

The largest entry in the inverse of a Vandermonde matrix

Original

The largest entry in the inverse of a Vandermonde matrix / Sanna, C., Shallit, J., Zhang, S.. - In: LINEAR & MULTILINEAR ALGEBRA. - ISSN 0308-1087. - STAMPA. - 70:20(2022), pp. 5634-5641.
[10.1080/03081087.2021.1922337]

Availability:

This version is available at: 11583/2898892 since: 2021-05-10T10:24:03Z

Publisher:

Taylor & Francis

Published

DOI:10.1080/03081087.2021.1922337

Terms of use:

This article is made available under terms and conditions as specified in the corresponding bibliographic description in the repository

Publisher copyright

(Article begins on next page)

The Largest Entry in the Inverse of a Vandermonde Matrix

Carlo Sanna*

Department of Mathematical Sciences
Politecnico di Torino
Corso Duca degli Abruzzi 24
10129 Torino
Italy

carlo.sanna.dev@gmail.com

Jeffrey Shallit and Shun Zhang

School of Computer Science
University of Waterloo
Waterloo, ON N2L 3G1
Canada

shallit@uwaterloo.ca

s385zhang@uwaterloo.ca

December 11, 2020

Abstract

We investigate the size of the largest entry (in absolute value) in the inverse of certain Vandermonde matrices. More precisely, for every real $b > 1$, let $M_b(n)$ be the maximum of the absolute values of the entries of the inverse of the $n \times n$ matrix $[b^{ij}]_{0 \leq i, j < n}$. We prove that $\lim_{n \rightarrow +\infty} M_b(n)$ exists, and we provide some formulas for it.

1 Introduction

Let $\mathbf{a} = (a_0, a_1, \dots, a_{n-1})$ be a list of n real numbers. The classical *Vandermonde matrix* $V(\mathbf{a})$ is defined as follows:

$$V(\mathbf{a}) := \begin{bmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^{n-1} \\ 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_{n-1} & a_{n-1}^2 & \cdots & a_{n-1}^{n-1} \end{bmatrix}.$$

As is well-known, the Vandermonde matrix $V(\mathbf{a})$ is invertible if and only if the a_i are pairwise distinct [3]. Formulas for the inverse $V(\mathbf{a})$ (when it exists) have been known at least since 1958 [5].

*C. Sanna is a member GNSAGA of the INdAM and of CrypTO, the Group of Cryptography and Number Theory of Politecnico di Torino.

In what follows, n is a positive integer and $b > 1$ is a fixed real number. Let us define the entries $c_{i,j,n}$ by

$$[c_{i,j,n}]_{0 \leq i,j < n} = V(b^0, b^1, b^2, \dots, b^{n-1})^{-1}, \quad (1)$$

and let $M_b(n) = \max_{0 \leq i,j < n} |c_{i,j,n}|$, the maximum of the absolute values of the entries of $V(1, b, b^2, \dots, b^{n-1})^{-1}$. The size of the entries of inverses of Vandermonde matrices have been studied for a long time (e.g., [1]). Recently, in a paper by the first two authors and Daniel Kane [2], we needed to estimate $M_2(n)$, and we proved that $M_2(n) \leq 34$. In fact, even more is true: the limit $\lim_{n \rightarrow \infty} M_2(n)$ exists and equals $3 \prod_{i \geq 2} (1 + \frac{1}{2^i - 1}) \doteq 5.19411992918 \dots$. In this paper, we generalize this result, replacing 2 with any real number greater than 1.

Our main results are as follows:

Theorem 1. *Let $b > 1$ and $n_0 = \lceil \log_b(1 + \frac{1}{b}) \rceil$. Then $|c_{i,j,n}| \leq |c_{n_0, n_0, n}|$ for $i, j \geq n_0$. Hence $M_b(n) \in \{|c_{i,j,n}| : 0 \leq i, j \leq n_0\}$.*

Theorem 2. *Let $b \geq \tau = (1 + \sqrt{5})/2$ and $n \geq 2$. Then $M_b(n) \in \{|c_{0,0,n}|, |c_{1,1,n}|\}$.*

Theorem 3. *For all real $b > 1$ the limit $\lim_{n \rightarrow \infty} M_b(n)$ exists.*

2 Preliminaries

For every real number x , and for all integers $0 \leq i, j < n$, let us define the power sum

$$\sigma_{i,j,n}(x) := \sum_{\substack{0 \leq h_1 < \dots < h_i < n \\ h_1, \dots, h_i \neq j}} x^{h_1 + \dots + h_i}.$$

The following lemma will be useful in later arguments.

Lemma 4. *Let i, j, n be integers with $0 \leq i < n$, $0 \leq j < n - 1$, and let x be a positive real number.*

(a) *If $x > 1$, then $\sigma_{i,j,n}(x) \geq \sigma_{i,j+1,n}(x)$.*

(b) *If $x < 1$, then $\sigma_{i,j,n}(x) \leq \sigma_{i,j+1,n}(x)$.*

Proof. We have

$$\sigma_{i,j+1,n}(x) - \sigma_{i,j,n}(x) = \sum_{(h_1, \dots, h_i) \in S_{i,j,n}} x^{h_1 + \dots + h_i} - \sum_{(h_1, \dots, h_i) \in T_{i,j,n}} x^{h_1 + \dots + h_i},$$

where

$$S_{i,j,n} := \{0 \leq h_1 < \dots < h_i < n : j \in \{h_1, \dots, h_i\}, j+1 \notin \{h_1, \dots, h_i\}\}$$

and

$$T_{i,j,n} := \{0 \leq h_1 < \dots < h_i < n : j \notin \{h_1, \dots, h_i\}, j+1 \in \{h_1, \dots, h_i\}\}.$$

Now there is a bijection $S_{i,j,n} \rightarrow T_{i,j,n}$ given by

$$(h_1, \dots, h_i) \mapsto (h_1, \dots, h_{i_0-1}, h_{i_0} + 1, h_{i_0+1}, \dots, h_i),$$

where i_0 is the unique integer such that $h_{i_0} = j$. Hence, it follows easily that $\sigma_{i,j,n}(x) \geq \sigma_{i,j+1,n}(x)$ for $x > 1$, and $\sigma_{i,j,n}(x) \leq \sigma_{i,j+1,n}(x)$ for $x < 1$. \square

Lemma 5. *Let i, j, n be integers with $0 \leq i, j < n$, and let x be a nonzero real number. Then*

$$\frac{\sigma_{n-i-1,j,n}(x)}{x^{n(n-1)/2-j}} = \sigma_{i,j,n}(x^{-1}).$$

Proof. Note that the subsets \mathcal{A} of $\{0, 1, \dots, n-1\} - \{j\}$ of cardinality $n-i-1$ are in one-to-one correspondence with the subsets \mathcal{A}' of cardinality i . The correspondence is given by the complement $\mathcal{A} \mapsto \mathcal{A}' = \{0, 1, \dots, n-1\} - \{j\} - \mathcal{A}$. In particular, we have

$$\sum_{a \in \mathcal{A}} a = \sum_{k \in \{0, 1, \dots, n\} - \{j\}} k - \sum_{a \in \mathcal{A}'} a = \frac{n(n-1)}{2} - j - \sum_{a \in \mathcal{A}'} a.$$

As a consequence, we get that

$$\sigma_{n-i-1,j,n}(x) = \sum_{\mathcal{A}} x^{\sum_{a \in \mathcal{A}} a} = x^{n(n-1)/2-j} \sum_{\mathcal{A}'} x^{-\sum_{a \in \mathcal{A}'} a} = x^{n(n-1)/2-j} \sigma_{i,j,n}(x^{-1}),$$

as claimed. \square

Recall the following formula for the entries of the inverse of a Vandermonde matrix (see, e.g., [4, §1.2.3, Exercise 40]).

Lemma 6. *Let a_0, \dots, a_{n-1} be pairwise distinct real numbers. If $V(a_0, a_1, \dots, a_{n-1})^{-1} = [d_{i,j}]_{0 \leq i, j < n}$ then*

$$d_{n-1,j} X^{n-1} + d_{n-2,j} X^{n-2} + \dots + d_{0,j} X^0 = \prod_{\substack{0 \leq i < n \\ i \neq j}} \frac{X - a_i}{a_j - a_i}.$$

For $0 \leq i, j < n$ define

$$\pi_{j,n} := \prod_{\substack{0 \leq h < n \\ h \neq j}} |b^j - b^h|.$$

We now obtain a relationship between the entries of $V(b^0, b^1, \dots, b^{n-1})^{-1}$ and $\sigma_{i,j,n}$ and $\pi_{j,n}$.

Lemma 7. Let $V(b^0, b^1, \dots, b^{n-1})^{-1} = [c_{i,j,n}]_{0 \leq i,j < n}$. Then

$$|c_{i,j,n}| = \frac{\sigma_{n-i-1,j,n}}{\pi_{j,n}} \quad (2)$$

for $0 \leq i, j < n$.

Proof. By Lemma 6, we have

$$\prod_{\substack{0 \leq h < n \\ h \neq j}} \frac{X - b^h}{b^j - b^h} = \sum_{0 \leq i < n} c_{i,j,n} X^i.$$

which in turn, by Vieta's formulas, gives

$$c_{n-i-1,j,n} = (-1)^i \left(\prod_{\substack{0 \leq h < n \\ h \neq j}} \frac{1}{b^j - b^h} \right) \sum_{\substack{0 \leq h_1 < \dots < h_i < n \\ h_1, \dots, h_i \neq j}} b^{h_1 + \dots + h_i} \quad (3)$$

for $0 \leq i < n$. The result now follows by the definitions of σ and π . \square

Next, we obtain some inequalities for π .

Lemma 8. Define $n_0 = \lceil \log_b(1 + \frac{1}{b}) \rceil$. Then

$$\pi_{j,n} \leq \pi_{j+1,n} \quad \text{for } n_0 \leq j < n.$$

Proof. For $0 \leq j < n - 1$, we have

$$\pi_{j+1,n} := \prod_{\substack{0 \leq h < n \\ h \neq j+1}} |b^{j+1} - b^h| = b^{n-1} \prod_{\substack{0 \leq h < n \\ h-1 \neq j}} |b^j - b^{h-1}| = \frac{b^{n+j-1} - b^{n-2}}{b^{n-1} - b^j} \pi_{j,n}.$$

A quick computation shows that the following inequalities are equivalent:

$$\frac{b^{n+j-1} - b^{n-2}}{b^{n-1} - b^j} \geq 1 \quad \iff \quad b^j \geq \frac{b^{n-1} + b^{n-2}}{b^{n-1} + 1}.$$

Let n_0 be the minimum positive integer such that $b^{n_0} \geq 1 + \frac{1}{b}$. Then $n_0 = \lceil \log_b(1 + \frac{1}{b}) \rceil$. Hence, for $n_0 \leq j < n$, we have

$$b^j \geq 1 + \frac{1}{b} > \frac{b^{n-1} + b^{n-2}}{b^{n-1} + 1},$$

so that

$$\pi_{j,n} \leq \pi_{j+1,n} \quad \text{for } n_0 \leq j < n. \quad (4)$$

\square

Finally, we have the easy

Lemma 9. For $0 \leq i, j < n$ we have $c_{i,j,n} = c_{j,i,n}$.

Proof. $V(b^0, b^1, \dots, b^{n-1})$ is a symmetric matrix, so its inverse is also. \square

3 Proof of Theorem 1

Proof. Suppose $i, j \geq n_0$. Then

$$\begin{aligned}
|c_{i,j,n}| &= \frac{\sigma_{n-i-1,j,n}}{\pi_{j,n}} \quad (\text{by (2)}) \\
&\leq \frac{\sigma_{n-i-1,n_0,n}}{\pi_{j,n}} \quad (\text{by Lemma 4 (a)}) \\
&\leq \frac{\sigma_{n-i-1,n_0,n}}{\pi_{n_0,n}} \quad (\text{by Lemma 8}) \\
&= |c_{i,n_0,n}| \quad (\text{by (2)}),
\end{aligned}$$

and so we get

$$|c_{i,j,n}| \leq |c_{i,n_0,n}|. \quad (5)$$

But

$$c_{i,n_0,n} = c_{n_0,i,n} \quad (6)$$

by Lemma 9. Make the substitutions n_0 for i and i for j in (5) to get

$$|c_{n_0,i,n}| \leq |c_{n_0,n_0,n}|. \quad (7)$$

The result now follows by combining Eqs. (5), (6), and (7). \square

4 Proof of Theorem 2

Proof. Since $b \geq \tau$, it follows that $b \geq 1 + 1/b$. Hence in Theorem 1 we can take $n_0 = 1$, and this gives $M_b(n) \in \{|c_{0,0,n}|, |c_{1,0,n}|, |c_{0,1,n}|, |c_{1,1,n}|\}$. However, by explicit calculation, we have

$$\begin{aligned}
\sigma_{n-1,1,n} &= b^{n(n-1)/2-1} \\
\sigma_{n-2,1,n} &= b^{n(n-1)/2-1} + \sum_{(n-1)(n-2)/2-1 \leq i \leq n(n-1)/2-3} b^i,
\end{aligned}$$

so that

$$\sigma_{n-1,1,n} \leq \sigma_{n-2,1,n}. \quad (8)$$

Hence

$$\begin{aligned}
|c_{1,0,n}| &= |c_{0,1,n}| \quad (\text{by Lemma 9}) \\
&= \frac{\sigma_{n-1,1,n}}{\pi_{1,n}} \quad (\text{by (2)}) \\
&\leq \frac{\sigma_{n-2,1,n}}{\pi_{1,n}} \quad (\text{by (8)}) \\
&= |c_{1,1,n}| \quad (\text{by (2)}),
\end{aligned}$$

and the result follows. \square

5 Proof of Theorem 3

Proof. We have

$$\begin{aligned}
|c_{i,j,n}| &= \frac{\sigma_{n-i-1,j,n}}{\pi_{j,n}} \\
&= \frac{\sigma_{n-i-1,j,n}(b)}{\prod_{\substack{0 \leq h < n \\ h \neq j}} |b^j - b^h|} \\
&= \frac{\sigma_{n-i-1,j,n}(b)}{\prod_{\substack{0 \leq h < n \\ h \neq j}} (b^h \cdot |b^{j-h} - 1|)} \\
&= \frac{\sigma_{n-i-1,j,n}(b)}{b^{n(n-1)/2-j}} \cdot \frac{1}{\prod_{\substack{0 \leq h < n \\ h \neq j}} |b^{j-h} - 1|} \\
&= \sigma_{i,j,n}(b^{-1}) \frac{1}{\prod_{\substack{0 \leq h < n \\ h \neq j}} |b^{j-h} - 1|},
\end{aligned}$$

where we used Lemma 5.

For $x < 1$ define

$$\sigma_{i,j,\infty}(x) = \sum_{\substack{0 \leq h_1 < \dots < h_i < \infty \\ h_1, \dots, h_i \neq j}} \frac{1}{x^{h_1 + \dots + h_i}},$$

with the convention $\sigma_{0,j,\infty}(x) := 1$.

Hence the limits

$$\begin{aligned}
\ell_{i,j} &:= \lim_{n \rightarrow +\infty} |c_{i,j,n}| \\
&= \lim_{n \rightarrow +\infty} \sigma_{i,j,n}(b^{-1}) \frac{1}{\prod_{\substack{0 \leq h < n \\ h \neq j}} |b^{j-h} - 1|} \\
&= \lim_{n \rightarrow +\infty} \sigma_{i,j,n}(b^{-1}) \prod_{0 \leq h < j} \frac{1}{b^{j-h} - 1} \prod_{j < h < n} \frac{1}{1 - b^{j-h}} \\
&= \lim_{n \rightarrow +\infty} \sigma_{i,j,n}(b^{-1}) \prod_{1 \leq s \leq j} \frac{1}{b^s - 1} \prod_{1 \leq t < n-j} \frac{1}{1 - b^{-t}} \\
&= \sigma_{i,j,\infty}(b^{-1}) \left(\prod_{1 \leq s \leq j} \frac{1}{b^s - 1} \right) \left(\prod_{t \geq 1} \frac{1}{1 - b^{-t}} \right) \tag{9}
\end{aligned}$$

exist and are finite.

From Theorem 1 we see that

$$\lim_{n \rightarrow +\infty} M_b(n) = \max_{0 \leq i \leq j < n_0} \lim_{n \rightarrow +\infty} |c_{i,j,n}| = \max_{0 \leq i \leq j \leq n_0} \ell_{i,j},$$

and the proof is complete. \square

From this theorem we can explicitly compute $\lim_{n \rightarrow +\infty} M_b(n)$ for $b \geq \tau$.

Corollary 10. Let $\alpha \doteq 2.324717957$ be the real zero of the polynomial $X^3 - 3X^2 + 2X - 1$.

(a) If $b \geq \alpha$, then $\lim_{n \rightarrow \infty} M_b(n) = \prod_{t \geq 1} (1 - b^{-t})^{-1}$.

(b) If $\tau \leq b \leq \alpha$, then $\lim_{n \rightarrow \infty} M_b(n) = \frac{b^2 - b + 1}{b(b-1)^2} \prod_{t \geq 1} (1 - b^{-t})^{-1}$.

Proof. From Theorem 2 we know that for $b \geq \tau$ we have $\lim_{n \rightarrow \infty} M_b(n) \in \{\ell_{0,0}, \ell_{1,1}\}$. Now an easy calculation based on (9) shows that

$$\begin{aligned} \ell_{0,0} &= \prod_{t \geq 1} (1 - b^{-t})^{-1} \\ \ell_{1,1} &= \frac{b^2 - b + 1}{b(b-1)^2} \prod_{t \geq 1} (1 - b^{-t})^{-1}. \end{aligned}$$

By solving the equation $\frac{b^2 - b + 1}{b(b-1)^2} = 1$, we see that for $b \geq \alpha$ we have $\ell_{0,0} \geq \ell_{1,1}$, while if $\tau \leq b \leq \alpha$ we have $\ell_{1,1} \geq \ell_{0,0}$. This proves both parts of the claim. \square

Remark 11. The quantity $M_b(n)$ converges rather slowly to its limit when b is close to 1. The following table gives some numerical estimates for $M_b(n)$.

b	$\lim_{n \rightarrow \infty} M_b(n)$
3	1.785312341998534190367486
$\alpha \doteq 2.3247$	2.4862447382651613433
2	5.194119929182595417
$\tau \doteq 1.61803$	26.788216012030303413
1.5	67.3672156
1.4	282.398
1.3	3069.44
1.2	422349.8

6 Final remarks

We close with a conjecture we have been unable to prove.

Conjecture 12. Let $b > 1$ and $n_0 = \lceil \log_b(1 + \frac{1}{b}) \rceil$. Then, for all sufficiently large n , we have $M_b(n) = |c_{i,i,n}|$ for some i , $0 \leq i \leq n_0$.

Acknowledgment

We thank the referee for a careful reading of the paper and several helpful suggestions.

References

- [1] W. Gautschi. On inverses of Vandermonde and confluent Vandermonde matrices. *Numer. Mathematik* **4** (1962), 117–123.
- [2] D. M. Kane, C. Sanna, and J. Shallit. Waring’s theorem for binary powers. *Combinatorica* **39** (2019), 1335–1350.
- [3] A. Klinger. The Vandermonde matrix. *Amer. Math. Monthly* **74** (1967), 571–574.
- [4] D. E. Knuth. *The Art of Computer Programming, Vol. 1, Fundamental Algorithms*. Addison-Wesley, third edition, 1997.
- [5] N. Macon and A. Spitzbart. Inverses of Vandermonde matrices. *Amer. Math. Monthly* **65** (1958), 95–100.