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IFAC PapersOnLine 54-7 (2021) 298–303

A concave approach to errors-in-variables sparse linear system identification

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Abstract Sparse linear system identification can be performed through convex optimization, by the minimization of an ℓ_1-norm functional. If an errors-in-variables model is considered, the problem is more challenging as inherently non-convex. The ℓ_1-norm approach for the errors-in-variables model is studied in recent literature. In this work, we propose to replace the ℓ_1-norm functional by a concave functional. Concave functionals have been shown to improve the performance in practical experiments of sparse linear regression; nevertheless, theoretical analyses of this improvement are missing in the errors-in-variables setting. The goal of this paper is to fill this gap, by studying conditions that guarantee that the concave approach is variable selection consistent. Moreover, we illustrate how to implement it through ℓ_1 reweighting techniques, and we present numerical simulations that show its effectiveness.

Keywords: Errors-in-variables identification, linear systems, sparse optimization, non-convex optimization, compressed sensing.

1. INTRODUCTION

Sparse system identification is the science of learning parsimonious models from data, that is, models that depend on a relatively small number of parameters. Usually, the available data are experimental measurements of the input and output of the system that one aims to characterize. The need for parsimony is crucial for many motivations. Nowadays, it is possible to acquire large amounts of data, which supports a better description of systems. Nevertheless, the abundance of data may induce problems of overfitting in system identification, which requires suitable strategies to prune the obtained solutions. Typically, an overfitted model does not provide an interpretable physical description of the considered system, and also may incur into an excessive numerical complexity, due to a large number of parameters, see Hastie et al. (2015); Brunton and Kutz (2019). This issue has propelled the application of sparse optimization techniques to trim solutions in system identification, statistical learning, and, more recently, in deep learning. The common idea in these different fields is to use optimization techniques that remove the redundant parameters and weights, by forcing their values to zero. Specifically, the problem is formulated with a large number of potential parameters, most of which are then nullified by using suitable cost functionals and penalties. In other terms, the problem consists in identifying the few most significant parameters, and discarding the others. The identification of the most significant parameters is also referred to as *variable selection*.

From a mathematical perspective, variable selection is an NP-hard problem, see (Foucart and Rauhut, 2013, Section 2.3). In the last decades, this fact has motivated the research of approximated, possibly convex, alternatives to the original formulation, which has lead to the use of the ℓ_1-norm as cost functional or penalty to promote sparsity, see Tibshirani (1996). Indeed, the ℓ_1-norm is the convex approximation of the ℓ_0-norm, which counts the number of non-zero parameters. Analyses of the ℓ_1-norm variable selection consistency have been provided, see, e.g., (Foucart and Rauhut, 2013, Chapter 4) and (Hastie et al., 2015, Chapter 11), and its use is very popular in many applications.

As to linear dynamical systems, some peculiar features are associated with sparse system identification. In particular, the identification task turns out to be a linear regression problem, and compressed sensing (CS, Donoho (2006)) can be applied to learn the significant parameters. Specifically, the theory of CS proves that sparse vectors, i.e., vectors with few non-zero components, can be recovered from a relatively small number of measurements. The reconstruction can be performed by using the ℓ_1-norm approach mentioned above. CS has been applied to sparse linear system identification, see, e.g., Tóth et al. (2012); Rojas et al. (2014); Fosson et al. (2020), with a relevant caveat: the theorems that guarantee an effective reconstruction in CS with tight conditions leverage the hypothesis that the regressors are random and independent, which is not the case of input/output linear systems. This yields to the necessity of a larger number of measurements in system identification with respect to CS, but still compressed with respect to the original dimension of the problem.

Furthermore, classical CS envisages the possible presence of measurement noise, but not on the regression matrix, which is assumed to be exactly known. Nevertheless, this is not the case in data-driven problems, where both input and output are experimentally measured, hence naturally affected by perturbations. In linear system identification, the regression matrix contains both input and output elements, then it is sufficient to have disturbances in the observation of either input or output to affect it. In

10.1016/j.ifacol.2021.08.375

the literature, this is known as errors-in-variables (EIV) problem, which is challenging since inherently non-convex.

In Cerone et al. (2012), the EIV linear system identification is tackled in a set-membership perspective, by using techniques of polynomial optimization. Nevertheless, this approach may be computationally demanding for sparse identification from high-dimensional data. In Fosson et al. (2020), an ℓ_1 approach is proposed for EIV linear regression, which does not depend on the structure of the matrix. This recasts the problem into convex optimization, provided that a preliminary estimate of the parameters' signs is obtained through ridge regression.

In the last years, concave alternatives to the ℓ_1-norm have been developed in sparse optimization, which are observed to be more precise than the ℓ_1-norm, see, e.g., Zhang (2010); Loh and Wainwright (2017); Fosson (2018); Woodworth and Chartrand (2016) and references therein. Basically, concave functionals are closer to ℓ_0-norm, then more favorable to support sparsity than the ℓ_1-norm, with respect to which they have less bias. Among the most popular ones, we mention the log functional, see Candès et al. (2008), the minimax concave penalty, see Zhang (2010), and the ℓ_p^p-norm, with $p \in (0,1)$, see Foucart and Laui (2009).

To the best of our knowledge, those non-convex techniques have not yet been applied and analyzed in the EIV framework. The goal of this paper is to fill this gap, by proposing and analyzing a concave approach for EIV sparse linear system identification. Specifically, the main contributions of this work are: the definition of a suitable concave optimization problem for EIV sparse linear system identification; the analysis of this approach in terms of variable selection consistency; the implementation of an effective algorithm to solve the proposed optimization problem; the illustration of numerical results.

The paper is organized as follows. In Section 2, we define a suitable optimization problem with concave cost functional; in Section 3, we analyze its effectiveness, by proving sufficient conditions for its variable selection consistency. In Section 4, we provide the solving algorithm, and test it through numerical experiments.

2. PROBLEM STATEMENT

In this paper, we consider a discrete-time, single-input single-output (SISO), linear-time-invariant (LTI) system ruled by following-input output relation:

$$y_t = \sum_{p=1}^{\mathrm{n_a}} a_p y_{t-p} + \sum_{q=1}^{\mathrm{n_b}} b_q u_{t-q} \tag{1}$$

where $u_t \in \mathbb{R}$ and $y_t \in \mathbb{R}$ respectively are the input and the output at time t. We consider an EIV model: u_t and y_t are experimentally measured, thus corrupted by additive noises ξ_t and η_t. Then, the available data are:

$$\overline{u}_t = u_t + \xi_t, \quad \overline{y}_t = y_t + \eta_t. \tag{2}$$

The noises are unknown, while we assume that their magnitudes do not overcome known bounds:

$$|\xi_t| \le \Delta_\xi, \quad |\eta_t| \le \Delta_\eta. \tag{3}$$

No prior information on the probabilistic distribution of the noises is considered, which envisages also quantization and truncation errors, possibly due to the transmission of the data over communication channels, see Cerone et al. (2019).

Our final aim is to perform a variable selection in $\theta = (a_1, \dots, a_{\mathrm{n_a}}, b_1, \dots, b_{\mathrm{n_b}})^T \in \mathbb{R}^n$, $n = \mathrm{n_a} + \mathrm{n_b}$, i.e., to identify the most significant parameters, by applying a suitable sparsity promoting technique in the estimation of θ.

The total dimension n might be large, and we aim to keep the number of observations as small as possible. If we set a time window of length m, the noise-free problem can be interpreted as a linear regression $(y_{t+1}, \dots, y_{t+m})^T = A\theta$ where

$$A = \begin{pmatrix} y_t & \cdots & y_{t-\mathrm{n_a}+1} & u_t & \cdots & u_{t-\mathrm{n_b}+1} \\ y_{t+1} & \cdots & y_{t-\mathrm{n_a}+2} & u_{t+1} & \cdots & u_{t-\mathrm{n_b}+2} \\ \vdots & & & & & \vdots \\ y_{t+m-1} & \cdots & y_{t+m-\mathrm{n_a}} & u_{t+m-1} & \cdots & u_{t+m-\mathrm{n_b}} \end{pmatrix}. \tag{4}$$

If $m < n$, we can recast the problem into CS, as illustrated in Tóth et al. (2011), and the minimization of the ℓ_1-norm can be tackled to estimate the sparsest θ that satisfies $(y_{t+1}, \dots, y_{t+m})^T = A\theta$, see Tóth et al. (2011); Rojas et al. (2014) for details.

When the EIV model is considered, the problem turns into non-convex, bilinear optimization, as A is not exactly known and is multiplied by the unknown θ. In the literature, this EIV linear regression problem has been tackled in different ways. In Cerone (1993), a feasible parameter set is estimated, by recasting the problem into convex sub-problems. In Cerone et al. (2012), such estimation is refined, by approximating the problem by convex relaxations of polynomial optimization problems; this approach is valuable for a reduced number of parameters, and is not focused on high-dimensional variable selection. In sparse optimization, the bilinear problem has been tackled by moving the weight of noise into a vector $e \in \mathbb{R}^m$ such that $(y_{t+1}, \dots, y_{t+m})^T = A\theta + e$, then by tackling it with a usual ℓ_1-norm approach, see Herman and Strohmer (2010). Nevertheless, this is not very effective, as illustrated in Fosson et al. (2020), where an ℓ_1 approach is proposed and analyzed, specific for the EIV model. By starting from the approach of Cerone (1993), in Fosson et al. (2020) a pre-processing, based on ridge regression, is proposed, that effectively evaluates the parameters' signs: this recasts the problem into linear programming (LP). We refer the reader to Fosson et al. (2020) for more details on this pre-processing. Here, we assume that the signs are correctly assessed by this pre-processing. Without loss of generality, in the following we assume that all the parameters θ_i, $i = 1, \dots, n$, are non-negative, so that $\|\theta\|_1 = \sum_{i=1}^n \theta_i$. Let

$$\overline{A} := \begin{pmatrix} \overline{y}_t & \cdots & \overline{y}_{t-\mathrm{n_a}+1} & \overline{u}_t & \cdots & \overline{u}_{t-\mathrm{n_b}+1} \\ \overline{y}_{t+1} & \cdots & \overline{y}_{t-\mathrm{n_a}+2} & \overline{u}_{t+1} & \cdots & \overline{u}_{t-\mathrm{n_b}+2} \\ \vdots & & & & & \vdots \\ \overline{y}_{t+m-1} & \cdots & \overline{y}_{t+m-\mathrm{n_a}} & \overline{u}_{t+m-1} & \cdots & \overline{u}_{t+m-\mathrm{n_b}} \end{pmatrix}. \tag{5}$$

Moreover, for notational simplicity, let us consider $\Delta = \max\{\Delta_\xi, \Delta_\eta\}$. Then, in Fosson et al. (2020) the variable selection is performed by solving the following LP:

Problem $\mathcal{P}_1$:

$$\begin{aligned} &\min_{\theta\in\mathbb{R}^n_+} \sum_{i=1}^n \theta_i \ \text{ s. t. } C\theta \preceq f \\ &\text{where} \\ &C = \begin{pmatrix} \overline{A} - \Delta \mathbf{1}_m \mathbf{1}_n^T \\ -\overline{A} - \Delta \mathbf{1}_m \mathbf{1}_n^T \end{pmatrix} \in \mathbb{R}^{2m,n} \\ &f = \begin{pmatrix} \overline{y} + \Delta \mathbf{1}_m \\ -\overline{y} + \Delta \mathbf{1}_m \end{pmatrix} \in \mathbb{R}^{2m} \end{aligned} \tag{6}$$

where $\preceq$ denotes the component-wise inequality between matrices. In Cerone et al. (2019), it is proposed to replace $\sum_{i=1}^n \theta_i$ with $\sum_{i=1}^n \frac{\alpha+\beta}{2}\theta_i - \theta_i^2$, given that each non-zero component of θ belongs to a compact interval $[\alpha, \beta]$ with $0 < \alpha < \beta$. Then, the solution is evaluated by polynomial optimization techniques. This approach is shown to perform better, in terms of variable selection, than the ℓ_1 approach in a sparse linear regression experiment, with independent regressors; however, theoretical guarantees of improvement are missing. Moreover, the computational complexity of that approach is intense, then not feasible in high-dimensional problems.

In this paper, we propose a concave approach to the EIV model, and we discuss its enhancement with respect to the ℓ_1 approach of $\mathcal{P}_1$. Moreover, we provide a low-complex algorithm to implement it. Specifically, we propose the following optimization problem:

Problem $\mathcal{P}_{\log}$:

$$\begin{aligned} &\min_{\theta\in\mathbb{R}^n_+} \sum_{i=1}^n \log(\theta_i + \epsilon) \ \text{ s. t. } C\theta \preceq f \\ &\text{where} \\ &C = \begin{pmatrix} \overline{A} - \Delta \mathbf{1}_m \mathbf{1}_n^T \\ -\overline{A} - \Delta \mathbf{1}_m \mathbf{1}_n^T \end{pmatrix} \in \mathbb{R}^{2m,n} \\ &f = \begin{pmatrix} \overline{y} + \Delta \mathbf{1}_m \\ -\overline{y} + \Delta \mathbf{1}_m \end{pmatrix} \in \mathbb{R}^{2m}. \end{aligned} \tag{7}$$

where $\epsilon > 0$ is an arbitrarily small design parameter. The log functional, studied, e.g., in Candès et al. (2008); Fosson (2018), is chosen as it is tractable for our purposes. However, the minimax convex penalty Zhang (2010) and ℓ_p^p, $p \in (0,1)$ Foucart and Laui (2009) may be valuable alternatives, and will be considered in future work.

3. ANALYSIS

In this section, we propose an analysis of problem $\mathcal{P}_{\log}$. Specifically, we evaluate and discuss conditions under which the solution of $\mathcal{P}_{\log}$ is variable selection consistent (VSC), that is, correctly identifies the support of the true vector of parameters $z \in \mathbb{R}^n$.

In Fosson et al. (2020), Theorem 2 provides a set of conditions, that we denote as $\mathcal{C}$, that are sufficient to guarantee that $\mathcal{P}_1$ is VSC. At present, $\mathcal{C}$ represents the state-of-the-art condition set for linear sparse identification in the EIV setting; for this motivation it can be considered as benchmark.

Let S be the the support of z, i.e., the set of indices of the non-zero components of z, and let S^c be its complementary. Let $\tau > 0$ a suitable threshold that discriminates between null and non-null parameters. For example, if we priorly know $c := \min_{i\in S} |z_i|$ and $d := \max_{i\in S} |z_i|$, then it might be natural to set $\tau = \frac{c}{2}$. Then, given a candidate solution $\beta \neq z$, (Fosson et al., 2020, Theorem 2) proves that $\|\beta - z\|_\infty < \tau$, hence $\mathcal{P}_1$ is VSC, under conditions on the maximum noise and on the coherence of the matrix A. More precisely, given $w := \beta - z$ and $\phi := \sqrt{m}[\Delta(2 + kd + k_\beta d)]$, if there exist $\nu \in (0,1)$ and $\psi \in (1,2)$ such that

$$\mathcal{C}: \begin{cases} \text{for each } i \in S^c: \ \displaystyle\sum_{l\in S} |A_i^T A_l| \leq \frac{1-\nu}{\psi}. \\ \text{for each } j \in S: \ \displaystyle\sum_{l\in S, l\neq j} |A_j^T A_l| \leq 1 - \frac{1+\nu}{\psi} \\ \tau\nu > \phi k \psi \end{cases}$$

then $\mathcal{P}_1$ is VSC.

This result is obtained by using the fact that

$$\|\beta\|_1 - \|z\|_1 \geq \|w_{S^c}\|_1 - \|w_S\|_1 \tag{8}$$

and by verifying whether the solution of the following LP is positive:

$$\begin{aligned} &\min_{|w|\in\mathbb{R}^n_+} \|w_{S^c}\|_1 - \|w_S\|_1 \\ &\text{s.t. } \begin{pmatrix} \Gamma \\ -e_j^T \end{pmatrix} |w| \preceq \begin{pmatrix} \phi \mathbf{1}_n \\ -\tau \end{pmatrix} \end{aligned} \tag{9}$$

where $\Gamma := I - |I - \overline{A}^T \overline{A}| \in \mathbb{R}^{n,n}$. The core of our analysis is to show that a similar LP approach can be used when the ℓ_1-norm functional is replaced by the log functional.

Let $z + w$, with $w \neq 0$ any vector in $[0,d]^n$, with support $\widehat{S} \neq S$. We remark that $w_i \in [0,d]$ when $i \in S^c$, and $w_i \in [-z_i, d - z_i]$ when $i \in S$. Then, $z + w$ is not a solution to problem $\mathcal{P}_{\log}$ if $\sum_{i=1}^n [\log(z_i + w_i + \epsilon) - \log(z_i + \epsilon)] > 0$. Since this functional is non-convex, we evaluate its linear lower bound. We have

$$\begin{aligned} \sum_{i=1}^n [\log(z_i + w_i + \epsilon) - \log(z_i + \epsilon)] &= \sum_{i=1}^n \log\left(1 + \frac{w_i}{z_i + \epsilon}\right) \\ &= \sum_{i\in S^c} \log\left(1 + \frac{w_i}{\epsilon}\right) + \sum_{i\in S} \log\left(1 + \frac{w_i}{z_i + \epsilon}\right) \end{aligned} \tag{10}$$

and

$$\begin{aligned} &\text{for each } i \in S^c: \ \log\left(1 + \frac{w_i}{\epsilon}\right) \geq r(w_i) := s w_i \\ &\text{for each } i \in S: \ \log\left(1 + \frac{w_i}{z_i + \epsilon}\right) \geq r(w_i) + g_i \\ &\text{where} \\ &s := \frac{1}{d} \log\left(1 + \frac{d}{\epsilon}\right) \\ &g_i := s z_i + \log\left(\frac{\epsilon}{z_i + \epsilon}\right). \end{aligned} \tag{11}$$

An illustration of these inequalities is depicted in Fig. 1.

At this point, we can state the following result. Let $G := \sum_{i\in S} \frac{|g_i|}{s}$. We observe that $\frac{g_i}{s} = z_i + d\frac{\log\epsilon - \log(z_i+\epsilon)}{\log(d+\epsilon) - \log\epsilon} \to z_i - d$ for $\epsilon \to 0$. Therefore, for sufficiently small ϵ, $G \leq k(d-c)$.

Proposition 1. If there exist $\nu \in (0,1)$ and $\phi \in (1,2)$ such that

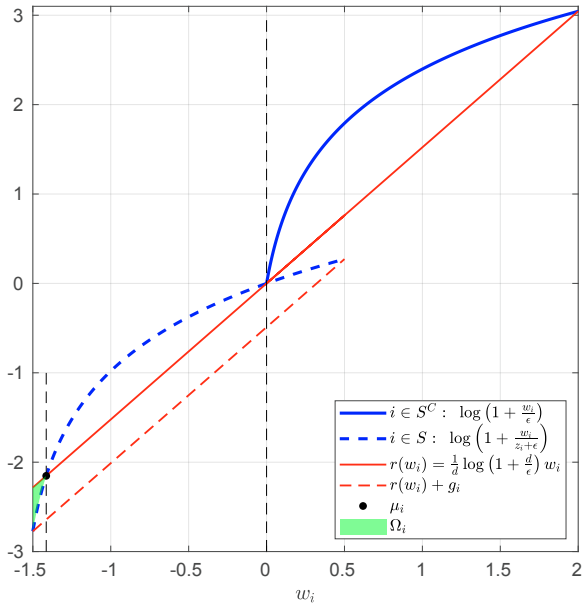

Figure 1. Illustration of (11); $[c,d]=[1,2]$, $z_i=1.5$.

$$\mathcal{C}_{\mathcal{P}_{\log},1}: \begin{cases} \text{for each } i\in S^c: & \sum_{l\in S}|A_i^TA_l| \le \dfrac{1-\nu}{\psi}. \\ \text{for each } j\in S: & \sum_{l\in S, l\neq j}|A_j^TA_l| \le 1-\dfrac{1+\nu}{\psi} \\ \tau\nu > \phi k\psi + G \end{cases}$$

then $\mathcal{P}_{\log}$ is VSC.

Moreover, if $c=d$, that is, if $z\in\{0,d\}^n$, then $\mathcal{C}_{\mathcal{P}_{\log},1}$ is equal to $\mathcal{C}$.

Proof. From (10) and (11), if the following LP has a positive solution, then $\mathcal{P}_{\log}$ is VSC:

$$\begin{aligned} &\min_{|w|\in\mathbb{R}^n_+} \|w_{S^c}\|_1 - \|w_S\|_1 - G \\ &\text{s.t.} \begin{pmatrix} \Gamma \\ -e_j^T \end{pmatrix} |w| \preceq \begin{pmatrix} \phi\mathbf{1}_n \\ -\tau \end{pmatrix}. \end{aligned} \tag{12}$$

By exploiting the proof of (Fosson et al., 2020, Theorem 2), (12) has positive solution if conditions $\mathcal{C}_{\mathcal{P}_{\log},1}$ hold. On the other hand, if $c=d$, then $G=0$, and the problem is equivalent to (9), and $\mathcal{C}_{\mathcal{P}_{\log},1}$ corresponds to $\mathcal{C}$.

***Remark** 1.* If $c<d$, $\mathcal{C}_{\mathcal{P}_{\log},1}$ is more demanding than $\mathcal{C}$. However, we notice that

$$\sum_{i=1}^n [\log(z_i+w_i+\epsilon)-\log(z_i+\epsilon)] \ge s\|w_{S^c}\|_1 - s\|w_S\|_1 \tag{13}$$

for $w_i\ge\mu_i$, for each $i\in S$, where μ_i is the intersection point depicted in Fig. 1. In other terms, except for a region Ω_i (highlighted in green in Fig. 1), $\mathcal{C}$ is sufficient for $\mathcal{P}_{\log}$ to be VSC. Moreover, Ω_i tends to zero when $\epsilon\to 0$.

***Remark** 2.* The linear upper bound in (13) is generally quite far from the original log function; this marks a difference to $\mathcal{P}_1$, where the approximation (8) is tight. For this motivation, we generally expect that those conditions are sufficient, but not necessary for $\mathcal{P}_{\log}$, and that the log approach has better performance in practice. Regarding the ℓ_1 approach, we also remark that the condition $\|w_{S^c}\|_1 > \|w_S\|_1$ for vectors in the feasible set of solutions is sufficient and necessary for perfect recovery in noise-free CS, where is known as null-space property, see Foucart and Rauhut (2013).

Now, we specify the results in the limit case $\epsilon\to 0$.

Corollary 1. If $\epsilon\to 0$, for each $i\in S$, $\mu_i\to -z_i$ and $\Omega_i\to 0$. This implies that, in the limit case, conditions $\mathcal{C}$ are sufficient for $\mathcal{P}_{\log}$ to be VSC.

Proof. μ_i is the solution of the following equation in $w_i<0$:

$$\log\left(1+\frac{w_i}{z_i+\epsilon}\right) = \frac{1}{d}\log\left(1+\frac{d}{\epsilon}\right)w_i.$$

If $\epsilon\to 0$, then $\frac{1}{d}\log\left(1+\frac{d}{\epsilon}\right)w_i\to-\infty$. Therefore, we must have $\log\left(1+\frac{w_i}{z_i+\epsilon}\right)\to-\infty$, which occurs if and only if $w_i\to -z_i$. This implies that the area $\Omega_i\to 0$.

Beyond the limit case $\epsilon\to 0$, we can prove that if ϵ is chosen sufficiently small, then the solution of $\mathcal{P}_{\log}$ has no components in Ω_i.

Proposition 2. Let $\mu=\min_{i\in S}|\mu_i|$. If there exist $\nu\in(0,1)$ and $\psi\in(1,2)$ such that

$$\mathcal{C}_{\mathcal{P}_{\log},2}: \begin{cases} \text{for each } i\in S^c: & \sum_{l\in S}|A_i^TA_l| \le \dfrac{1}{\psi} \\ \text{for each } j\in S: & \sum_{l\in S, l\neq j}|A_j^TA_l| \le 1-\dfrac{1+\nu}{\psi} \\ \mu\nu > \phi k\psi + G \end{cases}$$

then the solution to $\mathcal{P}_{\log}$ has no components $i\in S$ with distance from z_i larger than μ_i.

The proof, omitted for brevity, is based on the proof of (Fosson et al., 2020, Theorem 2).

We remark that the third condition in $\mathcal{C}_{\mathcal{P}_{\log},2}$ is feasible when μ is large with respect to G, provided that the noise is sufficiently small. Given that, for sufficiently small ϵ, $\mu\ge c$ and $G\le k(d-c)$, the conditions are favorable when the non-zero values of z are far from zero (i.e., c is large) and concentrated in a relatively small interval (i.e., $d-c$ is small).

4. ALGORITHMS AND NUMERICAL RESULTS

In this section, we illustrate how to implement the proposed $\mathcal{P}_{\log}$, and we present some numerical results.

4.1 Reweighting algorithm to solve $\mathcal{P}_{\log}$

The main challenge in the application of $\mathcal{P}_{\log}$ is its non-convexity. However, in the literature different approaches have been proposed that can be adapted to our purpose. In particular, in this work we focus on ℓ_1 reweigthing techniques, see Candès et al. (2008); Fosson (2018). Let us consider a problem $\min_{x\in\mathbb{R}^n}\sum_{i=1}^n\mathcal{F}(|x_i|)$ where $\mathcal{F}:\mathbb{R}_+\to\mathbb{R}_+$, is a concave, sparsity promoting functional, along with constraints provided by the data. The key idea of ℓ_1 reweigthing techniques is to iteratively minimize $\sum_{i=1}^n\lambda_i(t)|x_i|$ where $\lambda_i(t)=\mathcal{F}'(|x_i(t)|$. The rationale behind this algorithm is that $\sum_{i=1}^n\mathcal{F}(|x_i|)$ lies below the tangent line, due to the concavity. Then, one can improve

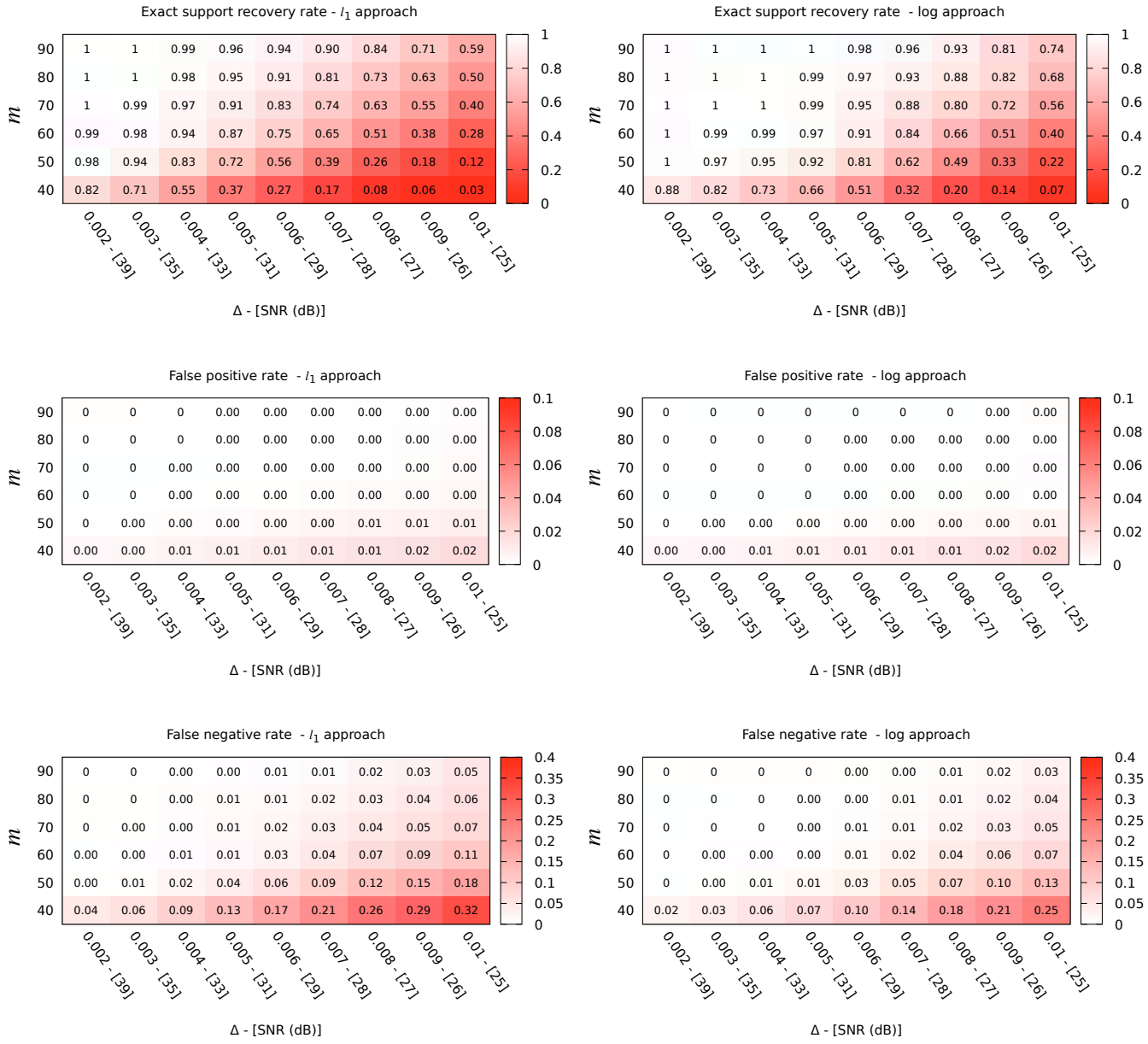

Figure 2. Identification of EIV ARX system (4): $\mathcal{P}_1$ (first column) vs $\mathcal{P}_{\log}$ (second column).

an estimate $v \in \mathbb{R}^n$ by locally minimizing a linearization around v. The term $\lambda_i(t)|x_i|$ comes from this linearization, and corresponds to updating the weights based on the previous estimate. Moreover, due to the concavity, the components that are closer to zero are penalized more than the others. For $\mathcal{P}_{\log}$, $\lambda_i = \frac{1}{|x_i|+\epsilon}$.

This method, known as ℓ_1-reweigthing, is proven to converge to a local minimum of the functional, see (Candès et al., 2008, Section 2.3). The convergence of the iterates of the algorithm is analyzed in Fosson (2018) in a Lasso-based setting. Even though not guaranteed to achieve the global minimum, ℓ_1-reweigthing has been shown to be effective in several applications, see Fosson (2018).

The algorithm is summarized in Algorithm 1. For the first iteration, it makes sense to initialize the weight in a "democratic" way, namely $\lambda(0) = (1, \dots, 1) \in \mathbb{R}^n$: this means that the first iteration corresponds to $\mathcal{P}_1$ in our setting. Each iteration requires the solution of an LP, which keeps the complexity low. Typically, a small number of iterations is sufficient to converge to a solution.

Algorithm 1 ℓ_1 reweigthing

1: Initialize: $\lambda(0) = (1, \dots, 1) \in \mathbb{R}^n$; $\epsilon > 0$
2: **for all** $t = 1, \dots, T_{stop}$ **do**
3: $\quad \lambda_i(t) = \frac{1}{\theta_i(t)+\epsilon}$ for each $i \in \{1, \dots, n\}$
4: $\quad \theta(t+1) = \underset{\theta \in \mathbb{R}^n_+}{\operatorname{argmin}} \sum_{i=1}^n \lambda_i(t)\theta_i$ s.t. $C\theta \preceq f$
5: **end for**

4.2 Numerical results

In this section, we present numerical results, in which we compare $\mathcal{P}_1$ and $\mathcal{P}_{\log}$, the latest one implemented with Algorithm 1. We consider the EIV ARX problem

as illustrated in Section 2. We set $\mathrm{n_a} = \mathrm{n_b} = 50$, then $n = 100$, and $k = 10$. The significant parameters have magnitude in $[c, d] = [0.2, 0.4]$. In particular, they are uniformly distributed in $[-d, -c] \cup [c, d]$. The input is generated according to a Gaussian distribution $\mathcal{N}(0, \frac{1}{10})$.

In Algorithm 1, we set to 3 the maximum number of reweigthing iterations; as noticed in Candès et al. (2008); Fosson (2018), a small number of reweigthing steps is usually sufficient to obtain a good estimation. At each iteration, an LP has to be solved, which is computationally affordable. In particular, in our simulations we solve the LP by the alternating direction method of multipliers (ADMM, Boyd et al. (2010)) implemented in C++, which requires a few seconds to solve $\mathcal{P}_{\log}$ on standard hardware.

In Fig. 2, we illustrate the rates of exact support recovery (i.e., the method is VSC), false positives (i.e., non-significant parameters identified as significant), and false negatives (i.e., significant parameters identified as non-significant), at different Δ and m. The results are averaged over 200 random runs. For each Δ, the measured signal-to-noise-ratio (SNR) is reported as well, defined as $10 \log_{10} \frac{\|y\|_2^2 + \|A\|_F^2}{\|\delta_y\|_2^2 + \|\delta_A\|_F^2}$, where $y = (y_{t+1}, \dots, y_{t+m})^T$, δ_y and δ_A denote the perturbations, and $\|\cdot\|_F$ is the Frobenius norm. The considered range for Δ is $[0.002, 0.01]$, which corresponds to an SNR between 39 and 25 dB. In the experiments, we assume to know the signs of the parameters, i.e., whether $\theta_i \geq 0$ or $\theta_i \leq 0$. This information can be obtained by using the algorithm proposed in (Fosson et al., 2020, Section 3.2).

In Fig. 2, we can see that the proposed method $\mathcal{P}_{\log}$ improves the accuracy in the variable selection with respect to $\mathcal{P}_1$, for all the considered m and Δ. By looking at false positive/negative rates, we get that the improvement is obtained by reducing the false negative rate. This makes sense, as in the EIV setting, the considered feasible set is larger than the true one, due to noise; this may yield too sparse solutions that do not well describe the system, if the number of measurements m is not sufficient. The use of the log functional provides a more precise recovery, which prevents the occurrence of too sparse solutions.

5. CONCLUSION

In this paper, we propose a novel concave approach to sparse linear system identification in the errors-in-variables setting. In particular, we focus on the use of a log functional and we analyze its variable selection consistency. The proposed method is guaranteed to be variable selection consistent under similar conditions proven for the classical ℓ_1 approach. These conditions are evaluated through an LP that approximates the original problem. This LP approximation is tight for the ℓ_1 case, while for the log case it is generally pejorative; thus, an enhancement by using the log approach is expected. This enhancement is attested by numerical simulations.

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