Practical central binomial coefficients

Original
Practical central binomial coefficients / Sanna, C.. - In: QUAESTIONES MATHEMATICAE. - ISSN 1607-3606. - ELETTRONICO. - 44:9(2021), pp. 1141-1144. [10.2989/16073606.2020.1775156]

Availability:
This version is available at: 11583/2883058 since: 2021-12-21T16:42:50Z

Publisher:
Taylor and Francis

Published
DOI:10.2989/16073606.2020.1775156

PRACTICAL CENTRAL BINOMIAL COEFFICIENTS

CARLO SANNA[†]

Abstract. A *practical number* is a positive integer n such that all positive integers less than n can be written as a sum of distinct divisors of n. Leonetti and Sanna proved that, as $x \to +\infty$, the central binomial coefficient $\binom{2n}{n}$ is a practical number for all positive integers $n \leq x$ but at most $O(x^{0.88097})$ exceptions. We improve this result by reducing the number of exceptions to $\exp\big(C(\log x)^{4/5} \log\log x\big)$, where $C > 0$ is a constant.

1. Introduction

A *practical number* is a positive integer n such that all positive integers less than n can be written as a sum of distinct divisors of n. Practical numbers were defined by Srinivasan [15], althought they were already used by Fibonacci to decompose rational numbers as sums of unit fractions [12, pag. 121]. Estimates for the counting function of practical numbers were given by Hausman and Shapiro [3], Tenenbaum [16], Margenstern [8], Saias [13], and, lastly, Weingartner [17], who proved that the number of practical numbers up to x is asymptotic to $cx/\log x$, as $x \to +\infty$, where $c = 1.33607\ldots$ [18], settling a conjecture of Margenstern [8].

In analogy with Goldbach's conjecture and prime triplet conjecture, Melfi [10] proved that every positive even integer is the sum of two practical numbers, and that there are infinitely many triples $(n, n+2, n+4)$ of practical numbers. Moreover, Melfi [9] proved that every Lucas sequence $(U_n(P,Q))$ satisfying some mild conditions contains infinitely many practical numbers, and Sanna [14] showed that $U_n(P,Q)$ is practical for at least $\gg_{P,Q} x/\log x$ positive integers $n \leq x$, as $x \to +\infty$; and asked for a nontrivial upper bound.

Leonetti and Sanna [7] studied binomial coefficients that are practical numbers. They proved that, for fixed $\varepsilon > 0$ and as $x \to +\infty$, all binomial coefficients $\binom{n}{k}$, with $0 \leq k \leq n \leq x$, are practical numbers but at most $O_\varepsilon\big(x^{2-(2^{-1}\log 2-\varepsilon)/\log\log x}\big)$ exceptions. Furthermore, they showed that the central binomial coefficient $\binom{2n}{n}$ is a practical number for all positive integers $n \leq x$ but at most $O(x^{0.88097})$ exceptions. In this note, we give the following improvement of the last result.

Theorem 1.1. *For $x \geq 3$ the central binomial coefficient $\binom{2n}{n}$ is a practical number for all positive integers $n \leq x$ but at most $\exp\big(C(\log x)^{4/5}\log\log x\big)$ exceptions, where $C > 0$ is a constant.*

We remark that (as already pointed out in [7]), likely, there are only finitely many positive integers n such that $\binom{2n}{n}$ is not a practical number, but proving so could be out of reach. In fact, if n is a power of 2 whose base 3 representation does not contain the digit 2, then $\binom{2n}{n}$ is not a practical number [7, Proposition 2.1]. However, establishing whether there are finitely or infinitely many such powers of 2 is an open problem [2, 4, 6, 11].

2. Preliminaries

We need some preliminary results.

Lemma 2.1. *If d is a practical number and n is a positive integer divisible by d and having all prime factors not exceeding $2d$, then n is a practical number.*

2010 *Mathematics Subject Classification.* Primary: 11B65, Secondary: 11N25.

Key words and phrases. central binomial coefficient; practical number.

† C. Sanna is a member of the INdAM group GNSAGA.

Proof. See [7, Lemma 2.2]. □

For every positive integer n, let $s_2(n)$ be the number of nonzero binary digits of n.

Lemma 2.2. *For every positive integer n, the exponent of 2 in the prime factorization of $\binom{2n}{n}$ is equal to $s_2(n)$.*

Proof. A result of Kummer [5] says that for every prime number p and for all positive integers m, n the exponent of p in the prime factorization of $\binom{m+n}{n}$ is equal to the number of carries in the addition $m + n$ done in base p. If $m = n$ and $p = 2$ then we get the desired claim. □

Lemma 2.3. *We have*

$$\#\big\{n \leq x : s_2(n) \leq \varepsilon(\log n/\log 2 + 1)\big\} \leq x^{\left(\frac{1}{\log 2}+o(1)\right)\varepsilon\log(1/\varepsilon)},$$

uniformly as $\varepsilon \log x \to +\infty$ and $\varepsilon \to 0^+$.

Proof. Put $N := \lfloor \log x/\log 2 + 1\rfloor$ and $k := \lceil \varepsilon(\log n/\log 2 + 1)\rceil$. Then

$$C := \#\big\{n \leq x : s_2(n) \leq \varepsilon(\log n/\log 2 + 1)\big\} \leq \#\big\{n < 2^N : s_2(n) \leq k\big\},$$

where the right-hand side is the number of binary strings of length N having at most k nonzero bits (including $n = 0$ to the count). Therefore,

$$C \leq \sum_{j=0}^{k}\binom{N}{j} \leq \sum_{j=0}^{k}\frac{N^j}{j!} = \sum_{j=0}^{k}\frac{k^j}{j!}\left(\frac{N}{k}\right)^j < \left(\frac{eN}{k}\right)^k < e^{(1-\log\varepsilon)(\varepsilon(\log x/\log 2+1)+1)},$$

and the claim follows recalling that $\varepsilon\log x \to +\infty$ and $\varepsilon \to 0^+$. □

The following result of Erdős and Kolesnik is the key to the proof of Theorem 1.1.

Theorem 2.4. *There exist constants $c_1, c_2 > 0$ such that, for all integers m, n, r with*

$$2 \leq m \leq n/2 \quad \text{and} \quad 1 \leq r \leq c_1\left(\frac{(\log m)^3}{(\log n)^2 \log\log n}\right)^{1/4},$$

there exist at least $c_2 r m^{1/r}/(4^r \log m)$ prime numbers $p \in [m^{1/r}, n^{1/r}]$ such that $p^r \mid\mid \binom{n}{m}$.

Proof. See [1, Theorem 2]. □

Corollary 2.1. *There exists a constant $c_3 > 0$ such that, for all integers n, r with*

$$n \geq 3 \quad \text{and} \quad 1 \leq r \leq c_3\left(\frac{\log n}{\log\log n}\right)^{1/4},$$

there exists a prime number $p \in [n^{1/r}, (2n)^{1/r}]$ such that $p^r \mid\mid \binom{2n}{n}$.

Proof. The claim follows by replacing m and n with n and $2n$, respectively, in Theorem 2.4. □

3. Proof of Theorem 1.1

Fix $C > \max\big((5\log 2)^{-1}, (2/c_3)^4\big)$, where c_3 is the constant of Corollary 2.1. Assume that x is sufficiently large and put $E := \exp\big(C(\log x)^{4/5}\log\log x\big)$ and $\varepsilon := (\log x)^{-1/5}$. Let $n \leq x$ be a positive integer and let v be the exponent of 2 in the prime factorization of $\binom{2n}{n}$. Since

$$\frac{1}{\log 2}\,\varepsilon\log(1/\varepsilon)\log x = \frac{1}{5\log 2}(\log x)^{4/5}\log\log x < C(\log x)^{4/5}\log\log x,$$

from Lemma 2.2 and Lemma 2.3 we get that $2^v \leq n^\varepsilon$ for less than $\frac{1}{2}E$ choices of n. Hence, we can assume that $2^v > n^\varepsilon$ and $n > \frac{1}{2}E$, which excludes at most E positive integers not exceeding x. Then, since $n > \frac{1}{2}E$ and x is sufficiently large, we have

$$\frac{\log n}{\log\log n} > \frac{\log(\frac{1}{2}E)}{\log\log(\frac{1}{2}E)} > C(\log x)^{4/5} > \left(\frac{2(\log x)^{1/5}}{c_3}\right)^4.$$

Therefore,

$$r := \left\lfloor c_3 \left(\frac{\log n}{\log\log n}\right)^{1/4} \right\rfloor > \frac{1}{\varepsilon}.$$

Thanks to Corollary 2.1, there exists a prime number $p \in [n^{1/r}, (2n)^{1/r}]$ such that p^r divides $\binom{2n}{n}$. Now 2^v is a practical number, because all powers of 2 are practical numbers. Morever, since

$$p \leq (2n)^{1/r} < (2n)^{\varepsilon} < 2^{v+1},$$

from Lemma 2.1 it follows that $2^v p^r$ is a practical number. Finally, $2^v p^r$ divides $\binom{2n}{n}$, $2^v p^r \geq 2n$, and all prime factors of $\binom{2n}{n}$ are not exceeding $2n$, hence Lemma 2.1 yields that $\binom{2n}{n}$ is a practical number. The proof is complete.

References

1. P. Erdős and G. Kolesnik, *Prime power divisors of binomial coefficients*, vol. 200, 1999, Paul Erdős memorial collection, pp. 101–117.
2. H. Gupta, *Powers of* 2 *and sums of distinct powers of* 3, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. (1978), no. 602-633, 151–158 (1979).
3. M. Hausman and H. N. Shapiro, *On practical numbers*, Comm. Pure Appl. Math. **37** (1984), no. 5, 705–713.
4. R. E. Kennedy and C. Cooper, *A generalization of a result by Narkiewicz concerning large digits of powers*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. **11** (2000), 36–40 (2001).
5. E. E. Kummer, *Über die Ergänzungssätze zu den allgemeinen Reciprocitätsgesetzen*, J. Reine Angew. Math. **44** (1852), 93–146.
6. J. C. Lagarias, *Ternary expansions of powers of 2*, J. Lond. Math. Soc. (2) **79** (2009), no. 3, 562–588.
7. P. Leonetti and C. Sanna, *Practical numbers among the binomial coefficients*, J. Number Theory **207** (2020), 145–155.
8. M. Margenstern, *Les nombres pratiques: théorie, observations et conjectures*, J. Number Theory **37** (1991), no. 1, 1–36.
9. G. Melfi, *A survey on practical numbers*, Rend. Sem. Mat. Univ. Politec. Torino **53** (1995), no. 4, 347–359, Number theory, II (Rome, 1995).
10. G. Melfi, *On two conjectures about practical numbers*, J. Number Theory **56** (1996), no. 1, 205–210.
11. W. Narkiewicz, *A note on a paper of H. Gupta concerning powers of two and three: "Powers of* 2 *and sums of distinct powers of* 3*" [Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 602-633 (1978), 151–158 (1979); MR 81g:10016]*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. (1980), no. 678-715, 173–174 (1981).
12. L. Pisano, *Fibonacci's Liber Abaci*, Sources and Studies in the History of Mathematics and Physical Sciences, Springer-Verlag, New York, 2002, A translation into modern English of Leonardo Pisano's Book of calculation, Translated from the Latin and with an introduction, notes and bibliography by L. E. Sigler.
13. E. Saias, *Entiers à diviseurs denses. I*, J. Number Theory **62** (1997), no. 1, 163–191.
14. C. Sanna, *Practical numbers in Lucas sequences*, Quaest. Math. **42** (2019), no. 7, 977–983.
15. A. K. Srinivasan, *Practical numbers*, Current Sci. **17** (1948), 179–180.
16. G. Tenenbaum, *Sur un problème de crible et ses applications*, Ann. Sci. École Norm. Sup. (4) **19** (1986), no. 1, 1–30.
17. A. Weingartner, *Practical numbers and the distribution of divisors*, Q. J. Math. **66** (2015), no. 2, 743–758.
18. A. Weingartner, *The constant factor in the asymptotic for practical numbers*, Int. J. Number Theory **16** (2020), no. 3, 629–638.

Politecnico di Torino, Department of Mathematical Sciences
Corso Duca degli Abruzzi 24, 10129 Torino, Italy
Email address: carlo.sanna.dev@gmail.com