

Cosmology of quadratic metric-affine gravity

Original

Cosmology of quadratic metric-affine gravity / Damianos, I., Ravera, L.. - In: PHYSICAL REVIEW D. - ISSN 2470-0010. - STAMPA. - 105:2(2022), pp. 1-24. [10.1103/PHYSREVD.105.024007]

Availability:

This version is available at: 11583/2949076 since: 2022-01-28T17:44:01Z

Publisher:

American Physical Society

Published

DOI:10.1103/PHYSREVD.105.024007

Terms of use:

This article is made available under terms and conditions as specified in the corresponding bibliographic description in the repository

Publisher copyright

(Article begins on next page)

Cosmology of quadratic metric-affine gravity

Damianos Iosifidis^{1,*} and Lucrezia Ravera^{2,3,†}

¹*Institute of Theoretical Physics, Department of Physics, Aristotle University of Thessaloniki, 54124 Thessaloniki, Greece*

²*DISAT, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy*

³*INFN, Sezione di Torino, Via P. Giuria 1, 10125 Torino, Italy*



(Received 8 November 2021; accepted 14 December 2021; published 4 January 2022)

We investigate the cosmological aspects of the most general parity preserving Metric-Affine Gravity theory quadratic in torsion and nonmetricity in the presence of a cosmological hyperfluid. The equations of motion are obtained by varying the action with respect to the metric and the independent affine connection. Subsequently, considering a Friedmann-Lemaître-Robertson-Walker background, we derive the most general form of the modified Friedmann equations for the full quadratic theory. We then focus on a characteristic subcase involving only two quadratic contributions given in terms of torsion and nonmetricity vectors. In this setup, studying the modified Friedmann equations along with the conservation laws of the perfect cosmological hyperfluid, we provide exact solutions both for purely dilation and for purely spin hypermomentum sources. We then discuss the physical consequences of our model and the prominent role of torsion and nonmetricity in this cosmological setup.

DOI: [10.1103/PhysRevD.105.024007](https://doi.org/10.1103/PhysRevD.105.024007)

I. INTRODUCTION

Modified gravity models [1] beyond general relativity (GR) have received considerable attention in recent years, as they might provide solutions to both small and large scale open issues that still afflict Einstein’s well-celebrated theory of gravity. GR is based on Riemannian geometry and, on the other hand, proposed generalizations and extensions of GR have their roots in non-Riemannian geometry [2], where the assumptions of metric compatibility and torsionlessness of the affine connection are released. In other words, in this framework nonvanishing torsion and nonmetricity are allowed, both having a clear geometric interpretation. The former ensures that no infinitesimal parallelogram can be formed (i.e., they crack into pentagons), while the latter causes dot products and lengths of vectors to change under parallel transport in spacetime.

A prominent class of theories developed in the non-Riemannian setup goes by the name of metric-affine gravity (MAG) [3–5]. The latter can be formulated as a “gauge” theory of gravity [3,4], but this is not mandatory, as the development of gravitational models out of the gauge realm already has turned out to be relevant (cf. e.g., [5–8]). In this context, the metric and the connection can be treated as independent objects without assuming any form or symmetry or compatibility condition for the general affine

connection from the very beginning. The latter is eventually obtained from the study of the field equations derived in the first order (i.e., Palatini) formalism.¹ Another characteristic feature of MAG is the fact that the matter Lagrangian is in principle allowed to depend on the general affine connection as well. The coupling is encoded into the so-called hypermomentum tensor, which encompasses the microstructure of matter [3,13,14]. In particular, it describes dilation, spin, and shear sources. This last intriguing characteristic of MAG, namely the microstructure-extended geometry interrelation, makes it especially interesting.

Under the perspective described above, MAG theories result to be rather general and powerful [11,12,15–23], and can possibly lead to diverse scenarios beyond GR, which may have relevant consequences, for instance, in the cosmological context. For some relevant results already obtained in MAG cosmology we refer the reader to, e.g., [24–34]. In particular, in this paper we are interested in the cosmological aspects, in a Friedmann-Lemaître-Robertson-Walker (FLRW) background, of the most general MAG theory in n spacetime dimensions involving all possible parity preserving quadratic terms in torsion and

¹The literature on the topic is huge. For a recent review of Einstein manifolds with torsion and nonmetricity see, e.g., [9]. Besides, let us also mention that Einstein-Cartan-Weyl spaces in three dimensions, involving torsion and nonmetricity, have recently proved to describe the near-horizon geometry of some supergravity BPS black holes solutions [10]. Furthermore, the $(3 + 1)$ formulation for gravity with torsion and nonmetricity has been recently developed in [11,12].

*diosifid@auth.gr

†lucrezia.ravera@polito.it

nonmetricity, in the presence of a cosmological hyperfluid. The latter is a classical continuous medium carrying hypermomentum [35–39]. A novel, general formulation of perfect hyperfluid generalizing the classical perfect fluid notion of GR has been recently presented in [40–42] by first giving its physical definition and later using the appropriate mathematical formulation in order to extract its energy tensors by demanding spatial isotropy. We will refer to the aforementioned perfect cosmological hyperfluid in this work.

Quadratic MAG theories have been previously considered in [43–52]. More precisely, in [47] a quadratic theory involving also a piece quadratic in the so-called homothetic curvature tensor was studied, while in [48] only contributions quadratic in the torsion tensor were taken into account. In the cosmological context, in [53] the cosmology of a quadratic metric-compatible torsionful gravity theory in the presence of a perfect hyperfluid was presented and analyzed for the first time. The current work also provides a generalization of the results presented in [53] to the case in which both torsion and nonmetricity are nonvanishing.

The remainder of this paper is structured as follows: In Sec. II we review the theoretical background of MAG, cosmological aspects of torsion and nonmetricity, and the perfect cosmological hyperfluid. In Sec. III we study the most general parity preserving MAG theory quadratic in torsion and nonmetricity in the presence of a perfect hyperfluid. The gravitational part of the action is given by the (non-Riemannian) Einstein-Hilbert contribution plus all possible parity preserving torsion and nonmetricity scalars. We derive the field equations in the first order formalism, treating the metric and the general affine connection as independent variables. Section IV is devoted to the study of the cosmology of the quadratic theory. First of all, we consider a FLRW background and obtain the modified Friedmann equations of the general quadratic theory. Then, we focus on a characteristic subcase involving only two quadratic contributions given in terms of torsion and nonmetricity vectors and, in this setup, from the study of the modified Friedmann equations and the conservation laws of the perfect cosmological hyperfluid, we are able to provide exact solutions in the cases of purely dilation and purely spin hypermomentum types. We conclude our work with some remarks and possible future developments. Useful formulas are collected in the Appendix.

II. THEORETICAL BACKGROUND

In this section we briefly give all the theoretical background necessary for a clearer understanding of the present work.

A. Geometric setup

Let us first review the geometric setup. We consider the framework of non-Riemannian geometry, and accordingly

introduce on the manifold a metric tensor $g_{\mu\nu}$ and a general affine connection ∇ with coefficients $\Gamma^\lambda_{\mu\nu}$. These geometric objects will be treated as independent, *a priori*. We consider n spacetime dimensions and our convention for the metric signature is mostly plus. The generic decomposition of a general affine connection is

$$\Gamma^\lambda_{\mu\nu} = \tilde{\Gamma}^\lambda_{\mu\nu} + N^\lambda_{\mu\nu}, \quad (1)$$

where

$$\tilde{\Gamma}^\lambda_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \quad (2)$$

is the Levi-Civita connection and

$$N^\lambda_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} (\underbrace{Q_{\mu\nu\rho} + Q_{\nu\rho\mu} - Q_{\rho\mu\nu}}_{\text{deflection (or disformation)}}) - \underbrace{g^{\rho\lambda} (S_{\rho\mu\nu} + S_{\rho\nu\mu} - S_{\mu\nu\rho})}_{\text{contorsion} := K^\lambda_{\mu\nu}} \quad (3)$$

is the distortion tensor, given in terms of torsion,

$$S_{\mu\nu}{}^\lambda := \Gamma^\lambda_{[\mu\nu]}, \quad S_{\mu\alpha} = N_{\alpha[\mu\nu]}, \quad (4)$$

and nonmetricity

$$\begin{aligned} Q_{\lambda\mu\nu} &:= -\nabla_\lambda g_{\mu\nu} = -\partial_\lambda g_{\mu\nu} + \Gamma^\rho_{\mu\lambda} g_{\rho\nu} + \Gamma^\rho_{\nu\lambda} g_{\mu\rho}, \\ Q_{\nu\alpha\mu} &= 2N_{(\alpha\mu)\nu}. \end{aligned} \quad (5)$$

Both torsion and nonmetricity have trace parts: $S_\lambda := S_{\lambda\sigma}{}^\sigma$ is the torsion vector, $Q_\lambda := Q_{\lambda\mu}{}^\mu$ and $q_\nu := Q^\mu_{\nu\mu}$ are the nonmetricity vectors, with the former oftentimes referred to as the Weyl vector. Furthermore, in the case $n = 4$ one can also define the torsion pseudovector according to $t^\rho := \epsilon^{\rho\lambda\mu\nu} S_{\lambda\mu\nu}$, where $\epsilon^{\rho\lambda\mu\nu}$ is the Levi-Civita tensor in four dimensions. The covariant derivative ∇ (associated with Γ) of, e.g., a vector v^λ is given by

$$\nabla_\nu v^\lambda = \partial_\nu v^\lambda + \Gamma^\lambda_{\mu\nu} v^\mu. \quad (6)$$

We define the curvature (Riemann) tensor as

$$\begin{aligned} R^\mu{}_{\nu\alpha\beta} &:= 2\partial_{[\alpha}\Gamma^\mu{}_{\nu|\beta]} + 2\Gamma^\mu{}_{\rho[\alpha}\Gamma^\rho{}_{\nu|\beta]} \\ &= \tilde{R}^\mu{}_{\nu\alpha\beta} + 2\tilde{\nabla}_{[\alpha}N^\mu{}_{\nu|\beta]} + 2N^\mu{}_{\lambda[\alpha}N^\lambda{}_{\nu|\beta]}, \end{aligned} \quad (7)$$

where $\tilde{\nabla}$ denotes the Levi-Civita covariant derivative. In particular, one can form the following contractions of the curvature tensor $R^\mu{}_{\nu\alpha\beta}$:

$$R_{\nu\beta} := R^\mu{}_{\nu\mu\beta}, \quad (8)$$

$$\hat{R}_{\alpha\beta} := R^\mu{}_{\mu\alpha\beta} = \partial_{[\alpha}Q_{\beta]}, \quad (9)$$

$$\check{R}^\lambda{}_\alpha := R^\lambda{}_{\mu\nu\alpha} g^{\mu\nu}, \quad (10)$$

where (8) defines the Ricci tensor of Γ , (9) the so-called homothetic curvature tensor, and (10) a third tensor referred to as the co-Ricci tensor in the literature. Finally,

$$R := R_{\mu\nu} g^{\mu\nu} = -\check{R}_{\mu\nu} g^{\mu\nu} \quad (11)$$

is the curvature scalar associated with Γ which is still uniquely defined. Note that the latter can be decomposed into its Riemannian part plus non-Riemannian contributions associated with torsion and nonmetricity. Having introduced the minimum theoretical setup for the formalism we are going to be using we can now move on to the physics part and elaborate a little bit on the sources of MAG.

B. Hypermomentum and energy-momentum tensors

Let us now recall the concepts of energy-momentum and hypermomentum tensors, following [40]. We will assume the full action to be a functional of the metric (and its derivatives), the general affine connection and the matter fields (here collectively denoted by φ), that is

$$S[g, \Gamma, \varphi] = S_G[g, \Gamma] + S_M[g, \Gamma, \varphi]. \quad (12)$$

In particular,

$$S_G[g, \Gamma] = \frac{1}{2\kappa} \int d^n x \sqrt{-g} \mathcal{L}_G(g, \Gamma) \quad (13)$$

is the gravitational part of the action ($\kappa = 8\pi G$ is the gravitational constant) and

$$S_M[g, \Gamma, \varphi] = \int d^n x \sqrt{-g} \mathcal{L}_M(g, \Gamma, \varphi) \quad (14)$$

the matter part. One can then define the metric energy-momentum tensor (MEMT),

$$T_{\mu\nu} := -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} \mathcal{L}_M)}{\delta g^{\mu\nu}}, \quad (15)$$

and the hypermomentum tensor [3,13,14],

$$\Delta_\lambda{}^{\mu\nu} := -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta \Gamma^\lambda{}_{\mu\nu}} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} \mathcal{L}_M)}{\delta \Gamma^\lambda{}_{\mu\nu}}. \quad (16)$$

On the other hand, working in the equivalent formalism based on the vielbeins $e_\mu{}^c$ and spin connection $\omega_{\mu|ab}$, where a, b, \dots are Lorentz (i.e., tangent) indices,² one may also introduce the canonical energy-momentum tensor (CEMT),

²Here we have the usual relation $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}$ connecting metric and vielbeins, η_{ab} being the tangent space flat Minkowski metric.

$$t^\mu{}_c := \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta e_\mu{}^c}. \quad (17)$$

Note that, while the MEMT is symmetric, the CEMT, in general, is not. One can then prove that the CEMT is not independent of the metric energy-momentum and hypermomentum tensors [3], as the following relation holds [40]³:

$$t^\mu{}_\lambda := \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta e_\lambda{}^c} e_\lambda{}^c = T^\mu{}_\lambda - \frac{1}{2\sqrt{-g}} \hat{\nabla}_\nu (\sqrt{-g} \Delta_\lambda{}^{\mu\nu}), \quad (18)$$

where we have also defined a modified covariant derivative

$$\hat{\nabla}_\nu := 2S_\nu - \nabla_\nu. \quad (19)$$

Observe that for matter such that $\Delta_{\alpha\mu\nu} \equiv 0$ (namely, for matter with no microstructure) the CEMT and MEMT coincide. As we will also recall in the following, Eq. (18) turns out to be a conservation law of the cosmological hyperfluid [40]. Moreover, let us notice that from Eq. (18) one can derive the conservation law for spin [54],

$$2t_{[\mu\nu]} = \frac{1}{\sqrt{-g}} \hat{\nabla}_\alpha (\sqrt{-g} \tau_{\mu\nu}{}^\alpha) - Q_{\alpha\beta[\mu} \Delta_{\nu]}{}^{\beta\alpha}, \quad (20)$$

with

$$\tau_{\mu\nu}{}^\alpha := \Delta_{[\mu\nu]}{}^\alpha. \quad (21)$$

As we can see, the conservation law for spin receives, in general, contributions from nonmetricity. Let us also note here that in a homogeneous cosmological setup, any antisymmetric two-index object vanishes identically. This means that in a homogeneous cosmological setting $t_{[\mu\nu]} \equiv 0$ and, therefore, one is left only with the symmetric part of $t_{\mu\nu}$, that is $t_{\mu\nu} = t_{[\mu\nu]}$. Finally, contracting Eq. (18) in μ, λ , we find the following relation:

$$t = T + \frac{1}{2\sqrt{-g}} \partial_\nu (\sqrt{-g} \Delta^\nu), \quad (22)$$

where

$$t := t^\mu{}_\mu, \quad T := T^\mu{}_\mu, \quad \Delta^\nu := \Delta_\lambda{}^{\lambda\nu} \quad (23)$$

are the CEMT and MEMT traces and a trace of the hypermomentum, respectively. The above equation implies

³In the explicit calculations one also exploits the identity $\nabla_\nu e_\mu{}^a = 0 = \partial_\nu e_\mu{}^a - \Gamma^\rho{}_{\mu\nu} e_\rho{}^a + \omega_\nu{}^a{}_b e_\mu{}^b$, which connects the two formalisms.

$$T = 0 \leftrightarrow 2t = \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} \Delta^\nu), \quad (24)$$

$$t = 0 \leftrightarrow 2T = -\frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} \Delta^\nu), \quad (25)$$

$$t = T \leftrightarrow \partial_\nu (\sqrt{-g} \Delta^\nu) = 0, \quad (26)$$

which can follow from considering specific matter and correspond, respectively, to conformally frame rescalings, and special projective transformations invariant theories (cf. also [55]). From the latter equation it immediately follows that for theories that respect the special projective symmetry⁴ the associated dilation current is always conserved. This is true for all classes of theories whose gravitational part depends only on the Riemann tensor.

C. Cosmological aspects of torsion and nonmetricity

In the following we recall key cosmological aspects of torsion and nonmetricity, which will be useful in the remainder of this paper. First of all, we will consider a homogeneous, flat (i.e., $K = 0$, K being a curvature parameter) FLRW spacetime with the usual Robertson-Walker line element

$$ds^2 = -dt^2 + a^2 \delta_{ij} dx^i dx^j, \quad (27)$$

where $a(t)$ is the scale factor of the Universe and $i, j = 1, 2, \dots, n-1$. In addition, we define the projector tensor

$$h_{\mu\nu} := g_{\mu\nu} + u_\mu u_\nu, \quad (28)$$

which projects objects on the space orthogonal to u^μ , the latter being the normalized n -velocity field of a given fluid which in comoving coordinates is expressed as $u^\mu = \delta_0^\mu = (1, 0, 0, \dots, 0)$, $u_\mu u^\mu = -1$. Accordingly, let us also introduce the temporal derivative

$$\dot{} = u^\alpha \nabla_\alpha. \quad (29)$$

The above constitutes a $1 + (n-1) = 1 + m$ space-time split.

1. Torsion and nonmetricity in FLRW spacetime

In a non-Riemannian FLRW spacetime in $1 + 3$ dimensions the general affine connection can be written as [40]

$$\Gamma^\lambda_{\mu\nu} = \tilde{\Gamma}^\lambda_{\mu\nu} + X(t) u^\lambda h_{\mu\nu} + Y(t) u_\mu h^\lambda_\nu + Z(t) u_\nu h^\lambda_\mu + V(t) u^\lambda u_\mu u_\nu + \epsilon^\lambda_{\mu\nu\rho} u^\rho W(t). \quad (30)$$

In particular, the nonvanishing components of the Levi-Civita connection are

⁴Also known as Einstein's λ transformations.

$$\tilde{\Gamma}^0_{ij} = \tilde{\Gamma}^0_{ji} = \dot{a} a \delta_{ij} = H g_{ij}, \quad \tilde{\Gamma}^i_{j0} = \tilde{\Gamma}^i_{0j} = \frac{\dot{a}}{a} \delta^i_j = H \delta^i_j, \quad (31)$$

where $H := \frac{\dot{a}}{a}$ is the Hubble parameter. On the other hand, the torsion and nonmetricity tensors can be written, respectively, in the following way [40,56]:

$$S_{\mu\nu\alpha}^{(n)} = 2u_{[\mu} h_{\nu]\alpha} \Phi(t) + \epsilon_{\mu\nu\alpha\rho} u^\rho P(t) \delta_4^n, \\ Q_{\alpha\mu\nu} = A(t) u_\alpha h_{\mu\nu} + B(t) h_{\alpha(\mu} u_{\nu)} + C(t) u_\alpha u_\mu u_\nu, \quad (32)$$

where δ_4^n is the Kronecker's delta ($\delta_4^{n=4} = 1$, otherwise it gives zero). Given (32), we may also compute

$$S_\alpha = (n-1) \Phi u_\alpha, \\ Q_\mu = (n-1) A u_\mu - C u_\mu, \\ q_\mu = \frac{(n-1)}{2} B u_\mu - C u_\mu. \quad (33)$$

The functions $X(t), Y(t), Z(t), V(t), W(t)$ in (30) and $\Phi(t), P(t), A(t), B(t), C(t)$ in (32) describe non-Riemannian cosmological effects and give, together with the scale factor, the cosmic evolution of non-Riemannian geometries.⁵ Moreover, using the definition of a general affine connection we have previously introduced, one can prove that

$$2(X+Y) = B, \quad 2Z = A, \quad 2V = C, \\ 2\Phi = Y - Z, \quad P = W. \quad (34)$$

The latter can also be inverted to get

$$W = P, \quad V = \frac{C}{2}, \quad Z = \frac{A}{2}, \\ Y = 2\Phi + \frac{A}{2}, \quad X = \frac{B}{2} - 2\Phi - \frac{A}{2}. \quad (35)$$

Here, let us also collect some useful formulas we have derived in the current cosmological context.

Torsion scalars.—Using the first of (32), we find the following relations⁶:

$$S_{\mu\nu\alpha} S^{\mu\nu\alpha} = -2(n-1) \Phi^2 + 6P^2 \delta_4^n, \\ S_{\mu\nu\alpha} S^{\alpha\mu\nu} = (n-1) \Phi^2 + 6P^2 \delta_4^n, \\ S_\mu S^\mu = -(n-1)^2 \Phi^2. \quad (36)$$

⁵Notice that $P(t)$ is a pseudoscalar and $\epsilon_{\mu\nu\alpha\beta} u^\beta$ is, by definition, purely spatial.

⁶ $P = P(t)$ is there only in $n = 4$ ($n = 1 + 3$, $m = 3$) space-time dimensions. For $m \neq 3$ it identically vanishes, i.e., $P \equiv 0$.

Besides, one might introduce the torsion scalar \mathcal{T} , given by

$$\mathcal{T} := S_{\mu\nu\alpha}S^{\mu\nu\alpha} - 2S_{\mu\nu\alpha}S^{\alpha\mu\nu} - 4S_\mu S^\mu, \quad (37)$$

and prove that

$$\mathcal{T} = 4(n-1)(n-2)\Phi^2 - 6P^2\delta_4^n \quad (38)$$

in our cosmological setup.

Mixed torsion-nonmetricity scalars.—Regarding mixed terms, we obtain

$$\begin{aligned} Q_{\mu\nu\alpha}S^{\mu\nu\alpha} &= -(n-1)A\Phi + \frac{(n-1)}{2}B\Phi, \\ Q_\mu S^\mu &= (n-1)\Phi[C - (n-1)A], \\ q_\mu S^\mu &= (n-1)\Phi\left[C - \frac{(n-1)}{2}B\right]. \end{aligned} \quad (39)$$

One could then define a $\mathcal{Q} * \mathcal{T}$ mixed torsion-nonmetricity scalar

$$\mathcal{Q} * \mathcal{T} := 2Q_{\alpha\mu\nu}S^{\alpha\mu\nu} + 2S_\mu(q^\mu - Q^\mu) \quad (40)$$

and prove that we have

$$\mathcal{Q} * \mathcal{T} = (n-1)(n-2)\Phi(2A - B) \quad (41)$$

in the current cosmological setting.

Nonmetricity scalars.—Concerning pure nonmetricity scalars, we compute

$$\begin{aligned} X_1 &:= Q_{\alpha\mu\nu}Q^{\alpha\mu\nu} = -\left[(n-1)A^2 + \frac{(n-1)}{2}B^2 + C^2\right], \\ X_2 &:= Q_{\alpha\mu\nu}Q^{\mu\nu\alpha} = -\left[(n-1)AB + \frac{(n-1)}{4}B^2 + C^2\right], \\ X_3 &:= Q_\alpha Q^\alpha = -[(n-1)A - C]^2, \\ X_4 &:= Q_\alpha q^\alpha = -[(n-1)A - C]\left[\frac{(n-1)}{2}B - C\right], \\ X_5 &:= q_\alpha q^\alpha = -\left[\frac{(n-1)}{2}B - C\right]^2. \end{aligned} \quad (42)$$

Finally, defining

$$\mathcal{Q} := \frac{1}{4}Q_{\alpha\mu\nu}Q^{\alpha\mu\nu} - \frac{1}{2}Q_{\alpha\mu\nu}Q^{\mu\nu\alpha} - \frac{1}{4}Q_\mu Q^\mu + \frac{1}{2}Q_\mu q^\mu, \quad (43)$$

and, consequently,

$$\mathcal{Z} := \mathcal{T} + \mathcal{Q} + \mathcal{Q} * \mathcal{T}, \quad (44)$$

we have that the Ricci scalar of Γ can be written as

$$R = \tilde{R} + \mathcal{Z} + \tilde{\nabla}_\mu(q^\mu - Q^\mu - 4S^\mu). \quad (45)$$

Then, as

$$\begin{aligned} \tilde{\nabla}_\mu(q^\mu - Q^\mu - 4S^\mu) \\ = (n-1)\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}g^{\mu\nu}\left(\frac{B}{2} - A - 4\Phi\right)\right], \end{aligned} \quad (46)$$

where we have used

$$\tilde{\nabla}_\mu(\sqrt{-g}\xi^\mu) = \partial_\mu(\sqrt{-g}\xi^\mu), \quad \forall \text{ vector } \xi^\mu, \quad (47)$$

we find

$$\begin{aligned} R &= \tilde{R} + \frac{(n-1)}{4}[(n-2)A^2 - (n-3)AB + BC] \\ &\quad + (n-1)(n-2)(4\Phi^2 + \Phi(2A - B)) - 6P^2\delta_4^n \\ &\quad + (n-1)\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}g^{\mu\nu}\left(\frac{B}{2} - A - 4\Phi\right)\right], \end{aligned} \quad (48)$$

which is the final cosmological decomposition of the Ricci scalar of Γ in terms of Riemannian (i.e., \tilde{R}) and non-Riemannian contributions. This decomposition is a key ingredient for the derivation of the modified Friedmann equations we are going to present in what follows.

2. Relations among the invariants

Note that in the highly symmetric FLRW background not all of the 11 quadratic torsion and nonmetricity invariants are independent. Regarding the pure torsion invariants, since as we have seen above they involve only two combinations and we have three scalars [see (36)], the latter must be dependent. In fact, quite trivially we find

$$(n-1)S_{\mu\nu\alpha}S^{\mu\nu\alpha} - (n-1)S_{\mu\nu\alpha}S^{\alpha\mu\nu} - 3S_\mu S^\mu = 0 \quad (49)$$

confirming that indeed only two out of the three pure torsion scalars are independent. Analogously, concerning the mixed terms we observe that the difference of the last two in (39) is $(n-1)$ times the first one, namely

$$Q_\mu S^\mu - q_\mu S^\mu - (n-1)Q_{\mu\nu\alpha}S^{\mu\nu\alpha} = 0, \quad (50)$$

indicating again their linear dependence. More tricky and complicated are the pure nonmetricity scalars in (42), as here we have five expressions but six combinations appearing. Then, considering their linear combinations, if the equation

$$\sum_{i=1}^5 \lambda_i X_i = 0 \quad (51)$$

has only the trivial solution $\lambda_i = 0$ for all i , then the invariants are independent. If not all λ 's vanish, then the

invariants will be dependent. In our case it happens that only four equations are linearly independent, so we have a free parameter, which we take to be λ_3 , and

$$\begin{aligned}\lambda_1 &= -\lambda_3(n-1), & \lambda_2 &= \lambda_3(n-1), \\ \lambda_4 &= -2\lambda_3, & \lambda_5 &= \lambda_3.\end{aligned}\quad (52)$$

Now, setting $\lambda_3 = 1$, the latter boils down to

$$\lambda_3 = 1, \quad \lambda_1 = 1 - n, \quad \lambda_2 = n - 1, \quad \lambda_4 = -2, \quad \lambda_5 = 1 \quad (53)$$

and we find that the nonmetricity scalars obey the following constraint in n spacetime dimensions:

$$(1-n)X_1 + (n-1)X_2 + X_3 - 2X_4 + X_5 = 0, \quad (54)$$

that is, only four out of the five nonmetricity scalars are independent. Note that in the case $n = 4$ we get

$$\lambda_1 = -3\lambda_3, \quad \lambda_2 = 3\lambda_3, \quad \lambda_4 = -2\lambda_3, \quad \lambda_5 = \lambda_3, \quad (55)$$

that is, setting $\lambda_3 = 1$,

$$\lambda_3 = 1, \quad \lambda_1 = -3, \quad \lambda_2 = 3, \quad \lambda_4 = -2, \quad \lambda_5 = 1. \quad (56)$$

Then, in $n = 4$ one is left with

$$-3X_1 + 3X_2 + X_3 - 2X_4 + X_5 = 0. \quad (57)$$

To recap, we have proved that in a cosmological background, out of the three torsion scalars only two are independent, out of the five nonmetricity scalars four are independent, and out of the mixed three torsion-nonmetricity scalars two are independent. That is, in a homogeneous cosmological setting, only 8 out of the 11 parity even quadratic scalars in torsion and nonmetricity are independent. This fact will be used later on in our analysis.

D. Perfect cosmological hyperfluid

A hyperfluid is a classical continuous medium carrying hypermomentum. The general formulation of perfect cosmological hyperfluid generalizing the classical perfect fluid notion can be found in Refs. [40,41]. In this paper we consider a perfect cosmological hyperfluid in a homogeneous cosmological setting, demanding also homogeneity. The perfect hyperfluid is described in terms of the following MEMT and CEMT tensors [40]:

$$T_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu}, \quad (58)$$

and

$$t_{\mu\nu} = \rho_c u_\mu u_\nu + p_c h_{\mu\nu}, \quad (59)$$

along with the hypermomentum tensor

$$\begin{aligned}\Delta_{\alpha\mu\nu}^{(n)} &= \phi(t)h_{\mu\alpha}u_\nu + \chi(t)h_{\nu\alpha}u_\mu + \psi(t)u_\alpha h_{\mu\nu} \\ &+ \omega(t)u_\alpha u_\mu u_\nu + \delta_4^n \epsilon_{\alpha\mu\nu\rho} u^\rho \zeta(t),\end{aligned}\quad (60)$$

where ρ and p are the usual density and pressure of the perfect fluid component of the hyperfluid, ρ_c and p_c are, respectively, the canonical (net) density and canonical pressure of the hyperfluid. In addition, the functions ϕ , χ , ψ , ω , ζ characterize the microscopic properties of the hyperfluid which, upon using the connection field equations, act as sources of the non-Riemannian background. The tensors (58), (59), and (60) all respect spatial isotropy and are subject to the following conservation laws⁷:

$$\frac{1}{\sqrt{-g}} \hat{\nabla}_\mu (\sqrt{-g} t^\mu{}_\alpha) = \frac{1}{2} \Delta^{\lambda\mu\nu} R_{\lambda\mu\nu\alpha} + \frac{1}{2} Q_{\alpha\mu\nu} T^{\mu\nu} + 2S_{\alpha\mu\nu} t^{\mu\nu}, \quad (61)$$

$$t^\mu{}_\lambda = T^\mu{}_\lambda - \frac{1}{2\sqrt{-g}} \hat{\nabla}_\nu (\sqrt{-g} \Delta_\lambda^{\mu\nu}). \quad (62)$$

Equation (61) follows from diffeomorphism invariance, while Eq. (62) originates from the $GL(n, \mathbb{R})$ invariance of the matter part of the action [40] when working in the exterior calculus formalism. They will be fundamental in the cosmological study of the theory we are going to introduce. Observe that, as already anticipated, (62) coincides with Eq. (18). Moreover, from (61) we can see that the canonical energy-momentum tensor naturally couples to torsion, while the metric one couples to nonmetricity. Note also that one can use the latter of the above equations in order to eliminate $t^{\mu\nu}$ from the former, yielding a variant conservation law

$$\begin{aligned}\sqrt{-g}(2\tilde{\nabla}_\mu T^\mu{}_\alpha - \Delta^{\lambda\mu\nu} R_{\lambda\mu\nu\alpha}) + \hat{\nabla}_\mu \hat{\nabla}_\nu (\sqrt{-g} \Delta_\alpha^{\mu\nu}) \\ + 2S_{\mu\alpha}{}^\lambda \hat{\nabla}_\nu (\sqrt{-g} \Delta_\lambda^{\mu\nu}) = 0\end{aligned}\quad (63)$$

from which one can clearly see the modifications the energy-momentum tensor receives (in comparison to GR) due to the intrinsic structure of matter (i.e., hypermomentum contributions).

III. THE QUADRATIC THEORY

We shall start with the most general parity preserving MAG theory quadratic in torsion and nonmetricity (cf. also [47]), which depends on 11 parameters, in the presence of a cosmological hyperfluid, that is

⁷Of course these conservation laws are fairly general and hold true irrespectively of the form of the fluid one considers.

$$\begin{aligned}
 S[g, \Gamma, \varphi] &= \frac{1}{2\kappa} \int d^n x \sqrt{-g} [R + b_1 S_{\alpha\mu\nu} S^{\alpha\mu\nu} + b_2 S_{\alpha\mu\nu} S^{\mu\nu\alpha} + b_3 S_\mu S^\mu \\
 &\quad + a_1 Q_{\alpha\mu\nu} Q^{\alpha\mu\nu} + a_2 Q_{\alpha\mu\nu} Q^{\mu\nu\alpha} + a_3 Q_\mu Q^\mu + a_4 q_\mu q^\mu + a_5 Q_\mu q^\mu \\
 &\quad + c_1 Q_{\alpha\mu\nu} S^{\alpha\mu\nu} + c_2 Q_\mu S^\mu + c_3 q_\mu S^\mu] + S_{\text{hyp}} \\
 &= \frac{1}{2\kappa} \int d^n x \sqrt{-g} [R + \mathcal{L}_2] + S_{\text{hyp}}.
 \end{aligned} \tag{64}$$

One can show that in vacuum the above theory reduces to GR. In [47] the inclusion of the homothetic curvature $\hat{R}_{\mu\nu}$ was considered to the above action, and there it was shown that in vacuum the theory is equivalent to GR plus a Proca field.⁸ However, in the case of FLRW cosmology that we are interested in, this term would vanish identically, so there is no need to include it in our subsequent discussion. In the following, it will be useful to introduce also the following ‘‘superpotentials’’ [5]:

$$\begin{aligned}
 \Omega^{\alpha\mu\nu} &:= a_1 Q^{\alpha\mu\nu} + a_2 Q^{\mu\nu\alpha} + a_3 g^{\mu\nu} Q^\alpha + a_4 g^{\alpha\mu} q^\nu + a_5 g^{\alpha\mu} Q^\nu, \\
 \Sigma^{\alpha\mu\nu} &:= b_1 S^{\alpha\mu\nu} + b_2 S^{\mu\nu\alpha} + b_3 g^{\mu\nu} S^\alpha, \\
 \Pi^{\alpha\mu\nu} &:= c_1 S^{\alpha\mu\nu} + c_2 g^{\mu\nu} S^\alpha + c_3 g^{\alpha\mu} S^\nu.
 \end{aligned} \tag{65}$$

Varying (64) with respect to the metric we derive

$$R_{(\mu\nu)} - \frac{R}{2} g_{\mu\nu} - \frac{\mathcal{L}_2}{2} g_{\mu\nu} + \frac{1}{\sqrt{-g}} \hat{\nabla}_\alpha [\sqrt{-g} (W^\alpha_{(\mu\nu)} + \Pi^\alpha_{(\mu\nu)})] + A_{(\mu\nu)} + B_{(\mu\nu)} + C_{(\mu\nu)} = \kappa T_{\mu\nu}, \tag{66}$$

where $\hat{\nabla}$ is defined in (19) and

$$W^\alpha_{(\mu\nu)} := 2a_1 Q^\alpha_{\mu\nu} + 2a_2 Q_{(\mu\nu)}^\alpha + (2a_3 Q^\alpha + a_5 q^\alpha) g_{\mu\nu} + (2a_4 q_{(\mu} + a_5 Q_{(\mu)} \delta_{\nu)}^\alpha), \tag{67}$$

along with

$$\begin{aligned}
 A_{\mu\nu} &:= a_1 (Q_{\mu\alpha\beta} Q_\nu^{\alpha\beta} - 2Q_{\alpha\beta\mu} Q^{\alpha\beta}_\nu) - a_2 Q_{\alpha\beta(\mu} Q^{\beta\alpha}_{\nu)} + a_3 (Q_\mu Q_\nu - 2Q^\alpha Q_{\alpha\mu\nu}) - a_4 q_\mu q_\nu - a_5 q^\alpha Q_{\alpha\mu\nu}, \\
 B_{\mu\nu} &:= b_1 (2S_{\nu\alpha\beta} S_\mu^{\alpha\beta} - S_{\alpha\beta\mu} S^{\alpha\beta}_\nu) - b_2 S_{\nu\alpha\beta} S_\mu^{\beta\alpha} + b_3 S_\mu S_\nu, \\
 C_{\mu\nu} &:= \Pi_{\mu\alpha\beta} Q_\nu^{\alpha\beta} - (c_1 S_{\alpha\beta\nu} Q^{\alpha\beta}_\mu + c_2 S^\alpha Q_{\alpha\mu\nu} + c_3 S^\alpha Q_{\mu\nu\alpha}) = c_1 (Q_\mu^{\alpha\beta} S_{\nu\alpha\beta} - S_{\alpha\beta\mu} Q^{\alpha\beta}_\nu) + c_2 (S_\mu Q_\nu - S^\alpha Q_{\alpha\mu\nu}).
 \end{aligned} \tag{68}$$

On the other hand, varying the action with respect to the general affine connection we get the field equations

$$P_\lambda^{\mu\nu} + \Psi_\lambda^{\mu\nu} = \kappa \Delta_\lambda^{\mu\nu}, \tag{69}$$

where

$$\begin{aligned}
 P_\lambda^{\mu\nu} &:= -\frac{\nabla_\lambda(\sqrt{-g} g^{\mu\nu})}{\sqrt{-g}} + \frac{\nabla_\sigma(\sqrt{-g} g^{\mu\sigma}) \delta_\lambda^\nu}{\sqrt{-g}} + 2(S_\lambda g^{\mu\nu} - S^\mu \delta_\lambda^\nu + g^{\mu\sigma} S_{\sigma\lambda}^\nu) \\
 &= \delta_\lambda^\nu \left(q^\mu - \frac{1}{2} Q^\mu - 2S^\mu \right) + g^{\mu\nu} \left(\frac{1}{2} Q_\lambda + 2S_\lambda \right) - (Q_\lambda^{\mu\nu} + 2S_\lambda^{\mu\nu})
 \end{aligned} \tag{70}$$

is the Palatini tensor and

$$\Psi_\lambda^{\mu\nu} = H^{\mu\nu}_\lambda + \delta_\lambda^\mu k^\nu + \delta_\lambda^\nu h^\mu + g^{\mu\nu} h_\lambda + f^{[\mu} \delta_\lambda^{\nu]}, \tag{71}$$

with

⁸Of course this is not surprising and has to do with the form of the homothetic curvature, which is the curl (field strength) of the nonmetricity vector Q_μ .

$$\begin{aligned}
H^{\mu\nu}{}_{\lambda} &:= 4a_1 Q^{\nu\mu}{}_{\lambda} + 2a_2(Q^{\mu\nu}{}_{\lambda} + Q_{\lambda}{}^{\mu\nu}) + 2b_1 S^{\mu\nu}{}_{\lambda} + 2b_2 S_{\lambda}{}^{[\mu\nu]} + c_1(S^{\nu\mu}{}_{\lambda} - S_{\lambda}{}^{\nu\mu} + Q^{[\mu\nu]}{}_{\lambda}), \\
k_{\mu} &:= 4a_3 Q_{\mu} + 2a_5 q_{\mu} + 2c_2 S_{\mu}, \\
h_{\mu} &:= a_5 Q_{\mu} + 2a_4 q_{\mu} + c_3 S_{\mu}, \\
f_{\mu} &:= c_2 Q_{\mu} + c_3 q_{\mu} + 2b_3 S_{\mu}.
\end{aligned} \tag{72}$$

Let us now elaborate a little bit on the metric field equations and derive some useful general results which we will then specialize in the case of FLRW cosmology. To start with, we first take the trace of (66) to arrive at

$$\left(1 - \frac{n}{2}\right)R + \left(1 - \frac{n}{2}\right)\mathcal{L}_2 - \frac{1}{\sqrt{-g}}\tilde{\nabla}_{\alpha}[\sqrt{-g}(\Pi^{\alpha} + W^{\alpha})] = \kappa T, \tag{73}$$

where we recall that $\tilde{\nabla}$ denotes the Levi-Civita covariant derivative and

$$\begin{aligned}
\Pi^{\alpha} &:= \Pi^{\alpha}{}_{\mu\nu}g^{\mu\nu} = (c_1 + nc_2 + c_3)S^{\alpha}, \\
W^{\alpha} &:= W^{\alpha}{}_{\mu\nu}g^{\mu\nu} = (2a_1 + 2na_3 + a_5)Q^{\alpha} + (2a_2 + 2a_4 + na_5)q^{\alpha}.
\end{aligned} \tag{74}$$

Expanding the Ricci scalar into its Riemannian part (i.e., \tilde{R}) plus non-Riemannian contributions, Eq. (73) becomes

$$\begin{aligned}
\tilde{R} &+ \left(a_1 + \frac{1}{4}\right)Q_{\alpha\mu\nu}Q^{\alpha\mu\nu} + \left(a_2 - \frac{1}{2}\right)Q_{\alpha\mu\nu}Q^{\mu\nu\alpha} + \left(a_3 - \frac{1}{4}\right)Q_{\mu}Q^{\mu} + a_4 q_{\mu}q^{\mu} + \left(a_5 + \frac{1}{2}\right)Q_{\mu}q^{\mu} \\
&+ (b_1 + 1)S_{\alpha\mu\nu}S^{\alpha\mu\nu} + (b_2 - 2)S_{\alpha\mu\nu}S^{\mu\nu\alpha} + (b_3 - 4)S_{\mu}S^{\mu} + (c_1 + 2)Q_{\alpha\mu\nu}S^{\alpha\mu\nu} + (c_2 - 2)Q_{\mu}S^{\mu} + (c_3 + 2)q_{\mu}S^{\mu} \\
&+ \frac{1}{\sqrt{-g}}\tilde{\nabla}_{\mu}[\sqrt{-g}((2a_2 + 2a_4 + na_5 + 1)q^{\mu} + (2a_1 + 2na_3 + a_5 - 1)Q^{\mu} + (c_1 + nc_2 + c_3 - 4)S^{\mu})] = \kappa T.
\end{aligned} \tag{75}$$

In addition, contracting (66) with $u^{\mu}u^{\nu}$ it follows that

$$R_{\mu\nu}u^{\mu}u^{\nu} + \frac{1}{2}(R + \mathcal{L}_2) = \kappa T_{\mu\nu}u^{\mu}u^{\nu} + \frac{u^{\mu}u^{\nu}}{\sqrt{-g}}(\nabla_{\alpha} - 2S_{\alpha})[\sqrt{-g}(\Pi^{\alpha}{}_{\mu\nu} + W^{\alpha}{}_{\mu\nu})] - (A_{\mu\nu} + B_{\mu\nu} + C_{\mu\nu})u^{\mu}u^{\nu}. \tag{76}$$

Then, using Eq. (73) in order to eliminate the $(R + \mathcal{L}_2)$ term and expanding the covariant derivative, after some algebra we finally arrive at

$$\begin{aligned}
R_{\mu\nu}u^{\mu}u^{\nu} &= \frac{\kappa}{(n-2)}[T + (n-2)T_{\mu\nu}u^{\mu}u^{\nu}] - (A_{\mu\nu} + B_{\mu\nu} + C_{\mu\nu})u^{\mu}u^{\nu} \\
&+ \frac{1}{\sqrt{-g}}\partial_{\alpha}\left[\sqrt{-g}\left(\frac{\Pi^{\alpha} + W^{\alpha}}{(n-2)} + M^{\alpha}\right)\right] - 2(\Pi^{\alpha}{}_{(\mu\nu)} + W^{\alpha}{}_{(\mu\nu)})u^{\mu}\nabla_{\alpha}u^{\nu},
\end{aligned} \tag{77}$$

with

$$\begin{aligned}
(A_{\mu\nu} + B_{\mu\nu} + C_{\mu\nu})u^{\mu}u^{\nu} &= a_1 u^{\mu}Q_{\mu\alpha\beta}Q_{\nu}{}^{\alpha\beta}u^{\nu} - Q_{\alpha\beta\mu}u^{\mu}(2a_1 Q^{\alpha\beta\nu}u_{\nu} + a_2 Q^{\beta\alpha\nu}u_{\nu}) + a_3(Q_{\mu}u^{\mu})^2 - a_4(q_{\nu}u^{\nu})^2 \\
&- Q_{\alpha\mu\nu}u^{\mu}u^{\nu}(2a_3 Q^{\alpha} + a_5 q^{\alpha}) + 2b_1 u^{\nu}S_{\nu\alpha\beta}u_{\mu}S^{\mu\alpha\beta} - b_1 S_{\alpha\beta\mu}u^{\mu}S^{\alpha\beta}{}_{\nu}u^{\nu} - b_2 u^{\nu}S_{\nu\alpha\beta}u_{\mu}S^{\mu\beta\alpha} \\
&+ b_3(S_{\mu}u^{\mu})^2 + c_1 u_{\nu}S^{\nu\alpha\beta}u^{\mu}Q_{\mu\alpha\beta} - c_1 S_{\alpha\beta\mu}u^{\mu}Q^{\alpha\beta\nu}u_{\nu} + c_2(S_{\mu}u^{\mu})(Q_{\nu}u^{\nu}) - c_2 S^{\alpha}Q_{\alpha\mu\nu}u^{\mu}u^{\nu},
\end{aligned} \tag{78}$$

and where we have defined

$$M^{\alpha} := (\Pi^{\alpha}{}_{\mu\nu} + W^{\alpha}{}_{\mu\nu})u^{\mu}u^{\nu}. \tag{79}$$

Note that all of the above was fairly general. In what follows we apply these considerations in cosmology.

IV. COSMOLOGY OF QUADRATIC MAG

We now wish to study the cosmology of the generalized quadratic action previously introduced. To this end, we shall consider a flat (i.e., $K = 0$) n -dimensional⁹ FLRW Universe with the usual Robertson-Walker line element as given in Eq. (27) and the homogeneous and isotropic affine connection (30). The various expressions in this cosmological setup have been presented earlier in Sec. II C. As already mentioned, the matter content we consider will be that of a perfect hyperfluid (cf. Sec. II D). Now, interestingly, due to the high symmetry of the FLRW spacetime, not all of the quadratic torsion and nonmetricity invariants are independent. Indeed, as we have previously shown in Sec. II and recall here, we have the following relations:

$$\begin{aligned} (n-1)S_{\mu\nu\alpha}S^{\mu\nu\alpha} - (n-1)S_{\mu\nu\alpha}S^{\alpha\mu\nu} - 3S_{\mu}S^{\mu} &= 0, \\ Q_{\mu}S^{\mu} - q_{\mu}S^{\mu} - (n-1)Q_{\mu\nu\alpha}S^{\mu\nu\alpha} &= 0, \\ (1-n)X_1 + (n-1)X_2 + X_3 - 2X_4 + X_5 &= 0, \end{aligned} \quad (80)$$

indicating that there is one relation for each of the pure torsion, pure nonmetricity, and mixed scalars. As a result, using the above relations, “only” 8 out of the 11 invariants are linearly independent. We may therefore set to zero one parameter¹⁰ from each set $\{a_I\}$, $\{b_J\}$, $\{c_J\}$, with $I = 1, 2, \dots, 5$, $J = 1, 2, 3$. We choose $\{a_2 = 0, b_2 = 0, c_1 = 0\}$, such that the generalized quadratic action boils down to

$$\begin{aligned} S[g, \Gamma, \varphi] &= \frac{1}{2\kappa} \int d^n x \sqrt{-g} [R + b_1 S_{\alpha\mu\nu} S^{\alpha\mu\nu} + b_3 S_{\mu} S^{\mu} \\ &+ a_1 Q_{\alpha\mu\nu} Q^{\alpha\mu\nu} + a_3 Q_{\mu} Q^{\mu} + a_4 q_{\mu} q^{\mu} + a_5 Q_{\mu} q^{\mu} \\ &+ c_2 Q_{\mu} S^{\mu} + c_3 q_{\mu} S^{\mu}] + S_{\text{hyp}} \end{aligned} \quad (81)$$

and, consequently, the first of (72) becomes

$$H^{\mu\nu}{}_{\lambda} = 4a_1 Q^{\nu\mu}{}_{\lambda} + 2b_1 S^{\mu\nu}{}_{\lambda}. \quad (82)$$

$$2(n-1)(2a_3 + a_5)A(t) + (n-1)\left(\frac{1}{2} + a_5 + 2a_4\right)B(t) - 4(a_1 + a_3 + a_4 + a_5)C(t) + 2(n-1)(c_2 + c_3)\Phi(t) = -\kappa\omega(t), \quad (86)$$

$$\left[\frac{(n-3)}{2} + (n-1)a_5\right]A(t) + [2a_1 + (n-1)a_4]B(t) - \left(\frac{1}{2} + 2a_4 + a_5\right)C(t) + [2(n-2) + (n-1)c_3]\Phi(t) = \kappa\psi(t), \quad (87)$$

We see that this observation has considerably simplified the analysis. The next step is to express the torsion and nonmetricity variables in terms of their sources (hypermomentum variables). To this end, we start by the connection field equations (69) and first take its totally antisymmetric part to arrive at

$$2(b_1 - 1)S^{[\alpha\mu\nu]} = \kappa\Delta^{[\alpha\mu\nu]}, \quad (83)$$

where we have used the fact that $P^{[\alpha\mu\nu]} = -2S^{[\alpha\mu\nu]}$ and also $H^{[\alpha\mu\nu]} = 2b_1 S^{[\mu\nu\alpha]} = 2b_1 S^{[\alpha\mu\nu]}$. Furthermore, using the cosmological forms of the hypermomentum and Palatini tensors [cf. Eqs. (60) and (A3), respectively] we find

$$2(b_1 - 1)P(t) = \kappa\zeta(t). \quad (84)$$

From the latter we see that the pseudoscalar degree of freedom completely decouples from the rest of the modes, as expected. Note that it is now essential to assume that $b_1 \neq 1$, since otherwise $P(t)$ would be completely undetermined and $\zeta(t)$ would be forced to vanish. In addition, for the special case $b_1 = 1$ it can be easily shown that the whole action above enjoys invariance under connection transformations of the form (in four dimensions)

$$\Gamma^{\lambda}{}_{\mu\nu} \rightarrow \Gamma^{\lambda}{}_{\mu\nu} + \epsilon^{\lambda}{}_{\mu\nu\alpha} K^{\alpha}, \quad (85)$$

where K^{α} is a generic pseudovector. As a result, only matter with the vanishing totally antisymmetric hypermomentum part is allowed, constraining the form of matter that one can couple. We shall therefore regard the special case $b_1 = 1$ as “unphysical” and assume that $b_1 \neq 1$ for the rest of our analysis. Now, continuing we consider independent operations on the connection field equations (69). First, we multiply (and contract) by $u^{\lambda}u_{\mu}u_{\nu}$, then by $u^{\lambda}h_{\mu\nu}$, subsequently in $h^{\lambda}{}_{\mu}u_{\nu}$, and finally in $h^{\lambda}{}_{\nu}u_{\mu}$, to derive the following four equations:

⁹In order to keep full generality we will do our calculations for arbitrary spacetime dimension n and only set $n = 4$ in Sec. IV B, when discussing in detail a particular subcase of the general quadratic theory.

¹⁰Obviously, this is always possible since we can collect the various parameters after using the dependence relations and a renaming would amount to setting one of each equal to zero.

$$\left[4a_1 + 4(n-1)a_3 - \frac{(n-1)}{2}c_2\right]A(t) + \left[(n-1)a_5 - \frac{(n-1)}{4}c_3 - \frac{1}{2}\right]B(t) + \left[\frac{1}{2}(c_2 + c_3) - 4a_3 - 2a_5\right]C(t) + [2(n-1)c_2 - 2b_1 - (n-1)b_3]\Phi(t) = \kappa\phi(t), \quad (88)$$

$$(n-1)\left(-\frac{1}{2} + a_5 + \frac{c_2}{2}\right)A(t) + \left[\frac{(n-2)}{2} + 2a_1 + (n-1)a_4 + \frac{(n-1)}{4}c_3\right]B(t) - \left[\frac{1}{2} + 2a_4 + a_5 + \frac{c_2 + c_3}{2}\right]C(t) + [-2(n-2) + 2b_1 + (n-1)c_3 + (n-1)b_3]\Phi(t) = \kappa\chi(t). \quad (89)$$

We may rewrite the above as

$$\alpha_{11}A + \alpha_{12}B + \alpha_{13}C + \alpha_{14}\Phi = -\kappa\omega, \quad (90)$$

$$\alpha_{21}A + \alpha_{22}B + \alpha_{23}C + \alpha_{24}\Phi = \kappa\psi, \quad (91)$$

$$\alpha_{31}A + \alpha_{32}B + \alpha_{33}C + \alpha_{34}\Phi = \kappa\phi, \quad (92)$$

$$\alpha_{41}A + \alpha_{42}B + \alpha_{43}C + \alpha_{44}\Phi = \kappa\chi, \quad (93)$$

respectively, where the relations between the α_{ij} coefficients and the a 's, b 's, c 's is obvious. For the sake of completeness, however, we include their exact relations in Appendix A 4. We then define the matrix M with elements α_{ij} and also the vectors $\mathcal{U} = (A, B, C, \Phi)^T$ and $\mathcal{W} = \kappa(-\omega, \psi, \phi, \chi)^T$. Hence, we can write the above equations in the matrix form

$$M\mathcal{U} = \mathcal{W}. \quad (94)$$

Furthermore, the determinant of M is nonzero *in general*, since otherwise this would imply that there is some symmetry under connection transformations that would make the trace equations linearly dependent [5,55]. Since there is no reason to demand that there is such a symmetry at play, we can safely assume that $\det(M) \neq 0$ and therefore that there exists the inverse matrix M^{-1} . Hence, we can formally write

$$\mathcal{U} = M^{-1}\mathcal{W}. \quad (95)$$

This equation allows to express the torsion and nonmetricity variables in terms of the hypermomentum sources as

$$\begin{aligned} A &= \kappa(-\lambda_{11}\omega + \lambda_{12}\psi + \lambda_{13}\phi + \lambda_{14}\chi), \\ B &= \kappa(-\lambda_{21}\omega + \lambda_{22}\psi + \lambda_{23}\phi + \lambda_{24}\chi), \\ C &= \kappa(-\lambda_{31}\omega + \lambda_{32}\psi + \lambda_{33}\phi + \lambda_{34}\chi), \\ \Phi &= \kappa(-\lambda_{41}\omega + \lambda_{42}\psi + \lambda_{43}\phi + \lambda_{44}\chi), \end{aligned} \quad (96)$$

where the λ 's are, in fact, the elements of M^{-1} and are therefore given in terms of the a 's, b 's, and c 's by inverting M . Generally, A, B, C, Φ depend on all four sources ϕ, χ, ψ, ω , while the pseudoscalar mode $P(t)$ only on $\zeta(t)$, as previously seen and actually expected. In the following, we will conclude the cosmological analysis of the (general) theory by deriving the associated Friedmann equations.

A. Friedmann equations with torsion and nonmetricity

Let us now derive the Friedmann equations in the presence of both torsion and nonmetricity. Starting from (75) and considering the cosmological forms (32) of torsion and nonmetricity, after some straightforward algebra we finally arrive at

$$\begin{aligned} &2(n-1)\left[\frac{\ddot{a}}{a} + \frac{(n-2)}{2}\left(\frac{\dot{a}}{a}\right)^2\right] - (n-1)\left[a_1 - \frac{(n-2)}{4} + (n-1)a_3\right]A^2 - \frac{(n-1)}{4}[2a_1 + a_2 + (n-1)a_4]B^2 \\ &- (a_1 + a_2 + a_3 + a_4 + a_5)C^2 - (n-1)[2b_1 - b_2 + (n-1)b_3 - 4(n-2)]\Phi^2 + 6(b_1 + b_2 - 1)P^2\delta_4^4 \\ &- (n-1)\left[a_2 + \frac{(n-1)}{2}a_5 + \frac{(n-3)}{4}\right]AB + (n-1)(2a_3 + a_5)AC + (n-1)\left(a_4 + \frac{1}{2}a_5 + \frac{1}{4}\right)BC \\ &- (n-1)[c_1 + (n-1)c_2 - 2(n-2)]A\Phi + \frac{(n-1)}{2}[c_1 - (n-1)c_3 - 2(n-2)]B\Phi + (n-1)(c_2 + c_3)C\Phi \\ &= -[\dot{f} + (n-1)Hf] - \kappa[-\rho + (n-1)p], \end{aligned} \quad (97)$$

where we have also used (58) and introduced

$$\begin{aligned} f &:= (2a_2 + 2a_4 + na_5 + 1)\frac{(n-1)}{2}B - [2(a_1 + a_2 + na_3 + a_4) + (n+1)a_5]C \\ &+ (2a_1 + 2na_3 + a_5 - 1)(n-1)A + (c_1 + nc_2 + c_3 - 4)(n-1)\Phi. \end{aligned} \quad (98)$$

Equation (97) is a variant of the first modified Friedmann equation. Now, the acceleration equation (also known as Raychaudhuri equation, that is the second Friedmann equation) for non-Riemannian universes has been derived in [56] and reads

$$\frac{\ddot{a}}{a} = -\frac{1}{(n-1)}R_{\mu\nu}u^\mu u^\nu + 2\left(\frac{\dot{a}}{a}\right)\Phi + 2\dot{\Phi} + \left(\frac{\dot{a}}{a}\right)\left(A + \frac{C}{2}\right) + \frac{\dot{A}}{2} - \frac{A^2}{2} - \frac{1}{2}AC - 2A\Phi - 2C\Phi. \quad (99)$$

In order to find its exact form in the present case, we have to compute the term (78) using the cosmological expressions of torsion and nonmetricity. After some lengthy algebra, we finally find (see Appendix A 3 for details)

$$\begin{aligned} \frac{\ddot{a}}{a} = & -\frac{\kappa}{(n-1)(n-2)}[(n-3)\rho + (n-1)p] \\ & + [a_1 + (n-1)a_3]A^2 - \frac{1}{4}[(2a_1 + a_2) + (n-1)a_4]B^2 + \frac{1}{(n-1)}(a_1 + a_2 + a_3 + a_4 + a_5)C^2 \\ & + \left(a_4 + \frac{a_5}{2}\right)BC + [2b_1 - b_2 + (n-1)b_3]\Phi^2 + [c_1 + (n-1)c_2]\Phi A \\ & - (H - Y)[\gamma_1 B + 2\gamma_2 A + 2\gamma_3 \Phi] - \frac{1}{2}C[(a_5 + 2a_4)B + 2(2a_3 + a_5)A + 2(c_2 + c_3)\Phi] \\ & + (2a_4 + a_5)(H - Y)C - \frac{\dot{l}}{(n-1)} - Hl \\ & + 2\left(\frac{\dot{a}}{a}\right)\Phi + 2\dot{\Phi} + \left(\frac{\dot{a}}{a}\right)\left(A + \frac{C}{2}\right) + \frac{\dot{A}}{2} - \frac{A^2}{2} - \frac{1}{2}AC - 2A\Phi - 2C\Phi, \end{aligned} \quad (100)$$

with $\gamma_1, \gamma_2, \gamma_3$, and l all respectively given in (A16) and (A18) of Appendix A 3, where we have also written explicitly the coefficients appearing inside l [see (A19)]. Recall also that $Y = 2\Phi + \frac{\dot{A}}{2}$ (cf. Sec. II). Obviously, to recover the Friedmann equations in the case of the theory with the choice $\{a_2 = 0, b_2 = 0, c_1 = 0\}$ we have previously made in a consistent way, it is sufficient to set the aforementioned coefficients to zero in the expressions above, as the latter have been derived for the general quadratic theory. We have therefore obtained the cosmological equations. As we are left with many free parameters, in order to give more details in the following we shall focus on the case in which only the two quadratic contributions $b_3 S_\mu S^\mu$ and $a_3 Q_\mu Q^\mu$ are there, setting all other coefficients in the general quadratic model to zero. We perform this particular choice as it involves two vectors that are particularly relevant in the context of non-Riemannian geometry, namely the torsion vector S_μ and the nonmetricity vector Q_μ . Additionally, as we will see in what follows, these two quadratic terms not only simplify the analysis greatly but also they are fairly representative and give reliable conclusions to be expected also for the full quadratic theory.

B. Subcase involving only two quadratic contributions

Let us now restrict ourselves to the following quadratic theory, subcase of the general one previously discussed, in $n = 4$ spacetime dimensions:

$$S[g, \Gamma, \varphi] = \frac{1}{2\kappa} \int d^4x \sqrt{-g} [R + b_3 S_\mu S^\mu + a_3 Q_\mu Q^\mu] + S_{\text{hyp}}, \quad (101)$$

that is we set $b_1 = a_1 = a_2 = a_4 = a_5 = c_1 = c_2 = c_3 = 0$ in the general quadratic model. The metric and connection field equations can be immediately derived by implementing these constraints in the corresponding equations derived for the general theory. We will now analyze the cosmology of (101). From the connection field equations we get the following relations:

$$\begin{aligned} P &= -\frac{\kappa}{2}\zeta, \\ 12a_3A + \frac{3}{2}B - 4a_3C &= -\kappa\omega, \\ \frac{1}{2}A - \frac{1}{2}C + 4\Phi &= \kappa\psi, \\ 12a_3A - \frac{1}{2}B - 4a_3C - 3b_3\Phi &= \kappa\phi, \\ -\frac{3}{2}A + B - \frac{1}{2}C + (-4 + 3b_3)\Phi &= \kappa\chi. \end{aligned} \quad (102)$$

The latter can be solved giving the expressions of the torsion and nonmetricity variables in terms of the hyper-momentum sources, namely

$$\begin{aligned}
P &= -\frac{\kappa}{2}\zeta, \\
A &= \frac{\kappa}{[16a_3(3b_3-32)-18b_3]} \{ (64a_3+3b_3+8)\omega + [-16a_3(3b_3-8)-9b_3]\psi + [64a_3+9b_3-24]\phi + (128a_3+9b_3)\chi \}, \\
B &= \frac{\kappa}{[8a_3(3b_3-32)-9b_3]} \{ 2[a_3(64-12b_3)+3b_3]\omega + 72a_3b_3\psi - [8a_3(3b_3-16)]\phi - 24a_3b_3\chi \}, \\
C &= \frac{\kappa}{[16a_3(3b_3-32)-18b_3]} \{ (192a_3+3b_3-8)\omega + [27b_3-48a_3(3b_3-8)]\psi + 3(64a_3+3b_3+8)\phi + (384a_3+9b_3)\chi \}, \\
\Phi &= \frac{\kappa}{[8a_3(3b_3-32)-9b_3]} \{ -(1-8a_3)\omega - 48a_3\psi + (8a_3+3)\phi + 16a_3\chi \}. \tag{103}
\end{aligned}$$

Furthermore, the first modified Friedmann equation (97) becomes

$$\begin{aligned}
6 \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 \right] - 3 \left(-\frac{1}{2} + 3a_3 \right) A^2 - a_3 C^2 - 3(3b_3 - 8)\Phi^2 - 6P^2 - \frac{3}{4}AB + 6a_3AC + \frac{3}{4}BC \\
+ 12A\Phi - 6B\Phi = -(\dot{f} + 3Hf) - \kappa(-\rho + 3p), \tag{104}
\end{aligned}$$

where now

$$f = \frac{3}{2}B - 8a_3C + 3(8a_3 - 1)A - 12\Phi, \tag{105}$$

while the second (i.e., acceleration) Friedmann equations reads

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{6}(\rho + 3p) + \frac{a_3}{3}(3A - C)^2 + 3b_3\Phi^2 - \frac{1}{3}(\dot{l} + Hl) + H \left(2\Phi + A + \frac{C}{2} \right) - \left(2\Phi + \frac{A}{2} \right) (A + C) + 2\dot{\Phi} + \frac{\dot{A}}{2}, \tag{106}$$

where now

$$l = 2a_3(3A - C). \tag{107}$$

It is worth emphasizing that the second and third terms in the right-hand side of Eq. (106) retain a fixed sign once the values of a_3 and b_3 are given. In particular, it is interesting to note that for $a_3 > 0$ and $b_3 > 0$ both of these non-metricity and torsion contributions accelerate the expansion. Incidentally, this also means that in order for the aforementioned terms to have an accelerating effect on the expansion, both “mass” terms for torsion and nonmetricity must have the correct (positive) sign.

We shall then consider also the conservation laws for our cosmological hyperfluid model. Here let us assume that the fluid is hypermomentum preserving, or, in other words, that the metrical energy-momentum tensor coincides with the canonical one, namely

$$t^{\mu\nu} = T^{\mu\nu}, \tag{108}$$

which implies [cf. Eqs. (58) and (59)]

$$\rho_c = \rho, \quad p_c = p. \tag{109}$$

This will be one of our assumptions when we will focus on a particular type of hypermomentum (i.e., more precisely, pure dilation hypermomentum) in the following. Observe that the same would also apply to the case of the unconstrained hyperfluid of [36]. In fact, recalling that we are considering a homogeneous cosmological setup, we have that any antisymmetric two index object vanishes identically. In particular, this means that $t_{[\mu\nu]} \equiv 0$ and, therefore, only its symmetric part survives, i.e., $t_{\mu\nu} = t_{(\mu\nu)}$. Hence we can consistently assume (108) to hold for our model. Consequently, the conservation laws (61) and (62) become

$$\tilde{\nabla}_\mu T^\mu{}_\nu = \frac{1}{2} \Delta^{\lambda\mu\nu} R_{\lambda\mu\nu\alpha}, \tag{110}$$

$$\hat{\nabla}_\nu (\sqrt{-g} \Delta_\lambda{}^{\mu\nu}) = 0, \tag{111}$$

respectively, where in order to reach the above result we have also expanded $\hat{\nabla}_\mu T^\mu{}_\nu$ into its Levi-Civita part plus non-Riemannian contributions. Now, plugging (58) and

$$\begin{aligned}
\Delta_{\alpha\mu\nu}^{(n)} &= \phi(t)h_{\mu\alpha}u_\nu + \chi(t)h_{\nu\alpha}u_\mu + \psi(t)u_\alpha h_{\mu\nu} + \omega(t)u_\alpha u_\mu u_\nu \\
&\quad + \epsilon_{\alpha\mu\nu\rho} u^\rho \zeta(t) \tag{112}
\end{aligned}$$

into the above conservation laws, and recalling that in a homogeneous cosmological setup the homothetic curvature tensor $\hat{R}_{\mu\nu}$ has to be identically vanishing, the above conservation laws respectively boil down to (see also [40])

$$\dot{\rho} + 3H(\rho + p) = -\frac{1}{2}u^\mu u^\nu (\chi R_{\mu\nu} + \psi \check{R}_{\mu\nu}), \quad (113)$$

where $\check{R}_{\mu\nu}$ is the co-Ricci tensor, and

$$\begin{aligned} & -\delta_\lambda^\mu \frac{\partial_\nu(\sqrt{-g}\phi u^\nu)}{\sqrt{-g}} - u^\mu u_\lambda \frac{\partial_\nu[\sqrt{-g}(\phi + \chi + \psi + \omega)u^\nu]}{\sqrt{-g}} + \left[\left(2S_\lambda + \frac{Q_\lambda}{2} \right) u^\mu - \nabla_\lambda u^\mu \right] \chi \\ & + [(2S^\mu + Q^\mu - q^\mu)u_\lambda - g^{\mu\nu}\nabla_\nu u_\lambda] \psi + u^\mu u_\lambda (\dot{\chi} + \dot{\psi}) - (\phi + \chi + \psi + \omega)(\dot{u}^\mu u_\lambda + u^\mu \dot{u}_\lambda) = 0. \end{aligned} \quad (114)$$

The former of the above equations is the modified perfect fluid continuity equation which receives corrections from the hypermomentum, while the latter provides the evolution laws for the hypermomentum currents (i.e., non-Riemannian degrees of freedom). Therefore, we are left with Eqs. (103), (104), (106), (113), and (114), ruling the cosmology of the theory. In the following, we will complete the analysis by focusing on the cases in which specific parts of the hypermomentum tensor are switched on in order to highlight the role of its irreducible parts and their contribution separately. To be more specific, the hypermomentum tensor can be decomposed as follows [42] (recall that we are considering four spacetime dimensions):

$$\Delta_{\alpha\mu\nu} = \Delta_{[\alpha\mu]\nu} + \frac{1}{4}g_{\alpha\mu}D_\nu + \check{\Delta}_{\alpha\mu\nu}, \quad (115)$$

where the first term on the right-hand side represents the spin part, $D^\nu := \Delta_{\mu}{}^{\mu\nu}$ is the dilation, and $\check{\Delta}_{\alpha\mu\nu}$ the shear (traceless symmetric part of the hypermomentum tensor). Then, given the most general form (60) of hypermomentum compatible with the cosmological principle (i.e., respecting

both isotropy and homogeneity), the spin, dilation, and shear parts read

$$\Delta_{[\alpha\mu]\nu} = (\psi - \chi)u_{[\alpha}h_{\mu]\nu} + \epsilon_{\alpha\mu\nu\rho}u^\rho\zeta, \quad (116)$$

$$D_\nu := \Delta_{\alpha\mu\nu}g^{\alpha\mu} = (3\phi - \omega)u_\nu, \quad (117)$$

$$\begin{aligned} \check{\Delta}_{\alpha\mu\nu} &= \Delta_{(\alpha\mu)\nu} - \frac{1}{4}g_{\alpha\mu}D_\nu = \frac{(\phi + \omega)}{4}(h_{\alpha\mu} + 3u_\alpha u_\mu)u_\nu \\ &+ (\psi + \chi)u_{(\mu}h_{\alpha)\nu}, \end{aligned} \quad (118)$$

respectively. Besides, for the case of pure dilation we are going to discuss in the following, we will assume that the perfect hyperfluid variables are related through a barotropic equation of state of the usual type, namely

$$p = w\rho, \quad (119)$$

where w is the associated barotropic index. Substituting this into the acceleration equation (106) we may then use the latter and eliminate the double derivative term (\ddot{a}) from (104), in such a way so as to obtain the final form of the first modified Friedmann equation for the model at hand, under the above assumptions, which reads

$$\begin{aligned} H^2 &= \frac{1}{2}(1 - 6a_3)A^2 - \frac{1}{6}a_3C^2 + \left(-4 - \frac{3}{2}b_3\right)\Phi^2 + P^2 + \frac{1}{8}AB + \left(\frac{1}{2} + a_3\right)AC - \frac{1}{8}BC \\ &+ 2A\Phi + B\Phi + 2C\Phi - 2a_3\dot{A} - \frac{1}{4}\dot{B} + \frac{2}{3}a_3\dot{C} + H \left[\left(\frac{1}{2} - 6a_3\right)A - \frac{3}{4}B + \left(2a_3 - \frac{1}{2}\right)C + 4\Phi \right] + \frac{\kappa}{3}\rho, \end{aligned} \quad (120)$$

where we recall that

$$H := \frac{\dot{a}}{a} \quad (121)$$

is the Hubble parameter. Furthermore, using (119) into (113), the latter becomes

$$\dot{\rho} + 3H(1 + w)\rho = -\frac{1}{2}u^\mu u^\nu (\chi R_{\mu\nu} + \psi \check{R}_{\mu\nu}). \quad (122)$$

To recap, the two Friedmann equations (103) and (120) supplemented with the conservations laws (114) and (122),

close the system of equations and we can now concentrate on the cosmological effects of dilation and spin.¹¹

1. Purely dilation hypermomentum

Let us now consider the case in which the hypermomentum tensors encode only dilation, namely $\Delta_{[\alpha\mu]\nu} = 0$ and $\check{\Delta}_{\alpha\mu\nu} = 0$. Specifically, this implies

¹¹We leave the case of pure shear hypermomentum to future work. The role of the latter but in the case where only the Einstein-Hilbert term was present in the gravitational action was investigated in [42].

$$\zeta = 0, \quad \psi = 0, \quad \chi = 0, \quad \omega = -\phi, \quad (123)$$

which gives

$$\Delta_{\alpha\mu\nu} = \frac{1}{4}g_{\alpha\mu}D_\nu = g_{\alpha\mu}\phi u_\nu. \quad (124)$$

Therefore, we are left with just one independent, non-vanishing source, that is ϕ . As a result, from (103) now we get

$$\begin{aligned} P &= 0, \\ A &= \kappa \left[\frac{(16 - 3b_3)}{8a_3(32 - 3b_3) + 9b_3} \right] \phi, \\ B &= \kappa \left[\frac{6b_3}{8a_3(32 - 3b_3) + 9b_3} \right] \phi, \\ C &= -\kappa \left[\frac{(16 + 3b_3)}{8a_3(32 - 3b_3) + 9b_3} \right] \phi, \\ \Phi &= -\kappa \left[\frac{4}{8a_3(32 - 3b_3) + 9b_3} \right] \phi. \end{aligned} \quad (125)$$

As we can see, the pseudoscalar mode P vanishes, while the other ones are given in terms of ϕ . It is interesting to note that when the ‘‘mass’’ term for the torsion vector is absent (i.e., $b_3 = 0$) it follows that $B = 0$ and $A + C = 0$, which is exactly the case of a Weyl nonmetricity. In other words, for $b_3 = 0$, interestingly enough, the geometry is fixed to be that of Weyl-Cartan type. Note also that there exist specific values for b_3 for which one can obtain a vanishing part for all (but one each time) of the three nonmetricity functions, while there is no parameter value for b_3 (or even a_3) that yields a vanishing torsion function Φ . Going back to our analysis, the conservation laws (114) and (122) respectively reduce to

$$\partial_\nu(\sqrt{-g}\phi u^\nu) = 0, \quad (126)$$

$$\dot{\rho} + 3H(1 + w)\rho = 0. \quad (127)$$

In particular, the latter provides the evolution of ρ , while from the former we get

$$\dot{\phi} + 3H\phi = 0. \quad (128)$$

Note that in this case the density decouples and evolves as in the usual perfect fluid continuity equation, while also for ϕ we observe an analogous evolution which, in particular, mimics that of dust. Moreover, plugging (125) into (120) we are left with

$$\begin{aligned} H^2 &= -\frac{\kappa}{6}\dot{\phi} - \frac{\kappa}{2}H\phi \\ &\quad - \frac{[3072 + 8a_3(32 - 3b_3)^2 - 288b_3 - 81b_3^2]}{12[9b_3 - 8a_3(3b_3 - 32)]^2} \\ &\quad \times \kappa^2\phi^2 + \frac{\kappa}{3}\rho, \end{aligned} \quad (129)$$

where, of course, we have $9b_3 - 8a_3(3b_3 - 32) \neq 0$. Furthermore, using the above continuity equation for ϕ the latter simplifies to

$$\begin{aligned} H^2 &= -\frac{[3072 + 8a_3(32 - 3b_3)^2 - 288b_3 - 81b_3^2]}{12[9b_3 - 8a_3(3b_3 - 32)]^2} \\ &\quad \times \kappa^2\phi^2 + \frac{\kappa}{3}\rho. \end{aligned} \quad (130)$$

Observe that for $\phi = 0$ or $a_3 = \frac{3(-1024 + 96b_3 + 27b_3^2)}{8(1024 - 192b_3 + 9b_3^2)}$ one would recover the usual (Riemannian) cosmological equation in the flat (that is, $K = 0$) case.¹² Concluding, let us also report the final form of the acceleration equation, that is

$$\frac{\ddot{a}}{a} = \frac{[4a_3(32 - 3b_3)^2 + 9(16 - 3b_3)b_3]}{3[9b_3 - 8a_3(3b_3 - 32)]^2} \kappa^2\phi^2 - \frac{\kappa}{6}(1 + 3w)\rho. \quad (131)$$

Again, note that for $\phi = 0$ or $a_3 = \frac{9(-16b_3 + 3b_3^2)}{4(1024 - 192b_3 + 9b_3^2)}$ one would recover the usual (purely Riemannian) acceleration equation. We are now in a position to provide solutions.

Purely dilation hypermomentum solutions.—Let us first of all collect all relevant cosmological equations obtained until now in the case of pure dilation. They are

$$H^2 = \frac{\kappa}{3}\rho + \mathcal{B}\frac{\kappa^2\phi^2}{4}, \quad (132)$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{6}(1 + 3w)\rho - \mathcal{C}\frac{\kappa^2\phi^2}{2}, \quad (133)$$

$$\dot{\rho} + 3H(1 + w)\rho = 0, \quad (134)$$

$$\dot{\phi} + 3H\phi = 0, \quad (135)$$

where, for simplicity, we have introduced

¹²On the other hand, considering a nonflat cosmological background, namely introducing a curvature parameter $K \neq 0$, one would get

$$H^2 = -\frac{K}{a^2} - \frac{[3072 + 8a_3(32 - 3b_3)^2 - 288b_3 - 81b_3^2]}{12[9b_3 - 8a_3(3b_3 - 32)]^2} \kappa^2\phi^2 + \frac{\kappa}{3}\rho,$$

where we can see that the usual term induced by the presence of a non-null curvature parameter K appears on the right-hand side.

$$\mathcal{B} := -\frac{[3072 + 8a_3(32 - 3b_3)^2 - 288b_3 - 81b_3^2]}{3[9b_3 - 8a_3(3b_3 - 32)]^2}, \quad (136)$$

$$\mathcal{C} := -\frac{[4a_3(32 - 3b_3)^2 + 9(16 - 3b_3)b_3]}{3[9b_3 - 8a_3(3b_3 - 32)]^2}, \quad (137)$$

with, of course, $9b_3 - 8a_3(3b_3 - 32) \neq 0$. Some comments are in order. First, notice that in Eqs. (132) and (133), in the end, there are no linear terms in H and these equations have a more clear physical interpretation. In fact, from the aforementioned equations we can see that ϕ^2 acts as an additional perfect fluid component with some peculiar associated density and pressure which can just have a net effect that accelerates the expansion. The two Friedmann equations are then supplemented with the conservation laws (134) and (135), and we have a closed system of equations. Besides, note from (133) that the effect of dilation hypermomentum on the cosmological expansion depends crucially on the sign of \mathcal{C} . More specifically, for $\mathcal{C} > 0$ the dilation hypermomentum slows down expansion, while for $\mathcal{C} < 0$ it accelerates the latter and for $\mathcal{C} = 0$ it has no effect on the acceleration equation. We shall therefore refer to these cases as repulsive ($\mathcal{C} > 0$), attractive ($\mathcal{C} < 0$), and static ($\mathcal{C} = 0$) dilation, respectively (following the same lines of [42], where, however, shear hypermomentum was considered). Secondly, from (132) it follows that if $\mathcal{B} > 0$ then dilation enhances the total density (and we have an effective amplification of the latter), while for $\mathcal{B} < 0$ it decreases the total density and for $\mathcal{B} = 0$ it does not affect the equation. This is also supported by the fact that only ϕ^2 terms appear in the two Friedmann equations, meaning that an identification $\rho \propto \phi^2$ is possible. Finally, differentiating (132) and employing both (134) and (135), comparing the result with (133), one obtains

$$\mathcal{C} = \mathcal{B}, \quad (138)$$

which, in turn, implies

$$a_3 = \frac{3[-512 + 9b_3(8 + b_3)]}{2(32 - 3b_3)^2}. \quad (139)$$

Hence, we are left with a single independent parameter, b_3 ,¹³ and the final system of equations reads as follows:

$$H^2 = \frac{\kappa}{3}\rho + \mathcal{B}\frac{\kappa^2\phi^2}{4}, \quad (140)$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{6}(1 + 3w)\rho - \mathcal{B}\frac{\kappa^2\phi^2}{2}, \quad (141)$$

¹³The relation between a_3 and b_3 we get as explained above is a peculiarity of the quadratic model we are considering under the assumptions (109) and (119) in the case of pure dilation hypermomentum.

$$\dot{\rho} + 3H(1 + w)\rho = 0, \quad (142)$$

$$\dot{\phi} + 3H\phi = 0, \quad (143)$$

with

$$\mathcal{B} = -\frac{(32 - 3b_3)^2[-1024 + 3b_3(64 + 3b_3)]}{9[2048 - 3b_3(128 + 9b_3)]^2}, \quad (144)$$

along with, of course, $2048 - 3b_3(128 + 9b_3) \neq 0$. We can finally recognize the following distinct cases: repulsive ($\mathcal{B} > 0$, for which dilation also enhances the total density), attractive ($\mathcal{B} < 0$, for which dilation also reduces the total density), and static [$\mathcal{B} = 0$, for which dilation does not affect Eq. (140) and (141)] dilation, respectively.

Now, as it is commonly accepted that non-Riemannian effects may have played a significant role in the early stages of the Universe, for early times one can assume dilation to be dominating with respect to the perfect fluid characteristics. Hence, in this regime, we can safely ignore the latter, namely ρ and p , and focus only on the cosmological effects of dilation. By doing so, we are left with

$$H^2 = \mathcal{B}\frac{\kappa^2\phi^2}{4}, \quad (145)$$

$$\frac{\ddot{a}}{a} = -\mathcal{B}\frac{\kappa^2\phi^2}{2}, \quad (146)$$

while Eq. (143) can be immediately integrated to get

$$\phi = \phi_0\left(\frac{a_0}{a}\right)^3, \quad (147)$$

given that for some fixed time $t = t_0$ we have $a(t_0) = a_0$ and $\phi(t_0) = \phi_0$. On the other hand, from (145) we can see that we must have $\mathcal{B} > 0$ [i.e., deceleration, from (146)]. Besides, solving (145) for H , we find

$$H = \mathcal{B}_1\phi, \quad \mathcal{B}_1 := \pm\frac{\kappa\sqrt{\mathcal{B}}}{2}. \quad (148)$$

Then, plugging the above expression for ϕ into this last equation, after integration we obtain

$$a(t) = a_0[\mathcal{B}_1\phi_0(t - t_0) + 1]^{\frac{1}{3}} = a_0\left[\pm\frac{\kappa\sqrt{\mathcal{B}}\phi_0}{2}(t - t_0) + 1\right]^{\frac{1}{3}}. \quad (149)$$

The latter describes the cosmological evolution for our model in the case of a dilation dominated Universe. Quite remarkably, we can notice that the effect of dilation is identical to that of stiff matter. To conclude the analysis regarding purely dilation hypermomentum, let us also

mention that, from the two modified Friedmann equations, exploiting the definition of the deceleration parameter q , one easily finds

$$q := -\frac{\ddot{a}a}{\dot{a}^2} = 2 > 0, \quad (150)$$

meaning, in accordance with the fact that here we must have $\mathcal{B} > 0$, that considering our model in the case of a dilation dominated (early) universe, one finds out that the latter is subject to a cosmic deceleration.

2. Purely spin hypermomentum

It is also interesting to study the case in which the hypermomentum is only of pure spin type. In this instance we have that $\Delta_{\alpha\mu\nu} = \Delta_{[\alpha\mu]\nu}$ and accordingly $D_\nu = 0$ as well as $\check{\Delta}_{\alpha\mu\nu} = 0$. The latter imply

$$\phi = 0, \quad \omega = 0, \quad \chi = -\psi, \quad (151)$$

along with

$$\Delta_{\alpha\mu\nu} = \Delta_{[\alpha\mu]\nu} = 2\psi u_{[\alpha} h_{\mu]\nu} + \epsilon_{\alpha\mu\rho} u^\rho \zeta, \quad (152)$$

and as a result the nonmetricity and torsion cosmological functions become

$$P = -\frac{\kappa}{2}\zeta, \quad (153)$$

$$A = -\frac{3b_3}{b_0}(3 + 8a_3)\kappa\psi, \quad (154)$$

$$B = \frac{96a_3b_3}{b_0}\kappa\psi, \quad (155)$$

$$C = \frac{b_3}{b_0}9(1 - 8a_3)\kappa\psi, \quad (156)$$

$$\Phi = -\frac{64a_3}{b_0}\kappa\psi, \quad (157)$$

where we have set

$$b_0 := \frac{1}{8a_3(3b_3 - 32) - 9b_3}. \quad (158)$$

From the above forms we can infer some quite interesting characteristics. In particular, we see that in order to have nonvanishing nonmetricity variables it must hold that $b_3 \neq 0$, that is the $S_\mu S^\mu$ term must be present. On the other hand, in order for the torsion function Φ not to vanish we must have that $a_3 \neq 0$, namely the $S_\mu S^\mu$ term must be there.¹⁴ From the above we also observe a highly nontrivial

¹⁴This is a very remarkable duality and of course highly nontrivial or to be expected by any speculation.

consequence: the spin part can source nonmetricity (along with torsion). Let us note that for the spin part to source these non-Riemannian degrees of freedom the inclusion of the quadratic torsion and nonmetricity invariants in the Lagrangian is absolutely essential.

Here, let us continue our analysis without imposing the assumptions (109) and (119), as releasing the latter, in this case, leads to interesting results. From the conservation law (61) we get the two relations,

$$p_c - p = \frac{1}{4}B\psi, \quad (159)$$

$$\rho_c - \rho = \frac{3}{4}B\psi, \quad (160)$$

which also imply that¹⁵

$$\rho_c - \rho = 3(p_c - p). \quad (161)$$

We notice, from the above equations, that it is quite crucial in this case to consider a generalized nonpreserving hypermomentum, namely $\rho_c \neq \rho$ and $p_c \neq p$ need to hold true. In the case of a hypermomentum preserving hyperfluid ($\rho_c = \rho$, $p_c = p$) from the above it immediately follows that either $\psi = 0$ or $B = 0$. For the former possibility ($\psi = 0$), the full hypermomentum would vanish, trivializing therefore the whole analysis. Also, in the latter case, given (155), the constraint $B = 0$ would then demand either a_3 or b_3 to vanish, producing, therefore, an inconsistent theory. With the above in mind, we can then combine Eqs. (159) and (160) with (155) to obtain the net density

$$\rho_c = \rho + \frac{72a_3b_3}{b_0}\kappa\psi^2 \quad (162)$$

as well as the net pressure

$$p_c = p + \frac{24a_3b_3}{b_0}\kappa\psi^2. \quad (163)$$

From these we see that it is important to include both quadratic terms in torsion and nonmetricity (i.e., $a_3 \neq 0$ and $b_3 \neq 0$) at the same time in order to observe modifications to the total pressure and density due to the hypermomentum degrees of freedom. In addition, as it is obvious from the above exposure, depending on the sign of the ratio a_3b_3/b_0 the hypermomentum contribution can either enhance or decrease the observed density and pressure. It is interesting to note that the hypermomentum contribution to the density might as well be negative since as long as it does not exceed

¹⁵Note that this seems to be a ‘‘radiationlike’’ equation of state $\hat{p} = \hat{\rho}/3$ for the modified pressure $\hat{p} = (p_c - p)$ and density $\hat{\rho} = (\rho_c - \rho)$.

ρ the net density will always be positive.¹⁶ However, in the very early Universe there might exist an exotic hyperfluid displaying a negative total density.

Coming back to our cosmological analysis, from the modified continuity equation (62), after some algebra and simplifications, it follows that

$$\dot{\rho}_c + 3H(\rho_c + p_c) = -\frac{3}{2}\mu_1\kappa\psi(\dot{\psi} + H\psi) - 3\psi\frac{\ddot{a}}{a}, \quad (164)$$

where

$$\mu_1 := \frac{28a_3 + 48a_3b_3 + \frac{9}{2}b_3}{b_0}. \quad (165)$$

Now, during a hypermomentum dominated era (very early universe), the main contributions in (162) and (163) would be the ones $\propto \psi^2$. In other words, the classical perfect fluid contributions ρ and p can be ignored. The same considerations apply to the acceleration equation (106), which can therefore be written in terms of ψ as follows:

$$\frac{\ddot{a}}{a} = \frac{\kappa}{2b_0^2}(\dot{\psi} + H\psi - 2304\kappa a_3^2 b_3^2 \psi^2). \quad (166)$$

$$H^2 + \frac{\ddot{a}}{a} = \frac{\kappa}{4b_0} \left\{ \frac{\kappa}{b_0}[-9b_3 + 8a_3(32 + 3b_3)][9b_3 + 8a_3(-32 + 9b_3)]\psi^2 + \frac{2}{b_0}(\dot{\psi} + 3H\psi) \right\} + \frac{\kappa^2}{4}\zeta^2. \quad (170)$$

Finally, plugging (166) into the above equation in order to eliminate $\frac{\ddot{a}}{a}$, we are left with

$$H^2 = \frac{\kappa}{2b_0^2} \left[\pi_1\dot{\psi} + (2 + \pi_1)H\psi + \frac{\kappa}{2}\pi_2\psi^2 \right] + \frac{\kappa^2}{4}\zeta^2, \quad (171)$$

where

$$\begin{aligned} \pi_1 &:= 1 + 9b_0b_3 - 8a_3b_0(-32 + 3b_3), \\ \pi_2 &:= -81b_3^2 - 144a_3b_3(-32 + 3b_3) + 64a_3^2(-1024 + 192b_3 + 99b_3^2). \end{aligned} \quad (172)$$

Now, even though at first sight the system might not look so manageable, as we will show with some simple observations we can get exact solutions quite easily.

Purely spin hypermomentum solutions.—To begin with, we note that the remaining spin part ζ will be related to ψ with a barotropic equation of state of the form $\zeta = w_\zeta\psi$, where w_ζ is the barotropic index related to this spin part (cf. also

¹⁶This situation shares a certain similarity with the negative temperature system in thermodynamics. It is known that in given thermodynamical systems some property of the system can possess negative temperature but the net temperature will always be positive.

From the right-hand side of the above equation, we note that the last term, keeping a fixed negative sign, always decelerates the expansion. However, there is also the presence of the first and second terms which can radically change the net result. As we will show shortly in the following, the actual effect of the spin part of hypermomentum on the acceleration depends on the parameters of the quadratic theory. Having obtained the acceleration equation above we can then use it in order to eliminate the double derivative term $\frac{\ddot{a}}{a}$ from the continuity equation, which results in

$$\dot{\psi} + (1 + \mu_2)H\psi + \mu_3\kappa\psi^2 = 0, \quad (167)$$

where we have also introduced

$$\begin{aligned} \mu_2 &:= 96a_3b_3\nu_1, \\ \mu_3 &:= -\frac{2304}{b_0}a_3^2b_3^2\nu_1, \end{aligned} \quad (168)$$

with

$$\nu_1 := \frac{1}{[-9b_3 + 8a_3(-32 + 15b_3) + b_0\mu_1]}. \quad (169)$$

On the other hand, the first modified Friedmann equation (104) now becomes

Appendix B of [40]). Next we may use (167) in order to eliminate the derivative term $\dot{\psi}$ from the modified first Friedmann equation above, ending up with

$$H^2 - \lambda_1H\psi - \lambda_2\psi^2 = 0, \quad (173)$$

where

$$\lambda_1 := \frac{\kappa}{2b_0^2}(2 - \mu_2\pi_1), \quad \lambda_2 := \frac{\kappa^2}{2b_0^2} \left(-\pi_1\mu_3 + \frac{\pi_2 + b_0^2w_\zeta^2}{2} \right). \quad (174)$$

Then, we observe that (173) is a simple quadratic equation, which, considering H as the unknown variable, admits the solutions (for $\lambda_1^2 + 4\lambda_2 > 0$)

$$H = \lambda_0 \psi, \quad \lambda_0 = \frac{\lambda_1 \pm \sqrt{\lambda_1^2 + 4\lambda_2}}{2}. \quad (175)$$

Substituting this back into the continuity equation for ψ it follows that

$$\dot{\psi} = -\mu_0 \psi^2, \quad \mu_0 = \lambda_0(1 + \mu_2) + \kappa\mu_3, \quad (176)$$

which trivially integrates to

$$\psi(t) = \frac{1}{c_1 + \mu_0 t}, \quad (177)$$

where c_1 is an arbitrary integration constant. Finally, substituting this form for ψ back into (175) and integrating, we find the following expression for the scale factor:

$$a(t) = c_2(c_1 + \mu_0 t)^{\frac{\lambda_0}{\mu_0}}, \quad (178)$$

where c_1 and c_2 are integration constants that can be expressed in terms of the values a_0 and ψ_0 for some fixed time t_0 . Therefore, we have established exact solutions for both the scale factor and the spin hypermomentum variable ψ . We see that we get power-law solutions for the scale factor and inverse time evolution for the spin part. As a result, ψ diminishes with the passing of time, while the scale factor goes like

$$a \propto t^{\frac{\lambda_0}{\mu_0}}. \quad (179)$$

It is also interesting to study the two limits $\mu_0 \rightarrow 0$ and $\mu_0 \rightarrow \infty$. In the former case the spin concentration (ψ) becomes constant and subsequently we get de Sitter-like expansion for the scale factor $a \propto e^{H_0 t}$. Hence we see that a constant spin distribution produces an exponential expansion. In the latter case (i.e., $\mu_0 \rightarrow \infty$) ψ essentially vanishes, resulting also in $H = 0$ and yielding, therefore, a static universe. These cover the two extreme cases and for the rest in between we have the nice power-law solutions we derived above.

V. CONCLUSIONS

We have investigated cosmological aspects of the most general parity preserving MAG theory quadratic in torsion and nonmetricity in the presence of a cosmological hyperfluid, deriving the modified Friedmann equations in a FLRW background. The theory exhibits, in principle, 11 parameters coming as coefficients of the 11 quadratic torsion and nonmetricity scalars supplementing the Einstein-Hilbert term in the gravitational action. For this general quadratic MAG theory we have subsequently derived the most general form of the modified Friedmann equations, along with the associated conservation laws for the sources, in the presence of a cosmological hyperfluid [40,41]. Then, in order to get a grasp of the

effect of the quadratic terms and also simplify the analysis, we have focused on a particular but fairly representative subcase involving only the two quadratic contributions with the parameters a_3 and b_3 , namely $a_3 Q_\mu Q^\mu$ and $b_3 S_\mu S^\mu$, which are respectively quadratic in the nonmetricity vector Q_μ and in the torsion vector S_μ . For this case we have studied the modified Friedmann equations along with the conservation laws of the perfect cosmological hyperfluid and we have been able to provide exact solutions for particular types of hypermomentum, that of purely dilation and purely spin matter. It is worth mentioning that for the pure dilation case if the torsion squared term is absent (that is for $b_3 = 0$), quite remarkably, the geometry is fixed to be that of Weyl-Cartan type. On the other hand, for the pure spin case, interestingly, the presence of $b_3 \neq 0$ is essential to guarantee nonvanishing nonmetricity, while nonvanishing a_3 is required in order for the torsion not to be zero. This is a quite compelling result showing a dualistic relation between torsion and nonmetricity, that is in order to have nonvanishing nonmetricity the pure torsion term must be present and the torsion is nonvanishing if the pure nonmetricity term is there, as seen from Eqs. (153)–(157).

Now, coming back to the hypermomentum parts, regarding pure dilation we have found that, for early times in which non-Riemannian effects are dominant, dilation causes a cosmic deceleration and its effect is indistinguishable from that of stiff matter.¹⁷ A very similar result was found for the purely shear counterpart of hypermomentum that was studied in [42]. With these two results, one could go as far as to say that the rather unconventional character of “stiff” matter component acquires a clear physical meaning as a specific part of the hypermomentum contribution (either shear or dilation) of the hyperfluid. On the other hand, concerning purely spin hypermomentum, we have obtained power-law solutions for the scale factor and inverse time evolution for the spin part ψ . In particular, ψ diminishes with the passing of time, while the scale factor behavior is given by Eq. (179). Here, two particular limits emerge: one in which we have an exponential expansion for the scale factor induced by a constant spin distribution and the other in which the result is a static universe ($H = 0$). We have therefore proven that the presence of torsion and nonmetricity driven by dilation and spin hypermomentum types in the context of quadratic MAG can have highly nontrivial effects on the cosmological evolution.

It would be interesting to study also the purely shear hypermomentum in this context. Another possible future development of the present work could consist in performing the cosmological analysis by including also parity odd quadratic terms into the gravitational part of the action which also include the so-called Hojman [57] (sometimes also referred to Holst [58]) parity violating term, whose contribution in the framework of MAG has been recently

¹⁷The same result was also reported in the analysis of [47].

presented and studied to some extent in [8]. Finally, it would be of much interest to investigate the possible changes implied by the inclusion of mixed torsion-nonmetricity terms such as $S_\mu Q^\mu$ on top of the pure torsion and pure nonmetricity terms we have considered here.

ACKNOWLEDGMENTS

D. I. acknowledge that this research is co-financed by Greece and the European Union [European Social Fund (ESF)] through the Operational Programme ‘‘Human Resources Development, Education and Lifelong Learning’’ in the context of the project ‘‘Reinforcement of Postdoctoral Researchers—2nd Cycle’’ (MIS-5033021), implemented by the State Scholarships Foundation (IKY). L.R. would like to thank the Department of Applied Science and Technology of the Polytechnic University of Turin, and in particular Laura Andrianopoli and Francesco Raffa, for financial support.

APPENDIX: USEFUL FORMULAS IN QUADRATIC MAG AND COSMOLOGICAL ASPECTS

In this Appendix we collect some formulas useful for the study of quadratic MAG and cosmological aspects of the latter that we have pursued in the main text.

1. Hypermomentum and Palatini tensor contractions

Considering the cosmological setup described in Sec. II and the hypermomentum in Eq. (60), we find the following contractions:

$$\begin{aligned} h^{\alpha\mu} \Delta_{\alpha\mu\nu} &= (n-1)\phi u_\nu, \\ h^{\alpha\nu} \Delta_{\alpha\mu\nu} &= (n-1)\chi u_\mu, \\ h^{\mu\nu} \Delta_{\alpha\mu\nu} &= (n-1)\psi u_\alpha, \\ \epsilon^{\alpha\mu\nu\lambda} \Delta_{\alpha\mu\nu} &= -6u^\lambda \zeta \delta_4^n, \\ u^\alpha u^\mu u^\nu \Delta_{\alpha\mu\nu} &= -\omega, \end{aligned} \quad (A1)$$

that is

$$\begin{aligned} \phi &= -\frac{1}{(n-1)} h^{\alpha\mu} u^\nu \Delta_{\alpha\mu\nu}, \\ \chi &= -\frac{1}{(n-1)} h^{\alpha\nu} u^\mu \Delta_{\alpha\mu\nu}, \\ \psi &= -\frac{1}{(n-1)} h^{\mu\nu} u^\alpha \Delta_{\alpha\mu\nu}, \\ \zeta &= \frac{1}{6} \epsilon^{\alpha\mu\nu\lambda} \Delta_{\alpha\mu\nu} u_\lambda \delta_4^n, \\ \omega &= -u^\alpha u^\mu u^\nu \Delta_{\alpha\mu\nu}. \end{aligned} \quad (A2)$$

On the other hand, regarding the Palatini tensor, whose definition is given in (70), we get

$$\begin{aligned} P_{\alpha\mu\nu} &= \left[\frac{(n-3)}{2} A + 2(n-2)\Phi - \frac{C}{2} \right] u_\alpha h_{\mu\nu} \\ &+ \left[\frac{(n-2)}{2} B - \frac{(n-1)}{2} A - 2(n-2)\Phi - \frac{C}{2} \right] u_\mu h_{\alpha\nu} \\ &- \frac{B}{2} u_\nu h_{\mu\alpha} - \frac{(n-1)}{2} B u_\alpha u_\mu u_\nu - 2\epsilon_{\alpha\mu\nu\rho} u^\rho P \delta_4^n. \end{aligned} \quad (A3)$$

Then, we have

$$\begin{aligned} h^{\alpha\mu} P_{\alpha\mu\nu} &= -\frac{(n-1)}{2} B u_\nu, \\ h^{\alpha\nu} P_{\alpha\mu\nu} &= (n-1) \\ &\times \left[\frac{(n-2)}{2} B - \frac{(n-1)}{2} A - 2(n-2)\Phi - \frac{C}{2} \right] u_\mu, \\ h^{\mu\nu} P_{\alpha\mu\nu} &= (n-1) \left[\frac{(n-3)}{2} A + 2(n-2)\Phi - \frac{C}{2} \right] u_\alpha, \\ \epsilon^{\alpha\mu\nu\lambda} P_{\alpha\mu\nu} &= 12P u^\lambda \delta_4^n, \\ u^\alpha u^\mu u^\nu P_{\alpha\mu\nu} &= \frac{(n-1)}{2} B \end{aligned} \quad (A4)$$

as fundamental contractions.

2. Variation of the quadratic nonmetricity terms with respect to the metric

Let us report here some calculations regarding the variation of the quadratic pure nonmetricity terms in the general quadratic theory (64), as it is slightly more complicated with respect to the variation of the other quadratic terms. First of all, note that the pure nonmetricity quadratic combinations in the general quadratic model can be written as

$$\begin{aligned} \mathcal{L}_Q &= a_1 Q_{\alpha\mu\nu} Q^{\alpha\mu\nu} + a_2 Q_{\alpha\mu\nu} Q^{\mu\nu\alpha} + a_3 Q_\mu Q^\mu \\ &+ a_4 q_\mu q^\mu + a_5 Q_\mu q^\mu \\ &= Q_\alpha^{\beta\gamma} Q_\sigma^{\lambda\rho} (a_1 g^{\sigma\alpha} g_{\lambda\beta} g_{\gamma\rho} + a_2 \delta_\beta^\sigma \delta_\rho^\alpha g_{\gamma\lambda} \\ &+ a_3 g_{\beta\gamma} g_{\lambda\rho} g^{\sigma\alpha} + a_4 \delta_\lambda^\sigma \delta_\beta^\alpha g_{\gamma\rho} + a_5 \delta_\lambda^\sigma \delta_\rho^\alpha g_{\beta\gamma}). \end{aligned} \quad (A5)$$

Then, defining

$$\begin{aligned} D^{\sigma\alpha}{}_{\lambda\beta\gamma\rho} &:= (a_1 g^{\sigma\alpha} g_{\lambda\beta} g_{\gamma\rho} + a_2 \delta_\beta^\sigma \delta_\rho^\alpha g_{\gamma\lambda} + a_3 g_{\beta\gamma} g_{\lambda\rho} g^{\sigma\alpha} \\ &+ a_4 \delta_\lambda^\sigma \delta_\beta^\alpha g_{\gamma\rho} + a_5 \delta_\lambda^\sigma \delta_\rho^\alpha g_{\beta\gamma}), \end{aligned} \quad (A6)$$

we have

$$\mathcal{L}_Q = Q_\alpha^{\beta\gamma} Q_\sigma^{\lambda\rho} D^{\sigma\alpha}{}_{\lambda\beta\gamma\rho}, \quad (A7)$$

and, introducing the quantities

$$\begin{aligned}
(1)\Omega^\alpha{}_{\beta\gamma} &:= Q_\sigma{}^{\lambda\rho} D^{\sigma\alpha}{}_{\lambda\beta\gamma\rho} = a_1 Q^\alpha{}_{\beta\gamma} + a_2 Q_{\beta\gamma}{}^\alpha + a_3 Q^\alpha g_{\beta\gamma} + a_4 \delta_\beta^\alpha q_\gamma + a_5 q^\alpha g_{\beta\gamma}, \\
(2)\Omega^\sigma{}_{\lambda\rho} &:= Q_\alpha{}^{\beta\gamma} D^{\sigma\alpha}{}_{\lambda\beta\gamma\rho} = a_1 Q^\sigma{}_{\lambda\rho} + a_2 Q_{\rho\lambda}{}^\sigma + a_3 Q^\sigma g_{\lambda\rho} + a_4 \delta_\lambda^\sigma q_\rho + a_5 \delta_\lambda^\sigma Q_\rho,
\end{aligned} \tag{A8}$$

it follows that

$$\mathcal{L}_Q = (1)\Omega^\alpha{}_{\beta\gamma} Q_\alpha{}^{\beta\gamma} = (2)\Omega^\sigma{}_{\lambda\rho} Q_\sigma{}^{\lambda\rho}. \tag{A9}$$

Accordingly, we may express the variation with respect to the metric as

$$\delta_g(\sqrt{-g}\mathcal{L}_Q) = -\frac{1}{2}\sqrt{-g}\mathcal{L}_Q g_{\mu\nu} \delta g^{\mu\nu} + \sqrt{-g}\delta_g \mathcal{L}_Q, \tag{A10}$$

and, after some algebra, we find

$$\begin{aligned}
\delta_g(\sqrt{-g}\mathcal{L}_Q) &= \delta g^{\mu\nu} \sqrt{-g} \left[-\frac{1}{2}\mathcal{L}_Q g_{\mu\nu} + \frac{1}{\sqrt{-g}} \hat{\nabla}_\alpha(\sqrt{-g}W^\alpha{}_{(\mu\nu)}) + a_1(Q_{\mu\alpha\beta}Q_\nu{}^{\alpha\beta} - 2Q_{\alpha\beta\mu}Q^{\alpha\beta}{}_\nu) - a_2 Q_{\alpha\beta(\mu}Q^{\beta\alpha}{}_{\nu)} \right. \\
&\quad \left. + a_3(Q_\mu Q_\nu - 2Q^\alpha Q_{\alpha\mu\nu}) - a_4 q_\mu q_\nu - a_5 q^\alpha Q_{\alpha\mu\nu} \right],
\end{aligned} \tag{A11}$$

where $\hat{\nabla}$ is defined in (19) and $W^\alpha{}_{(\mu\nu)}$ is given by (67).

3. Advantageous relations for FLRW cosmology of quadratic MAG

Let us now collect some important relations we used throughout our study of the FLRW cosmology of quadratic MAG. First, let us compute the form of the quadratic Lagrangian

$$\begin{aligned}
\mathcal{L}_2 &:= b_1 S_{\alpha\mu\nu} S^{\alpha\mu\nu} + b_2 S_{\alpha\mu\nu} S^{\mu\nu\alpha} + b_3 S_\mu S^\mu \\
&\quad + a_1 Q_{\alpha\mu\nu} Q^{\alpha\mu\nu} + a_2 Q_{\alpha\mu\nu} Q^{\mu\nu\alpha} + a_3 Q_\mu Q^\mu + a_4 q_\mu q^\mu + a_5 Q_\mu q^\mu \\
&\quad + c_1 Q_{\alpha\mu\nu} S^{\alpha\mu\nu} + c_2 Q_\mu S^\mu + c_3 q_\mu S^\mu.
\end{aligned}$$

Using the cosmological forms of torsion and nonmetricity tensors given by (32), after some straightforward algebra we find

$$\begin{aligned}
\mathcal{L}_2 &= -(n-1)[a_1 + (n-1)a_3]A^2 - \frac{(n-1)}{4}[2a_1 + a_2 + (n-1)a_4]B^2 - (a_1 + a_2 + a_3 + a_4 + a_5)C^2 \\
&\quad - (n-1)\left[a_2 + \frac{(n-1)}{2}a_5\right]AB + (n-1)(2a_3 + a_5)AC + (n-1)\left(a_4 + \frac{1}{2}a_5\right)BC \\
&\quad - (n-1)[2b_1 - b_2 + (n-1)b_3]\Phi^2 + 6(b_1 + b_2)P^2\delta_4^n \\
&\quad - (n-1)[c_1 + (n-1)c_2]A\Phi + \frac{(n-1)}{2}[c_1 - (n-1)c_3]B\Phi + (n-1)(c_2 + c_3)C\Phi.
\end{aligned} \tag{A12}$$

Notice that all A , B , C and Φ couple among each other, while the pseudoscalar mode P appears only as P^2 . Of course there is no surprise here, since P is a pseudoscalar, the total Lagrangian \mathcal{L}_2 must be a scalar, and the only combination in P that can yield a scalar result is P^2 (i.e., indeed, pseudoscalar \times pseudoscalar). Here we also compute the following quantities:

$$\begin{aligned}
 W^\alpha{}_{\mu\nu}u^\mu u^\nu &= 2(a_1 + a_2 + a_3 + a_4 + a_5)Cu^\alpha - (n-1)(2a_3 + a_5)Au^\alpha - (n-1)(a_5 + 2a_4)\frac{B}{2}u^\alpha, \\
 \Pi^\alpha{}_{\mu\nu}u^\mu u^\nu &= -(n-1)(c_2 + c_3)\Phi u^\alpha, \\
 W^\alpha{}_{(\mu\alpha)} &= \left[a_2 + 2a_3 + \frac{(n+1)}{2}a_5 \right] Q_\mu + [2a_1 + a_2 + 2a_3 + a_5 + (n+1)a_4]q_\mu \\
 &= \frac{1}{2}(n-1)[2a_2 + 4a_3 + (n+1)a_5]Au_\mu + \frac{1}{2}(n-1)[2a_1 + a_2 + a_5 + (n+1)a_4]Bu_\mu \\
 &\quad + \frac{1}{2}[-4a_1 - 4a_2 - 4a_3 - 2(n+1)a_4 - a_5(n+3)]Cu_\mu, \\
 \Pi^\alpha{}_{(\mu\alpha)} &= \left[-\frac{c_1}{2} + c_2 + \frac{(n+1)}{2}c_3 \right] S_\mu = [-c_1 + 2c_2 + (n+1)c_3]\Phi u_\mu, \\
 W_{\alpha\mu\nu}u^\alpha u^\mu u^\nu &= 2(a_1 + a_2)Q_{\alpha\mu\nu}u^\alpha u^\mu u^\nu - (2a_3 + a_5)Q_\mu u^\mu - (a_5 + 2a_4)q_\mu u^\mu \\
 &= -2(a_1 + a_2 + a_3 + a_4 + a_5)C + (n-1)(2a_3 + a_5)A + (n-1)(a_5 + 2a_4)\frac{B}{2}, \\
 \Pi_{\alpha\mu\nu}u^\alpha u^\mu u^\nu &= -(c_2 + c_3)S_\mu u^\mu = (n-1)(c_2 + c_3)\Phi.
 \end{aligned} \tag{A13}$$

Besides, given (78) and specializing to the FLRW cosmology case we considered in the main text [i.e., using (32)], we find

$$\begin{aligned}
 (A_{\mu\nu} + B_{\mu\nu} + C_{\mu\nu})u^\mu u^\nu &= (n-1)[a_1 + (n-1)a_3]A^2 - \frac{1}{4}[(2a_1 + a_2) + (n-1)a_4](n-1)B^2 \\
 &\quad - (a_1 + a_2 + a_3 + a_4 + a_5)C^2 + (n-1)\left(a_4 + \frac{a_5}{2}\right)BC \\
 &\quad + (n-1)[2b_1 - b_2 + (n-1)b_3]\Phi^2 + (n-1)[c_1 + (n-1)c_2]\Phi A.
 \end{aligned} \tag{A14}$$

Note that there are no AC , AB , ΦB , ΦC terms in the equation above, that is A and Φ couple only among themselves and so do B and C . Observe also that here P does not appear. Moreover, after some long calculations we find

$$\begin{aligned}
 -2(\Pi^\alpha{}_{(\mu\nu)} + W^\alpha{}_{(\mu\nu)})u^\mu \nabla_\alpha u^\nu &= (n-1)(H - Y)[\gamma_1 B + 2\gamma_2 A + 2\gamma_3 \Phi] \\
 &\quad + \frac{(n-1)}{2}C[(a_5 + 2a_4)B + 2(2a_3 + a_5)A + 2(c_2 + c_3)\Phi] \\
 &\quad - 2(a_1 + a_2 + a_3 + a_4 + a_5)C^2 - (n-1)(2a_4 + a_5)(H - Y)C,
 \end{aligned} \tag{A15}$$

where

$$\gamma_1 = 2a_1 + a_2 + (n-1)a_4, \quad \gamma_2 = a_2 + \frac{(n-1)}{2}a_5, \quad \gamma_3 = \frac{1}{2}[-c_1 + (n-1)c_3]. \tag{A16}$$

Gathering all the above results and combining them with (77), we finally arrive at

$$\begin{aligned}
 R_{\mu\nu}u^\mu u^\nu &= \frac{\kappa}{(n-2)}(T + (n-2)T_{\mu\nu}u^\mu u^\nu) \\
 &\quad - \left\{ (n-1)[a_1 + (n-1)a_3]A^2 - \frac{1}{4}[(2a_1 + a_2) + (n-1)a_4](n-1)B^2 \right. \\
 &\quad - (a_1 + a_2 + a_3 + a_4 + a_5)C^2 + (n-1)\left(a_4 + \frac{a_5}{2}\right)BC \\
 &\quad \left. + (n-1)[2b_1 - b_2 + (n-1)b_3]\Phi^2 + (n-1)[c_1 + (n-1)c_2]\Phi A \right\} \\
 &\quad + (n-1)(H - Y)[\gamma_1 B + 2\gamma_2 A + 2\gamma_3 \Phi] + \frac{(n-1)}{2}C[(a_5 + 2a_4)B + 2(2a_3 + a_5)A + 2(c_2 + c_3)\Phi] \\
 &\quad - 2(a_1 + a_2 + a_3 + a_4 + a_5)C^2 - (n-1)(2a_4 + a_5)(H - Y)C \\
 &\quad + \dot{l} + (n-1)Hl,
 \end{aligned} \tag{A17}$$

where

$$l = \beta_1 A + \beta_2 B + \beta_3 C + \beta_4 \Phi, \quad (\text{A18})$$

with

$$\begin{aligned} \beta_1 &= \left(\frac{n-1}{n-2}\right)[2a_1 + 4a_3 - a_5(n-3)], \\ \beta_2 &= \left(\frac{n-1}{n-2}\right)[a_2 - a_4(n-3) + a_5], \\ \beta_3 &= \frac{1}{(n-2)}[2a_1(n-3) + 2a_2(n-3) - 4a_3 + 2a_4(n-3) + a_5(n-5)], \\ \beta_4 &= \left(\frac{n-1}{n-2}\right)[c_1 + 2c_2 - c_3(n-3)]. \end{aligned} \quad (\text{A19})$$

All these formulas are particularly useful to reproduce the results we have presented in Sec. IV.

4. Coefficients in quadratic MAG cosmology

For the sake of completeness, in the following we report the explicit expressions of the coefficients α_{ij} appearing in Eqs. (90)–(93) in terms of the a 's, b 's, and c 's. They read

$$\begin{aligned} \alpha_{11} &= 2(n-1)(2a_3 + a_5), & \alpha_{12} &= (n-1)\left(\frac{1}{2} + a_5 + 2a_4\right), \\ \alpha_{13} &= -4(a_1 + a_3 + a_4 + a_5), & \alpha_{14} &= 2(n-1)(c_2 + c_3), \\ \alpha_{21} &= \frac{(n-3)}{2} + (n-1)a_5, & \alpha_{22} &= 2a_1 + (n-1)a_4, \\ \alpha_{23} &= -\left(\frac{1}{2} + 2a_4 + a_5\right), & \alpha_{24} &= 2(n-2) + (n-1)c_3, \\ \alpha_{31} &= 4a_1 + 4(n-1)a_3 - \frac{(n-1)}{2}c_2, & \alpha_{32} &= (n-1)a_5 - \frac{(n-1)}{4}c_3 - \frac{1}{2}, \\ \alpha_{33} &= \frac{1}{2}(c_2 + c_3) - 4a_3 - 2a_5, & \alpha_{34} &= 2(n-1)c_2 - 2b_1 - (n-1)b_3, \\ \alpha_{41} &= (n-1)\left(-\frac{1}{2} + a_5 + \frac{c_2}{2}\right), & \alpha_{42} &= \frac{(n-2)}{2} + 2a_1 + (n-1)a_4 + \frac{(n-1)}{4}c_3, \\ \alpha_{43} &= -\left[\frac{1}{2} + 2a_4 + a_5 + \frac{c_2 + c_3}{2}\right], & \alpha_{44} &= -2(n-2) + 2b_1 + (n-1)c_3 + (n-1)b_3. \end{aligned} \quad (\text{A20})$$

These are the elements of the matrix M we have introduced in Sec. IV.

-
- [1] E. N. Saridakis *et al.* (CANTATA Collaboration), Modified gravity and cosmology: An update by the CANTATA network, [arXiv:2105.12582](https://arxiv.org/abs/2105.12582).
 [2] L. P. Eisenhart, *Non-Riemannian Geometry*, American Mathematical Society Vol. VIII (Colloquium Publications, 1927), p. 184.

- [3] F. W. Hehl, J. D. McCrea, E. W. Mielke, and Y. Ne'eman, Metric affine gauge theory of gravity: Field equations, Noether identities, world spinors, and breaking of dilation invariance, *Phys. Rep.* **258**, 1 (1995).
 [4] F. W. Hehl and A. Macias, Metric affine gauge theory of gravity. 2. Exact solutions, *Int. J. Mod. Phys. D* **08**, 399 (1999).

- [5] D. Iosifidis, Metric-affine gravity and cosmology/aspects of torsion and non-metricity in gravity theories, [arXiv:1902.09643](#).
- [6] S. Klemm and L. Ravera, An action principle for the Einstein–Weyl equations, *J. Geom. Phys.* **158**, 103958 (2020).
- [7] S. Klemm and L. Ravera, Schrödinger connection with self-dual nonmetricity vector in $2 + 1$ dimensions, *Phys. Lett. B* **817**, 136291 (2021).
- [8] D. Iosifidis and L. Ravera, Parity violating metric-affine gravity theories, *Classical Quantum Gravity* **38**, 115003 (2021).
- [9] D. S. Klemm and L. Ravera, Einstein manifolds with torsion and nonmetricity, *Phys. Rev. D* **101**, 044011 (2020).
- [10] D. S. Klemm and L. Ravera, Supersymmetric near-horizon geometry and Einstein-Cartan-Weyl spaces, *Phys. Lett. B* **793**, 265 (2019).
- [11] S. Ariwahjoedi, A. Suroso, and F.P. Zen, $(3 + 1)$ -formulation for gravity with torsion and non-metricity: The stress-energy-momentum equation, *Classical Quantum Gravity* **38**, 155009 (2021).
- [12] S. Ariwahjoedi, A. Suroso, and F.P. Zen, $(3 + 1)$ -formulation for gravity with torsion and non-metricity II: The hypermomentum equation, *Classical Quantum Gravity* **38**, 225006 (2021).
- [13] F.W. Hehl, G.D. Kerlick, and P. Von Der Heyde, On hypermomentum in general relativity. 1. The notion of hypermomentum, *Z. Naturforsch. A* **31**, 111 (1976).
- [14] F.W. Hehl, G.D. Kerlick, and P. Von Der Heyde, On hypermomentum in general relativity. 2. The geometry of space-time, *Z. Naturforsch. A* **31**, 524 (1976).
- [15] D. Iosifidis, C. G. Tsagas, and A. C. Petkou, Raychaudhuri equation in spacetimes with torsion and nonmetricity, *Phys. Rev. D* **98**, 104037 (2018).
- [16] D. Iosifidis, Exactly solvable connections in metric-affine gravity, *Classical Quantum Gravity* **36**, 085001 (2019).
- [17] S. Capozziello and S. Vignolo, Metric-affine $f(R)$ -gravity with torsion: An overview, *Ann. Phys. (Berlin)* **19**, 238 (2010).
- [18] R. Percacci and E. Sezgin, New class of ghost- and tachyon-free metric affine gravities, *Phys. Rev. D* **101**, 084040 (2020).
- [19] J. B. Jiménez and A. Delhom, Instabilities in metric-affine theories of gravity with higher order curvature terms, *Eur. Phys. J. C* **80**, 585 (2020).
- [20] K. Aoki and K. Shimada, Scalar-metric-affine theories: Can we get ghost-free theories from symmetry?, *Phys. Rev. D* **100**, 044037 (2019).
- [21] F. Cabral, F. S. N. Lobo, and D. Rubiera-Garcia, Fundamental symmetries and spacetime geometries in gauge theories of gravity—prospects for unified field theories, *Universe* **6**, 238 (2020).
- [22] T. Helpin and M. S. Volkov, A metric-affine version of the Horndeski theory, *Int. J. Mod. Phys. A* **35**, 2040010 (2020).
- [23] S. Bahamonde and J. G. Valcarcel, New models with independent dynamical torsion and nonmetricity fields, *J. Cosmol. Astropart. Phys.* **09** (2020) 057.
- [24] J. B. Jimenez and T. S. Koivisto, Spacetimes with vector distortion: Inflation from generalized Weyl geometry, *Phys. Lett. B* **756**, 400 (2016).
- [25] J. B. Jiménez and T. S. Koivisto, Modified gravity with vector distortion and cosmological applications, *Universe* **3**, 47 (2017).
- [26] D. Kranas, C. G. Tsagas, J. D. Barrow, and D. Iosifidis, Friedmann-like universes with torsion, *Eur. Phys. J. C* **79**, 341 (2019).
- [27] C. Barragan, G. J. Olmo, and H. Sanchis-Alepuz, Bouncing cosmologies in Palatini $f(R)$ gravity, *Phys. Rev. D* **80**, 024016 (2009).
- [28] K. Shimada, K. Aoki, and K. I. Maeda, Metric-affine gravity and inflation, *Phys. Rev. D* **99**, 104020 (2019).
- [29] M. Kubota, K. Y. Oda, K. Shimada, and M. Yamaguchi, Cosmological perturbations in Palatini formalism, *J. Cosmol. Astropart. Phys.* **03** (2021) 006.
- [30] Y. Mikura, Y. Tada, and S. Yokoyama, Conformal inflation in the metric-affine geometry, *Europhys. Lett.* **132**, 39001 (2020).
- [31] Y. Mikura, Y. Tada, and S. Yokoyama, Minimal k -inflation in light of the conformal metric-affine geometry, *Phys. Rev. D* **103**, L101303 (2021).
- [32] F. Bombacigno and G. Montani, Big bounce cosmology for Palatini R^2 gravity with a Nieh-Yan term, *Eur. Phys. J. C* **79**, 405 (2019).
- [33] F. Bombacigno, S. Boudet, G. J. Olmo, and G. Montani, Big bounce and future time singularity resolution in Bianchi I cosmologies: The projective invariant Nieh-Yan case, *Phys. Rev. D* **103**, 124031 (2021).
- [34] D. Iosifidis, Riemann tensor and gauss-bonnet density in metric-affine cosmology, *Classical Quantum Gravity* **38**, 195028 (2021).
- [35] Y. N. Obukhov and R. Tresguerres, Hyperfluid: A model of classical matter with hypermomentum, *Phys. Lett. A* **184**, 17 (1993).
- [36] Y. N. Obukhov, On a model of an unconstrained hyperfluid, *Phys. Lett. A* **210**, 163 (1996).
- [37] O. V. Babourova and B. N. Frolov, Perfect hypermomentum fluid: Variational theory and equations of motion, *Int. J. Mod. Phys. A* **13**, 5391 (1998).
- [38] J. R. Ray and L. L. Smalley, Spinning fluids in the Einstein-Cartan theory, *Phys. Rev. D* **27**, 1383 (1983).
- [39] L. L. Smalley, Fluids with spin and twist, *J. Math. Phys. (N.Y.)* **36**, 778 (1995).
- [40] D. Iosifidis, Cosmological hyperfluids, torsion and non-metricity, *Eur. Phys. J. C* **80**, 1042 (2020).
- [41] D. Iosifidis, The perfect hyperfluid of metric-affine gravity: The foundation, *J. Cosmol. Astropart. Phys.* **04** (2021) 072.
- [42] D. Iosifidis, Non-Riemannian cosmology: The role of shear hypermomentum, *Int. J. Geom. Methods Mod. Phys.* **18**, 2150129 (2021).
- [43] R. W. Tucker and C. Wang, Black holes with Weyl charge and nonRiemannian waves, *Classical Quantum Gravity* **12**, 2587 (1995).
- [44] Y. N. Obukhov, E. J. Vlachynsky, W. Esser, R. Tresguerres, and F. W. Hehl, An exact solution of the metric affine gauge theory with dilation, shear, and spin charges, *Phys. Lett. A* **220**, 1 (1996).
- [45] E. J. Vlachynsky, R. Tresguerres, Y. N. Obukhov, and F. W. Hehl, An axially symmetric solution of metric affine gravity, *Classical Quantum Gravity* **13**, 3253 (1996).

- [46] T. Dereli, M. Onder, J. Schray, R. W. Tucker, and C. Wang, NonRiemannian gravity and the Einstein-Proca system, *Classical Quantum Gravity* **13**, L103 (1996).
- [47] Y. N. Obukhov, E. J. Vlachynsky, W. Esser, and F. W. Hehl, Irreducible decompositions in metric affine gravity models, [arXiv:gr-qc/9705039](https://arxiv.org/abs/gr-qc/9705039).
- [48] V. Vitagliano, T. P. Sotiriou, and S. Liberati, The dynamics of metric-affine gravity, *Ann. Phys. (Amsterdam)* **326**, 1259 (2011); *Ann. Phys. (Amsterdam)* **329**, 186(E) (2013).
- [49] P. Baekler and F. W. Hehl, Beyond Einstein-Cartan gravity: Quadratic torsion and curvature invariants with even and odd parity including all boundary terms, *Classical Quantum Gravity* **28**, 215017 (2011).
- [50] V. Vitagliano, The role of nonmetricity in metric-affine theories of gravity, *Classical Quantum Gravity* **31**, 045006 (2014).
- [51] D. Benisty, E. I. Guendelman, D. Vasak, J. Struckmeier, and H. Stoecker, Quadratic curvature theories formulated as covariant canonical gauge theories of gravity, *Phys. Rev. D* **98**, 106021 (2018).
- [52] D. Benisty, E. I. Guendelman, A. van de Venn, D. Vasak, J. Struckmeier, and H. Stoecker, The Dark side of the torsion: Dark Energy from kinetic torsion, [arXiv:2109.01052](https://arxiv.org/abs/2109.01052).
- [53] D. Iosifidis and L. Ravera, The cosmology of quadratic torsionful gravity, *Eur. Phys. J. C* **81**, 736 (2021).
- [54] Y. N. Obukhov and D. Puetzfeld, Conservation laws in gravity: A unified framework, *Phys. Rev. D* **90**, 024004 (2014).
- [55] D. Iosifidis and T. Koivisto, Scale transformations in metric-affine geometry, *Universe* **5**, 82 (2019).
- [56] D. Iosifidis, Cosmic acceleration with torsion and non-metricity in Friedmann-like Universes, *Classical Quantum Gravity* **38**, 015015 (2021).
- [57] R. Hojman, C. Mukku, and W. A. Sayed, Parity violation in metric-torsion theories of gravitation, *Phys. Rev. D* **22**, 1915 (1980).
- [58] S. Holst, Barbero's Hamiltonian derived from a generalized Hilbert-Palatini action, *Phys. Rev. D* **53**, 5966 (1996).