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ON THE L.C.M. OF SHIFTED FIBONACCI NUMBERS

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Abstract. Let $(F_n)_{n\geq 1}$ be the sequence of Fibonacci numbers. Guy and Matiyasevich proved that

$$\log \operatorname{lcm}(F_1, F_2, \ldots, F_n) \sim \frac{3\log\alpha}{\pi^2} \cdot n^2 \quad \text{as } n \to +\infty,$$

where lcm is the least common multiple and $\alpha := \big(1+\sqrt{5}\big)/2$ is the golden ratio.

We prove that for every periodic sequence $\mathbf{s} = (s_n)_{n\geq 1}$ in $\{-1, +1\}$ there exists an effectively computable rational number $C_{\mathbf{s}} > 0$ such that

$$\log \operatorname{lcm}(F_3 + s_3, F_4 + s_4, \ldots, F_n + s_n) \sim \frac{3\log\alpha}{\pi^2} \cdot C_{\mathbf{s}} \cdot n^2, \quad \text{as } n \to +\infty.$$

Moreover, we show that if $(s_n)_{n\geq 1}$ is a sequence of independent uniformly distributed random variables in $\{-1, +1\}$ then

$$\mathbb{E}\big[\log \operatorname{lcm}(F_3 + s_3, F_4 + s_4, \ldots, F_n + s_n)\big] \sim \frac{3\log\alpha}{\pi^2} \cdot \frac{15\operatorname{Li}_2(1/16)}{2} \cdot n^2, \quad \text{as } n \to +\infty,$$

where Li_2 is the dilogarithm function.

1. Introduction

Let $(F_n)_{n\geq 1}$ be the sequence of Fibonacci numbers, defined recursively by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$, for every integer $n \geq 1$. Guy and Matiyasevich [8] proved that, as $n \to +\infty$,

$$\log \operatorname{lcm}(F_1, F_2, \ldots, F_n) \sim \frac{3\log\alpha}{\pi^2} \cdot n^2, \tag{1}$$

where lcm denotes the least common multiple and $\alpha := \big(1+\sqrt{5}\big)/2$ is the golden ratio. This result was extended by Kiss–Mátyás [6], Akiyama [1], and Tropak [14] to more general binary recurrences, and by Akiyama [2, 3] to sequences satisfying some special divisibility properties (see also [4]).

We study what happens if each Fibonacci number F_k in (1) is replaced by a *shifted* Fibonacci number $F_k \pm 1$, for various choices of signs. Arithmetic properties of shifted Fibonacci have been studied before. For example, Bugeaud, Luca, Mignotte, and Siksek [5] determined all the shifted Fibonacci numbers that are perfect powers; Marques [7] gave formulas for the order of appearance of shifted Fibonacci numbers; and Pongsriiam [10] found all shifted Fibonacci numbers that are products of Fibonacci numbers.

Our first result concerns periodic sequences of signs.

Theorem 1.1. *For every periodic sequence $\mathbf{s} = (s_n)_{n\geq 1}$ in $\{-1, +1\}$, there exists an effectively computable rational number $C_{\mathbf{s}} > 0$ such that*

$$\log \operatorname{lcm}(F_3 + s_3, F_4 + s_4, \ldots, F_n + s_n) \sim \frac{3\log\alpha}{\pi^2} \cdot C_{\mathbf{s}} \cdot n^2,$$

as $n \to +\infty$. (The least common multiple starts from $F_3 + s_3$ to avoid zero terms.)

We computed the constant $C_{\mathbf{s}}$ for periodic sequences $\mathbf{s}$ with short period. We found that $C_{\mathbf{s}} = 1/2$ for most of such sequences. In particular, $C_{\mathbf{s}} = 1/2$ for all periodic sequences with period less than 5. Moreover, all the periodic sequences $\mathbf{s}$ with $C_{\mathbf{s}} \neq 1/2$ and period 5 or 6 are listed in Table 1 and Table 2, respectively.

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$\mathbf{s}$	$C_{\mathbf{s}}$	$\mathbf{s}$	$C_{\mathbf{s}}$	$\mathbf{s}$	$C_{\mathbf{s}}$	$\mathbf{s}$	$C_{\mathbf{s}}$
----+	43/96	-+--+	43/96	+----	43/96	+-+++	91/192
---+-	43/96	-+-+-	43/96	+--+-	43/96	++-+-	17/36
--+--	11/24	-+-++	91/192	+-+--	43/96	++-++	17/36
--+-+	11/24	-++-+	91/192	+-+-+	91/192	+++-+	91/192
-+---	43/96	-++++	91/192	+-++-	91/192	++++-	91/192

TABLE 1. All period-5 sequences $\mathbf{s}$ such that $C_{\mathbf{s}} \neq 1/2$.

$\mathbf{s}$	$C_{\mathbf{s}}$	$\mathbf{s}$	$C_{\mathbf{s}}$	$\mathbf{s}$	$C_{\mathbf{s}}$	$\mathbf{s}$	$C_{\mathbf{s}}$
-----+	13/32	--++-+	7/16	+-+-++	29/64	++--++	11/24
----++	13/32	--++++	29/64	+-++--	29/64	++-+--	13/32
---+--	7/16	-+----	13/32	+-+++-	29/64	+++-+-	11/24
---+-+	7/16	-+---+	13/32	+-++++	29/64	+++-++	11/24
--+-++	29/64	-+--++	13/32	++----	13/32	++++--	29/64
--++--	7/16	-+-+--	13/32	++--+-	11/24	+++++-	29/64

TABLE 2. All period-6 sequences $\mathbf{s}$ such that $C_{\mathbf{s}} \neq 1/2$.

Our second result regards random sequences of signs.

Theorem 1.2. *Let $(s_n)_{n\geq 1}$ be a sequence of independently uniformly distributed random variables in $\{-1,+1\}$. Then*

$$\mathbb{E}\big[\log \operatorname{lcm}(F_3+s_3, F_4+s_4, \dots, F_n+s_n)\big] \sim \frac{3\log\alpha}{\pi^2}\cdot\frac{15\operatorname{Li}_2(1/16)}{2}\cdot n^2,$$

as $n\to+\infty$, where $\operatorname{Li}_2(z) := \sum_{n=1}^{\infty} z^n/n^2$ denotes the dilogarithm.

Using the methods of the proofs of Theorem 1.1 and Theorem 1.2, it should be possible to prove similar results, where the sequence of Fibonacci numbers is replaced by the sequence of Lucas numbers or by a sequence of integers powers $(a^n)_{n\geq 1}$, with $a\geq 2$ a fixed integer. Also, one could consider what happens for a deterministic non-periodic sequence of signs $(s_n)_{n\geq 1}$. We leave these as problems for the interested reader.

Notation. We employ the Landau–Bachmann "Big Oh" notation O with its usual meaning. Any dependence of the implied constants is indicated with subscripts. We let φ denote the Euler's totient function. We reserve the letter p for prime numbers.

2. Preliminaries on Fibonacci and Lucas Numbers

Let $(L_n)_{n\geq 1}$ be the sequence of Lucas numbers, defined recursively by $L_1=1$, $L_2=3$, and $L_{n+2}=L_{n+1}+L_n$, for every integer $n\geq 1$. It is well known that the Binet's formulas

$$F_n = \frac{\alpha^n-\beta^n}{\alpha-\beta} \qquad \text{and} \qquad L_n = \alpha^n+\beta^n, \tag{2}$$

hold for every integer $n\geq 1$, where $\alpha := \big(1+\sqrt{5}\big)/2$ and $\beta := \big(1-\sqrt{5}\big)/2$. It is useful (proof of Lemma 2.3 later) to extend the sequences of Fibonacci and Lucas numbers to negative indices using (2). Let us define

$$\Phi_n := \prod_{\substack{1\leq k\leq n\\ \gcd(n,k)=1}} \Big(\alpha - \mathrm{e}^{\frac{2\pi \mathrm{i} k}{n}}\beta\Big), \tag{3}$$

for each integer $n \geq 2$, and put $\Phi_1 := 1$. It can be proved that each Φ_n is an integer [13, p. 428]. Moreover, from (2) and (3) it follows that

$$F_n = \prod_{n \in \mathcal{D}(n)} \Phi_d \qquad \text{and} \qquad L_n = \prod_{n \in \mathcal{D}'(n)} \Phi_d, \tag{4}$$

for every integer $n \geq 1$, where $\mathcal{D}(n) := \{d \in \mathbb{N} : d \mid n\}$ and $\mathcal{D}'(n) := \mathcal{D}(2n) \backslash \mathcal{D}(n)$. In particular, using (4) one can prove by induction that $\Phi_n > 0$ for every integer $n \geq 1$.

We need the following two results about Φ_n.

Lemma 2.1. *For all integers $m > n \geq 1$ we have $\gcd(\Phi_m, \Phi_n) \mid m$.*

Proof. For $m \geq 5$, $m \neq 6, 12$, and $n \geq 3$, it is known [13, Lemma 7] that $\gcd(\Phi_m, \Phi_n)$ divides the greatest prime factor of $m/\gcd(3, m)$, and consequently it divides m. The remaining cases follow easily since $\Phi_1 = \Phi_2 = 1$, $\Phi_3 = 2$, $\Phi_4 = 3$, $\Phi_5 = 5$, $\Phi_6 = 4$, and $\Phi_{12} = 6$. □

Lemma 2.2. *For all integers $n \geq 1$, we have $\log \Phi_n = \varphi(n) \log \alpha + O(1)$.*

Proof. See, e.g., [11, Lemma 2.1(iii)]. □

The next lemma belongs to the folklore and provides a way to write shifted Fibonacci numbers as products of Fibonacci and Lucas numbers.

Lemma 2.3. *For every integer k, we have*

$$\begin{aligned} F_{4k+1} - 1 &= F_{2k} L_{2k+1}, & F_{4k+1} + 1 &= F_{2k+1} L_{2k}, \\ F_{4k+2} - 1 &= F_{2k} L_{2k+2}, & F_{4k+2} + 1 &= F_{2k+2} L_{2k}, \\ F_{4k+3} - 1 &= F_{2k+2} L_{2k+1}, & F_{4k+3} + 1 &= F_{2k+1} L_{2k+2}, \\ F_{4k+4} - 1 &= F_{2k+3} L_{2k+1}, & F_{4k+4} + 1 &= F_{2k+1} L_{2k+3}. \end{aligned}$$

Proof. Employing (2) and $\alpha\beta = -1$, a quick algebraic manipulation yields

$$F_{a+b} + (-1)^b F_{a-b} = F_a L_b, \tag{5}$$

for all integers a, b. Each of the eight identities corresponds to a particular choice of a, b in (5), noting that $F_{-1} = 1$ and $F_{-2} = -1$. □

Finally, we need a lemma about the greatest common divisor of a Fibonacci number and a Lucas number.

Lemma 2.4. *For all integers m, n, we have that $\gcd(F_m, L_n)$ is equal to 1, 2, or $L_{\gcd(m,n)}$.*

Proof. See [9]. □

3. Further preliminaries

For every sequence $\mathbf{s} = (s_n)_{n \geq 1}$ in $\{-1, +1\}$ and for every integer $n \geq 5$, define

$$\ell_{\mathbf{s}}(n) = \operatorname{lcm}(F_5 + s_5, \ldots, F_n + s_n).$$

(Starting from F_5 instead of F_3 does not affect the asymptotic and simplifies a bit the next arguments.) Furthermore, define the sets

$$\begin{aligned} \mathcal{F}_{\mathbf{s}}(n) := \big\{ h \in [2, \tfrac{n}{2}] : \quad & s_{2h-2} = (-1)^h \quad \vee s_{2h-1} = (-1)^{h+1} \\ & \vee s_{2h+1} = (-1)^{h+1} \vee s_{2h+2} = (-1)^{h+1} \big\}, \end{aligned}$$

$$\begin{aligned} \mathcal{L}_{\mathbf{s}}(n) := \big\{ h \in [2, \tfrac{n}{2}] : \quad & s_{2h-2} = (-1)^{h+1} \vee s_{2h-1} = (-1)^h \\ & \vee s_{2h+1} = (-1)^h \quad \vee s_{2h+2} = (-1)^h \big\}, \end{aligned}$$

and

$$\mathcal{M}_{\mathbf{s}}(n) := \bigcup_{h \in \mathcal{F}_{\mathbf{s}}(n)} \mathcal{D}(h) \cup \bigcup_{h \in \mathcal{L}_{\mathbf{s}}(n)} \mathcal{D}'(h).$$

The next lemma is the key to the proofs of Theorem 1.1 and Theorem 1.2.

Lemma 3.1. *As $n \to +\infty$, we have*

$$\log \ell_{\mathbf{s}}(n) = \sum_{d \in \mathcal{M}_{\mathbf{s}}(n)} \varphi(d) \log \alpha + O\left(\frac{n^2}{\log n}\right).$$

Proof. Assume $n \geq 8$ and let $n = 4K + 4$, for some real number $K \geq 1$. Using Lemma 2.3, we can write each $F_i + s_i$ $(i = 5, \dots, n)$ as a product of a Fibonacci number and a Lucas number, which, in light of Lemma 2.4, have a greatest common divisor not exceeding 3. Therefore,

$$\log \ell_{\mathbf{s}}(n) = \log \operatorname{lcm}\left(\operatorname*{lcm}_{i \in \mathcal{F}'_{\mathbf{s}}(n)} F_i, \operatorname*{lcm}_{j \in \mathcal{L}'_{\mathbf{s}}(n)} L_j\right) + O(1), \tag{6}$$

where $\mathcal{F}'_{\mathbf{s}}(n), \mathcal{L}'_{\mathbf{s}}(n) \subseteq [2, 2K+3] \cap \mathbb{Z}$ are defined by

$$\begin{aligned}
2k \in \mathcal{F}'_{\mathbf{s}}(n) \quad &\iff \quad \big((1 \leq k \leq K) \ \wedge \ (s_{4k+1} = -1 \ \vee \ s_{4k+2} = -1)\big) \\
&\qquad \vee \ \big((2 \leq k \leq K+1) \ \wedge \ (s_{4k-1} = -1 \ \vee \ s_{4k-2} = +1)\big), \\
2k+1 \in \mathcal{F}'_{\mathbf{s}}(n) \quad &\iff \quad \big((1 \leq k \leq K) \ \wedge \ (s_{4k+1} = +1 \ \vee \ s_{4k+3} = +1 \ \vee \ s_{4k+4} = +1)\big) \\
&\qquad \vee \ \big((2 \leq k \leq K+1) \ \wedge \ s_{4k} = -1\big), \\
2k \in \mathcal{L}'_{\mathbf{s}}(n) \quad &\iff \quad \big((1 \leq k \leq K) \ \wedge \ (s_{4k+1} = +1 \ \vee \ s_{4k+2} = +1)\big) \\
&\qquad \vee \ \big((2 \leq k \leq K+1) \ \wedge \ (s_{4k-2} = -1 \ \vee \ s_{4k-1} = +1)\big), \\
2k+1 \in \mathcal{L}'_{\mathbf{s}}(n) \quad &\iff \quad \big((1 \leq k \leq K) \ \wedge \ (s_{4k+1} = -1 \ \vee \ s_{4k+3} = -1 \ \vee \ s_{4k+4} = -1)\big) \\
&\qquad \vee \ \big((2 \leq k \leq K+1) \ \wedge \ s_{4k} = +1\big),
\end{aligned} \tag{7}$$

for every integer $k \in [1, K+1]$. Since $F_i, L_i \leq 2^i$ for every integer $i \geq 1$, replacing all the bounds on k in (7) with $2 \leq k \leq n/4$ amount to an error at most $O(n)$ in (6), that is,

$$\log \ell_{\mathbf{s}}(n) = \log \operatorname{lcm}\left(\operatorname*{lcm}_{i \in \mathcal{F}_{\mathbf{s}}(n)} F_i, \operatorname*{lcm}_{j \in \mathcal{L}_{\mathbf{s}}(n)} L_j\right) + O(n). \tag{8}$$

Suppose that $p^v \mid\mid \ell_{\mathbf{s}}(n)$, for some prime number $p \leq n$ and some integer $v \geq 1$. Then $p^v \mid F_i + s_i$ for some integer $i \in [5, n]$, and consequently $p^v \leq F_n + 1 \leq 2^n$. Hence,

$$\log\Big(\prod_{\substack{p^v \mid\mid \ell_n \\ p \leq n}} p^v\Big) \leq \log\Big(\prod_{\substack{p^v \mid\mid \ell_n \\ p \leq n}} 2^n\Big) \leq \#\{p : p \leq n\} \cdot n \cdot \log 2 = O\left(\frac{n^2}{\log n}\right), \tag{9}$$

since the number of primes not exceeding x is $O(x/\log x)$.

Writing each F_i, L_j in (8) as a product of Φ_d's using (4), and taking into account Lemma 2.1 and (9), we obtain that

$$\log \ell_{\mathbf{s}}(n) = \log \prod_{d \in \mathcal{M}_{\mathbf{s}}(n)} \Phi_d + O\left(\frac{n^2}{\log n}\right).$$

Hence, by Lemma 2.2, we get

$$\log \ell_{\mathbf{s}}(n) = \sum_{d \in \mathcal{M}_{\mathbf{s}}(n)} \log \Phi_d + O\left(\frac{n^2}{\log n}\right) = \sum_{d \in \mathcal{M}_{\mathbf{s}}(n)} \varphi(d) \log \alpha + O\left(\frac{n^2}{\log n}\right),$$

since $\mathcal{M}_{\mathbf{s}}(n) \subseteq [2, n]$ and consequently $\#\mathcal{M}_{\mathbf{s}}(n) \leq n$. □

For all integers $r \geq 0$ and $m \geq 1$, and for every $x \geq 1$, let us define

$$\mathcal{A}_{r,m}(x) := \{n \leq x : n \equiv r \pmod{m}\}.$$

We need two lemmas about unions of $\mathcal{D}(n)$, respectively $\mathcal{D}'(n)$, with $n \in \mathcal{A}_{r,m}(x)$.

Lemma 3.2. *Let r, m be positive integers and let $\mathcal{S}$ be the set of $s \in \{1, \dots, m\}$ such that there exists an integer $t \geq 1$ satisfying $st \equiv r \pmod{m}$. For each $s \in \mathcal{S}$, let $t(s)$ be the minimal t. Then, for all $x \geq 1$, we have*

$$\bigcup_{n \in \mathcal{A}_{r,m}(x)} \mathcal{D}(n) = \bigcup_{s \in \mathcal{S}} \mathcal{A}_{s,m}\left(\frac{x}{t(s)}\right).$$

Proof. On the one hand, let $n \in \mathcal{A}_{r,m}(x)$ and pick $d \in \mathcal{D}(n)$. Clearly, $n = dt$ for some integer $t \geq 1$. Let $s \in \{1, \dots, m\}$ such that $d \equiv s \pmod{m}$. Then $st \equiv dt \equiv n \equiv r \pmod{m}$ and consequently $s \in \mathcal{S}$ and $t \geq t(s)$. Therefore, $d = n/t \leq x/t(s)$, so that $d \in \mathcal{A}_{s,m}(x/t(s))$.

On the other hand, suppose that $d \in \mathcal{A}_{s,m}(x/t(s))$ for some $s \in \mathcal{S}$. Letting $n := dt(s)$, we have $n \equiv st(s) \equiv r \pmod{m}$ and $n \leq x$, that is, $n \in \mathcal{A}_{r,m}(x)$. Finally, $d \in \mathcal{D}(n)$. □

Lemma 3.3. *Let r, m be positive integers and let $\mathcal{S}$ be the set of $s \in \{1, \dots, m\}$ such that there exists an odd integer $t \geq 1$ satisfying $st \equiv r \pmod{m}$. For each $s \in \mathcal{S}$, let $t(s)$ be the minimal t. Then, for all $x \geq 1$, we have*

$$\bigcup_{n \in \mathcal{A}_{r,m}(x)} \mathcal{D}'(n) = \bigcup_{s \in \mathcal{S}} \mathcal{A}_{2s,2m}\left(\frac{2x}{t(s)}\right).$$

Proof. On the one hand, let $n \in \mathcal{A}_{r,m}(x)$ and pick $d \in \mathcal{D}'(n)$. Then $2n = dt$ for some odd integer $t \geq 1$. In particular, d is even. Let $s \in \{1, \dots, m\}$ such that $\frac{d}{2} \equiv s \pmod{m}$. Then $st \equiv \frac{d}{2}t \equiv n \equiv r \pmod{m}$, and consequently $s \in \mathcal{S}$ and $t \geq t(s)$. Therefore, $d = 2n/t \leq 2x/t(s)$, so that $d \in \mathcal{A}_{2s,2m}(2x/t(s))$.

On the other hand, suppose that $d \in \mathcal{A}_{2s,2m}(2x/t(s))$ for some $s \in \mathcal{S}$. In particular, d is even and $\frac{d}{2} \equiv s \pmod{m}$. Letting $n := \frac{d}{2}t(s)$, we have $n \equiv st(s) \equiv r \pmod{m}$ and $n \leq x$, that is, $n \in \mathcal{A}_{r,m}(x)$. Finally, $2n = dt(s)$ and $t(s)$ is odd, so that $d \in \mathcal{D}'(n)$. □

Finally, we need two asymptotic formulas for sums of the Euler's function over an arithmetic progression.

Lemma 3.4. *Let r, m be positive integers. Then, for every $x \geq 2$, we have*

$$S_{r,m}(x) := \sum_{n \in \mathcal{A}_{r,m}(x)} \varphi(n) = \frac{3}{\pi^2} \cdot c_{r,m} x^2 + O_{r,m}(x \log x),$$

where

$$c_{r,m} := \frac{1}{m} \prod_{\substack{p \mid m \\ p \mid r}} \left(1 + \frac{1}{p}\right)^{-1} \prod_{\substack{p \mid m \\ p \nmid r}} \left(1 - \frac{1}{p^2}\right)^{-1}.$$

Proof. This is a special case of the asymptotic formula, given by Shapiro [12, Theorem 5.5A.2], for $\sum_{n \leq x} \varphi(f(n))$, where f a polynomial with integers coefficients, no multiple roots, and satisfying $f(n) \geq 1$ for every integer $n \geq 1$. □

Lemma 3.5. *Let r, m be positive integers and let $z \in (0, 1)$. Then, for every $x \geq 2$, we have*

$$\sum_{n \in \mathcal{A}_{r,m}(x)} \varphi(n)\big(1 - z^{\lfloor x/n \rfloor}\big) = \frac{3}{\pi^2} \cdot \frac{c_{r,m}(1-z)\operatorname{Li}_2(z)}{z} \cdot x^2 + O_{r,m}\big(x(\log x)^2\big).$$

Proof. For every integer $k \geq 1$, we have $\lfloor x/n \rfloor = k$ if and only if $x/(k+1) < n \leq x/k$. Hence,

$$\begin{aligned}
\sum_{n \in \mathcal{A}_{r,m}(x)} \varphi(n)\big(1 - z^{\lfloor x/n \rfloor}\big) &= \sum_{k \leq x} \big(1 - z^k\big) \left(S_{r,m}\Big(\frac{x}{k}\Big) - S_{r,m}\Big(\frac{x}{k+1}\Big) \right) \\
&= \sum_{k \leq x} \Big(\big(1 - z^k\big) - \big(1 - z^{k-1}\big) \Big) S_{r,m}\Big(\frac{x}{k}\Big) \\
&= (1-z) \sum_{k \leq x} z^{k-1} \left(\frac{3}{\pi^2} \cdot \frac{c_{r,m} x^2}{k^2} + O_{r,m}\Big(\frac{x \log x}{k}\Big) \right) \\
&= \frac{3}{\pi^2} \cdot c_{r,m}(1-z) \sum_{k=1}^{\infty} \frac{z^{k-1}}{k^2} \cdot x^2 + O_{r,m}\Big(\sum_{k > x} \frac{x^2}{k^2} \Big) + O_{r,m}\Big(\sum_{k \leq x} \frac{x \log x}{k} \Big) \\
&= \frac{3}{\pi^2} \cdot \frac{c_{r,m}(1-z)\operatorname{Li}_2(z)}{z} \cdot x^2 + O_{r,m}\big(x(\log x)^2\big),
\end{aligned}$$

where we employed Lemma 3.4. □

4. Proof of Theorem 1.1

Let $\mathbf{s} = (s_n)_{n \geq 1}$ be a periodic sequence in $\{-1, +1\}$, and let $T \geq 1$ be the length of its period. By the periodicity of $\mathbf{s}$, it follows that there exist $\mathcal{R}_1, \mathcal{R}_2 \subseteq \{1, \dots, m\}$, where $m := 2T$, such that

$$\mathcal{F}_{\mathbf{s}}(n) = \bigcup_{r \in \mathcal{R}_1} \mathcal{A}_{r,m}(n/2) \quad \text{and} \quad \mathcal{L}_{\mathbf{s}}(n) = \bigcup_{r \in \mathcal{R}_2} \mathcal{A}_{r,m}(n/2),$$

for every integer $n \geq 1$.

Then, by Lemma 3.2 and Lemma 3.3, we get that there exist $\mathcal{R} \subseteq \{1, \dots, 2m\}$ and positive rational numbers $(\theta_r)_{r \in \mathcal{R}}$ such that

$$\mathcal{M}_{\mathbf{s}}(n) = \bigcup_{r \in \mathcal{R}} \mathcal{A}_{r,2m}(\theta_r n)\,,$$

for every integer $n \geq 1$.

Therefore, Lemma 3.1 and Lemma 3.4 yield that

$$\log \ell_{\mathbf{s}}(n) = \sum_{r \in \mathcal{R}} \sum_{d \in \mathcal{A}_{r,2m}(\theta_r n)} \varphi(d) \log \alpha + O\Big(\frac{n^2}{\log n}\Big) = \frac{3 \log \alpha}{\pi^2} \cdot C_{\mathbf{s}} \cdot n^2 + O_{\mathbf{s}}\Big(\frac{n^2}{\log n}\Big)\,,$$

where

$$C_{\mathbf{s}} := \sum_{r \in \mathcal{R}} c_{r,2m}\, \theta_r^2$$

is a positive rational number effectively computable in terms of $s_1, \dots, s_T$.

The proof is complete.

5. Proof of Theorem 1.2

Let $\mathbf{s} = (s_n)_{n \geq 1}$ be a sequence of independent and uniformly distributed random variables in $\{-1, +1\}$, and let $n \geq 1$ be a sufficiently large integer. For every integer $k \in [2, n/2]$, we have that the event $k \notin \mathcal{F}_{\mathbf{s}}(n)$, respectively $k \notin \mathcal{L}_{\mathbf{s}}(n)$, depends only on $s_{2k-2}, s_{2k-1}, s_{2k+1}, s_{2k+2}$. In particular, if the integers $k_1, k_2 \in [2, n/2]$ satisfy $|k_1 - k_2| \geq 3$ then $\big(k_1 \notin \mathcal{F}_{\mathbf{s}}(n), k_2 \notin \mathcal{F}_{\mathbf{s}}(n)\big)$, $\big(k_1 \notin \mathcal{L}_{\mathbf{s}}(n), k_2 \notin \mathcal{L}_{\mathbf{s}}(n)\big)$, and $\big(k_1 \notin \mathcal{F}_{\mathbf{s}}(n), k_2 \notin \mathcal{L}_{\mathbf{s}}(n)\big)$ are pairs of independent events. Moreover, we have

$$\mathbb{P}\big[k \notin \mathcal{F}_{\mathbf{s}}(n)\big] = \mathbb{P}\big[k \notin \mathcal{L}_{\mathbf{s}}(n)\big] = 2^{-4} = 16^{-1}.$$

Therefore, it follows that

$$\begin{aligned}\mathbb{P}\big[d \notin \mathcal{M}_{\mathbf{s}}(n)\big] &= \mathbb{P}\Big[\bigwedge_{\substack{2\le k\le n/2\\ d\in\mathcal{D}(k)}} \big(k \notin \mathcal{F}_{\mathbf{s}}(n)\big) \ \wedge \bigwedge_{\substack{2\le h\le n/2\\ d\in\mathcal{D}'(h)}} \big(h \notin \mathcal{L}_{\mathbf{s}}(n)\big)\Big]\\ &= \prod_{\substack{2\le k\le n/2\\ d\in\mathcal{D}(k)}} \mathbb{P}\big[k \notin \mathcal{F}_{\mathbf{s}}(n)\big] \cdot \prod_{\substack{2\le h\le n/2\\ d\in\mathcal{D}'(h)}} \mathbb{P}\big[h \notin \mathcal{L}_{\mathbf{s}}(n)\big]\\ &= 16^{-\lfloor n/(2d)\rfloor} \cdot 16^{-\left(\lfloor n\gcd(2,d)/(2d)\rfloor - \lfloor n/(2d)\rfloor\right)}\\ &= 16^{-\lfloor n\gcd(2,d)/(2d)\rfloor},\end{aligned}$$

for every integer $d \ge 6$. Consequently, by Lemma 3.1, we get

$$\begin{aligned}\mathbb{E}\big[\log \ell_{\mathbf{s}}(n)\big] &= (\log\alpha) \sum_{d\in\mathcal{M}_{\mathbf{s}}(n)} \varphi(d)\,\mathbb{P}[d\in\mathcal{M}_{\mathbf{s}}(n)] + O\left(\frac{n^2}{\log n}\right)\\ &= (\log\alpha)\sum_{d\le n}\varphi(d)\left(1-16^{-\lfloor n\gcd(2,d)/(2d)\rfloor}\right) + O\left(\frac{n^2}{\log n}\right).\end{aligned} \tag{10}$$

In turn, by Lemma 3.5, we have

$$\begin{aligned}\sum_{d\le n}\varphi(d)\left(1-16^{-\lfloor n\gcd(2,d)/(2d)\rfloor}\right) &= \sum_{d\in\mathcal{A}_{1,2}(n/2)}\varphi(d)\left(1-16^{-\lfloor n/(2d)\rfloor}\right)\\ &\quad + \sum_{d\in\mathcal{A}_{2,2}(n)}\varphi(d)\left(1-16^{-\lfloor n/d\rfloor}\right)\\ &= \frac{3}{\pi^2}\cdot\left(\frac{c_{1,2}}{4}+c_{2,2}\right)15\,\mathrm{Li}_2(1/16)\cdot n^2 + O\big(n(\log n)^2\big)\\ &= \frac{3}{\pi^2}\cdot\frac{15\,\mathrm{Li}_2(1/16)}{2}\cdot n^2 + O\big(n(\log n)^2\big).\end{aligned} \tag{11}$$

Finally, putting together (10) and (11), we obtain

$$\mathbb{E}\big[\log \ell_{\mathbf{s}}(n)\big] \sim \frac{3\log\alpha}{\pi^2}\cdot\frac{15\,\mathrm{Li}_2(1/16)}{2}\cdot n^2,$$

as $n \to +\infty$.

The proof is complete.

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