Exact response theory and Kuramoto dynamics

EXACT RESPONSE THEORY AND KURAMOTO DYNAMICS

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Abstract. The dynamics of Kuramoto oscillators is investigated in terms of the exact response theory based on the Dissipation Function, which has been introduced in the field of nonequilibrium molecular dynamics. While linear response theory is a cornerstone of nonequilibrium statistical mechanics, it does not apply, in general, to systems undergoing phase transitions. Indeed, even a small perturbation may in that case result in a large modification of the state. An exact theory is instead expected to handle such situations. The Kuramoto dynamics, which undergoes synchronization transitions, is thus investigated analytically and numerically as a testbed for the exact theory mentioned above. A comparison between the two approaches shows how the linear theory fails, while the exact theory yields the correct response.

1. Introduction

The response of a system with many degrees of freedom to an external stimulus is a central topic in nonequilibrium statistical mechanics. Its investigation has greatly progressed with the works of Callen, Green, Kubo, and Onsager, in particular, who contributed to the development of linear response theory [32, 36]. In the '90s, the derivation of the Fluctuation Relations [19, 21, 26] provided the framework for a more general response theory, applicable to both Hamiltonian as well as dissipative deterministic particle systems [8, 10, 12, 13, 14, 24, 36, 40]. The study of response in stochastic processes, with a special focus on diffusion and Markov jump processes, has also been inspired by fluctuation relations, and has been studied *e.g.* in [2, 4, 7, 11, 15]. Moreover, the role of causality, expressed by the Kramers-Kronig relations, in nonlinear extensions of the linear response theory has been discussed in [35].

The introduction of the Dissipation Function, first made explicit in [22], and developed as the observable of interest in Fluctuation Relations [23, 41], paved the way to an exact response theory. A theory expected to hold in presence of arbitrarily large perturbations and modifications of states, which allows the study of the relaxation of particle systems to equilibrium or non-equilibrium steady states.

In this work we present and apply the Dissipation Function formalism to the Kuramoto model [33, 34], which is considered a prototype of many particle systems exhibiting *synchronization*, a phenomenon familiar in many physical and biological contexts [25, 27, 31, 37, 42, 43]. Furthermore, the Kuramoto model provides the stage for a large

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research endeavor, in applied mathematics, control theory and statistical physics [1, 3, 5, 18, 28, 39, 43]. See [29, 16] for recent reviews on the subject.

In this paper, our aim is two-fold. On the one hand, we probe the exact response theory on a dissipative system with many degrees of freedom undergoing nonequilibrium phase transitions, which is in fact a challenging open problem. On the other hand, while a vast mathematical literature exists on the Kuramoto model, it is interesting to analyze it from a new statistical mechanical perspective, in which some known results are reinterpreted, cf. *e.g.* Refs.[6, 17].

Our conclusion is that, while the linear response theory cannot characterize the Kuramoto synchronization process, the exact theory does. In particular, we obtain synchronization within the formalism of the Dissipation Function, thus showing how such a behaviour is captured by the exact response theory, while it is not evidenced by the linear theory. Synchronization corresponds indeed to the maximum value of the Dissipation Function, which we prove is attained in time. When the number of oscillators N is large, this maximum value is proportional to the oscillators coupling constant K.

This paper is organized as follows. In Sec. 2 we review some basic properties of the Kuramoto dynamics. In Sec. 3 we illustrate the main ingredients of the Dissipation Function response theory. In Sec. 4 we study the response theory for the Kuramoto dynamics of identical oscillators. In Sec. 5 we review the linear response theory, and we compare it, analytically and numerically, with the exact response formalism. We draw our conclusions in Sec. 6.

2. The Kuramoto system

The Kuramoto dynamics is defined on the N-dimensional torus, $\mathcal{T}^N = (\mathbb{R}/(2\pi\mathbb{Z}))^N$, with $N \geq 1$, by the following set of coupled first order ODEs, for the phases $\theta_i(t)$:

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) \qquad i = 1, \dots, N \tag{2.1}$$

where $K > 0$ is a constant, and the natural frequencies $\omega_i \in \mathbb{R}$ are drawn from some given distribution $g(\omega)$.

The N oscillators are represented by points rotating on the unit circle centered at the origin of the complex plane, more precisely by $e^{i\theta_j}$ with $j = 1, \dots, N$. By introducing the polar coordinates of the barycenter,

$$Re^{i\Phi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j} \tag{2.2}$$

with $R \in [0, 1]$ and $\Phi \in \mathbb{R}$ (defined if $R > 0$), one can rewrite Eq.(2.1) as follows:

$$\dot{\theta}_i = \omega_i + KR\sin(\Phi - \theta_i)\,, \qquad i = 1, \dots, N \tag{2.3}$$

where $R = R(\theta(t))$ is the *order parameter* and $\Phi = \Phi(\theta(t))$ the *collective phase*, with $\theta = (\theta_1, ..., \theta_N) \in \mathcal{M} = \mathcal{T}^N$, and $\mathcal{M}$ the phase space. It is to be remarked that the

Kuramoto dynamics (2.3) can be written as a gradient flow:

$$\dot{\theta} = -\nabla f(\theta) \tag{2.4}$$

with potential

$$f(\theta) = -\sum_{i=1}^{N} \omega_i \theta_i + \frac{K}{2N} \sum_{i,j=1}^{N} \Big(1 - \cos(\theta_j - \theta_i)\Big) \tag{2.5}$$

that is analytic in θ.

A *complete frequency synchronization* occurs when the differences $\theta_i(t) - \theta_j(t)$ tend to a constant for all i and j, and $R(\theta(t))$ tends to a given $R^\infty \in (0,1]$, as $t \to +\infty$. In case $R^\infty = 1$, all the N terms of the sum in (2.2) coincide, hence the Kuramoto system undergoes *phase synchronization*.

Identities for the order parameter. *We remark that equation* (2.2) *leads to the following identities,*

$$R = \frac{1}{N} \sum_{i=1}^{N} \cos(\Phi - \theta_i)\,, \tag{2.6}$$

$$0 = \frac{1}{N} \sum_{i=1}^{N} \sin(\Phi - \theta_i)\,, \tag{2.7}$$

$$R \sin(\Phi - \theta_i) = \frac{1}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i)\,, \qquad i = 1, \ldots, N \tag{2.8}$$

$$R \cos(\Phi - \theta_i) = \frac{1}{N} \sum_{j=1}^{N} \cos(\theta_j - \theta_i)\,, \qquad i = 1, \ldots, N\,. \tag{2.9}$$

From the identities (2.6) *and* (2.9)*, we find that:*

$$R^2 = \frac{1}{N^2} \sum_{i,j=1}^{N} \cos(\theta_j - \theta_i)\,. \tag{2.10}$$

For $\theta \in \mathcal{M}$, we can rewrite Eq.(2.3) as:

$$\dot{\theta} = W + V(\theta) = V_K(\theta) \tag{2.11}$$

where $W = (\omega_1, \ldots, \omega_N)$ is interpreted as an *equilibrium* vector field made of N natural frequencies, while V represents a *nonequilibrium* perturbation with components:

$$V_i(\theta) = \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i) = KR \sin(\Phi - \theta_i)\,, \qquad i = 1, \ldots, N\,. \tag{2.12}$$

For later use, we prove the following identity.

Lemma 2.1. *The divergence of the Kuramoto vector field V_K of Eq.*(2.11), i.e. *the associated phase space volumes variation rate Λ, satisfies:*

$$\Lambda := \mathrm{div}_\theta V = K\left(1 - NR^2\right)\,. \tag{2.13}$$

Proof. By means of (2.12), for $i = 1, \dots, N$ one has

$$\begin{aligned}\partial_{\theta_i} V_i &= \frac{K}{N} \partial_{\theta_i} \left(\sum_{i \neq j = 1}^{N} \sin(\theta_j - \theta_i) \right) \\ &= -\frac{K}{N} \left(\sum_{i \neq j = 1}^{N} \cos(\theta_j - \theta_i) \right) = -\frac{K}{N} \left(\sum_{j=1}^{N} \cos(\theta_j - \theta_i) - 1 \right) \\ &= -KR \cos(\Phi - \theta_i) + \frac{K}{N}\end{aligned}$$

where we used (2.9). Summing over i, and using (2.6) , Eq.(2.13) follows. □

Therefore, the Kuramoto dynamics do not preserve the phase space volumes, and Λ actually varies in time, since R is a function of the dynamical variables $\theta(t)$.

3. Mathematical framework of Response theory

Let us summarize the mathematical framework of exact response theory, as described in [8, 24, 30, 41]. The starting point is a flow $S^t : \mathcal{M} \to \mathcal{M}$, with phase space $\mathcal{M} \subset \mathbb{R}^N$, $N \geq 1$, that is usually determined by an ODE system

$$\dot{\theta} = V(\theta)\,, \quad \theta \in \mathcal{M} \tag{3.1}$$

with V defined on $\mathcal{M}$. Let $S^t\theta$ denote the solution at time $t \in \mathbb{R}$, with initial condition θ, of such ODEs. The second ingredient is a probability measure $d\mu_0(\theta) = f_0(\theta)d\theta$ on $\mathcal{M}$, with positive and continuously differentiable density f_0. A time evolution is induced on the simplex of probabilities on $\mathcal{M}$, defining the probability at a time $t \in \mathbb{R}$ as:

$$\mu_t(E) = \mu_0(S^{-t}E)$$

for each measurable set $E \subset \mathcal{M}$. This amounts to consider probability in a phase space, like the mass of a fluid in real space. The corresponding continuity equation for the probability densities is the (generalized) Liouville equation:

$$\frac{\partial f}{\partial t} + \mathrm{div}_\theta(fV) = 0 \tag{3.2}$$

Denoting by f_t the solution of Eq.(3.2) with initial datum f_0, we can write $d\mu_t = f_t d\theta$.

Introducing the *Dissipation Function* $\Omega^{f,V}$ [30, 41]:

$$\Omega^{f,V}(\theta) := -\Lambda(\theta) - V(\theta) \cdot \nabla \log f(\theta) \tag{3.3}$$

with $\nabla = (\partial_{\theta_1}, \dots, \partial_{\theta_N})$ and

$$\Lambda = \mathrm{div}_\theta\, V\,,$$

the Euler version (3.2) of the Liouville equation may also be written as:

$$\frac{\partial f}{\partial t} = f\ \Omega^{f,V} \tag{3.4}$$

Also, the equation (3.2) can be cast in the Lagrangian form

$$\frac{df}{dt} = -f\,\Lambda\,, \tag{3.5}$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + V \cdot \nabla_\theta$ is the total derivative along the flow (3.1).

Direct integration of Eq.(3.5) yields

$$f_{s+t}(S^t\theta) = \exp\Big\{ - \Lambda_{0,t}(\theta)\Big\} f_s(\theta)\,, \quad \forall\, t,\, s \geq 0 \tag{3.6}$$

where we used the notation

$$\mathcal{O}_{s,t}(\theta) := \int_s^t \mathcal{O}(S^\tau\theta)d\tau \tag{3.7}$$

for the phase functions, or *observables*, $\mathcal{O} : \mathcal{M} \to \mathbb{R}$, so that, in particular, $\Lambda_{0,t}(\theta) = \int_0^t \Lambda(S^\tau\theta)d\tau$. In the following Proposition, this notation is used with the observable $\mathcal{O} = \Omega^{f,V}(\theta)$ given in (3.3), so that the time integral in (3.7) will correspondingly be denoted by $\Omega^{f,V}_{s,t}$.

Proposition 3.1. *For all t, $s \in \mathbb{R}$, the following identity holds:*

$$f_{s+t}(\theta) = \exp\Big\{\Omega^{f_s,V}_{-t,0}(\theta)\Big\} f_s(\theta)\,. \tag{3.8}$$

Proof. We start by claiming that

$$\Omega^{f_t,V}_{0,s}(\theta) = \log\frac{f_t(\theta)}{f_t(S^s\theta)} - \Lambda_{0,s}(\theta)\,. \tag{3.9}$$

Indeed, one has:

$$V(S^u\theta)\cdot\nabla\log f_t(S^u\theta) = \frac{d}{du}\log f_t(S^u\theta) \tag{3.10}$$

because t is fixed and f_t does not depend explicitly on u, hence Eqs.(3.3) and (3.10) imply:

$$\begin{aligned}\Omega^{f_t,V}_{0,s}(\theta) &= -\int_0^s \big[\Lambda(S^u\theta) + V\cdot\nabla\log f_t(S^u\theta)\big]du \\ &= -\Lambda_{0,s}(\theta) - \int_0^s \frac{d}{du}\log f_t(S^u\theta)d\theta = -\Lambda_{0,s}(\theta) - \log\frac{f_t(S^s\theta)}{f_t(\theta)}\end{aligned}$$

which leads to Eq.(3.9). Next, Eqs.(3.6) and Eq.(3.9) lead to

$$\exp\Big\{\Omega^{f_s,V}_{s,s+t}(\theta)\Big\} f_s(S^{s+t}\theta) = \exp\Big\{ - \Lambda_{s,s+t}(\theta)\Big\} f_s(S^s\theta) = f_{s+t}(S^{s+t}\theta) \tag{3.11}$$

which yields (3.8). □

As a consequence of Proposition 3.1, a probability density f is *invariant* under the dynamics if and only if $\Omega^{f,V}$ identically vanishes:

$$\Omega^{f,V}(\theta) = 0\,, \quad \forall\,\theta\in\mathcal{M}\,. \tag{3.12}$$

In the sequel, we shall use the notation

$$\langle\mathcal{O}\rangle_t := \int_{\mathcal{M}} \mathcal{O}(\theta)f_t(\theta)d\theta \tag{3.13}$$

to denote the average of an observable with respect to the probability measure $\mu_t = f_t \, d\theta$. The response theory states that the average $\langle \mathcal{O} \rangle_t$ can be expressed in terms of the initial density f_0, which is known. More precisely:

Lemma 3.1. (Exact response): *Given $\{S^t\}_{t\in\mathbb{R}}$ and an integrable observable $\mathcal{O} : \mathcal{M} \to \mathbb{R}$, the following identity holds:*

$$\langle \mathcal{O} \rangle_t = \langle \mathcal{O} \rangle_0 + \int_0^t \langle (\mathcal{O} \circ S^\tau) \; \Omega_V^{f_0} \rangle_0 \; d\tau \, . \tag{3.14}$$

Proof. First of all, f_0 is smooth as a function of θ by assumption, and evolves according to the Liouville equation. Therefore, f_t is also smooth with respect to θ and t for every finite time t. In turn, $\Omega^{f_t,V}(\theta)$ is differentiable with respect to θ and t, if f_0 (that depends only on θ) is differentiable with respect to θ. These conditions are immediately verified for differentiable f_0, and smooth dynamics on a compact manifold. Therefore two identities can be derived for integrable $\mathcal{O}$:

$$\begin{aligned} \mathcal{O}_{0,s}(\theta) &= \int_0^s \mathcal{O}(S^u\theta)du = \int_\tau^{s+\tau} \mathcal{O}(S^{u-\tau}\theta)du = \int_\tau^{s+\tau} \mathcal{O}(S^{-\tau}S^u\theta)du \\ &= \mathcal{O}_{\tau,s+\tau}(S^{-\tau}\theta) \, , \end{aligned}$$

which is valid for every $\tau \in \mathbb{R}$, and

$$\begin{aligned} \langle \mathcal{O} \rangle_{t+s} &= \int \mathcal{O}(\theta) f_{t+s}(\theta) d\theta = \int \mathcal{O}(S^s(S^{-s}\theta)) f_{t+s}(S^s(S^{-s}\theta)) \left| \frac{\partial \theta}{\partial(S^{-s}\theta)} \right| d(S^{-s}\theta) \\ &= \int \mathcal{O}(S^s(S^{-s}\theta)) f_{t+s}(S^s(S^{-s}\theta)) \exp\left\{ \Lambda_{-s,0}(\theta) \right\} d(S^{-s}\theta) \\ &= \int \mathcal{O}(S^s(S^{-s}\theta)) f_{t+s}(S^s(S^{-s}\theta)) \exp\left\{ \Lambda_{0,s}(S^{-s}\theta) \right\} d(S^{-s}\theta) \\ &= \int \mathcal{O}(S^s\theta) f_{t+s}(S^s\theta) \exp\left\{ \Lambda_{0,s}(\theta) \right\} d\theta = \int \mathcal{O}(S^s\theta) f_t(\theta) d\theta \\ &= \langle \mathcal{O} \circ S^s \rangle_t \end{aligned} \tag{3.15}$$

to obtain [30]:

$$\frac{d}{ds} \langle \mathcal{O} \rangle_s = \langle \mathcal{O} \; (\Omega^{f_r,V} \circ S^{r-s}) \rangle_s \tag{3.16}$$

which holds $\forall r \in \mathbb{R}^+$. Note that in Eq. (3.15) we used the relation

$$\left| \frac{\partial \theta}{\partial(S^{-s}\theta)} \right| = \exp\left\{ \Lambda_{-s,0}(\theta) \right\} , \tag{3.17}$$

which is discussed in Appendix B, see Eq. (B.3). Choosing $r = 0$ one finds

$$\frac{d}{ds} \langle \mathcal{O} \rangle_s = \langle \mathcal{O} \; (\Omega^{f_0,V} \circ S^{-s}) \rangle_s = \langle (\mathcal{O} \circ S^s) \; \Omega^{f_0,V} \rangle_0 \tag{3.18}$$

where we used (3.15). Then, integrating over time from 0 to t, Eq.(3.18) yields (3.14). $\square$

It may be useful to endow the density f_0 with physical meaning. To this aim, we assume that, for $t \in (-\infty, 0)$, the dynamics is determined by a (non dissipative) vector field V_0,

$$\dot{\theta} = V_0(\theta) \tag{3.19}$$

and that it has reached an equilibrium state, described by the probability measure $d\mu_0 = f_0\, d\theta$, long before the time $t = 0$. In other words, we assume that μ_0 is invariant under the dynamics (3.19), which we call *unperturbed* or *reference* dynamics. At time $t = 0$, the dynamics (3.19) is perturbed and the perturbation remains in place for $t \geq 0$. In general, the density f_0 will not be invariant under the perturbed vector field V, Eq.(3.1) say. Therefore, it will evolve as prescribed by Eq.(3.4) into a different density, f_t, at time $t > 0$. Nevertheless, Eq.(3.14) expresses the average $\langle \mathcal{O} \rangle_t$ in terms of a correlation function computed with respect to f_0, the non-invariant density, which is only invariant under the unperturbed dynamics.

The full range of applicability of this theory is still to be identified. However, it obviously applies to smooth dynamics on smooth compact manifolds, such as the Kuramoto dynamics (2.1), which has $\mathcal{M} = \mathcal{T}^N$. One advantage of using the Dissipation Function, compared to other possible exact approaches to response, apart from molecular dynamics efficiency, is that it corresponds to a physically measurable quantity, *e.g.* proportional to a current, that is adapted to the initial state of the system of interest. Moreover, it provides necessary and sufficient conditions for relaxation of ensembles, as well as sufficient conditions for the single system relaxation, known as T-mixing [30, 41].

Henceforth, we shall denote by $\Omega^{f,0}$ the Dissipation Function (3.3) defined in terms of the unperturbed flow V_0. The analysis of the response theory for a specific example of the Kuramoto model is discussed in the next Section.

4. Response theory for identical oscillators

Let us focus on the case of identical oscillators, namely the Kuramoto dynamics in which all the natural frequencies ω_i in Eq.(2.1) equal the same constant $\omega \in \mathbb{R}$. In particular, let the unperturbed dynamics be defined by the vector field $V_0(\theta) = W = (\omega, \ldots, \omega)$, which corresponds to $K = 0$ in Eq.(2.1), *i.e.* to decoupled oscillators, equipped with same natural frequency. Such dynamics are conservative, since $\mathrm{div}_\theta V_0 = 0$. The corresponding steady state can then be considered an equilibrium state. At time $t = 0$ the perturbation V is switched on, and we can write:

$$\dot{\theta} = \begin{cases} W & t < 0 \\ W + V(\theta) & t > 0 \end{cases} \tag{4.1}$$

The perturbed dynamics corresponds to the Kuramoto dynamics (2.1), which is not conservative, cf. Eq.(2.13). As an initial probability density, invariant under the unperturbed dynamics, we may take the factorized density:

$$f_0(\theta) = (2\pi)^{-N} \tag{4.2}$$

which, indeed, yields:

$$\Omega_0^{f_0} = -\left(\mathrm{div} V_0 + V_0 \cdot \nabla \log f_0\right) \equiv 0\,, \quad \text{and} \quad \frac{\partial f}{\partial t} = 0\,. \tag{4.3}$$

After the perturbation, the Dissipation Function takes the form:

$$\Omega_V^{f_0} = -\left(\mathrm{div}_\theta V + V \cdot \nabla \log f_0\right) = K\left(NR^2 - 1\right) = \frac{K}{N}\sum_{i,j=1}^{N} \cos(\theta_j - \theta_i) - K \tag{4.4}$$

and the density evolves as:

$$f_t(\theta) = \frac{1}{(2\pi)^N} \exp\Big[-K\left(t - NR^2_{-t,0}(\theta)\right)\Big] \tag{4.5}$$

where $R_{-t,0}$ denotes the integral of R from time $-t$ to 0, cf. Eq.(3.7).

Remark 4.1. *The Dissipation Function Eq.*(4.4) *is of class* C^∞*.*

Using the formula (3.14) to compute the response for the observable $\mathcal{O} = \Omega_V^{f_0}$, we obtain:

$$\langle \Omega_V^{f_0} \rangle_t = \langle \Omega_V^{f_0} \rangle_0 + \int_0^t \langle (\Omega_V^{f_0} \circ S^\tau)\Omega_V^{f_0} \rangle_0 \; d\tau \tag{4.6}$$

that is

$$\int_{\mathcal{M}} \Omega_V^{f_0}(\theta) f_t(\theta) d\theta = (2\pi)^{-N} \int_{\mathcal{M}} \Omega_V^{f_0}(\theta) d\theta + (2\pi)^{-N} \int_0^t \int_{\mathcal{M}} \Omega_V^{f_0}(S^\tau(\theta)) \Omega_V^{f_0}(\theta) \, d\theta d\tau\,.$$

Moreover:

$$\langle R^2 \rangle_0 = \frac{1}{N}\,, \quad \text{hence} \quad \langle \Omega_V^{f_0} \rangle_0 = K\big(N\langle R^2 \rangle_0 - 1\big) = 0 \tag{4.7}$$

as expected.

Remark 4.2. *Note that the scalar field* $\Omega_0^{f_0}$ *is identically 0, while* $\Omega_V^{f_0}$ *is not, see* (4.4)*. However, the phase space average* $\left\langle \Omega_V^{f_0} \right\rangle_0$ *vanishes.*

Therefore, using Eqs.(3.14) and (4.4) we can write:

$$\begin{aligned}
\left\langle \Omega_V^{f_0} \right\rangle_t &= \int_0^t \left\langle (\Omega_V^{f_0} \circ S^\tau) \Omega_V^{f_0} \right\rangle_0 \; d\tau \\
&= KN \int_0^t \left\langle \Omega_V^{f_0} \left[R^2 \circ S^\tau \right] \right\rangle_0 d\tau - K \int_0^t \left\langle \Omega_V^{f_0} \right\rangle_0 \; d\tau \\
&= KN \int_0^t \left\langle \Omega_V^{f_0} \left[R^2 \circ S^\tau \right] \right\rangle_0 \; d\tau \\
&= K^2 N^2 \int_0^t \left\langle R^2 \left[R^2 \circ S^\tau \right] \right\rangle_0 \; d\tau - K^2 N \int_0^t \left\langle R^2 \circ S^\tau \right\rangle_0 \; d\tau
\end{aligned}$$

For the second integral we have:

$$\begin{aligned}\int_0^t \left\langle R^2 \circ S^\tau \right\rangle_0 d\tau &= \frac{1}{(2\pi)^N} \int_0^t \int_{\mathcal{M}} R^2(S^\tau\theta) d\theta d\tau \\ &= \frac{1}{(2\pi)^N} \int_0^t \int_{\mathcal{M}} R^2(S^\tau\theta) \left| \frac{\partial \theta}{\partial S^\tau \theta} \right| dS^\tau\theta d\tau \\ &= \frac{1}{(2\pi)^N} \int_0^t \int_{\mathcal{M}} R^2(S^\tau\theta) e^{\Lambda_{0,\tau}(\theta)} dS^\tau\theta\end{aligned}$$

Explicit calculations can be carried out for $N = 2$ and will be discussed in Sec. 4.1, while the study of the general case with $N > 2$ is deferred to Sec. 4.2.

4.1. **The case with two oscillators.** For $N = 2$ and $\omega \geq 0$, consider the system for two oscillators:

$$\begin{cases} \dot{\theta}_1 = \dfrac{\omega}{2} + \dfrac{K}{2} \sin(\theta_2 - \theta_1) \\ \dot{\theta}_2 = -\dfrac{\omega}{2} + \dfrac{K}{2} \sin(\theta_1 - \theta_2)\,. \end{cases} \tag{4.8}$$

For $\omega = 0$ we say that the two oscillators are *identical* (in the sense that their natural frequencies coincide). Setting $\psi = \theta_1 - \theta_2$, we obtain the following equation:

$$\frac{d\psi}{dt} = \omega - K \sin(\psi)\,. \tag{4.9}$$

With a slight abuse of notation, in the following we denote by $S^t\theta$, $S^t\psi$ the flows corresponding to (4.8), (4.9) respectively, with initial data $\theta = (\theta_1, \theta_2)$.

The solution of (4.9) can be explicity expressed as ([9, Appendix D]) :

$$\psi(t) = \psi \circ S^t = 2\arctan(g(\psi, t)) \tag{4.10}$$

where

$$g(\psi, t) = \begin{cases} \frac{\sqrt{K^2-\omega^2}}{\omega} \left[\tanh\left(h_1(\psi) - \frac{t\sqrt{K^2-\omega^2}}{2}\right) + \frac{K}{\sqrt{K^2-\omega^2}}\right] & \text{if} \quad K > \omega \\ \\ \frac{\sqrt{\omega^2-K^2}}{\omega} \tan\left(\frac{t\sqrt{\omega^2-K^2}}{2} + h_2(\psi)\right) + \frac{K}{\omega} & \text{if} \quad 0 \leq K < \omega \\ \\ e^{-Kt} \tan(\frac{\psi}{2}) & \text{if} \quad \omega = 0 \end{cases}$$

and the functions h_1 and h_2 are given by:

$$\begin{aligned} h_1(\psi) &= \tanh^{-1}\left(\frac{\omega}{\sqrt{K^2-\omega^2}} \tan\left(\frac{\psi}{2}\right) - \frac{K}{\sqrt{K^2-\omega^2}} \right), \\ h_2(\psi) &= \arctan \frac{\omega \tan\left(\frac{\psi}{2}\right) - K}{\sqrt{\omega^2 - K^2}}\,. \end{aligned}$$

Recalling Eq.(2.10), we find that $(R^2 \circ S^t)$ can be written as

$$R^2(S^t\theta) = \frac{1}{2}\left[1 + \cos(S^t\psi)\right] = \frac{1}{g^2(S^t\psi) + 1}\,. \tag{4.11}$$

Restricting our analysis to the identical oscillator case, $\omega = 0$, yields:

$$R^2(S^t\theta) = \left(\tan^2\left(\frac{\psi}{2}\right)e^{-2Kt} + 1\right)^{-1} \tag{4.12}$$

and

$$S^t\psi \to 0 \quad \text{for } t \to +\infty\,, \qquad \text{if } |\psi| \neq \pi$$

while

$$|S^t\psi| \to \pi \quad \text{for } t \to -\infty\,, \qquad \text{if } \psi \neq 0\,.$$

In particular, for $\theta_1 \neq \theta_2$ and θ_1, $\theta_2 \in [0, 2\pi)$, then $t \to -\infty$ yields $S^t\psi \to -\pi$ if $\theta_1 < \pi$, and $S^t\psi \to \pi$ if $\theta_1 > \pi$.

Then, the set

$$E_\infty = \{(\theta_1, \theta_2) \in \mathcal{T}^2 : \theta_1 = \theta_2\}$$

is invariant and attracting for the Kuramoto dynamics, while the set

$$E_{-\infty} = \{(\theta_1, \theta_2) \in \mathcal{T}^2 : |\theta_1 - \theta_2| = \pi\}$$

is invariant and repelling. This also implies that:

$$R^2(S^t\theta) \to 0\,, \quad \Omega_V^{f_0} \to -K\,, \qquad \text{for} \;\; \psi \neq 0\,,\; t \to -\infty$$

while

$$R^2(S^t\theta) \to 1\,, \quad \Omega_V^{f_0} \to K\,, \qquad \text{for} \;\; |\psi| \neq \pi\,,\; t \to \infty\,.$$

Consequently, Eq.(4.5) shows that the probability piles up on the zero Lebesgue measure sets E_∞ and $E_{-\infty}$, respectively for $t \to \infty$ and $t \to -\infty$.

For $\tau \geq 0$, the following relations also hold:

$$\left\langle R^2 \circ S^\tau \right\rangle_0 = \frac{1}{(2\pi)^2}\int_{\mathcal{M}} \frac{1}{\tan^2(\frac{\theta_1-\theta_2}{2})e^{-2K\tau} + 1} d\theta = \frac{1}{e^{-K\tau} + 1} \tag{4.13}$$

and

$$\left\langle R^2(R^2 \circ S^\tau) \right\rangle_0 = \frac{1}{8\pi^2}\int_{\mathcal{M}} \frac{1 + \cos(\theta_1 - \theta_2)}{\tan^2(\frac{\theta_1-\theta_2}{2})e^{-2K\tau} + 1} d\theta = \frac{2e^{-K\tau} + 1}{2(e^{-K\tau} + 1)^2}$$

which then yields

$$\int_0^t \left\langle R^2 \circ S^\tau \right\rangle_0 \, d\tau = t + \frac{\ln\left(e^{-Kt} + 1\right)}{K} - \frac{\ln(2)}{K}$$

and

$$\int_0^t \left\langle R^2(R^2 \circ S^\tau) \right\rangle_0 \, d\tau = \frac{t}{2} + \frac{1}{2K}\left[\frac{3}{2} + \ln\left(\frac{e^{-Kt} + 1}{2}\right) - \frac{2}{e^{Kt} + 1} - \frac{1}{e^{-Kt} + 1}\right]\,.$$

Thus, we finally obtain the explicit expressions

$$\left\langle \Omega_V^{f_0} \right\rangle_t = K \tanh\left(\frac{Kt}{2}\right) \tag{4.14}$$

and

$$\left\langle (\Omega_V^{f_0} \circ S^t)\Omega_V^{f_0} \right\rangle_0 = \frac{K^2}{1 + \cosh(Kt)}\,. \tag{4.15}$$

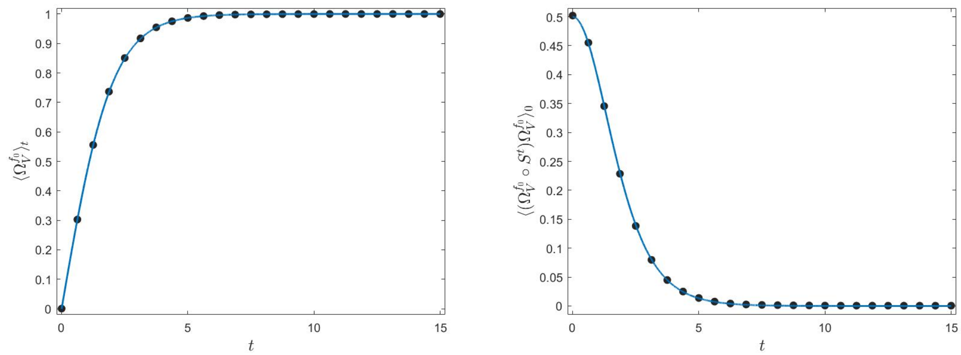

FIGURE 1. Behavior of $\langle \Omega_V^{f_0} \rangle_t$ and $\langle (\Omega_V^{f_0} \circ S^t)\Omega_V^{f_0} \rangle_0$ as functions of time, for $N = 2$, $K = 1$ and $\omega = 0$. Disks and solid lines correspond to the numerical and analytical results, respectively. The averages were taken over a set of 5000 trajectories with initial data sampled from the uniform distribution on $[0, 2\pi)$.

In the limit $t \to +\infty$, we thus find the asymptotic values

$$\left\langle \Omega_V^{f_0} \right\rangle_t \to K \qquad \text{and} \qquad \left\langle (\Omega_V^{f_0} \circ S^t)\Omega_V^{f_0} \right\rangle_0 \to 0 \tag{4.16}$$

In particular, the two-time autocorrelation of $\Omega_V^{f_0}$ is monotonic as also shown in the two panels of Fig.1. Indeed, Eq.(4.15) yields:

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\langle (\Omega_V^{f_0} \circ S^t)\Omega_V^{f_0} \right\rangle_0 = -K^2 \frac{\sinh Kt}{(1 + \cosh Kt)^2}$$

which is ≤ 0 for $t \geq 0$.

4.2. **The general case.** In this Subsection we assume $N \geq 2$ and $\omega = 0$, considering the following dynamics:

$$\dot{\theta}_i = \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i) = KR \sin(\Phi - \theta_i)\,, \qquad i = 1, \ldots, N\,. \tag{4.17}$$

where R and Φ are defined in Eq.(2.2). We are going to prove that the observable $\left\langle \Omega_V^{f_0} \right\rangle_t$ is a monotonic function of time, and we can estimate the asymptotic value it attains in the large time limit. We start by proving the following result.

Lemma 4.1. *For every $t > 0$, the time derivative of the expectation of the Dissipation Function obeys:*

$$\frac{d}{dt}\left(\Omega_V^{f_0}(S^t\theta)\right) \geq 0 \quad \textit{and} \quad \frac{d}{dt}\left\langle \Omega_V^{f_0} \right\rangle_t = \left\langle (\Omega_V^{f_0} \circ S^t)\Omega_V^{f_0} \right\rangle_0 \geq 0\,. \tag{4.18}$$

Proof. First, we note that by setting $\mathcal{O} = \Omega_V^{f_0}$ in Eq. (3.18), we find:

$$\frac{d}{dt}\langle \Omega_V^{f_0} \rangle_t = \langle (\Omega_V^{f_0} \circ S^t)\ \Omega_V^{f_0} \rangle_0\,. \tag{4.19}$$

Moreover, Eq. (3.15) with $t = 0$ and $\mathcal{O} = \Omega_V^{f_0}$ yields:

$$\left\langle \Omega_V^{f_0} \right\rangle_t = \left\langle \Omega_V^{f_0} \circ S^t \right\rangle_0 . \tag{4.20}$$

Therefore, we can write:

$$\begin{aligned} \frac{d}{dt} \left\langle \Omega_V^{f_0} \circ S^t \right\rangle_0 &= \frac{d}{dt} \int_{\mathcal{M}} \Omega_V^{f_0}(S^t\theta) f_0(\theta) d\theta = \\ &= \int_{\mathcal{M}} \frac{d}{dt} \left(\Omega_V^{f_0}(S^t\theta) \right) f_0(\theta) d\theta = \left\langle \frac{d}{dt} \left(\Omega_V^{f_0}(S^t\theta) \right) \right\rangle_0 . \end{aligned} \tag{4.21}$$

Then, using Eq.(2.5) in Ref.[6] we find:

$$\frac{d}{dt} R^2(S^t\theta) = \frac{2K}{N} R^2(S^t\theta) \sum_{j=1}^{N} \sin^2 \left(S^t\theta_j - \Phi\left(S^t\theta\right) \right) \tag{4.22}$$

where $S^t\theta_j$ denotes the $j-$th element of $S^t\theta$, and then

$$\frac{d}{dt} \left(\Omega_V^{f_0}(S^t\theta) \right) = 2K^2 R^2(S^t\theta) \left[\sum_{j=1}^{N} \sin^2 \left(S^t\theta_j - \Phi\left(S^t\theta\right) \right) \right] \geq 0 \tag{4.23}$$

for all $\theta \in \mathcal{M}$. By integrating over $\mathcal{M}$ we obtain (4.18). This completes the proof. □

Remark 4.3. *Unlike stationary current autocorrelations, that may fluctuate between positive and negative values, the two times $\Omega_V^{f_0}$ autocorrelation, computed with respect to the initial probability measure, is non-negative.*

Theorem 2.4 of Ref.[6] shows that non stationary solutions of the system (4.17) converge, as $t \to +\infty$, either to a complete frequency synchronized state Θ^*, *i.e.* to a state denoted by $(N, 0)$, that takes the form:

$$\Theta^* = (\varphi^*, \ldots, \varphi^*) \tag{4.24}$$

in which all phases are equal; or to a state denoted by $(N-1, 1)$, that takes the form:

$$\Theta^\dagger = (\varphi^* + k_1\pi, \varphi^* + k_2\pi, \varphi^* + k_3\pi, \varphi^* + k_4\pi, \ldots, \varphi^* + k_N\pi) \tag{4.25}$$

where $k_i \in \{-1, +1\}$ for a single $i \in \{1, 2, ..., N\}$, and all $k_j = 0$ with $j \neq i$. This can be understood also in terms of the Dissipation Function. In the first place, without loss of generality, let us consider a fixed point $\bar{\theta}$ of type $(N-1, 1)$ whose antipodal is in the N-component, *i.e.*

$$\bar{\theta} = (\varphi^*, \ldots, \varphi^*, (\varphi^* + \pi) \bmod 2\pi) \tag{4.26}$$

for a $\varphi^* \in [0, 2\pi)$. Then, the following holds:

Proposition 4.1. *The set of initial data such that the solution to* (4.17) *reaches a stationary* $(N-1, 1)$*-state for* $t \to +\infty$ *has 0-measure.*

Proof. For $V(\theta)$ as in (2.11), the Jacobian matrix $A(\theta) \doteq \nabla V(\theta)$ is given by

$$A_{ij} = \begin{cases} \frac{\partial V_j}{\partial \theta_i} = \frac{1}{N}\cos(\theta_i - \theta_j), & i \neq j \\ \\ \frac{\partial V_j}{\partial \theta_j} = -\frac{1}{N}\sum_{k \neq j}^{N} \cos(\theta_j - \theta_k) \end{cases}$$

For the fixed point $\bar{\theta}$ set in (4.26) we obtain a symmetric matrix $\bar{A} = A(\bar{\theta})$ whose entries are

$$\bar{A}_{ij} = \begin{cases} \frac{1}{N} & i \neq j \text{ and } i, j \neq N \\ -\frac{1}{N} & i \neq j \text{ and } i = N \text{ or } j = N \\ -\frac{N-3}{N} & i = j < N \\ \frac{N-1}{N} & i = j = N \end{cases} \tag{4.27}$$

By the symmetry of $\bar{A}$, the extremal representation of the eigenvalues $\{\lambda_k\}_{k=1}^{N}$ of $\bar{A}$ are given by the optimization problem:

$$\max_{1 \leq k \leq N} \lambda_k = \max_{\|x\|=1} \{x' \nabla \bar{A} x\}, \quad \min_{1 \leq k \leq N} \lambda_k = \min_{\|x\|=1} \{x' \bar{A} x\}$$

Setting x to be the standard-basis vectors $\mathbf{e}_i$, where $\mathbf{e}_i$ denotes the vector with a 1 in the ith coordinate and 0's elsewhere, we see that

$$\min_{1 \leq k \leq N} \lambda_k \leq \min_{1 \leq i \leq N} \{\bar{A}\}_{ii} = -\frac{N-3}{N} < 0, \qquad 0 < \frac{N-1}{N} = \max_{1 \leq i \leq N} \{\bar{A}\}_{ii} \leq \max_{1 \leq k \leq N} \lambda_k\,.$$

Therefore, there exists at least one positive eigenvalue and at least one negative eigenvalue. Indeed, the matrix $\bar{A}$ has the eigenvalues $\lambda_- = -(N-2)/N$ with algebraic multiplicity $N-2$, $\lambda_2 = 0$ and $\lambda_3 = 1$ with algebraic multiplicity 1. This can be checked considering the proposed subspaces of the center, stable and unstable subspace of the linearized system at $\bar{\theta}$

$$E^c = \left\{ \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{bmatrix} \right\},\ E^s = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ and } E^u = \left\{ \begin{bmatrix} -1 \\ -1 \\ \vdots \\ -1 \\ -1 \\ N-1 \end{bmatrix} \right\}.$$

Then, the Center Manifold Theorem [38, p.116] yields the existence of an $(N-2)$-dimensional stable manifold $W^s(\bar{\theta})$ tangent to the stable subspace E^s, and the existence of a 1-dimensional unstable manifold $W^u(\bar{\theta})$, and 1-dimensional center manifold $W^c(\bar{\theta})$ tangents to the E^u and E^c subspaces respectively. Consequently, the dimension of the center manifold conjoint with the stable manifold is smaller than N, which implies a null Lebesgue measure in $\mathbb{R}^n$. □

Moreover, we have:

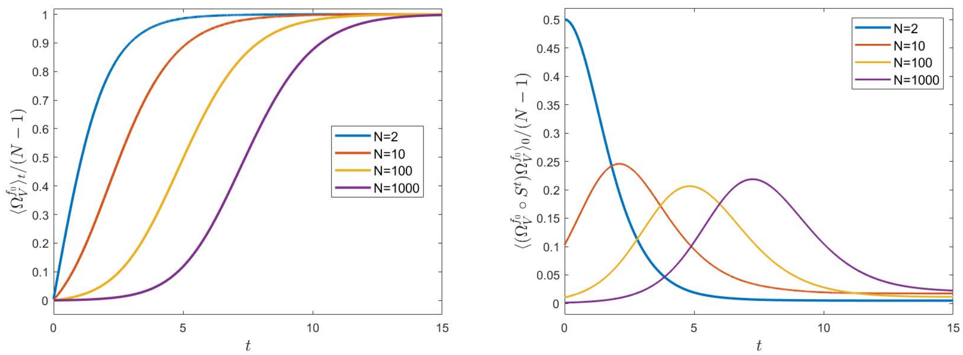

FIGURE 2. Behavior of $\langle\Omega_V^{f_0}\rangle_t$ (left panel) and $\langle(\Omega_V^{f_0}\circ S^t)\Omega_V^{f_0}\rangle_0$ (right panel), both rescaled by $(N-1)$, as functions of time, for $K=1$, $\omega=0$ and for different values of N. The curves on the right panel represent the time derivative of those in the left panel. In particular, $t=0$ in the right panel represents K^2/N, cf. Eq.(4.31).

Lemma 4.2. (Synchronization): *Irrespective of the initial condition $\theta\in\mathcal{T}$, the Dissipation Function obeys:*

$$\lim_{t\to\infty}\Omega_V^{f_0}(S^t\theta)=\begin{cases}K\,(N-1)\;, & \textit{for}\;\;\theta\neq\Theta^\dagger\\[2ex] K\,(N-1)\left(\frac{N-4}{N}\right) & \textit{for}\;\;\theta=\Theta^\dagger\end{cases}\tag{4.28}$$

where $K(N-1)$, the maximum of $\Omega_V^{f_0}$ in $\mathcal{T}^N$, corresponds to $(N,0)$ synchronization.

Proof. Because of Theorem 2.4 in Ref.[6] and of the continuity of $\Omega_V^{f_0}$, the long time limit of $\Omega_V^{f_0}\circ S^t$ in the case $\theta\neq\Theta^\dagger$ is given by $\Omega_V^{f_0}(\Theta^*)$. Then, Eq.(2.10) and Eq.(4.4), yield the first line of Eq.(4.28). The case $\theta=\Theta^\dagger$, gives, instead:

$$R^*e^{i\varphi^*}=\frac{1}{N}\left((N-1)e^{i\varphi^*}+e^{i(\varphi^*+\pi)}\right)=\frac{N-2}{N}e^{i\varphi^*}\,,\tag{4.29}$$

Substituting in Eq.(4.4) we obtain the second line of (4.28). □

Remark 4.4. *Equation* (4.28) *implies that*

$$\lim_{N\to\infty}\lim_{t\to\infty}\frac{\Omega_V^{f_0}(S^t\theta)}{N}=K\,.\tag{4.30}$$

In other words, the large t limit followed by the large N limit implies that the coupling constant K, which drives the synchronization process in the Kuramoto dynamics (2.1)*, equals the average Dissipation per oscillator. For fixed N, synchronization is also evident from the fact that Eq.*(4.23) *must converge to 0, for $\Omega_V^{f_0}$ to become constant.*

This also implies $R^2(S^t\theta)\to 1$, as $t\to\infty$. It suffices to consider the definition (4.4) of $\Omega_V^{f_0}$. For different values of N, Fig. 2 illustrates the behavior of $\langle\Omega_V^{f_0}\rangle_t$ and of its time derivative, which is $\langle(\Omega_V^{f_0}\circ S^t)\Omega_V^{f_0}\rangle_0$, as functions of time. The initial growth of the autocorrelation may look unusual, since autocorrelations are commonly found to decrease. However, unlike standard calculations that rely on an invariant distribution,[1] our autocorrelation is computed with respect to the transient probability measure μ_0. The figure portrays the result of numerical simulations. The right panel of Fig. 2, shows that for sufficiently large N the autocorrelation function $\langle(\Omega_V^{f_0} \circ S^t)\Omega_V^{f_0}\rangle_0$ reaches a maximum before it decreases, as required for convergence to a steady state. An interesting result is the following.

Lemma 4.3. *For $N \geq 2$, the derivative of the time dependent average of $\Omega_V^{f_0}$, computed at time $t = 0$ obeys:*

$$\frac{d}{dt}\left\langle \Omega_V^{f_0} \right\rangle_t \bigg|_{t=0} = \left\langle \left(\Omega_V^{f_0}\right)^2 \right\rangle_0 = K^2 \frac{N-1}{N} \,. \tag{4.31}$$

Note that the derivative of the mean Dissipation Function equals its autocorrelation function, as expressed by Eq.(4.18). Therefore, Eq.(4.31) gives the value of this autocorrelation function at $t = 0$, as shown in the right panel of Fig. 2.

Proof. Using (4.20), (4.21) and (4.23) we find that

$$\frac{d}{dt}\left\langle \Omega_V^{f_0} \right\rangle_t = \frac{d}{dt}\left\langle \Omega_V^{f_0} \circ S^t \right\rangle_0 = 2K^2 \left\langle R^2(S^t\theta) \sum_{j=1}^N \sin^2\left(S^t\theta_j - \Phi\left(S^t\theta\right)\right) \right\rangle_0 . \tag{4.32}$$

Thus, at $t = 0$, the integrand of (4.32) reads

$$\begin{aligned} R^2(\theta) \sum_{j=1} \sin^2(\Phi - \theta_j) &= \frac{1}{N^2} \sum_{j=1}^N \left(\sum_{l=1}^N \sin(\theta_l - \theta_j) \right)^2 \qquad (4.33) \\ &= \frac{1}{N^2} \sum_{j=1}^N \left[\sum_{l=1}^N \sin^2(\theta_j - \theta_l) + \sum_{l=1}^N \sum_{\substack{k=1 \\ k \neq l}}^N \sin(\theta_l - \theta_j)\sin(\theta_k - \theta_j) \right] . \end{aligned}$$

Furthermore, we have:

$$\begin{aligned} &\int_0^{2\pi} \int_0^{2\pi} \sin(\theta_l - \theta_j)\sin(\theta_k - \theta_j) d\theta_l d\theta_k \\ &\qquad = \int_0^{2\pi} \sin(\theta_l - \theta_j) d\theta_l \int_0^{2\pi} \sin(\theta_k - \theta_j) d\theta_k = 0 \,. \qquad (4.34) \end{aligned}$$

[1] In linear response the initial distribution is considered invariant to first order in the perturbation.

Therefore, considering (4.33) and (4.34) over (4.32) at time $t = 0$ we have:

$$
\begin{aligned}
\frac{d}{dt}\langle \Omega_V^{f_0} \rangle_t \Big|_{t=0} &= 2K^2 \int_{\mathcal{M}} R^2(\theta) \sum_{j=1}^{N} \sin^2(\Phi - \theta_j) f_0(\theta) d\theta \\
&= 2\frac{K^2}{N^2} \frac{1}{(2\pi)^N} \int_{\mathcal{M}} \sum_{j=1}^{N} \sum_{l=1}^{N} \sin^2(\theta_j - \theta_l) d\theta \\
&= 2\frac{K^2}{(2\pi)^2} \frac{N-1}{N} \int_0^{2\pi} \int_0^{2\pi} \sin^2(\theta_1 - \theta_2) d\theta_1 d\theta_2 \\
&= K^2 \frac{N-1}{N} .
\end{aligned}
$$

This completes the proof of (4.31). □

5. Comparison with linear response

In this Section we compare the foregoing exact response formalism with the standard linear response [20]. Consider a perturbed vector field V_ε, defined as

$$V_\varepsilon(\theta) = V_0(\theta) + \varepsilon V_p(\theta) \tag{5.1}$$

where the parameter ε expresses the strength of the perturbation. Following Section 4, we identify ε with K, and define:

$$V_0(\theta) = \omega \tag{5.2}$$

$$V_{p,j}(\theta) = R \sin(\Phi - \theta_j) , \quad j = 1, ..., N \tag{5.3}$$

Correspondingly, we denote by S_ε^t and S_0^t the perturbed and unperturbed flows, respectively. From Eq. (3.3), we obtain:

$$\Omega_\varepsilon^{f_0} = \Omega_0^{f_0} + \varepsilon \Omega_p^{f_0} = \varepsilon \Omega_p^{f_0} \tag{5.4}$$

where $\Omega_0^{f_0}$ and $\Omega_p^{f_0}$ denote the Dissipation Function (4.4) evaluated in terms of the vector fields V_0 and V_p, respectively. In particular, we have:

$$\Omega_p^{f_0} = \frac{1}{N} \sum_{i,j=1}^{N} \cos(\theta_j - \theta_i) - 1 \tag{5.5}$$

The last equality in Eq.(5.4) derives from the fact that $\Omega_0^{f_0} \equiv 0$ if, as assumed, f_0 is invariant under the unperturbed dynamics, cf. Eq.(4.3). We may then write the *exact* response Eq.(3.14) as:

$$\langle \mathcal{O} \rangle_{t,\varepsilon} = \langle \mathcal{O} \rangle_0 + \varepsilon \int_0^t \langle (\mathcal{O} \circ S_\varepsilon^\tau) \ \Omega_p^{f_0} \rangle_0 \ d\tau , \tag{5.6}$$

where $\mathcal{O} \circ S_\varepsilon^t$ denotes the observable $\mathcal{O}$ composed with the perturbed flow. Because this formula is exact, the parameter ε in it does not need to be small, and it appears both as a factor multiplying the integral and as a subscript indicating the perturbed flow S^t_ε. Next, using Eq. (3.8), we can write

$$f_t(\theta) = \exp\left\{ \varepsilon \int_{-t}^{0} \Omega_p^{f_0}(S^\tau_\varepsilon \theta)\, d\tau \right\} f_0(\theta)\,. \tag{5.7}$$

which can be expanded about $\varepsilon = 0$, and truncated to first order, to obtain the linear approximation of the evolving probability density:

$$\begin{aligned} \bar{f}_t(\theta;\varepsilon) &= f_0(\theta)\left(1 + \varepsilon \frac{d}{d\varepsilon} \exp\left\{ \varepsilon \int_{-t}^{0} \Omega_p^{f_0}(S^\tau_\varepsilon \theta)\, d\tau \right\}\Bigg|_{\varepsilon=0}\right) \qquad (5.8)\\ &= f_0(\theta)\left(1 + \varepsilon \int_{-t}^{0} \Omega_p^{f_0}(S^\tau_0 \theta)\, d\tau \right) \\ &= f_0(\theta)\left(1 + \varepsilon \int_{0}^{t} \Omega_p^{f_0}(S^{-\tau}_0 \theta)\, d\tau \right)\,. \qquad (5.9)\end{aligned}$$

Note that the expansion in the variable ε of the exponential in Eq.(5.7), requires computing the derivatives with respect to ε of the time integral in it. This, in turn, requires the derivatives of the Dissipation Function $\Omega_p^{f_0}(S^\tau_\varepsilon\theta)$, and of the evolved trajectory points $S^\tau_\varepsilon\theta$. Because both the Dissipation Function and the dynamics are smooth on a compact manifold, their derivatives are bounded, and their integral up to any time t computed at $\varepsilon = 0$ is also bounded. Multiplied by ε, this integral gives a vanishing contribution to the first derivative of the exponential in Eq.(5.7). There only remain the exponential and the integral computed at $\varepsilon = 0$, multiplied by the increment ε, which is the brackets in Eq.(5.9). We then define:

$$\overline{\langle \mathcal{O} \rangle}_{t,\varepsilon} = \int_{\mathcal{M}} \mathcal{O}(\theta) \bar{f}_t(\theta;\varepsilon)\, d\theta = \langle \mathcal{O} \rangle_0 + \varepsilon \int_0^t \left\langle \mathcal{O} \;\left(\Omega_p^{f_0} \circ S_0^{-\tau}\right)\right\rangle_0 \, d\tau \tag{5.10}$$

which is the linear response result. At the same time, the invariance of the correlation function under time translations of the unperturbed dynamics, which is proven in Appendix B, yields:

$$\overline{\langle \mathcal{O} \rangle}_{t,\varepsilon} = \langle \mathcal{O} \rangle_0 + \varepsilon \int_0^t \left\langle (\mathcal{O} \circ S_0^{\tau})\; \Omega_p^{f_0}\right\rangle_0 \, d\tau \tag{5.11}$$

It is interesting to note that, unlike the Green-Kubo formulae, which are obtained from small Hamiltonian perturbations, here the perturbation is not Hamiltonian. Therefore, we may call (5.11) a *generalized* GK formula. It is worth comparing it with the exact response formula (5.6), as follows:

$$\langle \mathcal{O} \rangle_{t,\varepsilon} - \overline{\langle \mathcal{O} \rangle}_{t,\varepsilon} = \varepsilon \int_0^t \left\langle \Big[(\mathcal{O} \circ S_\varepsilon^{\tau}) - (\mathcal{O} \circ S_0^{\tau}) \Big] \Omega_p^{f_0}\right\rangle_0 \, d\tau \tag{5.12}$$

which shows that the two formulae tend to be the same, in the small ε limit, as expected. Thanks to the use of the Dissipation Function, their difference lies only in the use of the perturbed rather than the unperturbed flow inside $\mathcal{O}$.

Let us dwell on the response of two relevant observables, in the case in which $V_0 = \omega = 0$, hence S_0^t is the identity operator, Id. First, taking $\mathcal{O} = \Omega_\varepsilon^{f_0} = \varepsilon \Omega_p^{f_0}$, we find

$$\begin{aligned} \langle \Omega_\varepsilon^{f_0} \rangle_{t,\varepsilon} - \overline{\langle \Omega_\varepsilon^{f_0} \rangle}_{t,\varepsilon} &= \int_0^t \left\langle \left[\left(\Omega_\varepsilon^{f_0} \circ S_\varepsilon^\tau \right) - \left(\Omega_\varepsilon^{f_0} \circ S_0^\tau \right) \right] \, \Omega_\varepsilon^{f_0} \right\rangle_0 \, d\tau \\ &= \int_0^t \left[\left\langle \left(\Omega_\varepsilon^{f_0} \circ S_\varepsilon^\tau \right) \Omega_\varepsilon^{f_0} \right\rangle_0 - \left\langle \left(\Omega_\varepsilon^{f_0} \right)^2 \right\rangle_0 \right] \, d\tau \end{aligned} \tag{5.13}$$

where we used the identity $\left(\Omega_\varepsilon^{f_0} \circ S_0^\tau \right) = \Omega_\varepsilon^{f_0}$, which derives from the fact that S_0^t =Id, and which yields, cf. Eq.(4.31):

$$\left\langle \left(\Omega_\varepsilon^{f_0} \right)^2 \right\rangle_0 = \varepsilon^2 \frac{N-1}{N} \tag{5.14}$$

For $N = 2$, we can also use the explicit expression (4.15) for the autocorrelation function:

$$\left\langle \left(\Omega_\varepsilon^{f_0} \circ S_\varepsilon^\tau \right) \Omega_\varepsilon^{f_0} \right\rangle_0 = \frac{\varepsilon^2}{1 + \cosh(\varepsilon\tau)} \tag{5.15}$$

which leads to:

$$\langle \Omega_\varepsilon^{f_0} \rangle_{t,\varepsilon} = \varepsilon \tanh\left(\frac{\varepsilon t}{2} \right), \quad \text{and} \quad \overline{\langle \Omega_\varepsilon^{f_0} \rangle}_{t,\varepsilon} = \frac{\varepsilon^2 t}{2} \tag{5.16}$$

so that

$$\langle \Omega_\varepsilon^{f_0} \rangle_{t,\varepsilon} = \overline{\langle \Omega_\varepsilon^{f_0} \rangle}_{t,\varepsilon} + o(\varepsilon^2) t \tag{5.17}$$

In other words, for any $\varepsilon > 0$, the difference of the two responses is small at small times, but it diverges linearly as time passes.

As a second instance, let us take $\mathcal{O} = \psi = \theta_1 - \theta_2$. From (4.4) and (4.11) we have:

$$\Omega_\varepsilon^{f_0} = 2\varepsilon R^2(\psi) - \varepsilon = \frac{2\varepsilon}{\tan^2\left(\frac{\psi}{2}\right) + 1} - \varepsilon = \varepsilon \cos(\psi) \tag{5.18}$$

Moreover, Eq.(4.10) yields:

$$(\psi \circ S_\varepsilon^t) = 2 \arctan\left[\tan\left(\frac{\psi}{2} \right) e^{-\varepsilon t} \right] \tag{5.19}$$

and we can write:

$$\begin{aligned} \langle \psi \rangle_{t,\varepsilon} - \overline{\langle \psi \rangle}_{t,\varepsilon} &= \int_0^t \left[\left\langle (\psi \circ S_\varepsilon^\tau) \, \Omega_\varepsilon^{f_0} \right\rangle_0 - \left\langle (\psi \circ S_0^\tau) \, \Omega_\varepsilon^{f_0} \right\rangle_0 \right] \, d\tau \\ &= \int_0^t \left[\left\langle (\psi \circ S_\varepsilon^\tau) \, \Omega_\varepsilon^{f_0} \right\rangle_0 - \left\langle \psi \; \Omega_\varepsilon^{f_0} \right\rangle_0 \right] \, d\tau \end{aligned} \tag{5.20}$$

where we used S_0^t =Id, which implies $(\psi \circ S_0^\tau) \equiv \psi$. Therefore, using (5.18) and (5.19) in (5.20), we obtain:

$$\begin{aligned} \langle \psi \rangle_{t,\varepsilon} - \overline{\langle \psi \rangle}_{t,\varepsilon} &= \frac{1}{(2\pi)^2} \int_0^t \int_{\mathcal{M}} 2 \arctan\left[\tan\left(\frac{\theta_1 - \theta_2}{2} \right) e^{-\varepsilon\tau} \right] \cos(\theta_1 - \theta_2) d\theta d\tau \\ &\quad - \frac{1}{(2\pi)^2} \int_0^t \int_{\mathcal{M}} (\theta_1 - \theta_2) \cos(\theta_1 - \theta_2) d\theta d\tau = 0 \end{aligned} \tag{5.21}$$

The last equality follows from the fact that the integrands in Eq. (5.21) are odd continuous and periodic functions, that are integrated over a whole period, so thatone actually:

$$\langle\psi\rangle_{t,\varepsilon} = \overline{\langle\psi\rangle}_{t,\varepsilon} \equiv 0 \quad , \quad \forall\, t > 0\,. \tag{5.22}$$

Clealry, there are observables for which the difference of responses is irrelevant, since they do not evolve in time, and others for which the difference is substantial, even under small perturbations. In any event, the exact response characterizes the synchronization transition, while the linear response does not.

6. Concluding remarks

We investigated the Kuramoto dynamics for identical oscillators through the statistical mechanics framework of response theory. As a reference (unperturbed) dynamics we took a system of uncoupled oscillators, with statistical properties given by a factorized N-body distribution with uniform marginal densities. Next, we interpreted the classical Kuramoto mean-field dynamics as a perturbation of the reference one. For any finite number N of oscillators, we then derived an exact response formula whose validity holds for arbitrarily large perturbations, and we computed, both analytically and numerically, the asymptotic value of the Dissipation Function. The latter is indeed the main ingredient of the exact response theory. Explicit analytical results are given for $N = 2$. We also investigated the autocorrelation function of the Dissipation Function, and highlighted its non-monotonic behavior, for sufficiently large N. Finally, we compared the exact response formalism with the linear response regime. We found that the two responses differ substantially, even for very small perturbations, and that only the exact response describes the transition to synchronized states.

This indicates that the exact response theory, which by definition must be capable of describing even systems undergoing non-equilibrium phase transitions, may actually be used in practice. Synchronization phenomena, which are ubiquitous in Nature, are indeed of that kind.

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Appendix A. Unstable fixed points for the identical case

In this section we show explicitly the existence of unstable points in any neighborhood of a fixed point of $(N-1,1)$ type.

Proposition A.1. *Let $\bar\theta$ be the stationary type $(N-1,1)$ solution set in (4.26) and $\delta > 0$. If $\theta = (\theta_1, \ldots, \theta_N)$ satisfy*

$$|\theta_j - \varphi^*| \leq \delta^2, \quad j = [1, \ldots, N-1] \tag{A.1}$$

$$\theta_N = \varphi^* + \pi + \delta \tag{A.2}$$

then there exists a δ_0 *such that for any* $0 < |\delta| < \delta_0$ *one has:*

$$R^2(\theta) > \left(\frac{N-2}{N}\right)^2 \tag{A.3}$$

and therefore $R(S^t\theta) \to 1$ *as* $t \to \infty$.

Proof. From the equation (2.10) we have that

$$\begin{aligned} R^2(\theta) - \left(\frac{N-2}{N}\right)^2 &= \frac{1}{N^2}\sum_{i,j=1}^{N} \cos(\theta_i - \theta_j) - \sum_{i,j=1}^{N-1} 1 - 2\sum_{i=1}^{N-1} 1 + 1 \\ &= \frac{1}{N^2}\left[\underbrace{\sum_{i,j=1}^{N-1}[\cos(\theta_i - \theta_j) - 1]}_{I_1} + \underbrace{2\sum_{j=1}^{N-1}[\cos(\theta_N - \theta_j) + 1]}_{I_2}\right] \\ &= \frac{1}{N^2}(I_1 + I_2) \end{aligned}$$

Next, we estimate the lower bounds of I_1 and I_2. We use the elementary inequality $\frac{x^2}{4} \le 1 - \cos(x) \le \frac{x^2}{2}$, which is valid for $|x| \le x_0$ where $x_0 \in (0, \frac{5\pi}{6})$. Then, by using (A.1), for I_1 we get

$$\begin{aligned} I_1 &= \sum_{i,j=1}^{N-1} \cos(\theta_i - \theta_j) - 1 \ge -\frac{1}{2}\sum_{i,j=1}^{N-1}(\theta_i - \theta_j)^2 \\ &= -\frac{1}{2}\sum_{i,j=1}^{N}[(\theta_i - \varphi^*) + (\varphi^* - \theta_j)]^2 \ge -2\delta^4(N-1)^2 \end{aligned} \tag{A.4}$$

if $2\delta^2 \le x_0$. On the other hand, for I_2 we first observe that for $1 \le j \le N-1$

$$\begin{aligned} |\theta_N - \pi - \theta_j| &\le |\theta_N - \pi - \varphi^*| + |\varphi^* - \theta_j| \\ &\le \delta^2 + |\delta| \\ &\le 2|\delta| \end{aligned}$$

if we take $|\delta| \le 1$.

Then we can use the inequality (A.1) to obtain that

$$\begin{aligned} \cos(\theta_N - \theta_j) + 1 &= 1 - \cos(\theta_N - \pi - \theta_j) \\ &\ge \frac{1}{4}(\delta^2 + |\delta|)^2 = \frac{1}{4}\delta^2(1 + |\delta|)^2 \\ &\ge \frac{\delta^2}{4} \end{aligned}$$

where we consider that $\delta^2 + |\delta| \le 2|\delta| \le x_0$. Therefore

$$I_2 \ge (N-1)\frac{\delta^2}{2} \tag{A.5}$$

and it follows from the equations (A.4) and (A.5) that

$$
\begin{aligned}
R^2(\theta) - \left(\frac{N-2}{N}\right)^2 &\geq \frac{1}{N^2}\left\{\frac{N-1}{2}\delta^2 - 2\delta^4(N-1)^2\right\} \\
&= \delta^2 \frac{N-1}{N^2}\left[\frac{1}{2} - 2\delta^2(N-1)\right] \\
&> 0
\end{aligned}
$$

for $\delta^2 < \frac{1}{2(N-1)}$. In summary, if we choose $\delta_0 = \min\{1, \frac{x_0}{2}, \frac{1}{2\sqrt{N-1}}\}$, then (A.3) holds.

Finally, to prove that $R(S^t\theta) \to 1$ as $t \to +\infty$, we use the fact that $t \to R(S^t\theta)$ is not decreasing and converges to a value $(N-2k)/N > 0$ for some integer $k \geq 0$.

By (A.3) and the monotonicity we deduce that $R(S^t\theta) > \frac{N-2k}{N}$ for all $k \geq 1$ and all $t \geq 0$, and therefore we conclude that, necessarily, the limiting value has $k = 0$. The proof is complete. □

Appendix B. Stationary correlation functions

Given a vector field V_0, let f_0 be an invariant probability density under the flow S_0^t generated by V_0. With the notation set by Eq.(3.7), let $\Lambda_{0,t}^0$ be the time integral over a trajectory segment, from time 0 to time t, of the phase space volume variation rate Λ, which is the divergence of the vector field V_0. Two-time correlation functions between two generic observables $\mathcal{A}, \mathcal{B} : \mathcal{M} \to \mathbb{R}$, evaluated with the density f_0, are invariant under the time translations determined by S_0^t. This can be shown as follows. First we note that, proceeding as in Eq. (3.9), one finds

$$
\begin{aligned}
\Omega_{-t,0}^{f_s,0} &= \int_{-t}^{0} \Omega_0^{f_s}(S_0^\tau\theta)d\tau = -\Lambda_{-t,0}^0 - \int_{-t}^{0} \frac{d}{d\tau}\left(\log f_s(S_0^\tau\theta)\right)d\tau \\
&= -\Lambda_{-t,0}^0 - \log\frac{f_s(\theta)}{f_s(S_0^{-t}\theta)}
\end{aligned} \tag{B.1}
$$

Upon setting $s = 0$ in (B.1) and using Eq.(3.12), we find $\Omega_{-t,0}^{f_0,0} \equiv 0$, from which we obtain the following useful relation

$$
f_0(\theta) = \exp\Big\{-\Lambda_{-t,0}^0(\theta)\Big\} f_0(S_0^{-t}\theta) \tag{B.2}
$$

where the exponential term is related to the Jacobian determinant of the dynamics as [30]:

$$
\left|\frac{\partial\left(S_0^{-t}\theta\right)}{\partial\theta}\right| = \exp\Big\{-\Lambda_{-t,0}^0(\theta)\Big\} \tag{B.3}
$$

Let us look, next, at time correlation functions of the form

$$
\langle\left(\mathcal{A}\circ S_0^{s+\tau}\right)\ \left(\mathcal{B}\circ S_0^t\right)\rangle_0 = \int_{\mathcal{M}} \mathcal{A}(S_0^{s+\tau}\theta)\ \mathcal{B}(S_0^t\theta) f_0(\theta)d\theta
$$

for any $s, t, \tau \in \mathbb{R}$. By a change of variables, one finds

$$\begin{aligned}
\langle \left(\mathcal{A} \circ S_0^{s+\tau}\right) \ \left(\mathcal{B} \circ S_0^{t}\right)\rangle_0 &= \int_{\mathcal{M}} \mathcal{A}(S_0^s\theta) \ \mathcal{B}(S_0^{t-\tau}\theta) f_0(S_0^{-\tau}\theta) d\left(S_0^{-\tau}\theta\right) \\
&= \int_{\mathcal{M}} \mathcal{A}(S_0^s\theta) \ \mathcal{B}(S_0^{t-\tau}\theta) f_0(S_0^{-\tau}\theta) \left| \frac{\partial\left(S_0^{-\tau}\theta\right)}{\partial\theta} \right| d\theta \\
&= \int_{\mathcal{M}} \mathcal{A}(S_0^s\theta) \ \mathcal{B}(S_0^{t-\tau}\theta) e^{-\Lambda^0_{-\tau,0}} f_0(S_0^{-\tau}\theta) d\theta \\
&= \int_{\mathcal{M}} \mathcal{A}(S_0^s\theta) \ \mathcal{B}(S_0^{t-\tau}\theta) f_0(\theta) d\theta \\
&= \langle (\mathcal{A} \circ S_0^s) \ \left(\mathcal{B} \circ S_0^{t-\tau}\right)\rangle_0 \qquad \text{(B.4)}
\end{aligned}$$

where we used (B.3) and, in the last line, the formula (B.2).

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