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The sunflower equation: novel stability criteria ⋆

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Abstract: In this paper, we consider a delayed counterpart of the mathematical pendulum model that is termed sunflower equation and originally was proposed to describe a helical motion (circumnutation) of the apex of the sunflower plant. The "culprits" of this motion are, on one hand, the gravity and, on the other hand, the hormonal processes within the plant, namely, the lateral transport of the growth hormone auxin. The first mathematical analysis of the sunflower equation was conducted in the seminal work by Somolinos (1978) who gave, in particular, a *sufficient condition* for the solutions' boundedness and for the existence of a periodic orbit. Although more than 40 years have passed since the publication of the work by Somolinos, the sunflower equation is still far from being thoroughly studied. It is known that a periodic solution may exist only for a sufficiently large delay, whereas for small delays the equation exhibits the same qualitative behavior as a conventional pendulum, and every solution converges to one of the equilibria. However, necessary and sufficient conditions for the stability of the sunflower equation (ensuring the convergence of all solutions) are still elusive. In this paper, we derive a novel condition for its stability, which is based on absolute stability theory of integro-differential pendulum-like systems developed in our previous work. As will be discussed, our estimate for the maximal delay, under which the stability can be guaranteed, improves the existing estimates and appears to be very tight for some values of the parameters.

Keywords: Sunflower equation, delays system, nonlinear system, stability

1. INTRODUCTION

The general theory of delay systems has been thoroughly studied in the recent decades (Bellman and Cooke, 1963; Halanay, 1966; Hale, 1977; Burton, 1985; Fridman, 2014). The most studied problems of delay systems theory are concerned with the existence and uniqueness of solutions, time-domain and frequency-domain stability conditions for linear time-invariant and periodic delay systems, and nonlinear stability theory based on the Lyapunov-Krasovskii functionals and Razumikhin's techniques.

In spite of enormous progress achieved in analysis and control of delay systems, dynamics of relatively simple delay systems remains insufficiently studied. In this paper, we consider one of such systems, being a delayed counterpart of the usual pendulum and known as the *sunflower equation*. This mathematical model was proposed to portray helical movements of the tip of a growing plant studied by botanists since the beginning of 19th century (Israelsson and Johnsson, 1967; Somolinos, 1978; Casal and Somolinos, 1982). The non-trivial plant's motion is caused by hormonal processes in the plant, namely, non-uniform accumulation of the growth hormone (auxin) under the influence of gravity. The key parameter of the equation is the *geotropic reaction time*, that is, the time lag between the plant's inclination from the vertical and the growth hormone gradient (Somolinos, 1978). For a sufficiently small time lag the asymptotic behavior of the sunflower equation mimics the behavior of the pendulum (Burton, 1985, Section 2.2.2): all solutions converge to equilibria and no periodic orbits exist. The periodic solution exists, however, if the delay belongs to some interval that depends on the equation's parameters (Somolinos, 1978). Periodic solutions can be found via e.g. the harmonic balance method (McDonald, 1995; Liu and Kalmár-Nagy, 2010).

The theory of sunflower equation has been developed in many different directions, we mention only a few of them. Casal and Somolinos (1982) examined the entrainment of plant oscillations by periodic exogenous forces. Kulenović and Ladas (1988) showed that the conditions for the periodic orbit existence, established in Somolinos (1978), in fact ensure the oscillatory behavior of every solution: the plant crosses the vertical infinitely many times (these oscillations, however, may decay over time). Lizana (1995) considered the counterpart of the sunflower equation on a cylinder and showed that such a system has a global

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attractor (the maximal compact invariant set that attracts every orbit). Bisconti and Spadini (2015) analyzed a generalized sunflower equation with periodic coefficients and found non-trivial properties of the manifold spanned by periodic trajectories. Recent works (Wei and Huang, 1997; Zhang and Zheng, 2006; Li, 2009) are devoted to advanced bifurcation analysis of the sunflower equation.

In this paper, we return to the original problem examined by Somolinos (1978), namely, the existence of periodic solutions. The criterion from (Somolinos, 1978) relates the properties of the nonlinear sunflower equation and its linearization about zero equilibrium. It is shown in (Somolinos, 1978) that the periodic solution is born through the Hopf bifurcation when the delay passes through some critical value, at which the linearized equation loses its stability. The results of numerical simulations reported in (Somolinos, 1978) suggest that the original sunflower equation is "stable" when the delay is below this critical margin (more formally, all its solutions converge and no periodic solution exists). This conjecture, to the best of the authors' knowledge, still has not been proved; and a gap between sufficient stability conditions (see e.g. Burton (1985)) and the necessary stability condition still exists. In this paper, we substantially reduce this gap by deriving a novel stability condition for the sunflower equation. This condition is based on our recent work (Smirnova and Proskurnikov, 2019), which, in turn, advances V.M. Popov's approach to absolute stability theory and provides sufficient stability conditions for a class of infinite-dimensional systems with periodic nonlinearities.

2. PRELIMINARIES. PROBLEM SETUP.

Following the notation from (Somolinos, 1978), the sunflower equation is written as follows

$$\dot{x}(t) = -\frac{b}{r}\int_{-\infty}^{t-r} \exp\left(-a\frac{t-r-\tau}{r}\right)\sin x(\tau)d\tau. \tag{1}$$

A more elegant form of the sunflower equation, usually examined in the literature, is obtained by differentiating (1) with respect to t and is as follows

$$r\ddot{x}(t) + a\dot{x}(t) + b\sin x(t-r) = 0. \tag{2}$$

As can be easily seen, every bounded solution of (2) defined for $t \in (-\infty, \infty)$ (in particular, all periodic solutions) obeys the integral equation (1), however, formally equation (2) is a more general than (1).

The equation (2) involves three positive constants $a, b, r > 0$. The constant $r > 0$ quantifies the delay in hormone transport. The problem first addressed in (Somolinos, 1978) is to find the range of r for which the system (2) has a periodic solution. More generally, one may be interested in the behavior of (2) for different triples a, b, r.

2.1 Stability under small delays

When $r \to 0$, the second-order equation (2) degenerates into the first-order equation

$$a\dot{x}(t) + b\sin x(t) = 0, \tag{3}$$

which is featured by the *gradient-like behavior* (Leonov et al., 1992): each solution converges to one of the equilibria $x = \pi k$. It can also be proved by using the quadratic Lyapunov function $V(x) = (x - 2\pi k)^2$ that every solution starting in the region of initial conditions $x(0) \in (2\pi k - \pi, 2\pi k + \pi)$ with k being integer, converges to $2\pi k$ (being a stable equilibrium of (3)) as $t \to \infty$. Constant solutions $x \equiv 2\pi k + \pi$ stand for the unstable equilibria.

One can expect that the gradient-like behavior is enjoyed by equation (2) for $r > 0$ being small. This statement appears to be correct, as shown by the following.

Lemma 1. (Burton, 1985, Section 2.2.2) Equation (2) is featured by the gradient-like behavior (all solutions converge to equilibria [1]) whenever $a, b > 0$ and r is such that

$$0 < r < r_* \triangleq \frac{a}{2b}. \tag{4}$$

Lemma 1 is proved (Burton, 1985) by using the Lyapunov-Krasovskii functionals. As we will show in Section 3, in fact estimate (4) is rather conservative and can be tightened. Unlike (Burton, 1985), our method does not rely on the Lyapunov methods and employs frequency-domain stability criteria established in (Leonov et al., 1992, 1996) and generalized in (Smirnova and Proskurnikov, 2019).

2.2 A linearized equation and periodic orbits

The properties of nonlinear equation (2) are closely related to the behavior of its linearization at the equilibrim $x \equiv 0$, that is, the linear delay equation

$$\ddot{x}(t) + \frac{a}{r}\dot{x}(t) + \frac{b}{r}x(t-r) = 0. \tag{5}$$

The linearized equation has been examined in (Somolinos, 1978). It was shown that a critical margin $r_0 = r_0(a, b)$ exists such that (5) is stable [2] when $r < r_0$ and unstable when $r > r_0$. The critical case $r = r_0$ corresponds to neutral stability: the characteristic equation

$$\lambda^2 + \frac{a}{r}\lambda + \frac{b}{r}e^{-r\lambda} = 0 \tag{6}$$

acquires a pair of imaginary roots $\lambda_{1,2} = \pm\imath\omega_0$, where $\omega_0 \in (0, \pi/2)$. Hence, r_0 and ω_0 are related by the equations

$$\omega_0^2 = br_0\cos\omega_0, \quad a\omega_0 = br_0\sin\omega_0. \tag{7}$$

Since $\omega/\sin\omega \in (1, \pi/2)$ whenever $\omega \in (0, \pi/2)$, we have

$$\frac{a}{b} < r_0 < \frac{\pi a}{2b}.$$

Using the seminal Hopf birufcation theorem (Somolinos, 1978; Li, 2009; Wei and Huang, 1997) that a periodic orbit is born as delay passes through the critical value $r = r_0$, and hence the periodic solutions exist when $r \to r_0, r > r_0$. A non-trivial result by Somolinos (1978) shows, furthermore, that a periodic orbit exists when

$$r_0 < r < r_1 \triangleq \frac{\pi a^2}{b(a+1)}. \tag{8}$$

Furthermore, it was shown in (Wei and Huang, 1997) that a solution of period $\geq 2r$ exists provided that

$$r_0 < r < r_2 \triangleq \frac{\pi a}{b} - \frac{1}{a} = \frac{(1+a)r_1 - 1}{a} \text{ and } a > b. \tag{9}$$

[1] It was also shown in (Burton, 1985) that (4) provides that $\dot{x} \in L_2[0, \infty)$ for every solution.

[2] It is evident that if (5) is stable, then the equilibria $2\pi k$, $k = 0, \pm1, \pm2, \ldots$ of the nonlinear equation are locally stable. It can be easily shown that the remaining equilibria $\pi + 2\pi k$ are exponentially unstable for any choice of the delay $r \geq 0$.

If $a > b$ and $r_1 > 1$, then $r_2 > r_1$ and hence the result by Wei and Huang (1997) outperforms the result by Somolinos (1978), giving a broader range of delays in which the periodic orbit's existence is ensured.

2.3 Problem in question: stability for "medium" delays

In this paper, we are interested in the behavior of sunflower equation (2) in the range of "medium" delays such that $r_* < r < r_0$. The numerical results reported in (Somolinos, 1978) (for special parameters a, b) suggest that the equation has no periodic orbits as $0 < r < r_0$, and the equation is featured by the gradient-like behavior. To the best of the authors' knowledge, the validity of this statement for $r \in [r_*, r_0)$ remains an open problem. In this paper, we take a step towards stability analysis in this delay range.

Using techniques developed in our previous work (Smirnova and Proskurnikov, 2019), we derive a frequency-domain inequality ensuring stability of (2) (which allows, in particular, to improve the result of Lemma 1). Whereas analytic validation of this frequency-domain inequality can be troublesome, it can be efficiently tested numerically. It appears, in particular, that for the numerical parameters reported in (Somolinos, 1978), the value of r_0 is indeed very close to the maximal delay under which the stability is guaranteed by our criterion.

3. NEW RESULTS: FREQUENCY-DOMAIN STABILITY CRITERION

For our purposes, it is convenient to consider an *integro-differential* equation including (2) as a special case

$$\dot{x}(t) = \alpha(t) - \int_0^t g(t-\tau)\sin x(\tau)d\tau \quad \forall t \geq 0,$$
$$g(t) \triangleq \begin{cases} \frac{b}{r}e^{-a\frac{t-r}{r}} & t \geq r, \\ 0, & t < r. \end{cases} \tag{10}$$

where $\alpha(t)$ can be an *arbitrary* exponentially decaying function, $|\alpha(t)| \leq ce^{-\lambda t}, c, \lambda > 0$. Henceforth, integral equation (10) is referred to as the *sunflower equation.*

Equation (10) has a structure typical for many control systems and may be considered as a *feedback superposition* [3] of the exponentially stable linear block

$$\dot{x}(t) = \alpha(t) - \int_0^t g(t-\tau)x(\tau)d\tau \quad \forall t \geq 0 \tag{11}$$

and the nonlinear feedback characteristics is $\varphi(\sigma) = \sin(\sigma)$. To examine equation (10), we employ a frequency-domain criterion from (Smirnova and Proskurnikov, 2019).

3.1 A frequency-domain stability criterion

First, we need to introduce some notation. Following (Smirnova and Proskurnikov, 2019), we introduce the *transfer* function of the linear block (11) from the input $(-\xi)$ to $\dot{x}$, which, in our situation, is given by

$$K(p) = \int_0^\infty g(t)e^{-pt}\,dt = \frac{b}{r}\frac{e^{-pr}}{p+\frac{a}{r}},\ p \in \mathbb{C}.$$

Lemma 2. (Smirnova and Proskurnikov, 2019, Lemma 3) Consider a system (11), where the function $\varphi(\cdot)$ is C^1-smooth, Δ-periodic ($\varphi(\sigma) = \varphi(\sigma+\Delta)$) with finite number of zeros over the period, obeys the slope restrictions $\alpha_1 \leq \varphi'(\sigma) \leq \alpha_2\,\forall\sigma$ and has the zero average over the period $\int_0^\Delta \varphi(\sigma)\,d\sigma = 0$. Assume that α, g exponentially decay as $t \to \infty$. Suppose also that inequality holds

$$Re\big\{\varkappa K(\imath\omega) - \varepsilon|K(\imath\omega)|^2 - \\ - \left[K(\imath\omega) + \imath\omega\alpha_1^{-1}\right]^* \tau \left[K(\imath\omega) + \imath\omega\alpha_2^{-1}\right]\big\} - \delta \geq 0\,\forall\omega$$

for some parameters $\varkappa \in \mathbb{R}, \delta > 0, \tau, \varepsilon \geq 0$. Then every solution of (11) converges to one of the equilibria

$$\varphi(x(t)) \xrightarrow[t\to\infty]{} 0,\ \dot{x}(t) \xrightarrow[t\to\infty]{} 0.$$

Furthermore, $\varphi(x(\cdot)) \in L_2[0,\infty)$, which also implies, in view of (11), that $\dot{x} \in L_2$.

Applying Lemma 2 to linear block (11) and the nonlinearity $\varphi(x) = \sin x$ (corresponding to $\alpha_1 = -1, \alpha_2 = 1$), one obtains the following convergence criterion.

Corollary 3. Assume that parameters $\varkappa \in \mathbb{R}, \delta > 0, \tau, \varepsilon \geq 0$ exist such that the frequency-domain inequality holds

$$\Pi(\omega) \triangleq Re\,\varkappa K(\imath\omega) - (\tau+\varepsilon)|K(\imath\omega)|^2 + \tau\omega^2 \geq \delta\,\forall\omega \geq 0. \tag{12}$$

Then, every solution of integral equation (10) converges to some equilibrium point, that is,

$$\dot{x}(t) \xrightarrow[t\to\infty]{} 0,\ x(t) \xrightarrow[t\to\infty]{} 2\pi k,\ k \in \{0, \pm1, \pm2, \ldots\}.$$

Furthermore, $\dot{x}(\cdot), \sin x(\cdot) \in L_2[0,\infty)$.

3.2 An equivalent form of the frequency-domain condition

To reduce the number of free parameters in the stability criterion, notice that

$$Re\,K(\imath\omega) = \frac{b(\cos\omega r - \imath\sin\omega r)}{\imath\omega r + a} = \frac{b(a\cos\omega r - \omega r\sin\omega r)}{(a^2 + r^2\omega^2)},$$
$$|K(\imath\omega)| = \frac{b}{\sqrt{a^2 + r^2\omega^2}}.$$

It is convenient to introduce a new variable $\bar\omega = \omega r$ and denote $B \triangleq br$. Then, (12) shapes into

$$\varkappa Br(a\cos\bar\omega - \bar\omega\sin\bar\omega) - (\tau+\varepsilon)B^2 + \tau\bar\omega^2(a^2+\bar\omega^2) \geq \delta.$$

Substituting $\bar\omega = 0$, it is obvious that the latter inequality can hold only for $\varkappa > 0$. Rescaling the quadruple $(\varkappa, \tau, \varepsilon, \delta)$, we may assume, without loss of generality, that $\varkappa r = 1$. Also, the left-hand side converges to ∞ as $\bar\omega \to \infty$. Hence, the latter inequality is satisfied for sufficiently small $\delta, \varepsilon > 0$ if and only if the condition is valid

$$\Pi_\tau(\omega) \triangleq \tau\omega^2(a^2+\omega^2) - \tau B^2 + B(a\cos\omega - \omega\sin\omega) > 0 \tag{13}$$

for some parameter $\tau > 0$ and all $\omega > 0$ (for brevity, we have redesignated $\bar\omega \mapsto \omega$). One arrives at the following.

Theorem 4. Suppose that for some parameters $a, b, r > 0$ a number $\tau > 0$ exists such that the frequency-domain inequality (13) is fulfilled. Then, the sunflower equation (10)

[3] Systems of such a structure are sometimes called "pendulum-like" (Leonov et al., 1992). They are also termed "synchronization systems" or "synchronous control" systems (Leonov, 2006; Lindsey, 1972; Hoppensteadt, 1983; Leonov and Kuznetsov, 2014), because they describe a broad class of control circuits (e.g. phase- and frequency-locked loops) providing synchronization of oscillators.

enjoys gradient-like behavior, that is, all its solutions converge to equilibria. In particular, the equation may not have (non-constant) periodic solutions.

Remark 5. One may notice that (13) cannot hold when $r \geq r_0$, where r_0 is delay corresponding to Hopf bifurcation (see Section 2). Indeed, finding $\omega_0 \in (0, \pi/2)$ from (7), one may easily notice that $a\cos\omega_0 - \omega_0 \sin\omega_0 = 0$ and $\omega_0^2(\omega_0^2 + a^2) = b^2 r_0^2 < B^2 = b^2 r^2$ due to (7). For this reason, (13) is violated at $\omega = \omega_0$, no matter how $\tau > 0$ is chosen. Our stability criterion thus allows to examine stability of the sunflower equation for $r < r_0$.

3.3 A sufficient analytic condition

Although analytic validation of the inequality (13) is problematic, Theorem 4 allows to derive a sufficient analytic condition that generalizes Lemma 1. Notice that

$$\sin\omega \leq \omega \quad \forall \omega \geq 0,$$

$$\cos\omega = 1 - 2\left(\sin\frac{\omega}{2}\right)^2 \geq 1 - \frac{\omega^2}{2} \; \forall \omega \geq 0.$$

Hence, the inequality (13) is entailed by the following condition

$$\tau\omega^2(a^2+\omega^2) - \tau B^2 + Ba\left(1 - \frac{\omega^2}{2}\right) - B\omega^2 > 0, \quad (14)$$

which is equivalent to

$$f_\tau(s) = \tau s^2 + \left[\tau a^2 - \frac{aB}{2} - B\right] s + (aB - \tau B^2) > 0 \quad \forall s \geq 0.$$

The latter condition, obviously, holds in two situations

(1) $\tau a^2 - \frac{aB}{2} - B \geq 0$ and $aB - \tau B^2 > 0$ (then $f_\tau(s) \geq f_\tau(0) > 0$);
(2) if $(\tau a^2 - \frac{aB}{2} - B)^2 < 4\tau(aB - \tau B^2)$ (then, the global minimum of f_τ is positive).

The first condition can be satisfied for some $\tau > 0$ only if

$$\frac{a}{B} > \frac{(a+2)B}{2a^2} \Longrightarrow B < a\sqrt{\frac{2a}{a+2}}.$$

The second situation is possible if $\tau > 0$ exists such that

$$\tau^2\left[a^4 + 4B^2\right] - \tau aB[a^2 + 2a + 4] + \\ + B^2\left(1 + \frac{a^2}{4} + a\right) < 0,$$

which is possible if and only if $a^2B^2[a^2+2a+4]^2 > 4[a^4+4B^2]B^2\left[1 + \frac{a^2}{4} + a\right] = (a^4+4B^2)B^2(a+2)^2$ or, equivalently,

$$8a(a^3+2a^2+2a) > 4B^2(a+2)^2 \Longrightarrow B < a\frac{\sqrt{2(a^2+2a+2)}}{a+2}.$$

It can be noted that in the second situation we get a broader range of parameters, because

$$a\sqrt{\frac{2a}{a+2}} < a\frac{\sqrt{2(a^2+2a+2)}}{a+2}.$$

Recalling that $B = br$, we arrive at the following analytic stability criteria.

Theorem 6. Suppose that the delay r is so small that

$$0 < r < r_+ \triangleq \frac{a}{b}\frac{\sqrt{2(a^2+2a+2)}}{a+2}. \quad (15)$$

Then, all solutions of equation (10) converge to equilibria.

As we know, r_0 is the critical delay margin beyond which the gradient-like property fails to hold, hence, we always have $r_+(a,b) \leq r_0(a,b)$. Generally (see the next subsection), the inequality is strict, so (15) is a much stronger condition than (13). However, it can be easily seen from (7) that for b being fixed and $a \to 0+$, one has

$$r_0(a,b) = \frac{a}{b} + o(a) = r_+(a,b) + o(a) \quad \text{as } a \to 0.$$

In other words, the estimate (15) becomes *asymptotically tight* as $a \to 0$. As will be reported in the next section, for parameters presented in (Somolinos, 1978), the gap $r_0 - r_+$ is less than 10% of r_0.

One can also see that Theorem 6 entails Lemma 1, because $r_* = a/2b < r_+$ whenever $a, b > 0$.

3.4 Numerical validation of the frequency-domain condition

Notice that the condition $r < r_+$ is only *sufficient* for validity of the frequency-domain condition (13), which can hold also for $r > r_+$. As we have already noticed, (13) can be valid only when $r \in (r_+, r_0)$. Besides this, $aB > \tau B^2$ (which is obtained by substituting $\omega = 0$). If the latter condition holds, then it suffices to check (13) for

$$\omega^2 \leq \Omega(a, B, \tau),$$

$$\Omega(a,B,\tau) \triangleq \frac{B - \tau a^2 + \sqrt{(\tau a^2 - B)^2 + 4\tau(\tau B^2 + aB)}}{2\tau}.$$

Indeed, in view of the relation $|\sin\omega| \leq |\omega|$, the left-hand side of (13) can be estimated as

$$\Pi_\tau(\omega) \geq \tau\omega^4 + (\tau a^2 - B)\omega^2 - (\tau B^2 + aB),$$

which expression is automatically positive for $\omega^2 > \Omega(a,B,\tau)$. These observations enable *numerical* validation of (13) by sweeping the parameters $r \in (r_+, r_0)$ and $\tau \in (0, a/B)$ and finding minimum of the left-hand side of (13) over all $\omega \in [0, \sqrt{\Omega(a,B,\tau)}]$. As demonstrated in the next section, in special situation this method actually gives a very tight estimate for the range of delays, in which stability is guaranteed.

The results on stability for different delay ranges are summarized in Table 1.

4. NUMERICAL EXPERIMENT

In this section we consider a numerical experiment, allowing to evaluate the conservatism of stability criteria.

We consider the parameters reported in (Somolinos, 1978): $a = 4.8, b = 0.186$. To solve (7), we first find ω_0 from

$$\omega_0 \tan\omega_0 = a,$$

obtaining that $\omega_0 \approx 1.3053$. This allows to find $r_0 \approx 34.908$ from the second equation [4]. As follows from results obtained in (Somolinos, 1978), for $r \in (r_0, r_1)$ the system has a periodic orbit. The simulation shows, in fact, that around each equilibrium $x(t) \equiv 2\pi k$ a periodic attractor exists that attracts the solutions. Fig. 1 shows [5] the behavior of solutions corresponding to delay $r = 34.908$ and the initial condition $x(t) = x_0 = 0.5, \dot{x}(t) = 0 \, \forall t \leq 0$. The simulation demonstrates that, in fact, in the vicinity of equilibria $2\pi k$ several locally stable periodic orbits exist.

[4] Somolinos (1978) reports an approximate value $r_0 = 35$.
[5] To solve the delay equation (2), we use `dde23` function.

Paper	Delay range	Type of result
Somolinos (1978)	$r < r_0$	Local stability of equilibria $2\pi k$;
	$r \in (r_0, r_1)$	Existence of a periodic solution
Burton (1985)	$r < r_* < r_0$	Global stability of the equilibrium set (solution convergence)
Wei and Huang (1997)	$r \in (r_0, r_2)$, $r_2 > r_1$	Existence of a solution with period $\geq 2r$
This work	$r_* < r < r_+$	Global stability of the equilibrium set (solution convergence)
	$r_+ \leq r < r_0$	Sufficient global stability condition: a frequency-domain numerical test
Remains an open problem:	$r > r_2$	

Table 1. Properties of the sunflower equation for different delay ranges.

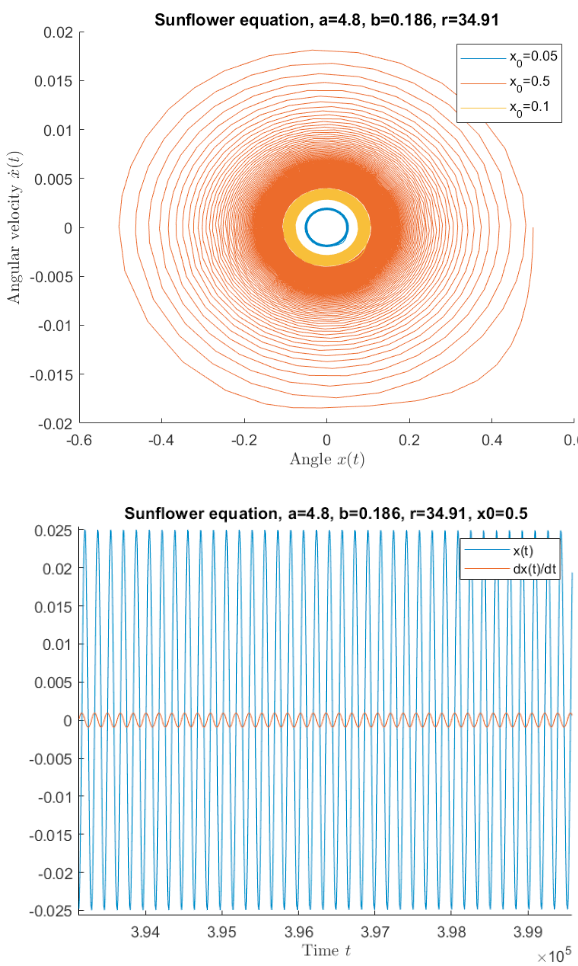

Fig. 1. The solutions converging to periodic attractors. Top section: the solutions corresponding to $x_0 = 0.1$ and $x_0 = 0.5$ approach a common periodic orbit (from, respectively, inside and outside), the solution corresponding to $x_0 = 0.05$ converges to another periodic orbit. Bottom section: the steady periodic motion corresponding to $x_0 = 0.5$: amplitude 0.025 rad for the angle and 0.0016 rad/s for the angular velocity. The duration of simulation is $T = 4 \cdot 10^5$s.

We first compare the two analytic stability criteria. Lemma 1 states that the equation is gradient-like if $r < r_* = a/(2b) \approx 12.903$. Theorem 6 substantially improves the latter estimate, ensuring stability for

$$r < r_+ \approx 31.588,$$

which is much closer to r_0. Actually, the difference between r_+ and r_0 is less than 10%. Fig. 2 demonstrates solutions converging to equilibria $2\pi k$, $k = 0, \pm 1$.

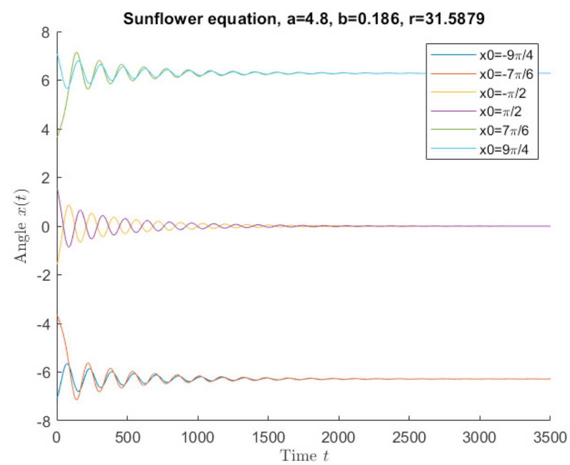

Fig. 2. Solutions converging to equilibria ($r < r_+$).

The numerical analysis of the frequency-domain equation (13) shows that, in fact, it holds for $r < 34.9079$, so in fact Theorem 4 gives an almost tight estimate for the critical delay margin. For instance, in the case of $r = 34.9078$, the frequency-domain condition (13) holds for $\tau = 0.5574$. Fig. 3 shows the plot of function $\Pi_\tau(\omega)$ in the left-hand side of (13), which attains its minimal value $\min \Pi_\tau \approx 1.37 \cdot 10^{-4}$ at $\omega \approx 1.3076$. It should be

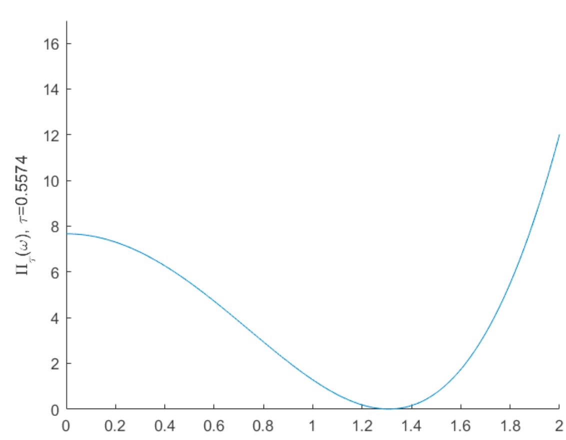

Fig. 3. Validation of the frequency-domain inequality with $r = 34.9078$ and $\tau = 0.5574$.

noticed, however, that as r approaches its critical value r_0, the convergence becomes very slow, so it is practically impossible to validate this result numerically using the standard DDE solvers. To obtain an explicit convergence rate and other characteristics of the transient behavior in pendulum-like systems is a non-trivial open problem.

Numerical simulation for $r = 34.9$ shows that the solutions starting at $x_0 = \pm\pi/2$ oscillates with a very slowly decaying amplitude (at time $t = 10^6$ seconds the amplitude of oscillations is ≈ 0.01), see Fig. 4

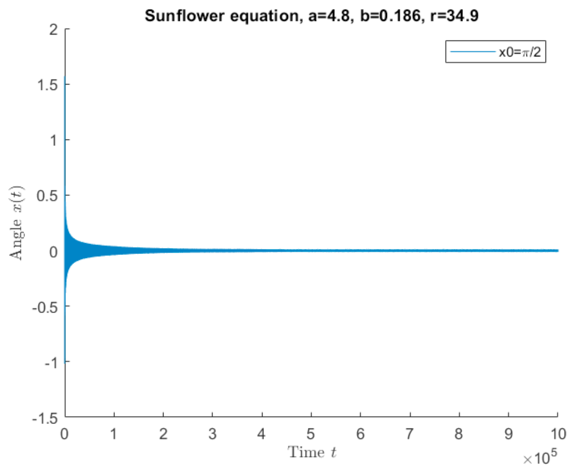

Fig. 4. $r = 34.9$: Theorem 4 predicts convergence, which is however very slow.

5. CONCLUSIONS

In this paper, we examine asymptotic properties of the sunflower equation for the delays smaller than the critical value, given by the Hopf bifurcation theorem. Numerical simulations show that for such delays the equation enjoys gradient-like behavior (all trajectories converge), however, this conjecture remains unproved. We obtain two efficient criteria for the gradient-like behavior. One of them requires to validate a frequency-domain inequality (similar to those arising in absolute stability theory). The second one is more conservative, however, gives an explicit estimate for the delay range in which solutions' convergence is ensured.

We have also performed a number of numerical experiments, showing that for parameters reported in (Somolinos, 1978), our frequency-domain stability criteria practically guarantees stability for all delays less than the critical value, even though the convergence becomes enormously slow. The analytic criterion ensures stability for the delay which differs from the critical one by less than 10%.

Finally, it should be noted that the sunflower equation is nowadays considered as a very simplistic model of circumnutation. Experiments with plants on the orbital space station have revealed that circumnutation persists even in the absence of gravity. The more realistic *two oscillator* model (Johnsson et al., 1999) is still waiting for mathematically rigorous analysis.

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