equation of the model. Therefore, one is left with the usual scalar field cosmology, namely the one that would have been obtained by considering the purely Riemannian case. The situation would obviously be different in the presence of non-vanishing scalar potential and nonmetricity. In the present case, the torsion pseudo-vector, which here is the only non-vanishing part of the torsion tensor, do not affect the cosmological equations of the theory, and we end up with a well-known result. This is essentially a consequence of the fact that we must fix the coupling constant $\lambda_0 = 0$ and the parameter $\beta = -1$ in order for the scalar field to have non-trivial dynamics and for the torsion to be non-vanishing, respectively.

The final form of the (metric but torsionful) affine connection for this model reads as follows:

$$\Gamma^\lambda{}_{\mu\nu} = \tilde{\Gamma}^\lambda{}_{\mu\nu} + \frac{1}{6}\varepsilon^\lambda{}_{\mu\nu\rho}t^\rho \,, \tag{157}$$

where $\tilde{\Gamma}^\lambda{}_{\mu\nu}$ is the Levi-Civita connection, defined in (A4). Please note that using (146), the final form of the connection given in (157) can be rewritten as

$$\Gamma^\lambda{}_{\mu\nu} = \tilde{\Gamma}^\lambda{}_{\mu\nu} + \varepsilon^\lambda{}_{\mu\nu\rho}u^\rho W(t) \,, \tag{158}$$

with

$$W(t) = P(t) \,. \tag{159}$$

Let us conclude this section by observing that in the sub-case in which $F = R + \beta T \to F = R$ one would obtain a completely vanishing torsion tensor and the scalar field cosmology would be the same retrieved above. On the other hand, in the sub-case $F = R + \beta T \to F = T$, where the dependence on the scalar curvature R in F is removed and $\beta = 1$ fixes the normalization of the theory, one is left with an imaginary coupling constant $\lambda_0 = \pm 2\mathrm{i}\sqrt{\frac{2}{3\kappa}}$. For the sake of completeness, we report this particular sub-case in Appendix C.

11. Conclusions

This review, where we have collected the MG-N models and their MAMG generalizations, also including their cosmological analysis, will be instrumental to the reader interested in these topics and also to further investigate mathematical, physical, and cosmological aspects and applications of the aforementioned theories, in view of further comparison with observations.

We have discussed the MAMG-N theories, offering a complete dictionary on the subject, and derived the modified Friedmann equations for the linear case in a homogeneous, non-Riemannian FLRW spacetime. In this context, let us stress that it would be interesting to release the assumptions of linearity of F, in particular to examine the cosmological consequences induced by the very presence of the divergence of the dilation current in F. Such a study might also shed some light on the phenomenological aspects of this peculiar contribution and on the energy-momentum trace counterpart as well.

Moreover, we have given a cosmological application of the results obtained so far to the case in which the matter action is given in terms of the Lagrangian for a scalar field directly coupled to torsion by means of the torsion vector S^μ, with coupling constant λ_0. In particular, focusing on the linear metric-affine $F = R + \beta T$ theory (i.e., the linear MAMG-I model), we have shown that we actually have to set $\lambda_0 = 0$ (and $\beta = -1$) in order to end up with a non-trivial theory. This implies that in this case, the torsion does not contribute to the cosmological evolution of the model and one ends up with the usual scalar field cosmology. Nevertheless, the torsion is non-vanishing in this case: it is completely antisymmetric (given in terms of the torsion pseudo-vector) and determines the final form of the affine connection.

As already stressed in [28], one could extend our analysis by considering also non-vanishing nonmetricity and hence direct coupling of the scalar field to both torsion and nonmetricity. In this context, one might take the following matter Lagrangian:

$$\mathcal{L}_\mathrm{m} = -\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi) + \lambda_0 S^\mu\nabla_\mu\phi + \lambda_1 Q^\mu\nabla_\mu\phi + \lambda_2 q^\mu\nabla_\mu\phi\,, \tag{160}$$

with constant parameters λ_0, λ_1, λ_2 and where Q^μ and q^μ are the nonmetricity vectors. In particular, the hypermomentum associated with (160) is (It can be derived by exploiting Equation (A18)).

$$\Delta_\lambda{}^{\mu\nu} = 2\left[\lambda_0\delta_\lambda^{[\mu}\nabla^{\nu]}\phi - 2\lambda_1\delta_\lambda^\mu\nabla^\nu\phi - \lambda_2(g^{\mu\nu}\nabla_\lambda\phi + \delta_\lambda^\nu\nabla^\mu\phi)\right]. \tag{161}$$

In this case, we expect the nonmetricity vectors, and possibly the torsion trace, to play a non-trivial role in cosmological solutions. We leave the complete study to a forthcoming paper.

It would be worth it to further elaborate on cosmological applications of MAMG theories, especially in the presence of a cosmological hyperfluid [93,94] (see also [95]). Possible future developments may also regard the inclusion of parity violating terms in MAMG, in particular of the so-called Hojman term [96] (most of the time referred to as the Holst term [97]), following the lines of [98]. Always in this context, one might also consider the inclusion of a parity violating coupling of a scalar field to torsion, adding the contribution $\lambda_3 t^\mu\nabla_\mu\phi$ to the matter Lagrangian (160), where λ_3 is another constant parameter of the theory. In this case, the hypermomentum tensor would be

$$\Delta_\lambda{}^{\mu\nu} = 2\left[\lambda_0\delta_\lambda^{[\mu}\nabla^{\nu]}\phi - 2\lambda_1\delta_\lambda^\mu\nabla^\nu\phi - \lambda_2(g^{\mu\nu}\nabla_\lambda\phi + \delta_\lambda^\nu\nabla^\mu\phi) - \lambda_3\varepsilon^{\rho\mu\nu}{}_\lambda\nabla_\rho\phi\right]. \tag{162}$$

Given this enlarged setup, we argue that here also the torsion pseudo-vector could play a non-trivial role in the study of cosmological solutions.

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Appendix A. Notation, Conventions and Useful Formulas in MAG

We consider four spacetime dimensions and our convention for the metric signature is mostly plus: $(-,+,+,+)$. We use minuscule Greek letters to denote spacetime indices, i.e., $\mu,\nu,\ldots = 0,1,2,3$.

The covariant derivative ∇ of, e.g., a vector v^λ is defined as

$$\nabla_\nu v^\lambda = \partial_\nu v^\lambda + \Gamma^\lambda{}_{\mu\nu} v^\mu \,, \tag{A1}$$

where we have also introduced the general affine connection $\Gamma^\lambda{}_{\mu\nu}$. The latter can be decomposed as follows:

$$\Gamma^\lambda{}_{\mu\nu} = \tilde{\Gamma}^\lambda{}_{\mu\nu} + N^\lambda{}_{\mu\nu} \,, \tag{A2}$$

where

$$N^\lambda{}_{\mu\nu} = \underbrace{\frac{1}{2} g^{\rho\lambda}\left(Q_{\mu\nu\rho} + Q_{\nu\rho\mu} - Q_{\rho\mu\nu}\right)}_{\text{deflection (or disformation)}} - \underbrace{g^{\rho\lambda}\left(S_{\rho\mu\nu} + S_{\rho\nu\mu} - S_{\mu\nu\rho}\right)}_{\text{contorsion} := K^\lambda{}_{\mu\nu}} \tag{A3}$$

is the distortion tensor and

$$\tilde{\Gamma}^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\rho\lambda}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right) \tag{A4}$$

is the Levi-Civita connection. The torsion and nonmetricity tensors in (A3) are defined as

$$\begin{aligned} S_{\mu\nu}{}^\rho &:= \Gamma^\rho{}_{[\mu\nu]} \,, \\ Q_{\lambda\mu\nu} &:= -\nabla_\lambda g_{\mu\nu} = -\partial_\lambda g_{\mu\nu} + \Gamma^\rho{}_{\mu\lambda} g_{\rho\nu} + \Gamma^\rho{}_{\nu\lambda} g_{\mu\rho} \,, \end{aligned} \tag{A5}$$

respectively. Their trace decomposition in four spacetime dimensions reads

$$\begin{aligned} S_{\lambda\mu}{}^\nu &= \frac{2}{3}\delta_{[\mu}{}^\nu S_{\lambda]} + \frac{1}{6}\varepsilon_{\lambda\mu\kappa\rho} g^{\kappa\nu} t^\rho + Z_{\lambda\mu}{}^\nu \,, \\ Q_{\lambda\mu\nu} &= \frac{5}{18} Q_\lambda g_{\mu\nu} - \frac{1}{9} q_\lambda g_{\mu\nu} + \frac{4}{9} g_{\lambda(\nu} q_{\mu)} - \frac{1}{9} g_{\lambda(\nu} Q_{\mu)} + \Omega_{\lambda\mu\nu} \,, \end{aligned} \tag{A6}$$

where $Q_\lambda := Q_{\lambda\mu}{}^\mu$ and $q_\nu := Q^\mu{}_{\mu\nu}$ are the nonmetricity vectors, $S_\lambda := S_{\lambda\sigma}{}^\sigma$ is the torsion vector, and $t^\rho := \varepsilon^{\rho\lambda\mu\nu} S_{\lambda\mu\nu}$ is the torsion pseudo-vector. On the other hand, $Z_{\lambda\mu}{}^\nu$ (with $Z_{\lambda\mu\nu} = \frac{4}{3} Z_{[\lambda(\mu]\nu)}$, $\epsilon^{\lambda\mu\nu\rho} Z_{\lambda\mu\nu} = 0$) and $\Omega_{\lambda\mu\nu}$ are the traceless parts of torsion and nonmetricity, respectively. We denote by $\varepsilon^{\mu\nu\alpha\beta} = \frac{1}{\sqrt{-g}} \epsilon^{\mu\nu\alpha\beta}$ the Levi-Civita tensor, while $\epsilon^{\mu\nu\alpha\beta}$ is the Levi-Civita symbol.

Exploiting the trace decomposition (A6), one can prove that the so-called Palatini tensor, whose definition reads

$$P_\lambda{}^{\mu\nu} := -\frac{\nabla_\lambda(\sqrt{-g}g^{\mu\nu})}{\sqrt{-g}} + \frac{\nabla_\sigma(\sqrt{-g}g^{\mu\sigma})\delta^\nu_\lambda}{\sqrt{-g}} + 2(S_\lambda g^{\mu\nu} - S^\mu \delta^\nu_\lambda + g^{\mu\sigma}S_{\sigma\lambda}{}^\nu)\,, \tag{A7}$$

can be written in terms of torsion and nonmetricity as [22]

$$\begin{aligned} P_\lambda{}^{\mu\nu} &= \delta^\nu_\lambda\left(q^\mu - \frac{1}{2}Q^\mu - 2S^\mu\right) + g^{\mu\nu}\left(\frac{1}{2}Q_\lambda + 2S_\lambda\right) - (Q_\lambda{}^{\mu\nu} + 2S_\lambda{}^{\mu\nu}) \\ &= -\Omega_\lambda{}^{\mu\nu} + \frac{1}{3}g^{\mu\nu}\left(\frac{2}{3}Q_\lambda + \frac{1}{3}q_\lambda + 4S_\lambda\right) + \frac{1}{9}\delta^\nu_\lambda(-4Q^\mu + 7q^\mu) + \frac{1}{9}\delta^\mu_\lambda\left(\frac{1}{2}Q^\mu - 2q^\nu\right) \\ &- \frac{1}{3}\varepsilon_\lambda{}^{\mu\nu\rho}t_\rho - 2Z_\lambda{}^{\mu\nu}\,. \end{aligned} \tag{A8}$$

Please note that the Palatini tensor is traceless in the indices μ, λ, i.e.,

$$P_\mu{}^{\mu\nu} = 0\,. \tag{A9}$$

Our definition of the Riemann tensor for the general affine connection $\Gamma^\lambda{}_{\mu\nu}$ is

$$R^\mu{}_{\nu\alpha\beta} := 2\partial_{[\alpha}\Gamma^\mu{}_{|\nu|\beta]} + 2\Gamma^\mu{}_{\rho[\alpha}\Gamma^\rho{}_{|\nu|\beta]}\,. \tag{A10}$$

Correspondingly, $R_{\mu\nu} := R^\rho{}_{\mu\rho\nu}$ and $R := g^{\mu\nu}R_{\mu\nu}$ are, respectively, the Ricci tensor and the scalar curvature of Γ. In four spacetime dimensions the Riemann tensor in (A10) can be decomposed into its Riemannian and non-Riemannian parts as follows:

$$\begin{aligned} R_{\lambda\mu\nu\kappa} &= \frac{1}{2}(g_{\lambda\nu}\tilde{R}_{\mu\kappa} - g_{\lambda\kappa}\tilde{R}_{\mu\nu} - g_{\mu\nu}\tilde{R}_{\lambda\kappa} + g_{\mu\kappa}\tilde{R}_{\lambda\nu}) - \frac{1}{6}\tilde{R}(g_{\lambda\nu}g_{\mu\kappa} - g_{\lambda\kappa}g_{\mu\nu}) + C_{\lambda\mu\nu\kappa} \\ &+ \tilde{\nabla}_\kappa N_{\lambda\mu\nu} - \tilde{\nabla}_\nu N_{\lambda\mu\kappa} + N_{\lambda\alpha\nu}N^\alpha{}_{\mu\kappa} - N_{\lambda\alpha\kappa}N^\alpha{}_{\mu\nu}\,, \end{aligned} \tag{A11}$$

where $\tilde{R}_{\mu\nu}$ and $\tilde{R} := g^{\mu\nu}\tilde{R}_{\mu\nu}$ are, respectively, the Ricci tensor and Ricci scalar of the Levi-Civita connection $\tilde{\Gamma}^\lambda{}_{\mu\nu}$, $\tilde{\nabla}$ denotes the covariant derivative of $\tilde{\Gamma}^\lambda{}_{\mu\nu}$, and $C^\lambda{}_{\mu\nu\kappa}$ is the Weyl tensor, fulfilling

$$C^\lambda{}_{\mu\lambda\kappa} = 0\,, \quad C^{(\lambda\mu)\nu\rho} = 0\,, \quad C^{\lambda\mu(\nu\rho)} = 0\,, \quad C^{\lambda\mu\nu\rho} = C^{\nu\rho\lambda\mu}\,, \quad C^{[\lambda\mu\nu\rho]} = 0\,, \quad g_{\rho\nu}C^{\rho\lambda\mu\nu} = 0\,. \tag{A12}$$

Furthermore, any tensor contraction between indices of the Weyl tensor vanishes.

We can also introduce the decomposition of the scalar curvature R in terms of the Riemannian scalar curvature $\tilde{R}$ plus the non-Riemannian contributions given by torsion and nonmetricity scalars, i.e.,

$$R = \tilde{R} + T + Q + 2Q_{\alpha\mu\nu}S^{\alpha\mu\nu} + 2S_\mu(q^\mu - Q^\mu) + \tilde{\nabla}(q^\mu - Q^\mu - 4S^\mu)\,, \tag{A13}$$

where we have defined the torsion and nonmetricity scalars as

$$\begin{aligned} T &:= S_{\mu\nu\alpha}S^{\mu\nu\alpha} - 2S_{\mu\nu\alpha}S^{\alpha\mu\nu} - 4S_\mu S^\mu\,, \\ Q &:= \frac{1}{4}Q_{\alpha\mu\nu}Q^{\alpha\mu\nu} - \frac{1}{2}Q_{\alpha\mu\nu}Q^{\mu\nu\alpha} - \frac{1}{4}Q_\mu Q^\mu + \frac{1}{2}Q_\mu q^\mu\,, \end{aligned} \tag{A14}$$

respectively. Moreover, notice that defining the "superpotentials"

$$\begin{aligned} \Xi^{\alpha\mu\nu} &:= \frac{1}{4}Q^{\alpha\mu\nu} - \frac{1}{2}Q^{\mu\nu\alpha} - \frac{1}{4}g^{\mu\nu}Q^\alpha + \frac{1}{2}g^{\alpha\mu}Q^\nu\,, \\ \Sigma^{\alpha\mu\nu} &:= S^{\alpha\mu\nu} - 2S^{\mu\nu\alpha} - 4g^{\mu\nu}S^\alpha \end{aligned} \tag{A15}$$

the torsion and nonmetricity scalars in (A14) can be rewritten in the following, more compact, form:

$$
\begin{aligned}
T &= S_{\alpha\mu\nu}\Sigma^{\alpha\mu\nu}\,, \\
Q &= Q_{\alpha\mu\nu}\Xi^{\alpha\mu\nu}\,.
\end{aligned} \tag{A16}
$$

Let us also report the useful formulas concerning the variation of the nonmetricity and torsion tensors with respect to the metric and the general affine connection $\Gamma^{\lambda}{}_{\mu\nu}$, i.e., [22]

$$
\begin{aligned}
\delta_g Q_{\rho\alpha\beta} &= \partial_\rho\left(g_{\mu\alpha}g_{\nu\beta}\delta g^{\mu\nu}\right) - 2g_{\lambda\mu}g_{\nu(\alpha}\Gamma^{\lambda}{}_{\beta)\rho}\delta g^{\mu\nu}\,, \\
\delta_g S_{\mu\nu}{}^{\alpha} &= 0\,, \\
\delta_\Gamma Q_{\rho\alpha\beta} &= 2\delta^{\nu}_{\rho}\delta^{\mu}_{(\alpha}g_{\beta)\lambda}\delta\Gamma^{\lambda}{}_{\mu\nu}\,, \\
\delta_\Gamma S_{\alpha\beta}{}^{\lambda} &= \delta^{[\mu}_{\alpha}\delta^{\nu]}_{\beta}\delta\Gamma^{\lambda}{}_{\mu\nu}\,.
\end{aligned} \tag{A17}
$$

In particular, it follows that

$$
\begin{aligned}
\delta_g Q_\rho &= \partial_\rho\left(g_{\mu\nu}\delta g^{\mu\nu}\right)\,, \\
\delta_g q_\beta &= \delta g^{\mu\nu}\left[g_{\nu\beta}g^{\rho\alpha}\left(\partial_\rho g_{\mu\alpha}\right) + \Gamma^{\lambda}{}_{\mu\nu}g_{\lambda\beta} - g^{\rho\sigma}\Gamma^{\alpha}{}_{\rho\sigma}g_{\mu\alpha}g_{\nu\beta}\right] + g_{\nu\beta}\left(\partial_\mu\delta g^{\mu\nu}\right)\,, \\
\delta_g S_\mu &= 0\,, \\
\delta_\Gamma Q_\rho &= 2\delta^{\nu}_{\rho}\delta^{\mu}_{\lambda}\delta\Gamma^{\lambda}{}_{\mu\nu}\,, \\
\delta_\Gamma q_\beta &= \left(g^{\mu\nu}g_{\beta\lambda} + \delta^{\mu}_{\beta}\delta^{\nu}_{\lambda}\right)\delta\Gamma^{\lambda}{}_{\mu\nu}\,, \\
\delta_\Gamma S_\alpha &= \delta^{[\mu}_{\alpha}\delta^{\nu]}_{\lambda}\delta\Gamma^{\lambda}{}_{\mu\nu}\,, \\
\delta_\Gamma t^\alpha &= \varepsilon^{\alpha\mu\nu}{}_{\lambda}\delta\Gamma^{\lambda}{}_{\mu\nu}\,.
\end{aligned} \tag{A18}
$$

Finally, regarding the matter content in MAG theories, moreover the usual energy-momentum tensor, which is defined as

$$
T_{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_{\rm m})}{\delta g^{\mu\nu}}\,, \tag{A19}
$$

we also have a non-trivial dependence of the matter Lagrangian on the general affine connection. In fact, the variation of the matter part of the action with respect to $\Gamma^{\lambda}{}_{\mu\nu}$ defines the hypermomentum tensor,

$$
\Delta_{\lambda}{}^{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_{\rm m})}{\delta\Gamma^{\lambda}{}_{\mu\nu}}\,. \tag{A20}
$$

The energy-momentum and hypermomentum tensors are not independent. In particular, they are subject to the conservation law

$$
\sqrt{-g}\left(2\tilde{\nabla}_\mu T^{\mu}{}_{\alpha} - \Delta^{\lambda\mu\nu}R_{\lambda\mu\nu\alpha}\right) + \check{\nabla}_\mu\check{\nabla}_\nu\left(\sqrt{-g}\Delta_{\alpha}{}^{\mu\nu}\right) + 2S_{\mu\alpha}{}^{\lambda}\check{\nabla}_\nu\left(\sqrt{-g}\Delta_{\lambda}{}^{\mu\nu}\right) = 0\,, \tag{A21}
$$

where we have introduced

$$
\check{\nabla}_\mu := 2S_\mu - \nabla_\mu = \sqrt{-g}\hat{\nabla}\,, \tag{A22}
$$

the derivative $\hat{\nabla}$ being defined in (23). Equation (A21) originates from the invariance under diffeomorphisms of the matter sector of the action (cf. [93]).

Appendix B. Cosmology in the Presence of Torsion and Nonmetricity

In this appendix we recall, following [93], the expressions of the general affine connection $\Gamma^{\lambda}{}_{\mu\nu}$, the torsion and nonmetricity tensors, the Palatini tensor, and the scalar curvature R in a homogeneous, non-Riemannian FLRW spacetime.

We consider a homogeneous, flat FLRW cosmology, where the line element is

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j \,, \tag{A23}$$

with $i, j, \ldots = 1, 2, 3$ and scale factor $a(t)$. The Hubble parameter is

$$H := \frac{\dot{a}}{a} \tag{A24}$$

and the projection tensor projecting objects on the space orthogonal to the normalized four-velocity u^μ (such that $u^\mu = \delta^\mu_0 = (1, 0, 0, 0)$ and $u_\mu u^\mu = -1$) is

$$h_{\mu\nu} := g_{\mu\nu} + u_\mu u_\nu = h_{\nu\mu} \,. \tag{A25}$$

In particular, we have

$$h^{\mu\nu}h_{\mu\nu} = 3 \,, \quad h_{\mu\alpha}h^{\nu\alpha} = \delta^\nu_\mu + u_\mu u^\nu \,, \quad h^\mu{}_\mu = 3 \,, \quad h_{\mu\nu}u^\mu u^\nu = 0 \,. \tag{A26}$$

We also define the temporal derivative

$$\dot{} = u^\alpha \nabla_\alpha \,. \tag{A27}$$

The projection operator (A25) and the temporal derivative (A27) constitute together a $1+3$ spacetime split.

In a non-Riemannian FLRW spacetime in $1+3$ dimensions the general affine connection can be written as

$$\Gamma^\lambda{}_{\mu\nu} = \tilde{\Gamma}^\lambda{}_{\mu\nu} + X(t)u^\lambda h_{\mu\nu} + Y(t)u_\mu h^\lambda{}_\nu + Z(t)u_\nu h^\lambda{}_\mu + V(t)u^\lambda u_\mu u_\nu + \varepsilon^\lambda{}_{\mu\nu\rho}u^\rho W(t) \,, \tag{A28}$$

where, in particular, the non-vanishing components of the Levi-Civita connection read

$$\tilde{\Gamma}^0{}_{ij} = \tilde{\Gamma}^0{}_{ji} = \dot{a}a\delta_{ij} = Hg_{ij} \,, \quad \tilde{\Gamma}^i{}_{j0} = \tilde{\Gamma}^i{}_{0j} = \frac{\dot{a}}{a}\delta^i_j = H\delta^i_j \,. \tag{A29}$$

The torsion and nonmetricity tensors can be written, respectively, in the following way:

$$\begin{aligned} S_{\mu\nu\alpha} &= 2u_{[\mu}h_{\nu]\alpha}\Phi(t) + \varepsilon_{\mu\nu\alpha\rho}u^\rho P(t) \,, \\ Q_{\alpha\mu\nu} &= A(t)u_\alpha h_{\mu\nu} + B(t)h_{\alpha(\mu}u_{\nu)} + C(t)u_\alpha u_\mu u_\nu \,. \end{aligned} \tag{A30}$$

The functions $X(t)$, $Y(t)$, $Z(t)$, $V(t)$, $W(t)$ in (A28) and $\Phi(t)$, $P(t)$, $A(t)$, $B(t)$, $C(t)$ in (A30) describe non-Riemannian cosmological effects and give, together with the scale factor, the cosmic evolution of non-Riemannian geometries. Moreover, recalling (A2) and plugging (A28) and (A30) into (A3), one can prove that the following linear relations hold among the functions introduced above:

$$2(X+Y) = B \,, \quad 2Z = A \,, \quad 2V = C \,, \quad 2\Phi = Y - Z \,, \quad P = W \,, \tag{A31}$$

which may also be inverted, obtaining

$$W = P \,, \quad V = \frac{C}{2} \,, \quad Z = \frac{A}{2} \,, \quad Y = 2\Phi + \frac{A}{2} \,, \quad X = \frac{B}{2} - 2\Phi - \frac{A}{2} \,. \tag{A32}$$

Therefore, one can prove that the torsion and nonmetricity scalars defined in (A14) become, respectively,

$$\begin{aligned} T &= 24\Phi^2 - 6P^2\,, \\ Q &= \frac{3}{4}\Big[2A^2 + B(C-A)\Big]\,. \end{aligned} \tag{A33}$$

Moreover, let us also mention that using (A30) into the explicit expression of the Palatini tensor in terms of torsion and nonmetricity, i.e., (A8), we obtain the cosmological expression of the Palatini tensor, which reads

$$\begin{aligned} P_{\alpha\mu\nu} = {} & \left(\frac{1}{2}A + 4\Phi - \frac{C}{2}\right)u_\alpha h_{\mu\nu} + \left(B - \frac{3}{2}A - 4\Phi - \frac{C}{2}\right)u_\mu h_{\alpha\nu} - \frac{B}{2}u_\nu h_{\mu\alpha} - \frac{3}{2}Bu_\alpha u_\mu u_\nu \\ & - 2\varepsilon_{\alpha\mu\nu\rho}u^\rho P\,, \end{aligned} \tag{A34}$$

or, in the form we use in the main text of this paper,

$$\begin{aligned} P_\lambda{}^{\mu\nu} = {} & \left(\frac{1}{2}A + 4\Phi - \frac{C}{2}\right)u_\lambda h^{\mu\nu} + \left(B - \frac{3}{2}A - 4\Phi - \frac{C}{2}\right)u^\mu h_\lambda{}^\nu - \frac{B}{2}u^\nu h^\mu{}_\lambda - \frac{3}{2}Bu_\lambda u^\mu u^\nu \\ & - 2\varepsilon_\lambda{}^{\mu\nu}{}_\rho u^\rho P\,. \end{aligned} \tag{A35}$$

Consequently, one can prove that the following relations hold:

$$\begin{aligned} h^{\alpha\mu}P_{\alpha\mu\nu} &= -\frac{3}{2}Bu_\nu\,, \\ h^{\alpha\nu}P_{\alpha\mu\nu} &= 3\left(B - \frac{3}{2}A - 4\Phi - \frac{C}{2}\right)u_\mu\,, \\ h^{\mu\nu}P_{\alpha\mu\nu} &= 3\left(\frac{1}{2}A + 4\Phi - \frac{C}{2}\right)u_\alpha\,, \\ \varepsilon^{\alpha\mu\nu\lambda}P_{\alpha\mu\nu} &= 12Pu^\lambda\,, \\ u^\alpha u^\mu u^\nu P_{\alpha\mu\nu} &= \frac{3}{2}B\,. \end{aligned} \tag{A36}$$

Finally, using (A33), we find that the scalar curvature R, once decomposed in its Riemannian and non-Riemannian parts (see Equation (A13)), acquires the following form:

$$R = \tilde{R} + 6\left[\frac{1}{4}A^2 + 4\Phi^2 + \Phi(2A - B)\right] + \frac{3}{4}B(C-A) - 6P^2 + \frac{3}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}u^\mu\left(\frac{B}{2} - A - 4\Phi\right)\right], \tag{A37}$$

where

$$\tilde{R} = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] \tag{A38}$$

is the Ricci scalar of the Levi-Civita connection $\tilde{\Gamma}^\lambda{}_{\mu\nu}$.

Appendix C. Cosmological Aspects of the Metric-Affine $F(T)$ Theory with a Scalar Field Coupled to Torsion

In this appendix we take the model of Section 10.1 and restrict ourselves to the subcase $F = R + \beta T \to F = T$ in the presence of a (free) scalar field coupled to torsion, namely we consider the action

$$\mathcal{S} = \frac{1}{2\kappa}\int\sqrt{-g}d^4x\left[T + 2\kappa\left(-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + \lambda_0 S^\mu\nabla_\mu\phi\right)\right]. \tag{A39}$$

The variation of (A39) with respect to the scalar field ϕ yields

$$\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}(\partial^\mu\phi - \lambda_0 S^\mu)\right] = 0\,. \tag{A40}$$

The hypermomentum tensor is given by (138) and, varying the action (A39) with respect to the general affine connection $\Gamma^\lambda{}_{\mu\nu}$, we obtain

$$2\left(S^{\mu\nu}{}_\lambda - 2S_\lambda{}^{[\mu\nu]} - 4S^{[\mu}\delta_\lambda^{\nu]}\right) = 2\kappa\lambda_0\delta_\lambda^{[\mu}\nabla^{\nu]}\phi\,. \tag{A41}$$

Taking the different contractions of (A41), one can prove that the latter is solved by

$$\begin{aligned} S^\mu &= \frac{3\kappa\lambda_0}{8}\partial^\mu\phi\,, \\ t_\lambda &= 0\,, \quad Z_\lambda{}^{\mu\nu} = 0\,. \end{aligned} \tag{A42}$$

Hence, only the torsion vector survives, and it is a pure gauge, while the torsion pseudo-vector and the traceless part of the torsion vanish. In particular, the first of (A42) indicates that it is the presence of the scalar field that produces spacetime torsion in this model. This is so because we have a non-vanishing hypermomentum.

Plugging this result back into the action (A39), we end up with

$$\mathcal{S} = \frac{1}{2\kappa}\int\sqrt{-g}d^4x\left[T - \kappa\left(1 - \frac{3\kappa\lambda_0^2}{4}\right)g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\right]. \tag{A43}$$

Interestingly, from Equation (A43) we can conclude that the interaction between the scalar and torsion modifies the factor of the kinetic term for the scalar field. Observe also that there is a peculiar value of the coupling constant, namely $|\lambda_0| = 2\sqrt{\frac{1}{2\kappa}}$, for which the kinetic term of the scalar disappears from (A43). Of course, we shall disregard this trivial case in the following.

Up to this point, the above considerations were general for the model at hand. We shall now focus on the homogeneous FLRW cosmology of this theory (c.f. Appendix B for the setup, definitions and useful formulas). Comparing the first equation of (A42) with the first of (A30), we immediately see that in this case, $P(t) = 0$. Hence, we find that the full torsion tensor is given by

$$S_{\mu\nu\alpha} = 2u_{[\mu}h_{\nu]\alpha}\Phi\,, \tag{A44}$$

with

$$\Phi = \Phi(t) = -\frac{\kappa\lambda_0}{8}\dot{\phi}\,. \tag{A45}$$

Moreover, inserting the first of (A42) into Equation (A40), we obtain

$$\left(1 - \frac{3\kappa\lambda_0^2}{4}\right)\partial_\mu\left[\sqrt{-g}\partial^\mu\phi\right] = 0\,, \tag{A46}$$

which for $|\lambda_0| \neq 2\sqrt{\frac{1}{2\kappa}}$, implies (148).

On the other hand, the metric field equations of the theory (147) read

$$-\frac{1}{2}g_{\mu\nu}T + \left(2S_{\nu\alpha\beta}S_\mu{}^{\alpha\beta} - S_{\alpha\beta\mu}S^{\alpha\beta}{}_\nu + 2S_{\nu\alpha\beta}S_\mu{}^{\beta\alpha}\right) = \kappa T_{\mu\nu}\,. \tag{A47}$$

Introducing the usual form of the energy-momentum tensor, i.e., (150), associated with the scalar field Lagrangian, taking the trace of (A47), and using Equation (A44), we obtain (recall that we are not dealing with a gauge theory of gravity)

$$\left[\left(\frac{\kappa\lambda_0}{4}\right)^2+\frac{\kappa}{6}\right]\dot{\phi}^2=0\,, \tag{A48}$$

which discarding the (trivial) case $\dot{\phi}^2=0$, is solved by

$$\lambda_0=\pm 2\mathrm{i}\sqrt{\frac{2}{3\kappa}}\,. \tag{A49}$$

Let us also observe that, contracting Equation (A47) with $u^\mu u^\nu$, we find the same Equation (A48). Thus, in the current model one is left with an imaginary coupling constant λ_0.

The final form of the (metric but torsionful) affine connection for the model at hand reads as follows:

$$\Gamma^\lambda{}_{\mu\nu}=\tilde{\Gamma}^\lambda{}_{\mu\nu}-\frac{2}{3}g_{\mu\nu}S^\lambda+\frac{2}{3}\delta^\lambda_\nu S_\mu\,, \tag{A50}$$

where $\tilde{\Gamma}^\lambda{}_{\mu\nu}$ is the Levi-Civita connection. Please note that using (A44), the final form of the connection in (A50) can be rewritten as

$$\Gamma^\lambda{}_{\mu\nu}=\tilde{\Gamma}^\lambda{}_{\mu\nu}+X(t)u^\lambda h_{\mu\nu}+Y(t)u_\mu h^\lambda{}_\nu\,, \tag{A51}$$

with

$$X(t)=-Y(t)=-2\Phi(t)\,. \tag{A52}$$

Finally, exploiting (A45), we are left with

$$\Gamma^\lambda{}_{\mu\nu}=\tilde{\Gamma}^\lambda{}_{\mu\nu}+\frac{\kappa\lambda_0}{4}\dot{\phi}\left(u^\lambda h_{\mu\nu}-u_\mu h^\lambda{}_\nu\right), \tag{A53}$$

written in terms of the temporal derivative of the scalar, $\dot{\phi}$. In fact, in this case, it is the scalar field that produces spacetime torsion (cf. the first equation in (A42)).

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