

# A note on the natural density of product sets

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## ABSTRACT

Given two sets of natural numbers  $\mathcal{A}$  and  $\mathcal{B}$  of natural density 1, we prove that their product set  $\mathcal{A} \cdot \mathcal{B} := \{ab : a \in \mathcal{A}, b \in \mathcal{B}\}$  also has natural density 1. On the other hand, for any  $\varepsilon > 0$ , we show there are sets  $\mathcal{A}$  of density  $> 1 - \varepsilon$  for which the product set  $\mathcal{A} \cdot \mathcal{A}$  has density  $< \varepsilon$ . This answers two questions of Hegyvári, Hennecart and Pach.

## 1. Introduction

Given two sets of natural numbers  $\mathcal{A}$  and  $\mathcal{B}$ , let  $\mathcal{A} \cdot \mathcal{B} := \{ab : a \in \mathcal{A}, b \in \mathcal{B}\}$  be their *product set*. Also, for any positive integer  $k$ , let  $\mathcal{A}^k$  denote the  $k$ -fold product  $\mathcal{A} \cdots \mathcal{A}$ .

The problem of studying the cardinality of product sets has long been of interest in mathematics. The classic *multiplication table problem* due to Erdős [2, 3] asks for bounds on the cardinality  $M_n$  of the  $n \times n$  multiplication table, that is, of the set  $\{1, \dots, n\}^2$ . Erdős showed that  $M_n = o(n^2)$  and Ford [5], following earlier results of Tenenbaum [11], determined the exact order of magnitude of  $M_n$ . More recently [7], the second author of the present paper provided uniform bounds for  $\#\{1, \dots, n_1\} \cdots \{1, \dots, n_s\}$  holding for a wide range of  $n_1, \dots, n_s \in \mathbb{N}$ .

For more general sets  $\mathcal{A}$ , the problem of estimating  $\#(\mathcal{A} \cap [1, x])^2$  was studied by Cilleruelo, Ramana, and Ramaré [1]. For example, they studied this problem when  $\mathcal{A}$  is the set of shifted primes, the set of sums of two squares, and the set of shifted sums of two squares. Moreover, they computed the (almost sure) asymptotic behavior for  $\#\mathcal{A}^2$  when  $\mathcal{A}$  is a random subset of  $\{1, \dots, n\}$  that contains each element of  $\{1, \dots, n\}$  independently with probability  $\delta \in (0, 1)$ . The third author of the present paper [10] extended this last result to the product of arbitrarily many sets, and Mastrostefano [9] gave a necessary and sufficient condition for having  $\#\mathcal{A}^2 \sim (\#\mathcal{A})^2/2$  almost surely.

Hegyvári, Hennecart and Pach [6] considered the analogous problem for infinite sets of natural numbers. In this context, the role of the cardinality is played by the *natural density*  $\mathbf{d}(\mathcal{A})$  of a set  $\mathcal{A}$ , defined as usual by

$$\mathbf{d}(\mathcal{A}) = \lim_{x \rightarrow \infty} \frac{\#\mathcal{A} \cap [1, x]}{x}.$$

They asked the following questions ([6, Questions 3 and 2], respectively):

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QUESTION 1. If  $\mathcal{A}$  is a set of natural numbers of density 1, is it true that  $\mathcal{A}^2$  also has density 1?

QUESTION 2. Is it true that  $\inf_{\mathcal{A} \subseteq \mathbb{N}: \mathbf{d}(\mathcal{A})=\alpha} \mathbf{d}(\mathcal{A}^2) = 0$  for any  $\alpha \in [0, 1)$ , or at least for  $\alpha \in [0, \alpha_0)$  for some  $\alpha_0 \in (0, 1)$ ?

Clearly, Question 1 has an affirmative answer if  $1 \in \mathcal{A}$ , and Hegyvári, Hennecart and Pach showed that it also suffices that  $\mathcal{A}$  contains an infinite subset of mutually coprime integers  $a_1 < a_2 < \dots$  such that  $\sum_{i=1}^{\infty} a_i^{-1} = +\infty$ . Here, we show that the answer is ‘yes’ in full generality.

THEOREM 1. *Let  $\mathcal{A}, \mathcal{B} \subseteq \mathbb{N}$ . If  $\mathbf{d}(\mathcal{A}) = \mathbf{d}(\mathcal{B}) = 1$ , then  $\mathbf{d}(\mathcal{A} \cdot \mathcal{B}) = 1$ .*

COROLLARY. *If  $\mathcal{A} \subseteq \mathbb{N}$  is such that  $\mathbf{d}(\mathcal{A}) = 1$ , then  $\mathbf{d}(\mathcal{A}^k) = 1$  for each  $k = 2, 3, \dots$*

REMARK. In fact, the case  $\mathcal{A} = \mathcal{B}$  of Theorem 1 implies easily the general case. Indeed, if  $\mathbf{d}(\mathcal{A}) = \mathbf{d}(\mathcal{B}) = 1$ , then  $\mathbf{d}(\mathcal{A} \cap \mathcal{B}) = 1$ . In addition, if  $(\mathcal{A} \cap \mathcal{B})^2$  has density 1, then so does  $\mathcal{A} \cdot \mathcal{B}$ .

As it will be clear from the proof, the difference in the density of  $\mathbf{d}(\mathcal{A}^2)$  with respect to Erdős’s multiplication table problem lies in the fact that many elements of  $\mathcal{A}^2$  come from very ‘unbalanced’ products, meaning products  $ab$  such that the sizes of  $a$  and  $b$  are completely different.

Let us now turn to Question 2. We will answer it in a strong form that shows, among other things, that the condition that  $\mathbf{d}(\mathcal{A}) = 1$  in Theorem 1 cannot be relaxed.

THEOREM 2. *For  $\alpha \in [0, 1]$ , we have*

$$\inf_{\mathcal{A} \subseteq \mathbb{N}: \mathbf{d}(\mathcal{A})=\alpha} \mathbf{d}(\mathcal{A}^2) = \begin{cases} 0 & \text{if } \alpha < 1, \\ 1 & \text{if } \alpha = 1. \end{cases}$$

## 2. Preliminaries

*Notation.* We employ Landau’s notation  $f = O(g)$  and Vinogradov’s notation  $f \ll g$  both to mean that  $|f| \leq C|g|$  for a some constant  $C > 0$ . Moreover, we write  $f \asymp g$  to mean that  $f \ll g$  and  $g \ll f$ . The notation  $f = o(g)$  as  $x \rightarrow a$  (respectively,  $f \sim g$  as  $x \rightarrow a$ ) means that  $\lim_{x \rightarrow a} f(x)/g(x) = 0$  (respectively,  $= 1$ ). Given an integer  $n$ , we write  $P^-(n)$  and  $P^+(n)$  for its smallest and largest prime factors, respectively, with the convention that  $P^-(1) = \infty$  and  $P^+(1) = 1$ . If  $P^+(n) \leq y$ , we say that  $n$  is  $y$ -smooth, and if  $P^-(n) > y$ , we say that it is  $y$ -rough. As usual, we let  $\Phi(x, y)$  denote the number of  $y$ -rough numbers in  $[1, x]$ . Given any integer  $n$ , we may write it uniquely as  $n = ab$  with  $P^+(a) \leq y < P^-(b)$ . We then call  $a$  and  $b$  the  $y$ -smooth and  $y$ -rough part of  $n$ , respectively. Finally, we let  $\Omega(n)$  denote the number of prime factors of  $n$  counted with multiplicity.

We need some standard lemmas. We give their proofs for the sake of completeness.

LEMMA 2.1. *For  $x \geq y > 1$ , we have  $\Phi(x, y) \ll x/\log y$ .*

*Proof.* This follows for example from [8, Theorem 14.2] with  $f(n) = 1_{P^-(n) > y}$ .  $\square$

LEMMA 2.2. *Uniformly for  $x \geq y^2 \geq 1$  and  $u \geq 1$ , we have*

$$\#\{n \leq x : \exists d|n \text{ such that } P^+(d) \leq y^{1/u} \text{ and } d > y\} \ll x \cdot (e^{-u} + y^{-1/3}).$$

*Proof.* Without loss of generality,  $u \geq 4$ . Let  $\mathcal{B}$  denote the set of  $n \in \mathbb{Z} \cap [1, x]$  that have a  $y^{1/u}$ -smooth divisor  $d > y$ . Given  $n \in \mathcal{B}$ , let  $p_1 \leq p_2 \leq \dots \leq p_k$  be the sequence of prime factors of  $n$  of size  $\leq y^{1/u}$  listed in increasing order and according to their multiplicity. By our assumption on  $n$ , we must have  $p_1 \cdots p_j > y$ . Let  $j$  be the smallest integer such that  $p_1 \cdots p_j > y$ . We must have  $j \geq 5$  because all factors  $p_i$  are  $\leq y^{1/u} \leq y^{1/4}$ . We then set  $a = p_1 \cdots p_{j-2}$ ,  $p = p_{j-1}$ , and  $b = n/(ap)$ , so that  $a > y/(p_{j-1}p_j) \geq \sqrt{y}$ ,  $ap \leq y$ , and  $P^+(a) \leq p \leq P^-(b)$ . Consequently,

$$\#\mathcal{B} \leq \sum_{p \leq y^{1/u}} \sum_{\substack{P^+(a) \leq p \\ \sqrt{y} < a \leq y/p}} \sum_{\substack{b \leq x/(ap) \\ P^-(b) \geq p}} 1 \ll \sum_{p \leq y^{1/u}} \sum_{\substack{P^+(a) \leq p \\ a > \sqrt{y}}} \frac{x}{ap \log p} \tag{1}$$

by Lemma 2.1. If we let  $\varepsilon_p = \min\{2/3, 2/\log p\}$ , then Rankin’s trick implies

$$\frac{\#\mathcal{B}}{x} \ll \sum_{p \leq y^{1/u}} \sum_{P^+(a) \leq p} \frac{(a/\sqrt{y})^{\varepsilon_p}}{ap \log p} = \sum_{p \leq y^{1/u}} \frac{y^{-\varepsilon_p/2}}{p \log p} \sum_{P^+(a) \leq p} \frac{1}{a^{1-\varepsilon_p}}.$$

The sum over  $a$  equals  $\prod_{q \leq p} (1 - q^{-1+\varepsilon_p})^{-1}$  with  $q$  denoting a prime number. Since  $q^{\varepsilon_p} = 1 + O(\log q / \log p)$  for  $q \leq p$ , Mertens’ estimates [8, Theorem 3.4] imply that the sum over  $a$  is  $\ll \log p$ . We conclude that

$$\begin{aligned} \frac{\#\mathcal{B}}{x} &\ll y^{-1/3} + \sum_{100 < p \leq y^{1/u}} \frac{e^{-\log y / \log p}}{p} \leq y^{-1/3} + \sum_{j \geq 1} \sum_{y^{1/(u(j+1))} < p \leq y^{1/(uj)}} \frac{e^{-ju}}{p} \\ &\ll y^{-1/3} + \sum_{j \geq 1} e^{-ju} \ll y^{-1/3} + e^{-u} \end{aligned}$$

using Mertens’ estimates once again. This completes the proof. □

LEMMA 2.3. *Let  $y \geq 2$  and  $\lambda \in [0, 1.99]$ , and set  $Q(\lambda) = \lambda \log \lambda - \lambda + 1$  for  $\lambda > 0$  and  $Q(0) = 0$ . If  $0 \leq \lambda \leq 1$ , then*

$$\prod_{p \leq y} \left(1 - \frac{1}{p}\right) \sum_{\substack{P^+(m) \leq y \\ \Omega(m) \leq \lambda \log \log y}} \frac{1}{m} \ll (\log y)^{-Q(\lambda)},$$

whereas if  $1 \leq \lambda \leq 1.99$ , then

$$\prod_{p \leq y} \left(1 - \frac{1}{p}\right) \sum_{\substack{P^+(m) \leq y \\ \Omega(m) \geq \lambda \log \log y}} \frac{1}{m} \ll (\log y)^{-Q(\lambda)}.$$

*Proof.* The result is trivial if  $\lambda = 0$  by Mertens’ estimates [8, Theorem 3.4], so assume  $\lambda > 0$ . If  $0 < \lambda \leq 1$ , then

$$\begin{aligned} \sum_{\substack{P^+(m) \leq y \\ \Omega(m) \leq \lambda \log \log y}} \frac{1}{m} &\leq \sum_{P^+(m) \leq y} \frac{\lambda^{\Omega(m) - \lambda \log \log y}}{m} = (\log y)^{-\lambda \log \lambda} \prod_{p \leq y} \left(1 - \frac{\lambda}{p}\right)^{-1} \\ &\asymp (\log y)^{-Q(\lambda)} \prod_{p \leq y} \left(1 - \frac{1}{p}\right)^{-1} \end{aligned}$$

where we used Mertens' estimates once again. Similarly, if  $1 \leq \lambda \leq 1.99$ , then

$$\sum_{\substack{P^+(m) \leq y \\ \Omega(m) \geq \lambda \log \log y}} \frac{1}{m} \leq \sum_{P^+(m) \leq y} \frac{\lambda^{\Omega(m) - \lambda \log \log y}}{m} \asymp (\log y)^{-Q(\lambda)} \prod_{p \leq y} \left(1 - \frac{1}{p}\right)^{-1}.$$

This completes the proof. □

LEMMA 2.4. *Let  $\mathcal{P}$  be a set of primes such that  $\sum_{p \in \mathcal{P}} 1/p < \infty$ . Then*

$$\mathbf{d}(\{n \in \mathbb{N} : p|n \Rightarrow p \notin \mathcal{P}\}) = \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p}\right).$$

*Proof.* The number of integers  $n \leq x$  with a prime divisor  $p > \log x$  from  $\mathcal{P}$  is

$$\leq \sum_{p > \log x, p \in \mathcal{P}} \frac{x}{p} = o(x) \quad \text{as } x \rightarrow \infty,$$

because  $\sum_{p \in \mathcal{P}} 1/p$  converges. Hence, if we write  $\mathcal{P}' = \mathcal{P} \cap [1, \log x]$ , then

$$\#\{n \leq x : p|n \Rightarrow p \notin \mathcal{P}\} = \#\{n \leq x : p|n \Rightarrow p \notin \mathcal{P}'\} + o(x) = x \prod_{p \in \mathcal{P}'} \left(1 - \frac{1}{p}\right) + o(x)$$

from the inclusion–exclusion principle that has  $\leq 2^{\#\mathcal{P}'} \leq 2^{\log x} = o(x)$  steps (for example, see [8, Theorem 2.1]). Since  $\prod_{p \in \mathcal{P} \setminus \mathcal{P}'} (1 - 1/p) \sim 1$  by our assumption that  $\sum_{p \in \mathcal{P}} 1/p < \infty$ , the proof is complete. □

### 3. Proof of Theorem 1

Assume  $x$  is sufficiently large and let  $y = y(x)$  and  $u = u(x)$  to be chosen later, with  $y, u \rightarrow +\infty$  slowly as  $x \rightarrow +\infty$ . In particular,  $y \leq \sqrt{x}$ . In the following, for the sake of notation, we will often omit the dependence on  $x, y, u$ .

With a small abuse of notation, given an integer  $n$ , let  $n_{\text{smooth}}$  denote its  $y^{1/u}$ -smooth part and let  $n_{\text{rough}}$  denote its  $y^{1/u}$ -rough part. We then set

$$\mathcal{N} = \{n \leq x : n_{\text{smooth}} \leq y\}.$$

By Lemma 2.2, we have  $\#\mathcal{N} \sim x$  as  $x \rightarrow \infty$ . Therefore, in order to prove Theorem 1, it is enough to show that

$$\#\mathcal{C} = o(x), \quad \text{where } \mathcal{C} := \mathcal{N} \setminus (\mathcal{A} \cdot \mathcal{B}).$$

Let  $n \in \mathcal{C}$ . Since  $n = n_{\text{smooth}} \cdot n_{\text{rough}}$ , we must have that either  $n_{\text{smooth}} \notin \mathcal{A}$  or  $n_{\text{rough}} \notin \mathcal{B}$ . Consequently,

$$\#\mathcal{C} \leq S_1 + S_2$$

with

$$S_1 := \#\{n \in \mathcal{N} : n_{\text{smooth}} \notin \mathcal{A}\} \quad \text{and} \quad S_2 := \#\{n \in \mathcal{N} : n_{\text{rough}} \notin \mathcal{B}\}.$$

Let us first bound  $S_1$ . Letting  $m = n_{\text{smooth}}$ , we have

$$S_1 \leq \sum_{m \leq y, m \notin \mathcal{A}} \Phi(x/m, y^{1/u}) \ll \frac{ux}{\log y} \sum_{m \leq y, m \notin \mathcal{A}} \frac{1}{m}$$

by Lemma 2.1. Since we have assumed that  $\mathbf{d}(\mathcal{A}) = 1$ , we must have that  $\mathbf{d}(\mathbb{N} \setminus \mathcal{A}) = 0$  and thus

$$\alpha(t) := \frac{1}{\log t} \sum_{m \leq t, m \notin \mathcal{A}} \frac{1}{m} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Hence, setting  $u = u(y) := \alpha(y)^{-1/2}$ , we have  $u \rightarrow +\infty$  and  $S_1 = o(x)$  as  $x \rightarrow +\infty$ .

Let us now bound  $S_2$ . Writing  $m' = n_{\text{rough}}$ , we have

$$S_2 \leq \sum_{m \leq y} \#\{m' \leq x/m : m' \notin \mathcal{B}\}.$$

By hypothesis, we have  $\mathbf{d}(\mathcal{B}) = 1$ , so that  $\mathbf{d}(\mathbb{N} \setminus \mathcal{B}) = 0$ . Thus

$$\beta(t) := \sup_{s \geq t} \frac{\#((\mathbb{N} \setminus \mathcal{B}) \cap [1, s])}{s} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Hence, setting  $y := \min(x^{1/2}, \exp(\beta(x^{1/2})^{-1/2}))$ , we have  $y \rightarrow +\infty$  as  $x \rightarrow +\infty$  and

$$S_2 \leq \sum_{d \leq y} \beta(x/d) \cdot \frac{x}{d} \leq x\beta(x/y) \sum_{d \leq y} \frac{1}{d} \ll x\beta(x^{1/2}) \log y \leq x\beta(x^{1/2})^{1/2} = o(x).$$

In conclusion,  $\#\mathcal{C} = o(x)$ , as desired.

REMARK. The proof of Theorem 1 can be made quantitative. For example, if one has  $\#\{n \leq x : n \notin \mathcal{A}\}, \#\{n \leq x : n \notin \mathcal{B}\} \ll x(\log x)^{-a}$  for some fixed  $0 < a < 1$ , then taking  $y = \exp((\log x)^{\frac{a}{1+a}})$  and  $u = \log \log x$  in the above argument yields

$$\#\{n \leq x : n \notin \mathcal{A} \cdot \mathcal{B}\} \ll xe^{-u} + \frac{xu}{(\log y)^a} + \frac{x \log y}{(\log x)^a} \ll x(\log x)^{-\frac{a^2}{1+a} + o(1)}.$$

An interesting question is to determine the optimal exponent of  $\log x$  in this upper bound.

#### 4. Proof of Theorem 2

The case  $\alpha = 1$  follows from Theorem 1, whereas for the case  $\alpha = 0$  one can just observe that  $\mathbf{d}(\emptyset) = \mathbf{d}(\emptyset^2) = 0$ . We may thus assume  $\alpha \in (0, 1)$ . Given any  $\varepsilon > 0$ , we need to construct a set  $\mathcal{A}$  of density  $\alpha$  such that the density of  $\mathcal{A}^2$  exists and is smaller than  $\varepsilon$ .

Let  $k \in \mathbb{N}$ ,  $y \geq 1$  and a set of primes  $\mathcal{P} \subset (y, +\infty)$  with  $\sum_{p \in \mathcal{P}} 1/p < \infty$  to be chosen later. Using the notation  $\Omega_y(n) = \sum_{p^a | n, p \leq y} 1$ , let us consider the sets

$$\mathcal{B}_{y,k,\mathcal{P}} := \{n \in \mathbb{N} : \Omega_y(n) \geq k, (n, p) = 1 \forall p \in \mathcal{P}\}.$$

The key property these sets have is that  $\mathcal{B}_{y,k,\mathcal{P}}^2 = \mathcal{B}_{y,2k,\mathcal{P}}$ .

Now, using Lemma 2.4 twice (once, with  $\mathcal{P}_{\text{Lemma 2.4}} = \mathcal{P} \cup \{p \leq y\}$  and once with  $\mathcal{P}_{\text{Lemma 2.4}} = \{p \leq y\}$ ), we find that

$$\mathbf{d}(\mathcal{B}_{y,k,\mathcal{P}}) = \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p}\right) \prod_{p \leq y} \left(1 - \frac{1}{p}\right) \sum_{\substack{P^+(m) \leq y \\ \Omega(m) \geq k}} \frac{1}{m} = \mathbf{d}(\mathcal{B}_{y,k,\emptyset}) \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p}\right).$$

Similarly,

$$\mathbf{d}(\mathcal{B}_{y,k,\mathcal{P}}^2) = \mathbf{d}(\mathcal{B}_{y,2k,\mathcal{P}}) = \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p}\right) \mathbf{d}(\mathcal{B}_{y,2k,\emptyset}).$$

Now, take  $y := \exp(\exp(4k/3))$ , so that  $k = \frac{3}{4} \log \log y$ . For any fixed  $\varepsilon > 0$ , Lemma 2.3 implies that if  $k$  is sufficiently large in terms of  $\alpha$  and  $\varepsilon$ , then  $\mathbf{d}(\mathcal{B}_{y,k,\emptyset}) > \alpha$  and  $\mathbf{d}(\mathcal{B}_{y,2k,\emptyset}) < \varepsilon$ . Let us fix for the remainder of the proof such a choice of  $k$ . We then construct  $\mathcal{P}$  in the following way: we take  $p_1 > y$  to be the smallest prime such that  $(1 - 1/p_1)\mathbf{d}(\mathcal{B}_{y,k,\emptyset}) > \alpha$ ,  $p_2 > p_1$  the smallest prime such that  $(1 - 1/p_1)(1 - 1/p_2)\mathbf{d}(\mathcal{B}_{y,k,\emptyset}) > \alpha$  and so on. Taking  $\mathcal{P} := \{p_1, p_2, \dots\}$  we clearly have  $\mathbf{d}(\mathcal{B}_{y,k,\emptyset}) \prod_{p \in \mathcal{P}} (1 - 1/p) = \alpha$ . Thus,  $\mathbf{d}(\mathcal{B}_{y,k,\mathcal{P}}) = \alpha$  and  $\mathbf{d}(\mathcal{B}_{y,k,\mathcal{P}}^2) < \varepsilon$ , as desired.

REMARK. If  $\mathbf{d}(\mathcal{A}^2)$  in Theorem 2 is replaced by the upper density  $\bar{\mathbf{d}}(\mathcal{A}^2)$ , then one could just take  $\mathcal{A}$  to be any density  $\alpha$  subset of  $\{n \in \mathbb{N} : \Omega_y(n) \geq \frac{3}{4} \log \log y\}$  for  $y$  large enough. However, in general there is no guarantee that  $\mathcal{A}^2$  has asymptotic density. For this reason, in order to prove Theorem 2, it is more convenient to construct explicit suitable sets  $\mathcal{A}$ .

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