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Zenodo - 2 March 2021 - DOI 10.5281/zenodo.4573493

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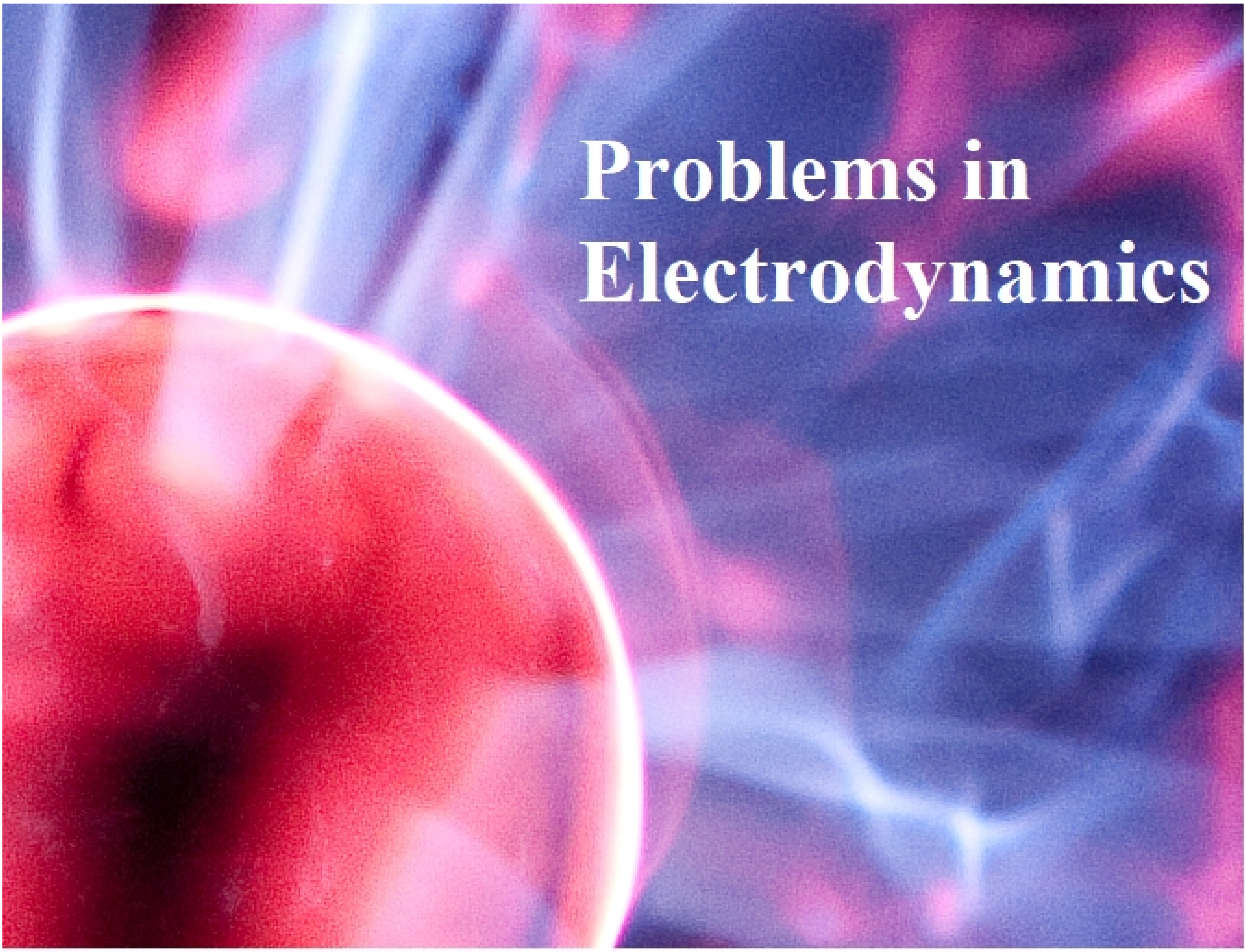

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Problems in Electrodynamics

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Some problems in electrodynamics are here proposed. Subjects are vector analysis, electrostatic fields, dielectric materials, magnetostatics and induction.

On the cover a detail of an image showing a plasma globe operating in a darkened room. Exposure time is 1/3 second. Image courtesy by Chocolateoak for Wikipedia.

Electric field $\boldsymbol{E}$ and magnetic induction $\boldsymbol{B}$ characterize the electromagnetic field. On charges and currents, fields are acting in the following manners:

$$\boldsymbol{F}=e\boldsymbol{E} \quad ; \quad d\boldsymbol{F}=I\,[d\boldsymbol{l}\times\boldsymbol{B}]$$

Two supplementary fields are conventionally introduced. They are $\boldsymbol{D}$, the displacement vector, and $\boldsymbol{H}$, the magnetic field strength. Being $\boldsymbol{P},\boldsymbol{M}$ the polarization and magnetization of the material, we have:

$$\boldsymbol{D}=\varepsilon_o\boldsymbol{E}+\boldsymbol{P} \quad ; \quad \boldsymbol{H}=\frac{\boldsymbol{B}}{\mu_o}-\boldsymbol{M}$$

Permittivity and permeability of the empty space are: $\varepsilon_o=8.854\times10^{-12}F/m$;

$\mu_o = 4\pi \times 10^{-7}\, H/m$; they are related by $\varepsilon_o \mu_o = c^{-2}$, where c is the speed of light in the empty space.

In an anisotropic medium, and in the linear approximation:

$$P_i = \varepsilon_o \alpha_{ik} E_k \quad ; \quad M_i = \chi_{ik} H_k$$

α_{ik}, χ_{ik} are the tensors of the polarizability and magnetic susceptibility. Then:

$$D_i = \varepsilon_o \epsilon_{ik} E_k \quad ; \quad B_i = \mu_o \mu_{ik} H_k \quad ; \quad \epsilon_{ik} = \delta_{ik} + \alpha_{ik} \quad ; \quad \mu_{ik} = \delta_{ik} + \chi_{ik}$$

For an isotropic medium:

$$\boldsymbol{P} = \varepsilon_o \alpha \boldsymbol{E} \quad ; \quad \boldsymbol{M} = \chi \boldsymbol{H} \quad ; \quad \boldsymbol{D} = \varepsilon_o \epsilon \boldsymbol{E} \quad ; \quad \boldsymbol{B} = \mu_o \mu \boldsymbol{H}$$

Maxwell equations are:

$$\text{curl}\, \boldsymbol{H} = \boldsymbol{j} + \frac{\partial \boldsymbol{D}}{\partial t} \quad ; \quad \text{curl}\, \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \quad ; \quad \text{div}\, \boldsymbol{D} = \rho \quad ; \quad \text{div}\, \boldsymbol{B} = 0$$

where $\boldsymbol{j}$ is the conduction current density and ρ the electric charge density of unbounded charges only.

Boundary conditions at the boundary between two media are:

$$D_{2n} - D_{1n} = \sigma \quad ; \quad E_{1t} = E_{2t} \quad ; \quad B_{1n} = B_{2n} \quad ; \quad [\boldsymbol{n} \times (\boldsymbol{H_2} - \boldsymbol{H_1})] = \boldsymbol{i}$$

$\boldsymbol{n}$ is a unit vector normal to the boundary between media 1 and 2, directed from medium 1 to medium 2. $\boldsymbol{t}$ is a unit vector tangent the boundary. σ is the surface charge density due to unbounded charges only. $\boldsymbol{i}$ is the surface current per unit length due to unbounded charges, flowing along the boundary.

Electrostatics

In electrostatics $\mathrm{curl}\,\boldsymbol{E}=0$, $\mathrm{div}\,\boldsymbol{D}=\rho$, so that $\boldsymbol{E}=-\mathrm{grad}\,\varphi$. Here, φ is the potential. Therefore:

$$\Delta\varphi=-\frac{\rho}{\varepsilon_o\epsilon}$$

At the boundary of two media, with ϵ_1,ϵ_2 , which are the relative permittivity or dielectric constant of the materials:

$$\varphi_1=\varphi_2 \quad ; \quad \epsilon_1\left(\frac{\partial\varphi}{\partial n}\right)_1-\epsilon_2\left(\frac{\partial\varphi}{\partial n}\right)_2=\frac{\sigma}{\varepsilon_o}$$

$\partial/\partial n$ indicates the derivative in the direction perpendicular to the boundary.
The second formula can be written as:

$$D_{1n}-D_{2n}=\sigma_{unbounded}$$

In electrostatics field $\boldsymbol{E}$ does not penetrate a conductor. Then, inside, $\varphi=constant$.

Near the surface *S* of a conductor in electrostatic condition:

$$\sigma=-\varepsilon_o\epsilon\left(\frac{\partial\varphi}{\partial n}\right)_S$$

The total charge of a conductor is $e=-\oint\varepsilon_o\epsilon\left(\frac{\partial\varphi}{\partial n}\right)\,dS$. Integration is over the conductor surface.

Many problems in electrostatics are proposed as two-dimensional cases. To solve them, let us introduce the complex function:

$$W(z)=\varphi+i\psi$$

where $z=x+iy$. Real and the imaginary parts are the solutions of Laplace equations:

$$\Delta\varphi=0 \quad ; \quad \Delta\psi=0$$

φ,ψ are restricted by the condition:

$$\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial y}=0$$

If the distribution of body and surface charges is given, the potential at point $\boldsymbol{r}$ is:

$$\varphi(\boldsymbol{r})=\frac{1}{4\pi\varepsilon_o\epsilon}\int\frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}dV'+\frac{1}{4\pi\varepsilon_o\epsilon}\int\frac{\sigma(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}dS'$$

At great distances: $\varphi(\boldsymbol{r})=\varphi^{(0)}(\boldsymbol{r})+\varphi^{(1)}(\boldsymbol{r})+\varphi^{(2)}(\boldsymbol{r})+\ldots$, where:

$$\varphi^{(0)}(\boldsymbol{r})=\frac{e}{4\pi\varepsilon_o\epsilon r} \quad ; \quad \varphi^{(1)}(\boldsymbol{r})=\frac{(\boldsymbol{p}\cdot\boldsymbol{r})}{4\pi\varepsilon_o\epsilon r^3} \quad ;$$

$$\varphi^{(2)}(\boldsymbol{r})=\frac{1}{4\pi\varepsilon_o\epsilon}\sum_{i,k}\frac{Q_{ik}}{2!}\frac{\partial^2}{\partial x_i \partial x_k}\left(\frac{1}{r}\right)$$

where: $\boldsymbol{p}=\int \rho(\boldsymbol{r})\boldsymbol{r}\,dV$, $Q_{ik}=\int \rho(\boldsymbol{r})(3x_i x_k - r^2\delta_{ik})dV$. Therefore, we have introduced dipoles and quadrupoles.

The energy of the field is given by: $W=\frac{1}{2}\rho\,\varphi\,dV=\frac{1}{2}\int(\boldsymbol{D}\cdot\boldsymbol{E})dV$.

The interaction energy of two electrically charged systems with charge densities ρ_1,ρ_2 is:

$$W=\frac{1}{4\pi\varepsilon_o\epsilon}\int\frac{\rho_1(\boldsymbol{r_1})\rho_2(\boldsymbol{r_2})}{|\boldsymbol{r_2}-\boldsymbol{r_1}|}dV_1\,dV_2$$

After some problems in vector analysis, we will start to discuss problems in electrostatics.

Vector analysis

Let us remember some relations which are useful to solve calculations.

$$\text{curl grad} f=0 \quad (1) \quad ; \quad \text{div curl}\boldsymbol{A}=0 \quad (2)$$

Eq. (1) is referring to conservative fields (f is a scalar quantity). Eq. (2) is referring to solenoidal fields. Here other equations from [1] (φ,ψ are scalar quantities):.

$$\text{curl curl}\boldsymbol{A}=\text{grad div}\,\boldsymbol{A}-\Delta\boldsymbol{A} \quad (3) \quad ; \quad \text{div}(f\boldsymbol{A})=f\,\text{div}\,\boldsymbol{A}+\text{grad} f\cdot\boldsymbol{A} \quad (4)$$

$$\text{div}[\boldsymbol{A}\times\boldsymbol{B}]=\boldsymbol{B}\cdot\text{curl}\,\boldsymbol{A}-\boldsymbol{A}\cdot\text{curl}\,\boldsymbol{B} \quad (5) \quad ; \quad \text{curl}(f\boldsymbol{A})=[\text{grad} f\times\boldsymbol{A}]+f\,\text{curl}\,\boldsymbol{A} \quad (6)$$

$$\Delta(\varphi\,\psi)=\psi\,\Delta\,\varphi+\varphi\,\Delta\,\psi+2\,\mathrm{grad}\,\varphi\cdot\mathrm{grad}\,\psi \quad (7)$$

We have also $\Delta\,\varphi=\mathrm{div}\,\mathrm{grad}\,\varphi$; (3) is useful to have the wave equation from Maxwell equations. (5) is an interesting expression for the use of the Poynting vector (see for instance [2]). Using relations (1-7), we can solve the problems in [3]. Let us add some relations given in the Appendix of [3].

$$\mathrm{grad}(\varphi\,\psi)=\varphi\,\mathrm{grad}\,\psi+\psi\,\mathrm{grad}\,\varphi \quad (8)$$

$$\mathrm{grad}(\boldsymbol{A}\cdot\boldsymbol{B})=(\boldsymbol{A}\cdot\nabla)\boldsymbol{B}+(\boldsymbol{B}\cdot\nabla)\boldsymbol{A}+[\boldsymbol{A}\times\mathrm{curl}\,\boldsymbol{B}]+[\boldsymbol{B}\times\mathrm{curl}\,\boldsymbol{A}] \quad (9)$$

$$\frac{1}{2}\mathrm{grad}(\boldsymbol{A}\cdot\boldsymbol{A})=(\boldsymbol{A}\cdot\nabla)\,\boldsymbol{A}+[\boldsymbol{A}\times\mathrm{curl}\,\boldsymbol{A}] \quad (10)$$

$$\mathrm{curl}\,[\boldsymbol{A}\times\boldsymbol{B}]=(\boldsymbol{B}\cdot\nabla)\,\boldsymbol{A}-(\boldsymbol{A}\cdot\nabla)\,\boldsymbol{B}+\boldsymbol{A}\,\mathrm{div}\,\boldsymbol{B}-\boldsymbol{B}\,\mathrm{div}\,\boldsymbol{A} \quad (11)$$

Here a second order relation:

$$\Delta(f\,\boldsymbol{A})=\Delta(f\,A_x\boldsymbol{i}+f\,A_y\boldsymbol{j}+f\,A_z\boldsymbol{k})=\Delta(f\,A_x)\,\boldsymbol{i}+\Delta(f\,A_y)\,\boldsymbol{j}+\Delta(f\,A_z)\,\boldsymbol{k}$$

If use (7), applying it to $\Delta(f\,A_x),\Delta(f\,A_y),\Delta(f\,A_z)$, we have:

$$\Delta(f\,\boldsymbol{A})=f\,\Delta\,\boldsymbol{A}+\boldsymbol{A}\,\Delta f+2(\mathrm{grad}\,f\cdot\nabla)\,\boldsymbol{A}$$

Second order relations useful in the elastic theory of liquid crystals are given in [4].

We have also relations concerning integrals.

$$\oiint \varphi\,\boldsymbol{n}\,dS=\iiint \mathrm{grad}\,\varphi\;dV \quad (12)$$

$\boldsymbol{n}$ is the unit vector, normal to the surface, pointing outside.

$$\oiint (\boldsymbol{A}\cdot\boldsymbol{n})\, dS = \iiint \text{div}\,\boldsymbol{A}\; dV \quad (13) \qquad ; \qquad \oiint [\boldsymbol{n}\times\boldsymbol{A}]\, dS = \iiint \text{curl}\,\boldsymbol{A}\; dV \quad (14)$$

$$\oint A_n\, dl = \iint \text{curl}_n \boldsymbol{A}\; dS \quad (15) \qquad ; \qquad \oint \varphi\, dl = \iint [\boldsymbol{n}\times\text{grad}\,\varphi]\; dS \quad (16)$$

Let us start problems in [3].

1) Calculate the gradient of function $f(r)$, which is depending only on the absolute value of the radius vector $\boldsymbol{r}$. Solution: $\text{grad} f(r) = \frac{df}{dr}\frac{\boldsymbol{r}}{r}$.

The gradient is:

$$\text{grad} f(r) = \frac{\partial f(r)}{\partial x}\mathbf{i} + \frac{\partial f(r)}{\partial y}\mathbf{j} + \frac{\partial f(r)}{\partial z}\mathbf{k} \;=\; \frac{df}{dr}\{\frac{\partial r}{\partial x}\mathbf{i} + \frac{\partial r}{\partial y}\mathbf{j} + \frac{\partial r}{\partial z}\mathbf{k}\} \;=$$

$$\frac{df}{dr}\{\frac{x}{r}\mathbf{i} + \frac{y}{r}\mathbf{j} + \frac{z}{r}\mathbf{k}\}$$

$$\text{grad} f(r) = \frac{df}{dr}\frac{\boldsymbol{r}}{r} \quad (17)$$

Let us consider an example. $\text{grad}\frac{1}{r^3} = -3\frac{1}{r^4}\frac{\boldsymbol{r}}{r} = -3\frac{\boldsymbol{r}}{r^5}$

Let us add a similar calculus for the divergence of function $\boldsymbol{A}(r)$, which is dependent on the absolute value of radius vector $\boldsymbol{r}$.

$$\operatorname{div}\boldsymbol{A}(r)=\frac{\partial A_x(r)}{\partial x}+\frac{\partial A_y(r)}{\partial y}+\frac{\partial A_z(r)}{\partial z} =$$

$$\frac{dA_x}{dr}\frac{\partial r}{\partial x}+\frac{dA_y}{dr}\frac{\partial r}{\partial y}+\frac{dA_z}{dr}\frac{\partial r}{\partial z} = \frac{dA_x}{dr}\frac{x}{r}+\frac{dA_y}{dr}\frac{y}{r}+\frac{dA_z}{dr}\frac{z}{r}=\frac{1}{r}\boldsymbol{r}\cdot\frac{d\boldsymbol{A}}{dr} \quad (18)$$

2) Calculate (a) $\operatorname{div}\boldsymbol{r}$, (b) $\operatorname{curl}\boldsymbol{r}$, (c) $\operatorname{curl}(f(r)\boldsymbol{r})$.

Let us note that $\boldsymbol{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. Therefore:

(a) $$\operatorname{div}\boldsymbol{r}=\frac{\partial x}{\partial x}\mathbf{i}+\frac{\partial y}{\partial y}\mathbf{j}+\frac{\partial z}{\partial z}\mathbf{k}=3$$

(b) $$\operatorname{curl}\boldsymbol{r}=(\frac{\partial z}{\partial y}-\frac{\partial y}{\partial z})\mathbf{i}+(-\frac{\partial z}{\partial x}+\frac{\partial x}{\partial z})\mathbf{j}+(\frac{\partial y}{\partial x}-\frac{\partial x}{\partial y})\mathbf{k}=0$$

(c) To calculate $\operatorname{curl}(f(r)\boldsymbol{r})$ we can use (6) and then (17):

$$\operatorname{curl}(f\boldsymbol{r})=[\operatorname{grad}f\times\boldsymbol{r}]+f\operatorname{curl}\boldsymbol{r}=[\operatorname{grad}f\times\boldsymbol{r}]=\frac{df}{dr}\frac{\boldsymbol{r}}{r}\times\boldsymbol{r}=0$$

3) Calculate (a) $\operatorname{grad}(\boldsymbol{P}\cdot\boldsymbol{r})$, (b) $\operatorname{grad}[(\boldsymbol{P}\cdot\boldsymbol{r})/r^3]$, (c) $(\boldsymbol{P}\cdot\nabla)\boldsymbol{r}$, (d) $\operatorname{div}(\boldsymbol{P}\times\boldsymbol{r})$, and (e) $\operatorname{curl}(\boldsymbol{r}\times\boldsymbol{P})$ where $\boldsymbol{P}$ is a constant vector.

(a) $$\operatorname{grad}(\boldsymbol{P}\cdot\boldsymbol{r})=\frac{\partial P_x x}{\partial x}\mathbf{i}+\frac{\partial P_y y}{\partial y}\mathbf{j}+\frac{\partial P_z z}{\partial z}\mathbf{k}=P_x\frac{\partial x}{\partial x}\mathbf{i}+P_y\frac{\partial y}{\partial y}\mathbf{j}+P_z\frac{\partial z}{\partial z}\mathbf{k}=\boldsymbol{P}$$

(b) We use the previous result to have $\operatorname{grad}[(\boldsymbol{P}\cdot\boldsymbol{r})/r^3]$, with (8) and (17).

$$\mathrm{grad}[(\boldsymbol{P}\cdot\boldsymbol{r})/r^3]=\frac{1}{r^3}\mathrm{grad}(\boldsymbol{P}\cdot\boldsymbol{r})+(\boldsymbol{P}\cdot\boldsymbol{r})\mathrm{grad}\frac{1}{r^3}$$

$$\mathrm{grad}[(\boldsymbol{P}\cdot\boldsymbol{r})/r^3]=\boldsymbol{P}/r^3-(\boldsymbol{P}\cdot\boldsymbol{r})(3\boldsymbol{r}/r^5)$$

(c) $$(\boldsymbol{P}\cdot\nabla)\boldsymbol{r}=(P_x\frac{\partial}{\partial x}+P_y\frac{\partial}{\partial y}+P_z\frac{\partial}{\partial z})\boldsymbol{r}=P_x\frac{\partial x}{\partial x}\mathbf{i}+P_y\frac{\partial y}{\partial y}\mathbf{j}+P_z\frac{\partial z}{\partial z}\mathbf{k}=\boldsymbol{P}$$

(d) To calculate $\mathrm{div}(\boldsymbol{P}\times\boldsymbol{r})$, we use (5) and consider that vector $\boldsymbol{P}$ is constant:

$$\mathrm{div}(\boldsymbol{P}\times\boldsymbol{r})=\boldsymbol{r}\cdot\mathrm{curl}\,\boldsymbol{P}-\boldsymbol{P}\cdot\mathrm{curl}\,\boldsymbol{r}=0$$

(e) For $\mathrm{curl}(\boldsymbol{r}\times\boldsymbol{P})$, let us apply the definition of cross product and curl. We find $\mathrm{curl}(\boldsymbol{r}\times\boldsymbol{P})=-2\boldsymbol{P}$.

Before continuing with problems in [3], let us remember the Gauss Law. Let us consider $\boldsymbol{A}(r)=\boldsymbol{r}/r^3$ and calculate $\mathrm{div}\,\boldsymbol{A}(r)$, using (4).

$$\mathrm{div}(\boldsymbol{r}/r^3)=\frac{1}{r^3}\,\mathrm{div}\,\boldsymbol{r}+\mathrm{grad}\frac{1}{r^3}\cdot\boldsymbol{r}=\frac{3}{r^3}-3\frac{\boldsymbol{r}\cdot\boldsymbol{r}}{r^5}=0$$

Actually, as we have seen, $\mathrm{grad}\frac{1}{r^3}=-3\frac{\boldsymbol{r}}{r^5}$. The field of Coulomb possesses a divergence which is zero in the space without the point where there is the charge.

4) Calculate (a) $\text{grad}(\boldsymbol{A}(r)\cdot\boldsymbol{B}(r))$, (b) $\text{div}(\varphi(r)\boldsymbol{A}(r))$, (c) $\text{curl}(\varphi(r)\boldsymbol{A}(r))$. Functions $\varphi(r), \boldsymbol{A}(r), \boldsymbol{B}(r)$ are depending only on the absolute value of radius vector $\boldsymbol{r}$.

(a) To calculate $\text{grad}(\boldsymbol{A}(r)\cdot\boldsymbol{B}(r))$, we can use the result of problem 1, that is $\text{grad} f(r)=\frac{df}{dr}\frac{\boldsymbol{r}}{r}$. We have:

$$\text{grad}(\boldsymbol{A}(r)\cdot\boldsymbol{B}(r))=\frac{d}{dr}(\boldsymbol{A}(r)\cdot\boldsymbol{B}(r))\,\frac{\boldsymbol{r}}{r}=(\frac{d\boldsymbol{A}}{dr}\cdot\boldsymbol{B}+\boldsymbol{A}\cdot\frac{d\boldsymbol{B}}{dr})\,\frac{\boldsymbol{r}}{r}$$

(b) To calculate $\text{div}(\varphi(r)\boldsymbol{A}(r))$ we use (4) , (17) and (18).

$$\text{div}(\varphi\boldsymbol{A})=\varphi\,\text{div}\,\boldsymbol{A}+\text{grad}\,\varphi\cdot\boldsymbol{A} \;=\; \frac{\varphi}{r}\,\boldsymbol{r}\cdot\frac{d\boldsymbol{A}}{dr} \;+\; \frac{1}{r}\frac{d\varphi}{dr}\,\boldsymbol{r}\cdot\boldsymbol{A}$$

(c) To have $\text{curl}(\varphi(r)\boldsymbol{A}(r))$, let us start from (6) and use definitions:

$$\text{curl}(\varphi\boldsymbol{A})=\text{grad}\,\varphi\times\boldsymbol{A}+\varphi\,\text{curl}\,\boldsymbol{A} \;=\; \frac{1}{r}\frac{d\varphi}{dr}\,\boldsymbol{r}\times\boldsymbol{A} \;+\; \frac{\varphi}{r}\,\boldsymbol{r}\times\frac{d\boldsymbol{A}}{dr}$$

5) Using **Ostrogradski theorem**, let us calculate $\boldsymbol{I}=\oiint\boldsymbol{r}(\boldsymbol{A}\cdot\boldsymbol{n})\,dS$ and $\boldsymbol{I}=\oiint(\boldsymbol{A}\cdot\boldsymbol{r})\,\boldsymbol{n}\,dS$, if V is the volume enclosed by the surface and $\boldsymbol{A}$ is a constant vector.

Let us consider an arbitrary constant vector $\boldsymbol{p}$:

$$\boldsymbol{p}\cdot\boldsymbol{I}=\boldsymbol{p}\cdot\oiint\boldsymbol{r}(\boldsymbol{A}\cdot\boldsymbol{n})\,dS=\oiint(\boldsymbol{p}\cdot\boldsymbol{r})(\boldsymbol{A}\cdot\boldsymbol{n})\,dS \;=\; \oiint(\boldsymbol{p}\cdot\boldsymbol{r}\,A_n)\,dS=\iiint\text{div}(\boldsymbol{p}\cdot\boldsymbol{r}\,\boldsymbol{A})\,dV$$

We can use relation $\text{div}(f\,\boldsymbol{A}) = f\,\text{div}\,\boldsymbol{A} + \text{grad} f \cdot \boldsymbol{A}$, that, in the case of a constant vector, becomes $\text{div}(f\,\boldsymbol{A}) = \text{grad} f \cdot \boldsymbol{A}$.

$$\iiint \text{div}(\boldsymbol{p}\cdot\boldsymbol{r}\,\boldsymbol{A})\,dV = \iiint \boldsymbol{A}\cdot\text{grad}(\boldsymbol{p}\cdot\boldsymbol{r})\,dV$$

We have $\text{grad}(\boldsymbol{p}\cdot\boldsymbol{r}) = \boldsymbol{p}$:

$$\iiint \boldsymbol{A}\cdot\text{grad}(\boldsymbol{p}\cdot\boldsymbol{r})\,dV = \iiint \boldsymbol{A}\cdot\boldsymbol{p}\;dV = (\boldsymbol{A}\cdot\boldsymbol{p})\;V$$

Since $\boldsymbol{p}$ is arbitrary: $\boldsymbol{I} = \boldsymbol{A}\,V$.

Now, let us calculate $\boldsymbol{I} = \oint\!\!\oint (\boldsymbol{A}\cdot\boldsymbol{r})\,\boldsymbol{n}\,dS$ in the same conditions as before.

$$\boldsymbol{p}\cdot\boldsymbol{I} = \boldsymbol{p}\cdot\oint\!\!\oint (\boldsymbol{A}\cdot\boldsymbol{r})\,\boldsymbol{n}\,dS = \oint\!\!\oint (\boldsymbol{p}\cdot\boldsymbol{n})(\boldsymbol{A}\cdot\boldsymbol{r})\,dS \;=\; \oint\!\!\oint (\boldsymbol{A}\cdot\boldsymbol{r}\,p_n)\,dS = \iiint \text{div}(\boldsymbol{A}\cdot\boldsymbol{r}\,\boldsymbol{p})\,dV$$

Let us use (4):

$$\iiint \text{div}(\boldsymbol{A}\cdot\boldsymbol{r}\,\boldsymbol{p})\,dV = \iiint \boldsymbol{p}\cdot\text{grad}(\boldsymbol{A}\cdot\boldsymbol{r})\,dV$$

We have that: $(\boldsymbol{A}\cdot\nabla)\boldsymbol{r} = \boldsymbol{A}$. Using (9):

$$\text{grad}(\boldsymbol{A}\cdot\boldsymbol{r}) = (\boldsymbol{A}\cdot\nabla)\boldsymbol{r} + (\boldsymbol{r}\cdot\nabla)\boldsymbol{A} + [\boldsymbol{A}\times\text{curl}\,\boldsymbol{r}] + [\boldsymbol{r}\times\text{curl}\,\boldsymbol{A}] = \boldsymbol{A} \quad .$$

$$\boldsymbol{p}\cdot\boldsymbol{I} \;=\; \iiint \boldsymbol{p}\cdot\text{grad}(\boldsymbol{A}\cdot\boldsymbol{r})\,dV = \iiint \boldsymbol{p}\cdot\boldsymbol{A}\;dV$$

And we can continue as in the previous calculation.

6) Show that $\iiint \boldsymbol{A}\, dV = 0$ if, inside the volume, it is $\operatorname{div} \boldsymbol{A} = 0$ and on the boundary $A_n = 0$.

As we did before, let us use an arbitrary constant vector $\boldsymbol{p}$. Moreover, let us assume $\boldsymbol{p} = \operatorname{grad} \varphi$.

$$\boldsymbol{p} \cdot \boldsymbol{I} = \iiint \boldsymbol{p} \cdot \boldsymbol{A}\, dV$$

We have already seen that: $\boldsymbol{p} \cdot \boldsymbol{A} = \operatorname{grad} \varphi \cdot \boldsymbol{A} = \operatorname{div} \varphi \boldsymbol{A}$.

$$\boldsymbol{p} \cdot \boldsymbol{I} = \iiint \boldsymbol{p} \cdot \boldsymbol{A}\, dV = \iiint \operatorname{div} \varphi \boldsymbol{A}\, dV = \oiint \varphi A_n dS$$

On the boundary $A_n = 0$ and being $\boldsymbol{p} = \operatorname{grad} \varphi$ arbitrary, we have $\iiint \boldsymbol{A}\, dV = 0$.

Field having a divergence which is zero, $\operatorname{div} \boldsymbol{A} = 0$, can be electric or magnetic fields. An interesting case is discussed in [2], where we have magnetic and electric fields as in the case of the problem here considered.

7) Show that the divergence of the following vector is zero:

$$\boldsymbol{A} + \frac{1}{4\pi} \operatorname{grad} \iiint \frac{\operatorname{div} \boldsymbol{A}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|}\, dV' \quad (20)$$

Let us assume $\operatorname{div} \boldsymbol{A} = \rho(\boldsymbol{r})$. In this manner, integral takes a familiar form, that of a potential of a charge distribution (besides the dielectric constant of course):

$$\varphi=\frac{1}{4\pi}\iiint\frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,dV'$$

Let us use a negative sign: $\varphi=-\frac{1}{4\pi}\iiint\frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,dV'$. This function is the solution of Poisson equation: $\Delta\varphi=\operatorname{div}\operatorname{grad}\varphi=0$. Immediately, we have that the divergence of (20) must be zero.

A vector calculus identity

Let us remember:

$$\boldsymbol{A}\cdot\boldsymbol{B}=(A_i\boldsymbol{e_i})\cdot(B_j\boldsymbol{e_j})=A_iB_j(\boldsymbol{e_i}\cdot\boldsymbol{e_j})=A_iB_j\delta_{ij}=A_iB_i$$
$$\boldsymbol{A}\times\boldsymbol{B}=(A_i\boldsymbol{e_i})\times(B_j\boldsymbol{e_j})=A_iB_j(\boldsymbol{e_i}\times\boldsymbol{e_j})=\boldsymbol{e_k}[\epsilon_{klm}A_lB_m]$$

ϵ_{klm} is the Levi-Civita tensor.

Using the relation: $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})=\boldsymbol{B}(\boldsymbol{A}\cdot\boldsymbol{C})-\boldsymbol{C}(\boldsymbol{A}\cdot\boldsymbol{B})$, and the Cartesian components, show that:

$$\begin{aligned}\nabla\times(\mathbf{A}\times\mathbf{B}) &= \mathbf{A}(\nabla\cdot\mathbf{B})-\mathbf{B}(\nabla\cdot\mathbf{A})+(\mathbf{B}\cdot\nabla)\mathbf{A}-(\mathbf{A}\cdot\nabla)\mathbf{B}\\ &= (\nabla\cdot\mathbf{B}+\mathbf{B}\cdot\nabla)\mathbf{A}-(\nabla\cdot\mathbf{A}+\mathbf{A}\cdot\nabla)\mathbf{B}\end{aligned}$$

$$\nabla\times(\boldsymbol{A}\times\boldsymbol{B})=\boldsymbol{e_i}\times\partial_i(A_j\boldsymbol{e_j}\times B_k\boldsymbol{e_k})$$
$$=\ \partial_i(A_jB_k)\ \boldsymbol{e_i}\times(\boldsymbol{e_j}\times\boldsymbol{e_k})$$
$$=\ [B_k\partial_iA_j+A_j\partial_iB_k]((\boldsymbol{e_i}\cdot\boldsymbol{e_k})\boldsymbol{e_j}-(\boldsymbol{e_i}\cdot\boldsymbol{e_j})\boldsymbol{e_k})$$
$$=\ [B_k\partial_iA_j+A_j\partial_iB_k](\delta_{ik}\boldsymbol{e_j}-\delta_{ij}\boldsymbol{e_k})$$
$$=\ [B_k\partial_iA_j+A_j\partial_iB_k](\delta_{ik}\boldsymbol{e_j}-\delta_{ij}\boldsymbol{e_k})$$

$$= B_i \partial_i A_j \boldsymbol{e_j} - B_k \partial_i A_i \boldsymbol{e_k} + A_j \partial_i B_i \boldsymbol{e_j} - A_i \partial_i B_k \boldsymbol{e_k}$$

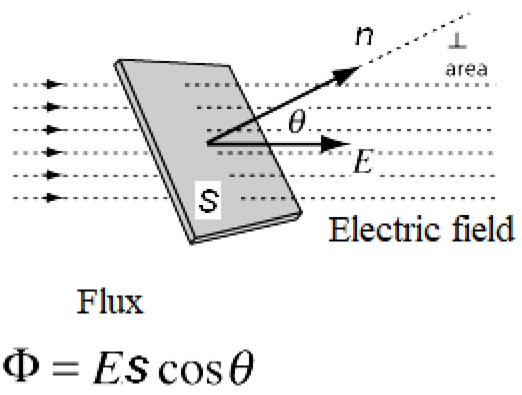
$$\text{curl}[\boldsymbol{A}\times\boldsymbol{B}] = (\boldsymbol{B}\cdot\nabla)\boldsymbol{A} - (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} - (\text{div}\,\boldsymbol{A})\boldsymbol{B} + (\text{div}\,\boldsymbol{B})\boldsymbol{A}$$

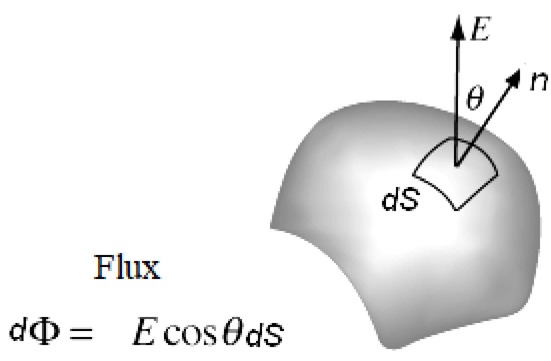
Gauss Law

Let us continue with problems on electrostatics from [3]. We will find problems about the calculation of an electric field by means of Gauss Law. Let us remember that the flux of an electric uniform field $\boldsymbol{E}$ through a plane surface $\boldsymbol{S}$ is:

$$\Phi_S(\boldsymbol{E}) = \boldsymbol{E}\cdot\boldsymbol{S}$$

The surface vector $\boldsymbol{S}$ has a direction perpendicular to the surface, so that $\boldsymbol{S} = S\,\boldsymbol{u}_n$. $\boldsymbol{u}_n$ is the unit vector (dimensionless) perpendicular to the surface. If the surface is not a planar one, we have to consider an element $d\boldsymbol{S} = dS\,\boldsymbol{u}_n$.





If we have a generic surface, we will subdivide it into small elements, that we can approximate as small planar elements, where locally the electric field $\boldsymbol{E}$ is constant.

We can calculate the flux through each element and add the results. Therefore:

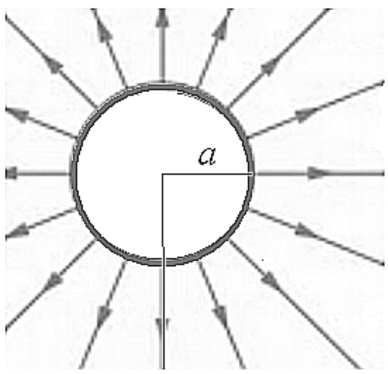
$$\Phi_S(\boldsymbol{E}) = \int d\Phi = \int \boldsymbol{E} \cdot d\boldsymbol{S} = \int \boldsymbol{E} \cdot \boldsymbol{u}_n \, dS = \int E \cos\theta \, dS$$

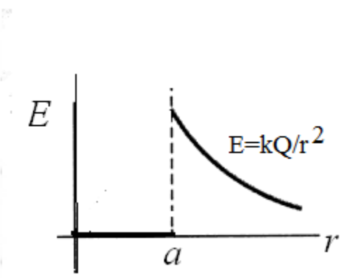
Gauss Law tells that the flux of the electric field through a closed surface *S* is equal to the algebraic sum of the charges Q_i inside surface *S*, divided by ε_o :

$$\Phi_S(\boldsymbol{E}) = \frac{Q_{\text{int}}}{\varepsilon_o}$$

Near the surface of a conductor, using the Gauss Law, we can see that $E = \sigma / \varepsilon_o$, where σ is the charge surface density.

8) Draw the lines of the field created by a positive charge uniformly distributed on a hollow sphere having radius *a*.





This problem possesses a spherical symmetry, since the charge is uniformly distributed on the surface of the sphere. On each spherical surface, which is containing the sphere, and having the same centre, the electric field is perpendicular and has a constant modulus which is decreasing as the radius increases. The electric field is equal to that produced by the charge as if it were concentrated in the centre of the hollow sphere. Inside the hollow sphere, and due to the symmetry, the field is zero.

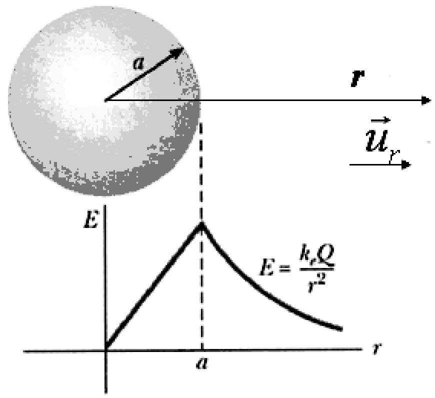
Let us now consider a solid sphere having a charge +Q, uniformly distributed in its volume, with radius a . Also in this case, the electric field possesses a spherical symmetry. The field is given as in the following figure.

$\boldsymbol{Q}$ total charge

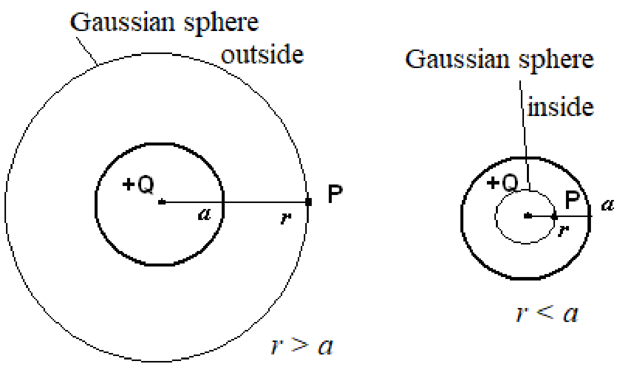
$\boldsymbol{a}$ radius

$$r \geq a \qquad \vec{E} = \frac{Q}{4\pi\varepsilon_o r^2}\vec{u}_r$$

$$r \leq a \qquad \vec{E} = \frac{Q\,r}{4\pi\varepsilon_o a^3}\vec{u}_r$$



$\vec{u}_r$ is the unit vector in the radial direction, pointing outside. To have the field given above, we can use the Gauss Law, applying it to two different virtual Gaussian surfaces, inside and outside the solid sphere.



If we are inside the solid sphere, only the charges inside the virtual Gaussian surface that we are considering will contribute to the electric field. If we are outside the solid sphere, all charge Q will contribute to the field.

9) Find the electric field intensity vector inside and outside a uniformly charged sphere of radius *R*. The body charge density of the sphere is ρ [3].

Let us apply the Gauss Law, using the virtual Gaussian surfaces given above.

If $r<R$. we have: $$E\epsilon_o 4\pi r^2 = Q' = \frac{4}{3}\pi\rho r^3$$

$$\boldsymbol{E} = \frac{1}{3\epsilon_o}\rho\boldsymbol{r}$$

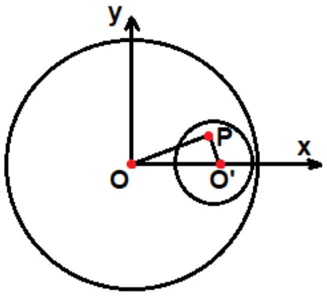
If $r>R$. we have:

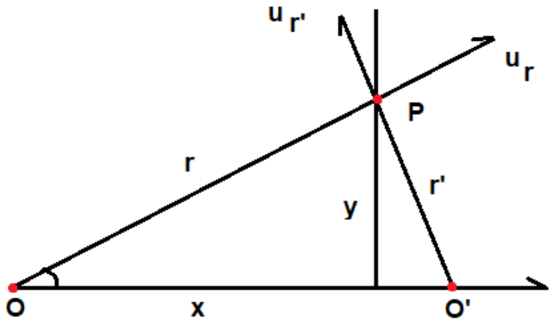
$$E\epsilon_o 4\pi r^2 = Q = \frac{4}{3}\pi\rho R^3$$

$$\boldsymbol{E} = \frac{1}{3\epsilon_o}\rho\frac{R^3}{r^3}\boldsymbol{r}$$

10) A sphere of radius *R* is uniformly charged with a charge density ρ . Inside it, there is a spherical cavity of radius *R*', whose centre is at a distance *a* from the centre of the sphere (in the figure |OO'|=*a*, |OP|=*r*, |O'P|=*r'*). Find the electric field intensity vector inside the cavity and inside and outside the sphere.

In the following figure $\boldsymbol{u}_r$ and $\boldsymbol{u}_{r'}$ are unit vectors [3].





Let us consider the field inside the cavity. From the superposition principle for fields it follows that the field in P is that created by the solid sphere having a charge uniformly distributed in it with body density ρ minus the field that we would have if the cavity had been filled by charges having the same density ρ .

Therefore: $\boldsymbol{E}=\boldsymbol{E_O}-\boldsymbol{E_{O'}}$

Using the figure given above:

$$\boldsymbol{E}=\frac{\rho}{3\epsilon_o}r\,\boldsymbol{u_r}-\frac{\rho}{3\epsilon_o}r'\,\boldsymbol{u_{r'}}$$

Then: $E_x=\frac{\rho}{3\epsilon_o}r\frac{x}{r}-\frac{\rho}{3\epsilon_o}r'\frac{(x-a)}{r'}=\frac{\rho}{3\epsilon_o}a$; $E_y=\frac{\rho}{3\epsilon_o}r\frac{y}{r}-\frac{\rho}{3\epsilon_o}r'\frac{y}{r'}=0$

The electric field inside the cavity is uniform: $\boldsymbol{E}=\frac{\rho}{3\epsilon_o}\boldsymbol{a}$. where $\boldsymbol{a}$ is the vector from O to O'.

Inside the sphere but outside the cavity: $\boldsymbol{E}=\frac{\rho}{3\epsilon_o}\,\boldsymbol{r}-\frac{\rho}{3\epsilon_o}\frac{R'^3}{|\boldsymbol{r}-\boldsymbol{a}|^3}(\boldsymbol{r}-\boldsymbol{a})$

Outside the sphere of radius R: $\boldsymbol{E}=\frac{\rho}{3\epsilon_o}\frac{R^3}{r^3}\,\boldsymbol{r}-\frac{\rho}{3\epsilon_o}\frac{R'^3}{|\boldsymbol{r}-\boldsymbol{a}|^3}(\boldsymbol{r}-\boldsymbol{a})$

11) Find the electric field vector inside and outside a sphere having a body density of charges varying as follow: $\rho=\alpha r^n$ con $n>-2$. The radius of the sphere is R [3].

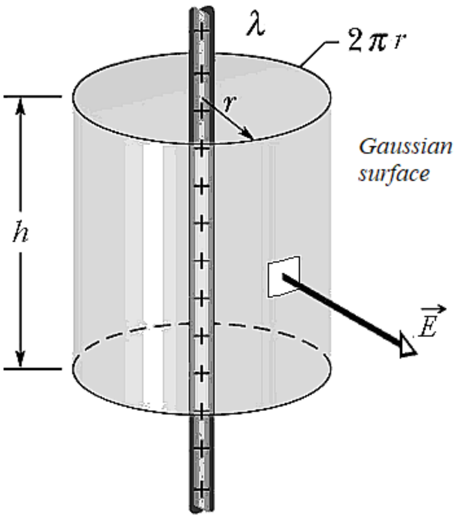
If $r<R$: $$E\epsilon_o 4\pi r^2=Q'=4\pi\int \alpha r^{n+2}dr$$

then: $$\boldsymbol{E}=\frac{\alpha}{\epsilon_o}\frac{r^n}{n+3}\boldsymbol{r}$$

If $r>R$, then: $$\boldsymbol{E}=\frac{\alpha}{\epsilon_o(n+3)}\frac{R^{n+3}}{r^3}\boldsymbol{r}$$

12) Find the electric field vector of an infinity straight line uniformly charged.

The problem has a cylindrical symmetry. Let us use the Gauss Law. The virtual Gaussian surface is a cylinder coaxial with the charged line. Due to the symmetry of the field, we have no flux through the bases of the Gaussian (virtual) surface.



We have:

$$\Phi=\sum \boldsymbol{E}\cdot\boldsymbol{n}\ \Delta S=E\sum \Delta S=E\ 2\pi r\ h\ .$$

Then: $\Phi=\frac{Q}{\varepsilon_o}=\frac{\lambda h}{\varepsilon_o}$., with $\lambda=Q/h$.

Therefore the field is: $E=\frac{\lambda}{2\pi\varepsilon_o r}$.

13) Find the electric field vector inside and outside a uniformly charged solid cylinder of radius R. The charge per unit length of the cylinder is λ [3].

The charge density is given by (*h* is the length of cylinder): $\rho=\frac{\lambda h}{\pi R^2 h}$

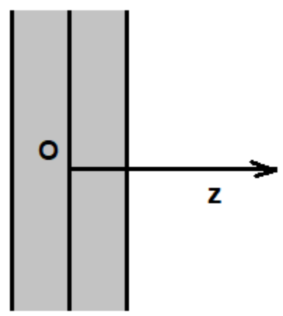
If $r<R$: $E\epsilon_o 2\pi r h=Q'=\rho\pi r^2 h=\frac{\lambda h}{\pi R^2 h}\ \pi r^2 h=\lambda h\ \frac{r^2}{R^2}$

Then: $\boldsymbol{E}=\frac{\lambda}{2\pi\epsilon_o R^2}\ \boldsymbol{r}$

If $r>R$: $E\epsilon_o 2\pi r h=Q=\lambda h$

Then: $\boldsymbol{E}=\frac{\lambda}{2\pi\epsilon_o r^2}\ \boldsymbol{r}$.

14) A layer of non-conducting material is put between two parallel planes. The thickness of the layer is d . The material is charged to a density ρ . Find the electric field vector inside and outside the layer [3].



Inside the material. Let us consider as Gaussian (virtual) surface a cylinder with radius a and height $2z$, having its axis parallel to z, and symmetric with respect to the plane in O perpendicular to z :

$$E\epsilon_o 2\pi a^2 = Q' = 2\pi a^2 z\rho$$

then:

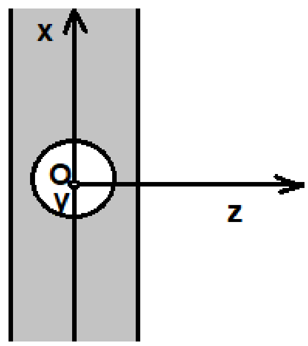
$$\mathbf{E} = \frac{\rho}{\epsilon_o}\mathbf{z}$$

Outside the material:

$$E\epsilon_o 2\pi a^2 = Q = \pi a^2 d\rho$$

$$\mathbf{E} = \frac{\rho d}{2\epsilon_o}\frac{\mathbf{z}}{|z|}$$

15) Let us consider the layer of the previous problem. Inside the layer there is a spherical cavity of radius *R*, smaller than *d/2*. The cavity has its centre in O. Evaluate the electric field vector inside and outside the layer. Use the frame of reference given in the following figure.



From the superposition principle for fields it follows that the field in a given point P is that created by the layer having a charge uniformly distributed in it with body density

ρ minus the field that we would have if the cavity had been filled by charges having the same density ρ . The field created by the layer is given in **7)**, that create by a sphere in **2)**. Using the solution given in **2)**, let us write the radius vector $\boldsymbol{r}$ as $x\boldsymbol{i}+y\boldsymbol{i}+z\boldsymbol{k}$ where $\boldsymbol{i},\boldsymbol{i},\boldsymbol{k}$ are the unit vectors of the frame of reference shown in the figure.

16) Find the capacitance for the following capacitors: spherical, plane-parallel, cylindrical. [3].

Spherical (R_1, R_2 are the radiuses of spheres, with $R_1 < R_2$):

$$C=\frac{4\pi\epsilon R_1 R_2}{R_2-R_1}$$

Plane-parallel (S surface of layer and d thickness): $C=\frac{\epsilon S}{d}$

Cylindrical (l length of capacitor): $C=\frac{4\pi\epsilon l}{\ln(R_2/R_1)}$

In the expressions given above, ϵ is the permittivity of the medium.

17) A plane surface has an area of 3.2 m^2 . Let us turn the surface in a uniform electric field of modulus $E = 6.2 \times 10^5$ *N/C*. Find the flux through the surface, a) when the field is perpendicular, b) when the field is parallel to the surface.

This problem allows us to discuss dimensions and units of measurement.

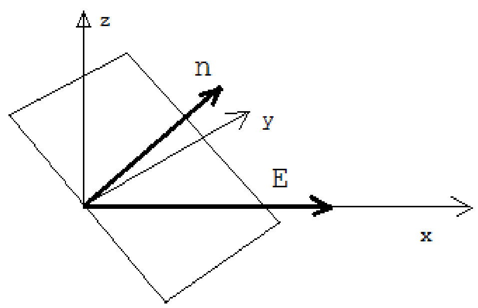
$$[\Phi_S(E)]=[\boldsymbol{E}\cdot\boldsymbol{S}]=[E][S]=[force/charge][L^2]=[energy/charge][L]$$

[L] means "length". Units of measurement are $Volt\cdot meter$.
In the case (a):

$$\Phi_S(E)=6.2\times10^5\,N/C\cdot3.2m^2=1.98\times10^6\;V\cdot m$$

In the case (b), the flux is zero.

18) An electric field vector, modulus 3.50 *kN/C,* is applied along the x-axis. Find the flux through a rectangular surface having sides of lengths 0.350 *m* and 0.700 *m* respectively, in the cases (a) the rectangular surface is parallel to the plane *yz*, and (b) the surface contains the *y*- axis, and its normal forms an angle of 40.0° with *x*- axis.



19) The electric field close to the surface of a hollow sphere having radius 0.750 m is equal to 890 N/C , directed outside the sphere. The field has a spherical symmetry. (a) What is the total charge inside the sphere? (b) What is possible to conclude about the nature and the distribution of the charges?

We can apply the Gauss Law and evaluate the total charge. However we are not able to tell how this total charge is distributed inside or on the surface of the sphere.

20) A point-like charge of $+5.00\ \mu$C is put at the centre of a sphere having radius of 12.0 cm. Find the flux of the electric field vector through the surface of the sphere.

The flux is the charge dividend by the permittivity of the empty space.

Electric field, charge distribution and potential

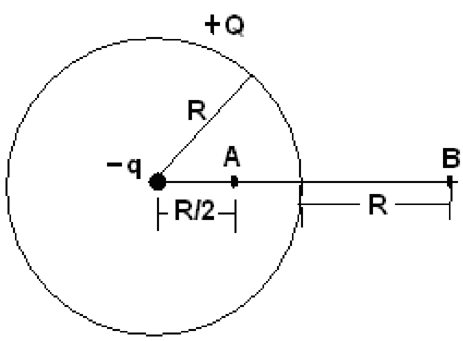
21) A small dielectric sphere is put inside a hollow dielectric sphere having radius R. The two bodies have the same centre. The sphere possesses a negative charge q, the hollow sphere a positive charge Q, uniformly distributed on it. Let us assume that the absolute value of q is greater than that of Q.

a) Find the lines of the field.

b) Find the modulus of the electric field in A, with a distance $R/2$ from q, and in B, with a distance $2R$ from q.

c) Find the difference of potential $V(A)-V(B)$ **.**

Here a figure to show the geometry of the problem.



Let us remember that the field created by the charge on the hollow sphere is zero inside it and outside is a Coulomb field. From the superposition principle for fields it follows that the field in a given point P of the space is that created by the hollow sphere having *Q* charge uniformly distributed in it and by the *q* charge of the small sphere. Let us consider this last charge as produced by a point-like charge.

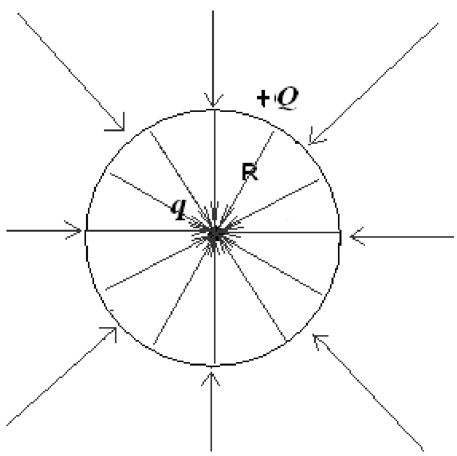
In *A*, the modulus of the field is: $E=\frac{1}{4\pi\varepsilon_o}\frac{q}{(R/2)^2}$,

with radial direction towards the centre of the hollow sphere.

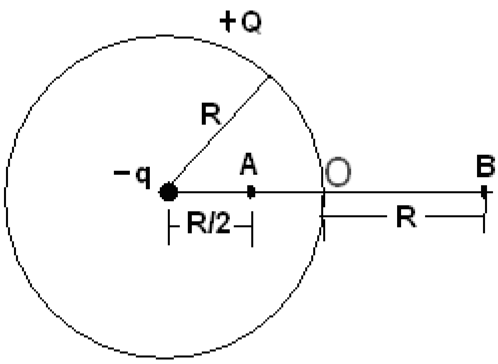
In *B*: $E=\frac{1}{4\pi\varepsilon_o}\frac{q}{(2R)^2}-\frac{1}{4\pi\varepsilon_o}\frac{Q}{(2R)^2}$,

with radial direction toward the centre of the sphere.

The following figure shows the lines of the field. The density of the lines is greater inside the hollow sphere. Fields are different: this is due to the fact that we have a surface distribution of charge between the two regions of the space, inside and outside the surface at radius R.



Let us calculate the difference of potential, remembering that is defined as $V(P_1)-V(P_2)=\int_{P_1}^{P_2} \boldsymbol{E}\cdot d\boldsymbol{l}$. Let us use the following figure.

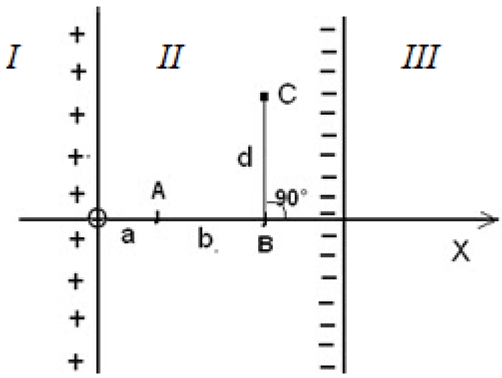


$$V(A)-V(B)=\int_A^B \boldsymbol{E}\cdot d\boldsymbol{l}=\int_A^O \boldsymbol{E}\cdot d\boldsymbol{l}+\int_O^B \boldsymbol{E}\cdot d\boldsymbol{l}=\int_A^O \boldsymbol{E}_{in}\cdot d\boldsymbol{l}+\int_O^B \boldsymbol{E}_{out}\cdot d\boldsymbol{l}=$$

$$=\int_A^O \frac{-q}{4\pi\varepsilon_o r^2}dr+\int_O^B \frac{Q-q}{4\pi\varepsilon_o r^2}dr=-\frac{-q}{4\pi\varepsilon_o}\left(\frac{1}{R}-\frac{1}{R/2}\right)-\frac{Q-q}{4\pi\varepsilon_o}\left(\frac{1}{2R}-\frac{1}{R}\right)=$$

$$=\frac{|q|}{4\pi\varepsilon_o}\left(\frac{1}{R}-\frac{1}{R/2}\right)+\frac{|q|-|Q|}{4\pi\varepsilon_o}\left(\frac{1}{2R}-\frac{1}{R}\right)$$

22) Two planes with charges uniformly distributed on them are parallel. The distance between the planes is è D. The plane shown in the figure is perpendicular to the charged planes. Find the electric field, if the charge densities are +σ and −2σ. Find also the difference of potential between A and B and between A and C.



We use the superposition principle and calculate $\boldsymbol{E}=\boldsymbol{E}_1+\boldsymbol{E}_2$, where 1 indicates the field created by the plane positively charged and 2 the field created by the plane negatively charged.

In region I:

$$\boldsymbol{E}=-E_1\boldsymbol{u}_x+E_2\boldsymbol{u}_x=-\frac{\sigma}{2\varepsilon_o}\boldsymbol{u}_x+\frac{2\sigma}{2\varepsilon_o}\boldsymbol{u}_x$$

In region II:

$$\boldsymbol{E}=E_1\boldsymbol{u}_x+E_2\boldsymbol{u}_x=\frac{\sigma}{2\varepsilon_o}\boldsymbol{u}_x+\frac{2\sigma}{2\varepsilon_o}\boldsymbol{u}_x$$

In region III:

$$\boldsymbol{E}=E_1\boldsymbol{u}_x-E_2\boldsymbol{u}_x=\frac{\sigma}{2\varepsilon_o}\boldsymbol{u}_x-\frac{2\sigma}{2\varepsilon_o}\boldsymbol{u}_x$$

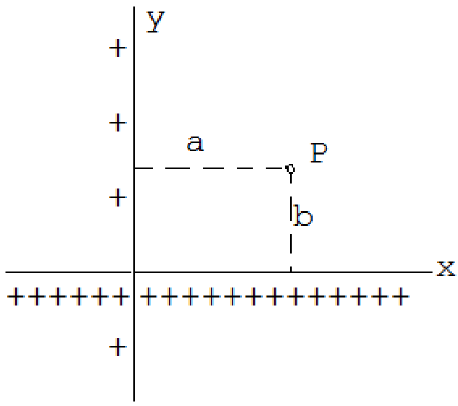
The difference of potential is defined as: $V(A)-V(B)=\int_A^B \boldsymbol{E}\cdot d\boldsymbol{l}$

$$V(A)-V(B)=\int_A^B\left(\frac{\sigma}{2\varepsilon_o}\boldsymbol{u}_x+\frac{2\sigma}{2\varepsilon_o}\boldsymbol{u}_x\right)\cdot\boldsymbol{u}_x dx=\frac{3\sigma}{2\varepsilon_o}b$$

Moreover: $V(A)-V(C)=\int_A^B \boldsymbol{E}\cdot d\boldsymbol{l}+\int_B^C \boldsymbol{E}\cdot d\boldsymbol{l}=V(A)-V(B)$.

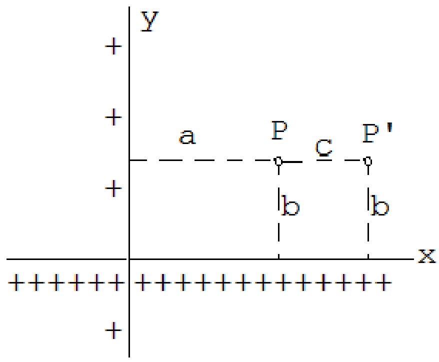
The second integral is equal to zero, because displacement is perpendicular to the field.

23) Two uniformly charged straight wires are coincident to the x- and y- axes. The wires are made of a material which does not allow the motion of charges. Therefore, the distributions of charges are not affecting each other. Neglecting any polarization effect, find the electric field in the point P given in the figure. Suppose P in the plane x,y.



The field is: $$\boldsymbol{E}=\boldsymbol{E}_1+\boldsymbol{E}_2=\frac{\lambda_y}{2\pi\varepsilon_o a}\,\boldsymbol{u}_x+\frac{\lambda_x}{2\pi\varepsilon_o b}\,\boldsymbol{u}_y$$

24) The wires are the same as before. Find the difference of potential between P and P' as given in the figure, in the plane x,y.



By definition:

$$V(P)-V(P')=\int_P^{P'} \boldsymbol{E}\cdot d\boldsymbol{l}=\int_P^{P'} \boldsymbol{E}\cdot\boldsymbol{u}_x dx$$

where $\boldsymbol{E}=\boldsymbol{E}_1+\boldsymbol{E}_2=\frac{\lambda_y}{2\pi\varepsilon_o a}\boldsymbol{u}_x+\frac{\lambda_x}{2\pi\varepsilon_o b}\boldsymbol{u}_y$.

$$V(P)-V(P')=\int_P^{P'} \boldsymbol{E}\cdot d\boldsymbol{l}=\int_a^{a+c} \frac{\lambda_y}{2\pi\varepsilon_o x}\ \boldsymbol{u}_x\cdot\boldsymbol{u}_x dx$$

Therefore: $$V(P)-V(P')=\int_a^{a+c} \frac{\lambda_y}{2\pi\varepsilon_o x}dx=\frac{\lambda_y}{2\pi\varepsilon_o}\ \ln\frac{a+c}{a}$$

25) Two long cylindrical conductors are arranged parallel to each other at a distance *d*. Calculate the capacitance per unit length of the system, provided $d \gg R_1$ and $d \gg R_2$, where $R_{1,} R_2$ are radius of the two cylinders [3].

Let us assume that the charge per unit length of the first conductor is λ and for the second $-\lambda$. The potential of each conductor is composed by the potential φ_1

created by the conductor's charges and the potential φ_2 created by the charges of the other conductor. This second term can be considered to be the same at every point of the conductor being the distance between conductors large.

For the first conductor:

$$\varphi_{first}=-\frac{\lambda}{2\pi\varepsilon_o}\ln R_1+\frac{\lambda}{2\pi\varepsilon_o}\ln d$$

For the second conductor:

$$\varphi_{second}=\frac{\lambda}{2\pi\varepsilon_o}\ln R_2-\frac{\lambda}{2\pi\varepsilon_o}\ln d$$

The capacitance for unit length of this system is:

$$C=\frac{\lambda}{\varphi_{first}-\varphi_{second}}=\frac{2\pi\varepsilon_o}{\ln\frac{d^2}{R_1 R_2}}=\pi\varepsilon_o\left(\ln\frac{d}{R}\right)^{-1}$$

26) Find the potential and the electric field intensity vector on the axis of a flat ring that has a surface charge density σ (the inner radius is R_1 and the outer radius R_2). Consider the following limiting (a) the field of a flat disk ($R_1\rightarrow 0$) and (b) the field of a plane ($R_1\rightarrow 0, R_2\rightarrow\infty$). [3]

Let us call *z* the axis of the flat ring.

$$\varphi=\int_0^{2\pi} d\phi \int_{R_1}^{R_2}\frac{\sigma\, r\, dr}{4\pi\varepsilon_o\sqrt{r^2+z^2}}=\frac{\sigma}{2\varepsilon_o}\left(\sqrt{R_2^2+z^2}-\sqrt{R_1^2+z^2}\right)$$

The field is:

$$E_x=E_y=0 \quad ; \quad E_z=\frac{\sigma}{2\varepsilon_o}\left(\frac{z}{\sqrt{R_2^2+z^2}}-\frac{z}{\sqrt{R_1^2+z^2}}\right)$$

In the limiting cases:

$$E_z=\frac{\sigma}{2\varepsilon_o}\left(\frac{z}{|z|}-\frac{z}{\sqrt{R_2^2+z^2}}\right) \quad ; \quad E_z=\frac{\sigma}{2\varepsilon_o}\frac{z}{|z|}$$

27) Evaluate the following integral when ρ is a constant, and the variable *r* is ranging from 0 to *R*.

$$\varphi=\frac{1}{4\pi\varepsilon_o}\int\frac{\rho}{d(r)}d\tau \quad ; \quad d\tau=r^2\sin\theta\, dr\, d\theta\, d\phi \quad ; \quad d(r)=\sqrt{z^2+r^2-2rz\cos\theta}$$

$$\varphi=\frac{\rho}{4\pi\varepsilon_o}\int\frac{r^2\sin\theta\, dr\, d\theta\, d\phi}{\sqrt{z^2+r^2-2rz\cos\theta}}$$

$$\int_0^{2\pi}d\phi=2\pi \quad ; \quad \int_0^{\pi}\frac{\sin\theta\, d\theta}{\sqrt{z^2+r^2-2rz\cos\theta}}=\frac{1}{rz}\left(\sqrt{r^2+z^2-2rz\cos\theta}\right)_0^{\pi}$$

$$\frac{1}{rz}\left(\sqrt{r^2+z^2+2rz}-\sqrt{r^2+z^2-2rz}\right)=\frac{1}{rz}(r+z-|r-z|)$$

If $r<z$ the integral is equal to $2/z$, if $r>z$ it is equal to $2/r$.

$$\varphi=\frac{\rho}{4\pi\varepsilon_o}4\pi\left(\int_0^z\frac{1}{z}r^2dr+\int_z^R\frac{1}{r}r^2dr\right) = \frac{\rho}{\varepsilon_o}\left(\frac{z^2}{3}+\frac{R^2-z^2}{2}\right)=\frac{\rho}{2\varepsilon_o}\left(r^2-\frac{z^2}{3}\right)$$

28) Determine the potential of the electric field produced by the electron of a hydrogen atom, assuming that the electric charge in the ground state is distributed with a charge density:

$$\rho = \frac{e}{\pi a^3} \exp(-2r/a) \quad ,$$

where a is a constant [3]

The problem possesses a spherical symmetry. Let us consider the equation $\Delta\varphi = -\frac{\rho}{\varepsilon_o}$. The solution is:

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}' \quad .$$

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o} \int_0^\infty r'^2 dr' \int_0^\pi \sin\theta' d\theta' \int_0^{2\pi} d\phi \left(\frac{\rho(r')}{(r^2 + r'^2 - 2rr'\cos\theta')^{1/2}} \right)$$

The integration on the radius can be performed, as in the example given before, by splitting the integral into two parts:

$$\varphi(r) = \frac{1}{\varepsilon_o r} \int_0^r \rho(r') r'^2 dr' + \frac{1}{\varepsilon_o} \int_r^\infty \rho(r') r' dr'$$

$$\varphi(r) = \frac{e}{4\pi\varepsilon_o r} [1 - \exp(-2r/a)] - \frac{e}{4\pi\varepsilon_o a} \exp(-2r/a)$$

29) Determine the Fourier transform for the potential of a point charge [3].

The potential of a point charge is given by the solution of equation:

$$\Delta\varphi=-\frac{e}{\varepsilon_o}\delta(\boldsymbol{r})$$

Let us represent $\varphi(\boldsymbol{r})$ and $\delta(\boldsymbol{r})$ in the form of Fourier expansions:

$$\varphi(\boldsymbol{r})=\int\varphi(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{r}}d\boldsymbol{k} \quad ; \quad \delta(\boldsymbol{r})=\frac{1}{(2\pi)^3}\int e^{i\boldsymbol{k}\cdot\boldsymbol{r}}d\boldsymbol{k}$$

$$\Delta\int\varphi(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{r}}d\boldsymbol{k}=-\frac{e}{\varepsilon_o(2\pi)^3}\int e^{i\boldsymbol{k}\cdot\boldsymbol{r}}d\boldsymbol{k}$$

$$-k^2\varphi(\boldsymbol{k})=-\frac{e}{\varepsilon_o(2\pi)^3} \rightarrow \varphi(\boldsymbol{k})=\frac{e}{\varepsilon_o(2\pi)^3k^2}$$

30) Find the potential of the electric field produced by a charge that is distributed in an infinite medium by the law $\rho = \rho_o \sin ax \sin by \sin cz$ [3].

$$\Delta\varphi=-\frac{\rho(\boldsymbol{r})}{\varepsilon_o} \quad ; \quad \frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}=-\frac{\rho_o}{\varepsilon_o}\sin ax \sin by \sin cz$$

$$\varphi=-\frac{\rho_o}{\varepsilon_o(a^2+b^2+c^2)}\sin ax \sin by \sin cz$$

Equation of the lines of the field

To represent a vector field we can use arrows. However it is possible to use curves called the "field lines". Let us discuss the two-dimensional case. A curve $y=y(x)$ is a field line for vector field $\boldsymbol{F}(x,y)$ if at each point (x_o,y_o) on the curve,

$\boldsymbol{F}(x_o, y_o)$ is tangent to the curve.

The field lines $y=y(x)$ of the field $\boldsymbol{F}(x,y)=P(x,y)\boldsymbol{i}+Q(x,y)\boldsymbol{j}$ are the solution of the differential equation:

$$\frac{dx}{P}=\frac{dy}{Q}$$

If $y=y(x)$ is a field line, we have that for every *x*, the field is collinear to the derivative of $(x,y(x))$, which is equal to $(1,y'(x))$:

$$(P(x,y),Q(x,y(x)))=\alpha(x)(1,y'(x))$$

Then:

$$P(x,y(x))=\alpha(x) \quad ; \quad Q(x,y(x))=\alpha(x)y'(x)=P(x,y(x)) \;\; y'(x)$$

$$\frac{dy}{dx}=y'(x)=\frac{Q(x,y(x))}{P(x,y(x))}$$

Complex variables

Many problems in electrostatics can be solved through the use of functions of a complex variable. Let us use the real and the imaginary parts of an analytic function $W(\zeta)=\varphi+i\psi$ of a complex variable $\zeta=x+iy$, where we find the solutions of Laplace equations $\Delta\varphi=0$ and $\Delta\psi=0$. Besides, φ and ψ are restricted by the condition:

$$\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial y}=0$$

In a two-dimensional problem, we have $\boldsymbol{E}=-\text{grad}\,\varphi$. In the free space $\text{div}\,\boldsymbol{E}=0$. Suppose to have a function ψ so that:

$$E_x=-\frac{\partial\psi}{\partial y} \quad ; \quad E_y=\frac{\partial\psi}{\partial x}$$

$$\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}=-\frac{\partial^2\psi}{\partial x\partial y}+\frac{\partial^2\psi}{\partial y\partial x}=0$$

$$\boldsymbol{E}=E_x\boldsymbol{i}+E_y\boldsymbol{j}=-\frac{\partial\psi}{\partial y}\boldsymbol{i}+\frac{\partial\psi}{\partial x}\boldsymbol{j}=\boldsymbol{k}\times\text{grad}\,\psi=-\text{grad}\,\psi\times\boldsymbol{k}$$

The functions φ and ψ satisfy the Cauchy-Riemann conditions:

$$\frac{\partial\psi}{\partial y}=\frac{\partial\varphi}{\partial x} \quad ; \quad -\frac{\partial\psi}{\partial x}=\frac{\partial\varphi}{\partial y}$$

$$\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial x}=-\frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial y}=-\frac{\partial\psi}{\partial y}\frac{\partial\varphi}{\partial y}$$

Therefore:

$$\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial y}=0$$

The curves $\varphi=const$ and $\psi=const$ are mutually orthogonal. φ and ψ can be the solutions of the electrostatic problem. If $\varphi(x,y)$ is the potential, $\psi=const$ are the lines of the field, orthogonal to the equipotential surfaces. The same we can find if we consider $\psi(x,y)$ as the potential, $\varphi=const$ are the lines of the field.

The complex potential is defined as: $W=\varphi+i\psi$.

Let us add shortly some discussion that we can find in [5]. In this book, it is stressed that for two-dimensional problems, where the variations of the fields in one direction is

zero or can be neglected, we have that the fields are solution of the Laplace equation. This condition allows the use of complex functions. In [5], we can find proposed three cases: $W(\zeta)=\zeta^2$, $W(\zeta)=1/\zeta^2$ and $W(\zeta)=\zeta\log\zeta$.
The first case is considered also in [6]. "The amazing conclusion is that any (differentiable) function" $W=\varphi+i\psi$ of a complex variable, provides two real functions $\varphi=\varphi(x,y)$ and $\psi=\psi(x,y)$ which are possible electric potential functions for some problem.
Let us consider $W=\varphi+i\psi=z=x+iy$. "What electrostatic problem is this the solution to?" [6]. If we take the potential to be ψ , then equipotential surfaces are surfaces of constant *y*, and the field lines have constant *x*. This is the elementary problem of a uniform field in the *x*- direction.
Let us consider $W=\varphi+i\psi=\zeta^2=(x+iy)^2=x^2-y^2+2ixy$.
We have: $\varphi=x^2-y^2$; $\psi=2xy$. To see the electrostatic problem of which W is the complex potential, let us look for the surfaces: $\varphi=x^2-y^2=A$; $\psi=2xy=B$. The corresponding graphs of these functions are hyperbolas. We can see them in [5] and [6], that this is the case of of the "quadrupole" system of conductors.

31) For the case of a homogeneous electric field with a field vector $\boldsymbol{E}$, write the corresponding complex-valued potential *W*. Consider the special case of the electric field of a charged plane with a surface charge density σ [3].

Let *x*-axis be directed along the field:

$$W=-|\boldsymbol{E}|(x+iy)=\varphi+i\psi \quad \text{where} \quad \varphi=-|\boldsymbol{E}|x \ ; \ \psi=-|\boldsymbol{E}|y$$

In the case that the field is created by a uniformly distribute charge surface, *y*=0. In this case:

$$|\boldsymbol{E}|=\frac{\sigma}{2\varepsilon_o} \quad .$$

32) Determine the potential near a grounded angle formed by two planes *x*=0 and *y*=0 [3].

Let us use a conformal mapping $W=A\zeta^2 i$ to transform the angle into a half-plane. The potential is:

$$W=\varphi+i\psi=A\zeta^2 i=A(x+iy)^2 i$$

So we have: $\varphi=-2Axy \;\; ; \;\; \psi=A(x^2+y^2)$. This is the case also discussed in [5].

33) Determine the potential surfaces and the lines of field if the potential is $\varphi=\Re(\sqrt{\zeta})$. What grounded contour has such a potential? [4].

The complex values potential is: $W=\varphi+i\psi=\Re(\sqrt{\zeta})+\Im(\sqrt{\zeta})=\sqrt{(\zeta)}$.

$$\zeta=x+iy=\varphi^2-\psi^2+2i\varphi\psi \quad ; \quad \varphi^2-\psi^2=x \quad ; \quad 2\varphi\psi=y$$

We have equations:

$$\frac{y^2}{4\varphi^2}=\varphi^2-x \quad ; \quad \frac{y^2}{4\psi^2}=\psi^2-x$$

Let us assume $\varphi=const=C$, then we have: $y^2=4C^2(C^2-x)$, which is a parabola. If *C*=0, the parabola turns into a straight line given by *y*=0, *x*<0. The function $\varphi=\Re\sqrt{\zeta}$ gives the potential near a grounded half-line. The equipotential surfaces are a system of parabolas.

34) Determine the potential and the equipotential surfaces if the complex values potential is $W = \ln x$ [3].

Writing ζ in the form $\zeta = r e^{i\theta}$, we find that $\varphi = \ln r$. The equipotential surfaces are circles with radiuses *r=const.* In the three-dimensional case, this is a potential created by a charged straight line along the z-axis.

35) Find the potential near a grounded parabola $y^2 = 4a(x+a)$. [3]

Let us write the equation of the parabola in a parametric form:

$$x = a p^2 - a \quad ; \quad y = 2ap \qquad -\infty < p < +\infty$$

$x = \varphi$; $y = \psi$ then $\zeta = aW^2 - a + 2iaW = (W+i)^2 a$. We have:

$$W = \sqrt{(\zeta / a)} - i = \varphi + i\psi$$

In [3], it is also discussed the case of a grounded ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Image charges

The method of image charges (also known as the method of images and method of mirror charges) is a basic tool in electrostatics. The name originates from the replacement of certain elements in the original layout of the problem with imaginary charges, which replicates the boundary conditions of the problem. The method of image charges is based on the uniqueness theorem, which states that the electric potential in a given volume is uniquely determined if both the charge density throughout the region and the value of the electric potential on all boundaries are specified. In the case that we know the electric potential or the electric field and the corresponding boundary conditions we can substitute the charge distribution which is involved with a charge configuration easier to analyse.

The simplest case where we can apply the method of image charges is that of a point charge, with charge *q*, located at $(0,0,a)$ above an infinite grounded conducting plate in *xy*-plane. To simplify this problem, we can replace the charge distribution which is induced on the ground plane with a plate (equipotential) with a charge –*q*, located at $(0,0,-a)$. This situation is equivalent to the original setup, and so the force on the real charge can now be calculated with Coulomb's law between two point charges [7].

The potential at any point in space, due to these two point charges is given in cylindrical coordinates as

$$\varphi(\mathbf{r})=\varphi(\rho,\phi,z)=\frac{1}{4\pi\varepsilon_o}\left(\frac{q}{\sqrt{\rho^2+(z-a)^2}}-\frac{q}{\sqrt{\rho^2+(z+a)^2}}\right)$$

The surface charge density on the grounded plane is therefore given by:

$$\sigma=-\varepsilon_o\left(\frac{\partial\varphi}{\partial z}\right)_{z=0}=-\frac{qa}{2\pi(\rho^2+a^2)^{3/2}}$$

The total charge induced on the conducting plane is:

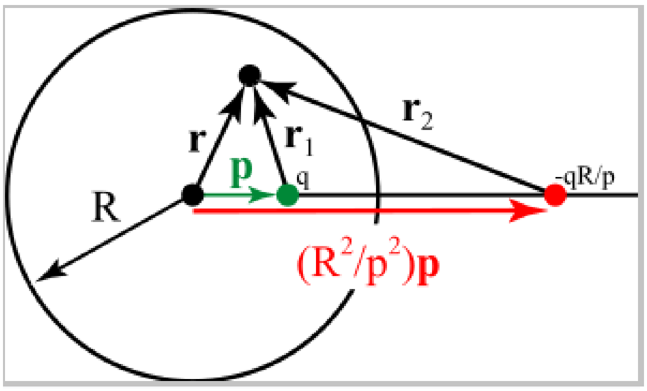
$$\int_0^{2\pi}\int_0^{\infty}\sigma(\rho)\rho\,d\rho\,d\phi=\frac{q}{2\pi}\int_0^{2\pi}d\phi\int_0^{\infty}\frac{1}{(\rho^2+a^2)^{3/2}}\rho\,d\rho=q$$

$$\int_0^{2\pi} d\phi \int_0^{\infty} \frac{1}{(\rho^2+a^2)^{3/2}} \rho\, d\rho = 2\pi \left[\frac{1}{\sqrt{\rho^2+a^2}} \right]_0^{\infty} = \frac{2\pi}{a}$$

The total charge induced on the plane turns out to be simply –q.

Because of the superposition principle, a conducting plane below multiple point charges can be replaced by the mirror images of each of the charges individually, with no other modifications necessary.

A grounded sphere



In the previous image we can see a diagram illustrating the image method for a grounded sphere of radius *R*. The green point is a charge q inside the sphere at a distance p from the origin, the red point is the image of that point, having charge $-qR/p$, outside the sphere at a distance of R^2/p from the origin. The potential produced by the two charges is zero on the surface of the sphere. (Image courtesy: PAR).

The method of the image of charge can be applied to a conducting sphere [8]. Let us consider *q,* the charge at the green point. The image of this charge with respect to the grounded sphere is shown in red. It has a charge of $q'=-qR/p$ at the distance given in the figure. The potential at a given radius vector $\boldsymbol{r}$ is given by the sum of the potentials:

$$4\pi\varepsilon_o\varphi(\boldsymbol{r})=\frac{q}{|\boldsymbol{r_1}|}+\frac{(-qR/p)}{|\boldsymbol{r_2}|}$$

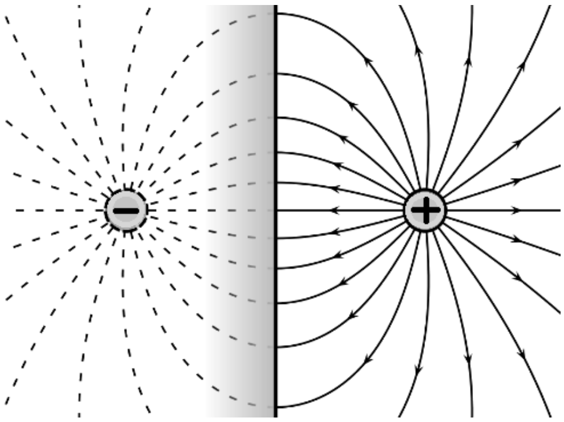
$$4\pi\varepsilon_o\varphi(\boldsymbol{r})=\frac{q}{(r^2+p^2-2\boldsymbol{r}\cdot\boldsymbol{p})^{1/2}}+\frac{(-qR/p)}{(r^2+R^4/p^2-2R^2\boldsymbol{r}\cdot\boldsymbol{p}/p^2)^{1/2}}$$

$$\varphi(\boldsymbol{r})=\frac{1}{4\pi\varepsilon_o}\left(\frac{q}{\sqrt{r^2+p^2-2\boldsymbol{r}\cdot\boldsymbol{p}}}-\frac{q}{\sqrt{r^2p^2/R^2+R^2-2\boldsymbol{r}\cdot\boldsymbol{p}}}\right)$$

The potential on the surface of the sphere vanishes. Note that, if the image of charge is outside the sphere, the potential given above will not be valid, since the image charge does not actually exist, the same if the charge image is inside.

Let us consider the problem with a charge outside a grounded sphere. If we have a charge *q* at vector position $\boldsymbol{p}$ outside the sphere of radius *R*, the potential outside of the sphere is given by the sum of the potentials of the charge and its image charge inside the sphere. Just as in the previous case, the image charge will have charge $-qR/p$ and will be located at vector position $R^2\boldsymbol{p}/p^2$.

The potential inside the sphere will be dependent only upon the true charge distribution inside the sphere.



Courtesy image: Geek3.

36) A point charge e is situated at a distance d from a conducting plane that is grounded. Find the potential and the electric field intensity vector of the system. Determine the surface density of the charge that is induced on the grounded surface. Show that the total induced charge is equal to $-e$ [3].

Let us assume that the grounded plane is plane $x=0$. The radius vector $\boldsymbol{r}$ is drawn from the charge to the observation point, and $\boldsymbol{r}'$ from the image of the chrge to the observation point.

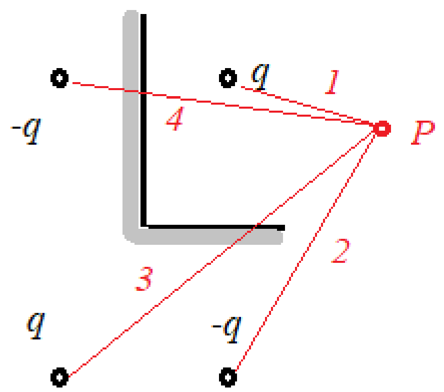
$$\varphi=\frac{e}{4\pi\varepsilon_o r}-\frac{e}{4\pi\varepsilon_o r'} \quad ; \quad \boldsymbol{E}=\frac{e\boldsymbol{r}}{4\pi\varepsilon_o r^3}-\frac{e\boldsymbol{r}'}{4\pi\varepsilon_o r'^3}$$

$$r=\sqrt{(x-d)^2+y^2+z^2} \quad ; \quad r'=\sqrt{(x+d)^2+y^2+z^2}$$

The surface density of the induced charge is: $\sigma=-\frac{ed}{2\pi\left(\sqrt{y^2+z^2+d^2}\right)^3}$.

37) Using the method of images, find the potential when a charge q that is placed inside an angle (right angle) formed by two grounded conducting planes [3].

We need three images, of the charge as given in the following figure.



$$\varphi=\frac{1}{4\pi\varepsilon_o}\left(\frac{q}{r_1}-\frac{q}{r_2}+\frac{q}{r_3}-\frac{q}{r_4}\right)$$

$$r_1=\sqrt{(x-a)^2+(y-b)^2} \quad ; \quad r_2=\sqrt{(x-a)^2+(y+b)^2}$$
$$r_3=\sqrt{(x+a)^2+(y+b)^2} \quad ; \quad r_4=\sqrt{(x+a)^2+(y-b)^2}$$

When y=0, $\varphi=0$; the same for x=0.

38) A point charge e is at a distance d from the centre of a conducting sphere of radius R. Using the method of images, determine the electric potential of the system. The sphere is grounded.

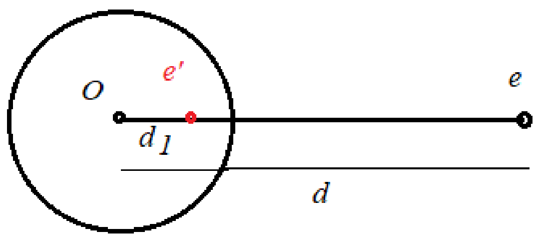
The potential inside the sphere and on it is zero. Outside the sphere, it must satisfy the equation:

$$\Delta\varphi = -\frac{e}{\varepsilon_o}\delta(\mathbf{r})$$

The origin of coordinate coincides with the position of the point charge. The solution is:

$$\varphi = \frac{e}{4\pi\varepsilon_o r} - \frac{e'}{4\pi\varepsilon_o r'}$$

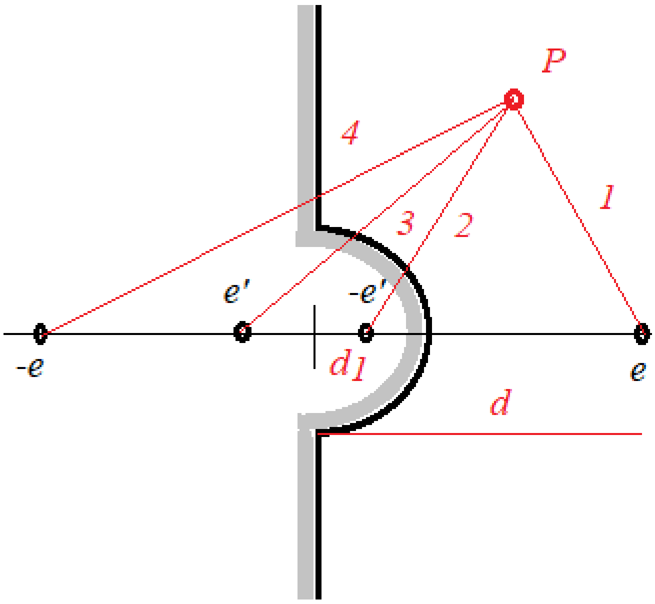
The second term in the potential may be thought of as the potential of a charge e', the charge's image, inside the sphere. Actually, there is no such charge. But the real charge induced on the surface of the sphere exists and acts on the same way as a certain charge would without the sphere. The symmetry of the problem implies that charge e' must lie on the line that connect the centre of the sphere with the charge e, placed at a distance d_1 for the centre.



$$\frac{e^2}{e'^2} = \frac{r^2}{r'^2} = \frac{R^2 + d^2 - 2dR\cos\theta}{R^2 + d_1^2 - 2d_1 R\cos\theta}$$

For any angle θ : $$d_1 = R^2/d \quad ; \quad e' = e\sqrt{\frac{d_1}{d}} = e\frac{R}{d}$$

39) A point charge *e* is at a distance *d* from the centre of a spherical projection of a conducting plane. The centre of the project lies on the plane, and the charge is situated opposite the point on the projection that is farthest from the plane. Determine the potential [3].



$$\varphi=\frac{1}{4\pi\varepsilon_o}\left(\frac{e}{r_1}-\frac{e'}{r_2}+\frac{e'}{r_3}-\frac{e}{r_4}\right)$$

$$e'=e\sqrt{\frac{d_1}{d}}\quad ;\quad d_1=R^2/d$$

40) At a distance *d* from the centre of a conducting grounded sphere .of radius *R* an electric dipole *p* is placed with its positive charge closest to the sphere. Find the electric potential of the system [3].

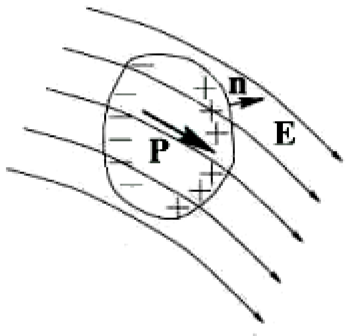
Each charge of the dipole induces an image charge. Since the distances from the charges to the sphere are different, the magnitudes of the image charges will be different. For this reason we must place a dipole ***p'***, oriented in the same way as ***p***, and a charge e' at a point that lies at a distance $d'=R^2/d$ from the sphere.

$$\varphi=\frac{\boldsymbol{p}\cdot\boldsymbol{r}}{4\pi\varepsilon_o r^3}+\frac{\boldsymbol{p}'\cdot\boldsymbol{r}'}{4\pi\varepsilon_o r'^3}+\frac{e'}{4\pi\varepsilon_o r'}$$

See Ref.3 for the details of calculation.

Dielectrics

Let us remember some formulas.



$$\boldsymbol{P}=\alpha\varepsilon_o\boldsymbol{E} \;\; ; \;\; \rho_{pol}=-\mathrm{div}\,\boldsymbol{P} \;\; ; \;\; \sigma_{pol}=\boldsymbol{P}\cdot\boldsymbol{n}$$

$\boldsymbol{n}$ is the unit vector perpendicular to the surface of the dielectric material.

$$\boldsymbol{D}=\varepsilon_o\boldsymbol{E}+\boldsymbol{P}=\epsilon\,\boldsymbol{E} \;\; ; \;\; \mathrm{div}\,\boldsymbol{E}=\frac{1}{\varepsilon_o}(\rho_{cond}+\rho_{pol}) \;\; ; \;\; \mathrm{div}\,\boldsymbol{D}=\rho_{cond}$$

where ρ_{cond},ρ_{pol} are the densities of charges on conductors (unbounded charged) and induced by polarization (bounded charges).

Boundary conditions are:

$$D_{1n}-D_{2n}=\sigma_{unbounded} \;\; ; \;\; E_{1t}=E_{2t}$$

The surface density of charges between the media is due to the unbounded charges only, not coming from polarization of the materials.

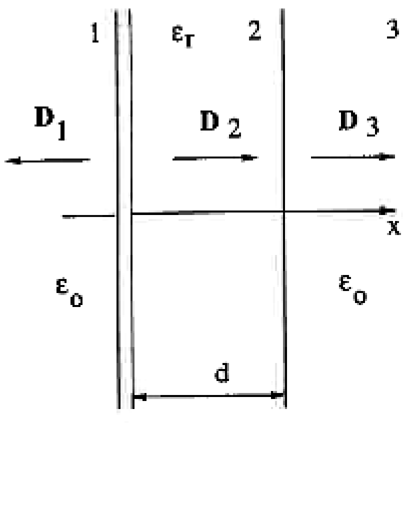
41) A point charge Q ($3\times10^{-10}C$) is at the centre of a sphere having radius R (10 cm), made of an isotropic and homogeneous dielectric medium with permittivity $\epsilon_r=4$. Outside the sphere there is the empty space. Find the surface charge density on the sphere [10].

Let us calculate:

$$\int \boldsymbol{D}\cdot\boldsymbol{u_n} dS=\int D\boldsymbol{u_r}\cdot\boldsymbol{u_n}=\int D dS=4\pi r^2 D=Q \quad ; \quad D(r)=\frac{Q}{4\pi r^2}$$

$$\boldsymbol{D}=\varepsilon_o\epsilon_r\boldsymbol{E}$$

$$\sigma=\boldsymbol{P}\cdot\boldsymbol{u_r}=\varepsilon_o(\epsilon_r-1)\boldsymbol{E}\cdot\boldsymbol{u_r}=\frac{\epsilon_r-1}{\epsilon_r}\frac{Q}{4\pi R^2}=2.\times10^{-9}\ \frac{C}{m^2}$$



42) An electric charge is uniformly distributed on a plane with a surface density σ .

Close to the plane, there is a lamina of dielectric material with permittivity ϵ_r and thickness *d*. Find the density of the polarization charge [10].

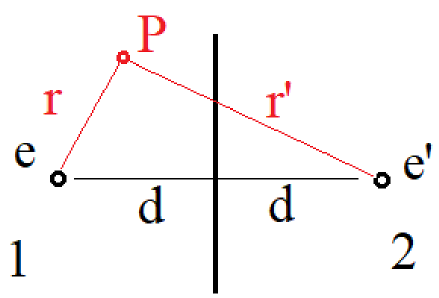
Using the boundary conditions for field $\boldsymbol{D}$ we have: $|D_1|=|D_2|=|D_3|=\frac{\sigma}{2}$

$$\mathbf{D}_1 = \epsilon_o \mathbf{E}_1 \quad \mathbf{D}_2 = \epsilon_o \epsilon_r \mathbf{E}_2 \quad \mathbf{D}_3 = \epsilon_o \mathbf{E}_3$$

$$E_1 = \frac{\sigma}{2\epsilon_o} = E_3 \quad E_2 = \frac{\sigma}{2\epsilon_o \epsilon_r}$$

$$|\sigma_p| = |\boldsymbol{P} \cdot \boldsymbol{u_n}| = P = \varepsilon_o (\epsilon_r - 1) E_2 \quad \text{then:} \quad |\sigma_p| = \frac{\epsilon_r - 1}{2\epsilon_r} \sigma$$

43) An electric charge *e* is placed at a distance *d* from the flat surface of an infinite dielectric with permittivity ϵ_2 . The permittivity of the medium where the charge is located is ϵ_1 . Determine the potential φ and the electric displacement vector $\boldsymbol{D}$ in the two media [3].



Discussion in given in [9]. Let us call O the point where the charge is placed, and O' the point where the image charge is place e'. In the medium 1, we have:

$$\varphi_1=\frac{e}{\epsilon_1 r}+\frac{e'}{\epsilon_1 r'}$$

Field in medium 2 is given by a charge e'' in O: $\varphi_2=\frac{e''}{\epsilon_2 r}$.

Boundary conditions are: $\varphi_1=\varphi_2$; $\epsilon_1\frac{\partial\varphi_1}{\partial n}=\epsilon_2\frac{\partial\varphi_2}{\partial n}$.

For $r=r'$: $e-e'=e''$; $\frac{e+e'}{\epsilon_1}=\frac{e''}{\epsilon_2}$. Therefore we have:

$$e'=e\,\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\quad ;\quad e''=e\,\frac{2\epsilon_2}{\epsilon_1+\epsilon_2}$$

In the case that $\epsilon_2\rightarrow\infty$, $e'=-e$, $\varphi_2=0$. We find the result for a conducting plane.

44) The centre of a conducting sphere of radius R is on a flat boundary between two dielectrics with permittivity ϵ_1,ϵ_2 . The sphere possesses charge q. Discuss the system [3].

Let us start from vector $\boldsymbol{D}$ as we did in [10]:

$$\int_S \boldsymbol{D}\cdot\boldsymbol{n}\,dS=q\quad ;\quad D=\epsilon\,\boldsymbol{E}=-\epsilon\frac{\partial\varphi}{\partial r}$$

Potential φ must be constant on the surface of the conductor. It must be zero to infinity. Let us assume $\varphi = C/r$.

$$C = \frac{q}{2\pi(\epsilon_1 + \epsilon_2)} \quad ; \quad \varphi = \frac{q}{2\pi(\epsilon_1 + \epsilon_2) r}$$

The electric field near the conductor is $E = \sigma/\varepsilon_o$. $\sigma = \sigma_s + \sigma_p$, charge densities on the sphere and of polarization [10].

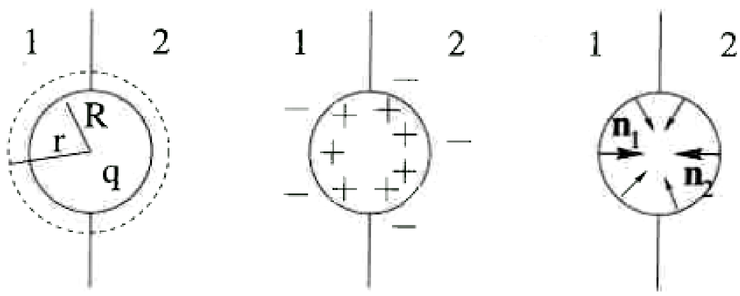
$$\sigma_{p1} = \mathbf{P} \cdot \mathbf{n}_1 = -P_1 = -\epsilon_o(\epsilon_1^r - 1)E_1 = -\frac{(\epsilon_1^r - 1)q}{2\pi(\epsilon_1^r + \epsilon_2^r)} \frac{1}{R^2}$$

$$\sigma_{p2} = \mathbf{P} \cdot \mathbf{n}_2 = -P_2 = -\epsilon_o(\epsilon_2^r - 1)E_2 = -\frac{(\epsilon_2^r - 1)q}{2\pi(\epsilon_1^r + \epsilon_2^r)} \frac{1}{R^2}$$

where $\epsilon_1^r = \epsilon_1/\varepsilon_o$, $\epsilon_2^r = \epsilon_2/\varepsilon_o$. σ_{p1}, σ_{p2} are the surface charge densities due to polarization. σ_{s1}, σ_{s2} are the densities on the sphere.

$$\sigma_{s1} = \sigma_1 - \sigma_{p1} = \varepsilon_o E_1 - \sigma_{p1} = \frac{\epsilon_1^r q}{2\pi(\epsilon_1^r + \epsilon_2^r)} \frac{1}{R^2}$$

$$\sigma_{s2} = \sigma_2 - \sigma_{p2} = \varepsilon_o E_2 - \sigma_{p2} = \frac{\epsilon_2^r q}{2\pi(\epsilon_1^r + \epsilon_2^r)} \frac{1}{R^2}$$



From Legendre Polynomials to Bessel Functions

Legendre polynomials are given by: $P_l(x)=\frac{1}{2^l l!}\frac{d^l(x^2-1)^l}{dx^l}$.

Here some of the polynomials: $P_o(x)=1$ $P_1(x)=x$ $P_2(x)=\frac{1}{2}(3x^2-1)$

$P_2(x)=\frac{1}{2}(5x^3-3x)$

45) A point charge *e* is at a distance *d* from the centre of a conducting sphere of radius *R*. Solve the problem by an expansion of the potential in a series of solutions of the Laplace equation in spherical coordinates [3].

Let us place the coordinate origin in the centre of the sphere and direct the polar axis along the line connecting the charge and the centre of the sphere.

$$\Delta\varphi=-\frac{e}{\varepsilon_o}\delta(\boldsymbol{r}-\boldsymbol{d})$$

$$\varphi(r,\theta)=\frac{e}{4\pi\varepsilon_o|\boldsymbol{r}-\boldsymbol{d}|}+\sum_{l=0}^{\infty}\frac{b_l}{r^{l+1}}P_l(\cos\theta)$$

$P_l(\cos\theta)$ are the Legendre polynomials. Coefficients b_l are determined by means of the boundary conditions. We know that (θ is the angle between $\boldsymbol{r}$ and $\boldsymbol{r}'$):

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}=\sum_{l=0}^{\infty}\frac{r^l}{(r')^{l+1}}P_l(\cos\theta) \quad \text{for} \quad r<r' \quad (*)$$

Boundary condition is: $\varphi=0,\ r=R$.

$$\sum_{l=0}^{\infty}\left(\frac{e}{4\pi\varepsilon_o}\frac{R^l}{(r')^{l+1}}+\frac{b_l}{R^{l+1}}\right)P_l(\cos\theta)=0$$

$$b_l=-\frac{e}{4\pi\varepsilon_o}\frac{R^{2l+1}}{d^{l+1}}$$

$$\varphi=\frac{e}{4\pi\varepsilon_o|\boldsymbol{r}-\boldsymbol{d}|}-\frac{eR}{4\pi\varepsilon_o d}\sum_{l=0}^{\infty}\left(\frac{R^2}{d}\right)^l\frac{P_l(\cos\theta)}{r^{l+1}} \qquad (**)$$

Using expansion (*), we can write the last member in (**) as the potential of a point charge $e'=eR/d$, placed at a distance $d_1=R^2/d$.

$$\varphi=\frac{e}{4\pi\varepsilon_o|\boldsymbol{r}-\boldsymbol{d}|}-\frac{|e'|}{4\pi\varepsilon_o|\boldsymbol{r}-\boldsymbol{d_1}|}$$

This solution coincides with the solution that we have obtained using the image charge.

Let us consider the charge density on the surface of the sphere:

$$\sigma(R,\theta)=-\frac{e}{4\pi}\sum_{l=0}^{\infty}(2l+1)\frac{R^{l-1}}{d^{l+1}}P_l(\cos\theta)$$

46) A point charge *e* is placed at a distance *d* from a sphere of radius *R* that has a potential *V*. Find the potential outside the sphere and the surface charge density on it [3].

$$\varphi=\frac{e}{4\pi\varepsilon_o|\boldsymbol{r}-\boldsymbol{d}|}+\frac{VR}{r}-\frac{eR}{4\pi\varepsilon_o d}\sum_{l=0}^{\infty}\left(\frac{R^2}{d}\right)^l\frac{P_l(\cos\theta)}{r^{l+1}}$$

$$\sigma(r,\theta)=\frac{V}{4\pi R}-\frac{e}{4\pi}\sum_{l=0}^{\infty}(2l+1)\frac{R^{l-1}}{d^{l+1}}P_l(\cos\theta)$$

47) Determine the potential of a charged sphere of radius *R*. The sphere charge density varies according to the law $\sigma=\sigma_o\cos\theta$ [3].

Inside: $\varphi_1=\sum_{l=0}^{\infty}A_l r^l P_l(\cos\theta)$; Outside: $\varphi_1=\sum_{l=0}^{\infty}B_l r^{-(l+1)}P_l(\cos\theta)$

At the surface of the sphere: $\varphi_1=\varphi_2$; $\frac{\partial\varphi_1}{\partial r}-\frac{\partial\varphi_2}{\partial r}=\frac{\sigma}{\varepsilon_o}$. Then:

$$A_1=\frac{\sigma_o}{3\varepsilon_o}\ ;\ B_1=\frac{\sigma_o}{3\varepsilon_o}R^2\ ;\ A_l=B_l=0,\ \ l\neq 1$$

The potential is:

$$\varphi_1=\frac{\sigma_o}{3\varepsilon_o}r\cos\theta\ ;\ \varphi_2=\frac{\sigma_o}{3\varepsilon_o}\frac{R^3}{r^2}\cos\theta$$

48) Determine the potential and the electric field vector of a uniformly polarized ball of radius *R*. The polarization vector of the balls is $\boldsymbol{P}$ [3].

Since we know the polarization vector, we can determine the bound surface and the bound body charge densities: $\rho'=0$; $\sigma'=P\cos\theta$ .. Therefore we can use the previous discussion:

$$\varphi_1 = \frac{P}{3\varepsilon_o} r\cos\theta \quad ; \quad \varphi_2 = \frac{P}{3\varepsilon_o}\frac{R^3}{r^2}\cos\theta$$

where φ_1 is the potential for $r < R$ and φ_2 for $r > R$. The corresponding electric fields are:

$$\boldsymbol{E_1} = -\frac{1}{3\varepsilon_o}\boldsymbol{P} \quad ; \quad \boldsymbol{E_2} = \frac{R^3}{r^5}\frac{(\boldsymbol{P}\cdot\boldsymbol{r})\ \boldsymbol{r}}{3\varepsilon_o} - \frac{R^3}{3\varepsilon_o}\frac{\boldsymbol{P}}{r^3}$$

49) A conducting sphere of radius R is located in a non-uniform electric field. Determine the potential around the sphere [3].

When there is no sphere, the potential of an external field in a region without charges satisfies the Laplace equation, so that the solution is:

$$\varphi_{ext} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm} r^l P_{lm}(\cos\theta) e^{im\alpha}$$

If we have the field, we know coefficients A_{lm} .

If we place in the field a conducting sphere, the potential becomes: $\varphi = \varphi_{sphere} + \varphi_{ext}$.

$$\varphi_{sphere} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} B_{lm} r^{-(l+1)} P_{lm}(\cos\theta) e^{im\alpha}$$

This is the potential of the charge induced on the sphere by the external potential.

Putting $\varphi = 0,\ r = R$:

$$B_{lm} = -A_{lm} R^{2l+1}$$

50) A conducting sphere that is grounded is located in a uniform electric field $\boldsymbol{E_o}$. Find the potential and the surface charge density [3].

Inside and on the surface of the sphere $\varphi=0$. Outside we have:

$$\varphi=\sum_l b_l r^{-(l+1)} P_l(\cos\theta)-E_o r P_1(\cos\theta)$$

Boundary conditions give: $b_1=E_o R^2$; $b_l=0$ for $l\neq 1$.

$$\varphi=-E_o r\cos\theta+\frac{E_o R^3}{r^2}\cos\theta \quad ; \quad \sigma=3\varepsilon_o E_o\cos\theta$$

51) A disc of radius R that has a surface charge density σ is placed coaxially in a hollow cylinder of radius r_o with conducting walls. Find the potential inside the cylinder [3].

Let us consider the Laplace equation in cylindrical coordinates for $\varphi(\rho,z)=R(\rho)Z(z)$.

$$\frac{d^2 Z}{dz^2}=k^2 Z \quad ; \quad \frac{d^2 R}{d\rho^2}+\frac{1}{\rho}\frac{dR}{d\rho}+k^2 R=0$$

Solution is:

$$\varphi(\rho,z)=e^{k_l z} J_o(k_l\rho) \quad , \quad for \quad z<0$$

$$\varphi(\rho,z)=e^{-k_l z} J_o(k_l\rho) \quad , \quad for \quad z>0$$

J_o is the Bessel function of the first kind. On the surface of the cylinder $\varphi=0$. In this manner $J_o(k_l r_o)=0$ determines the values of k_l .

Potentials are:

$$\varphi_1(\rho,z)=\sum_{l=1}^{\infty} A_l e^{k_l z} J_o(k_l \rho) \quad , \quad \textit{for} \quad z<0$$

$$\varphi_2(\rho,z)=\sum_{l=1}^{\infty} A_l e^{-k_l z} J_o(k_l \rho) \quad , \quad \textit{for} \quad z>0$$

Using boundary condition: $\left(\frac{\partial \varphi}{\partial n}\right)_1 - \left(\frac{\partial \varphi}{\partial n}\right)_2 = \frac{\sigma}{\varepsilon_o}$, we have:

$$2\sum_{l=1}^{\infty} A_l e^{k_l z} J_o(k_l \rho)=\frac{\sigma}{\varepsilon_o} \quad .$$

$$A_l=\frac{\sigma}{2\varepsilon_o k_l} \frac{\int_0^R \rho J_o(k_l \rho) d\rho}{\int_0^{r_o} \rho J_o^2(k_l \rho) d\rho}$$

Bessel equation and function

Bessel functions, first defined Daniel Bernoulli and then generalized by Friedrich Bessel, are canonical solutions y(x) of Bessel's differential equation:

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - \alpha^2\right) y = 0$$

In the equation, we cna find an arbitrary complex number α, the order of the Bessel function. Although α and −α produce the same differential equation, it is conventional to define different Bessel functions for these two values in such a way that the Bessel functions are mostly smooth functions of α.

The most important cases are when α is an integer or half-integer. Bessel functions for integer α are also known as cylinder functions or the cylindrical harmonics because they appear in the solution to Laplace's equation in cylindrical coordinates. Spherical Bessel functions with half-integer α are obtained when the Helmholtz equation is solved in spherical coordinates.

Bessel's equation arises when finding separable solutions to Laplace's equation and the Helmholtz equation in cylindrical or spherical coordinates. Bessel functions are therefore especially important for many problems of wave propagation and static potentials. In solving problems in cylindrical coordinate systems, one obtains Bessel functions of integer order (α = n); in spherical problems, one obtains half-integer orders (α = n + 1/2). For example: Electromagnetic waves in a cylindrical waveguide - Pressure amplitudes of inviscid rotational flows - Heat conduction in a cylindrical object - Diffusion problems on a lattice - Solutions to the radial Schrödinger equation (in spherical and cylindrical coordinates) for a free particle - Solving for patterns of acoustical radiation - Frequency-dependent friction in circular pipelines, and so on.

History of Legendre Polynomials and Spherical Harmonics

From [11]. "Una delle formule più semplici e utili della fisica-matematica fu trovata nel 1782 da Adrien-Marie Legendre e quindi estesa pochi mesi dopo da Pierre-Simon de Laplace. I due fisici-matematici erano interessati a calcolare il potenziale gravitazionale associato ad una distribuzione di masse μ_i poste nei punti $\boldsymbol{r_i}'$. Considerarono il

potenziale gravitazionale di questa distribuzione."
The function is:

$$\varphi(\boldsymbol{r})=\sum_i \frac{\mu_i}{R_i}=\sum_i \frac{\mu_i}{|\boldsymbol{r}-\boldsymbol{r}_{\boldsymbol{i}}'|}$$

Legendre calculated, for each site:

$$\frac{1}{R} = \frac{1}{(r^2+r'^2-2r'r\cos\alpha)^{1/2}} = \frac{1}{r}\sum_l \frac{r'^l}{r^l}P_l(\cos\alpha) \qquad (*)$$

α is the angle between vectors $\boldsymbol{r}$, $\boldsymbol{r}'$.
Actually, we can calculate polynomials as a Taylor series of the generator:

$$\frac{1}{(1+z^2-2zu)^{1/2}}=\sum_{l=0}^{\infty} z^l\, P_l(u)$$

"Laplace, sempre nel 1782, fece un passo in più: usando coordinate sferiche, si rese conto che esistevano funzioni, ... , con “buone” proprietà rispetto alle rotazioni, in termini delle quali era possibile sviluppare le funzioni di Legendre:"

$$P_l(\cos\alpha) = \frac{4\pi}{l+1}\sum_{m=-l}^{l}(-1)^m Y_l^{-m}(\theta,\phi)Y_l^m(\theta',\phi')$$

$(\theta,\phi),(\theta',\phi')$ are spherical coordinates concerning vectors $\boldsymbol{r}$, $\boldsymbol{r}'$. Y_l^m are the spherical harmonic functions.
Laplace wrote:

$$\varphi(\boldsymbol{r})=\sum_i \frac{\mu_i}{R_i}=\sum_i \frac{\mu_i}{|\boldsymbol{r}-\boldsymbol{r}_{\boldsymbol{i}}'|}=\sum_{l=0}^{\infty}\frac{4\pi}{l+1}\sum_{m=-l}^{l} p_{lm}\frac{Y_l^{-m}}{r^{l+1}}$$

where

$$p_{lm}=\sum_i (-1)^m \mu_i Y_l^m(\theta'_i,\phi'_i) r'^l$$

"Correva l'anno 1782 e già era chiaro che le soluzioni dell'equazione di Laplace si raggruppavano in multipletti: fissato l, l'insieme delle combinazioni lineari delle $(2\ l + 1)$ armoniche sferiche si trasforma in se stesso per rotazioni. Nel linguaggio moderno, le armoniche sferiche di ordine l sono una base per la rappresentazione irriducibile del gruppo delle rotazioni di dimensione l . Questa matematica la si ritrova in elettrostatica e in meccanica quantistica".

Here some information from Wikipedia[1].

Spherical harmonics were first investigated in connection with the Newtonian potential of Newton's law of universal gravitation in three dimensions. In 1782, Pierre-Simon de Laplace had, in his Mécanique Céleste, determined that the gravitational potential at an observation point associated with a set of point masses located at given points, as we have previously seen.

Laplace used the Legendre polynomials, that can be derived as a special case of spherical harmonics. Subsequently, in his 1782 memoire, Laplace investigated these coefficients using spherical coordinates to represent the angle between the radius vectors.

In 1867, William Thomson (Lord Kelvin) and Peter Guthrie Tait introduced the "solid spherical harmonics" in their Treatise on Natural Philosophy, and also first introduced the name of "spherical harmonics" for these functions. The "solid harmonics" were homogeneous[2] polynomial solutions of Laplace equation.

By means of the study of Laplace equation in spherical coordinates, Thomson and Tait recovered Laplace spherical harmonics. Wikipedia tells that "The term "Laplace's coefficients" was employed by William Whewell to describe the particular system of

1 https://en.wikipedia.org/wiki/Spherical_harmonics

2 In mathematics, a homogeneous function is one with multiplicative scaling behaviour: if all its arguments are multiplied by a factor, then its value is multiplied by some power of this factor.

solutions introduced along these lines, whereas others reserved this designation for the zonal spherical harmonics that had properly been introduced by Laplace and Legendre". "The prevalence of spherical harmonics already in physics set the stage for their later importance in the 20th century birth of quantum mechanics". In fact, the (complex-valued) spherical harmonics are the eigenfunctions of the square of the orbital angular momentum operator $-i\hbar \boldsymbol{r}\times\nabla$. Therefore they represent the different quantized configurations of atomic orbitals.

$$\mathbf{L} = -i\hbar(\mathbf{x}\times\nabla) = L_x\mathbf{i} + L_y\mathbf{j} + L_z\mathbf{k}.$$

Laplace's spherical harmonics are the joint eigenfunctions of the square of the orbital angular momentum (**) and the generator of rotations about the azimuthal axis (***):

$$\boldsymbol{L}^2 = -r^2\nabla^2 + \left(r\frac{\partial}{\partial r}+1\right) r\frac{\partial}{\partial r} = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \quad (**)$$

$$L_z = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = -i\frac{\partial}{\partial\phi} \quad \textbf{(***)}$$

Magnetostatics

Phenomena that take place in a constant magnetic field, that is when the magnetic field strength $\boldsymbol{H}$ and the magnetic induction $\boldsymbol{B}$ are independent of time, are governed by the system of equations which are:

$$\mathrm{curl}\,\boldsymbol{H} = \boldsymbol{j} \quad ; \quad \mathrm{div}\,\boldsymbol{B} = 0$$

Let us introduce $\boldsymbol{B}=\text{curl}\,\boldsymbol{A}$. We have: $\Delta\boldsymbol{A}=-\mu\mu_o\boldsymbol{j}$. Vector $\boldsymbol{j}$ is the current density vector. μ is the permittivity of the medium.

$$\boldsymbol{A}(\boldsymbol{r})=\frac{\mu\mu_o}{4\pi}\int\frac{\boldsymbol{j}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,dV'$$

At a distance considerably greater than the dimensions of the system:

$$\boldsymbol{A}(\boldsymbol{r})=\frac{\mu\mu_o[\boldsymbol{m}\times\boldsymbol{r}]}{4\pi r^3}$$

$\boldsymbol{m}$ is the magnetic moment, that is $\boldsymbol{m}=\frac{1}{2}\int[\boldsymbol{r}\times\boldsymbol{j}(\boldsymbol{r})]dV$.

The energy of a stationary magnetic field is:

$$W=\frac{1}{2}\int(\boldsymbol{B}\cdot\boldsymbol{H})dV=\frac{1}{2}\int(\boldsymbol{A}\cdot\boldsymbol{j})dV=\frac{\mu\mu_o}{8\pi}\int\frac{\boldsymbol{j}(\boldsymbol{r})\boldsymbol{j}'(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}dV\,dV'$$

For a system of conductors: $W=\frac{1}{2}\sum_{i,k}L_{ik}I_iI_k$, L_{ik} is the mutual inductance.

Show that for a constant and uniform magnetic field $\boldsymbol{B}$, the vector potential can be chosen in the form: $\boldsymbol{A}=\frac{1}{2}[\boldsymbol{B}\times\boldsymbol{r}]$.

Use the vector relations and problems previously discussed.

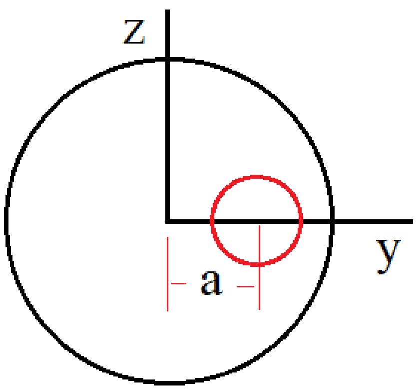
---

52) Find the magnetic field strength inside and outside a cylindrical conductor with an electric current whose density $\boldsymbol{j}$ is the same through any section of the conductor. The radius of the cylinder is R [3].

$$\boldsymbol{H}=\frac{1}{2}[\boldsymbol{j}\times\boldsymbol{r}] \quad for \quad r<R$$

$$\boldsymbol{H}=\frac{R^2}{2r^2}[\boldsymbol{j}\times\boldsymbol{r}] \quad for \quad r>R$$

The distance from the axis of the cylinder to the observation point is *r*.



53) Find the magnetic field strength inside a cylindrical cavity in a cylindrical conductor with an electric current whose density is the same through the section of the conductor. The axes of the cavity and the conductor are parallel and separated by a distance *a* [3]. This problem is also proposed in [12], where it is the magnetic induction which is required.

To solve the problem let us consider the superposition principle, with a current flowing in all the cylindrical conductor, plus an opposite current flowing in the cavity. Note that y-axis is given in the previous image.

$For \quad y>0$:

First current: $\boldsymbol{B}=\frac{1}{2}\mu_o j y \boldsymbol{u_z}$, $0<y<R$ and $\boldsymbol{B}=\frac{1}{2}\mu_o j \frac{R^2}{y}\boldsymbol{u_z}$, $y>R$

Second current: $\boldsymbol{B}=\frac{1}{2}\mu_o j \frac{r^2}{a-y}\boldsymbol{u_z}$, $0<y<a-r,\ y>a+r$ and

$$\boldsymbol{B}=\frac{1}{2}\mu_o j(a-y)\boldsymbol{u_z} \quad , \quad a-r<y<a+r \quad .$$

Adding the fields, we have that, in the cavity: $\boldsymbol{B}=\frac{1}{2}\mu j a \boldsymbol{u_z}$. Then: $\boldsymbol{H}=\frac{1}{2}[\boldsymbol{j}\times\boldsymbol{a}]$.

54) An electric current flows through an infinitely long conductor of radius *R*. The current density is a/ρ for $\rho\leq R$, where ρ is the distance from the axis of the conductor. Find the vector potential and the magnetic field strength inside and outside the conductor [3].

Let the z-axis be the axis of the conductor. The symmetry of the problem yields: $A_x=A_y=0$.

$$\Delta A_{1z}=-\mu_o j \quad for \quad \rho<R \quad ; \quad \Delta A_{2z}=0 \quad for \quad \rho>R$$

The A_z - component depends only on the distance from the axis. In cylindrical coordinates:

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dA_{1z}}{d\rho}\right)=-\mu_o\frac{a}{\rho} \quad \text{for} \quad \rho<R \quad ;$$

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dA_{2z}}{d\rho}\right)=0 \qquad \text{for} \quad \rho>R$$

$$A_{1z}=-\mu_o a\rho+C_1\ln\rho+C_2 \quad ; \quad A_{2z}=C_3\ln\rho+C_4$$

Let us put $C_1=0$. The continuity of the function and its derivative gives:

$$-\mu_o aR+C_2=C_3\ln R+C_4 \quad ; \quad -\mu_o a=C_3\frac{1}{R}$$

We have $C_3=-\mu_o aR$, $C_4=-\mu_o aR(1-lnR)+C_2$. The relations given previously define the vector potential up to a constant term C_2 .

The magnetic strength is defined as: $\boldsymbol{H}=\frac{1}{\mu_o}\,\text{curl}\,\boldsymbol{A}$.

Let us remember that:

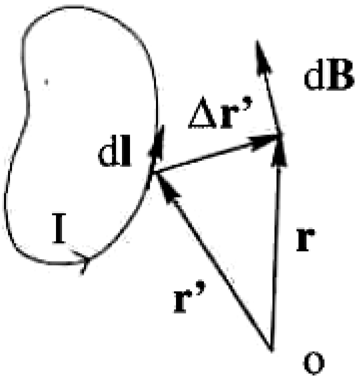
$$\text{curl}\,\boldsymbol{A}=\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi}-\frac{\partial A_\phi}{\partial z}\right)\boldsymbol{e}_\rho+\left(\frac{\partial A_\rho}{\partial z}-\frac{\partial A_z}{\partial\rho}\right)\boldsymbol{e}_\phi+\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\phi)-\frac{1}{\rho}\frac{\partial A_\rho}{\partial\phi}\right)\boldsymbol{e}_z$$

Inside: $A_z=-\mu_o a\rho+C_2$; $\boldsymbol{H}=a\boldsymbol{e}_\phi$

Outside: $A_z=-\mu_o aR\ln\rho+C_4$; $\boldsymbol{H}=aR\frac{1}{\rho}\boldsymbol{e}_\phi$.

55) Find the magnetic field strength of a plane with a surface current density i that is the same in any point of the plane [3].

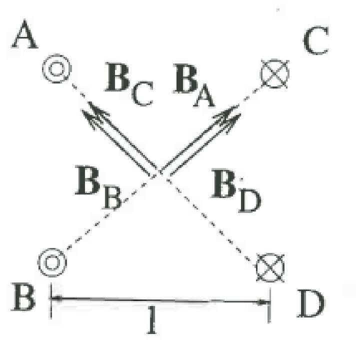
Let us chose the system of coordinates so that the z-axis is directed along the current and the x-axis along the normal to the plane: $H_y = -i/2 \quad x<0$, $H_y = i/2 \quad x>0$.



The element of the magnetic induction in an observation point O is given by the Laplace formula:

$$d\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_o}{4\pi} I \frac{d\boldsymbol{l} \times \Delta \boldsymbol{r}'}{|\Delta \boldsymbol{r}'|^3} \quad .$$

In the formula, $d\boldsymbol{l}$ is an element of the loop, where the current I is flowing.
Let us consider some problems about this formula.



56) Four straight infinitely long parallel wires are carrying the same current I as in the figure. Give the magnetic induction at the centre of the square ABCDA [10].

$$|\mathbf{B}_A| = |\mathbf{B}_B| = |\mathbf{B}_C| = |\mathbf{B}_D| = \frac{\mu_o}{2\pi}\frac{I}{l'} = \frac{\mu_o}{\sqrt{2}\pi}\frac{I}{l}$$

$$\mathbf{B}_A + \mathbf{B}_D = 2\frac{\mu_o}{\sqrt{2}\pi}\frac{I}{l}\mathbf{u}_{BC} \qquad \mathbf{B}_C + \mathbf{B}_B = 2\frac{\mu_o}{\sqrt{2}\pi}\frac{I}{l}\mathbf{u}_{DA}$$

$$|\boldsymbol{B}| = \sqrt{(\boldsymbol{B_A} + \boldsymbol{B_D})^2 + (\boldsymbol{B_C} + \boldsymbol{B_B})^2} = 2\frac{\mu_o}{\pi}\frac{I}{l}$$

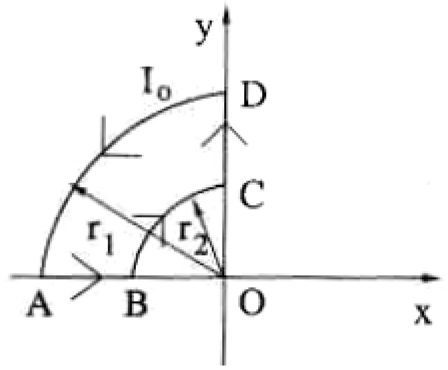


Figure of problem 57.

57) Give the magnetic induction created by by wire ABCDA shown in the figure, carrying a current I_o. The wire is in the *xy*-plane.

We know that: $\boldsymbol{B} = \frac{\mu_o}{4\pi}\oint\frac{d\boldsymbol{l}\times\boldsymbol{r}}{r^3}$. Let us subdivide the wire in four pieces. The pieces AB and CD have $d\boldsymbol{l}$ parallel to $\boldsymbol{r}$, therefore the contribution to the magnetic induction is zero. In the pieces BC and DA, $d\boldsymbol{l}$ is perpendicular to $\boldsymbol{r}$, then $d\boldsymbol{B}$ had the direction of unit vector $\boldsymbol{k}$.

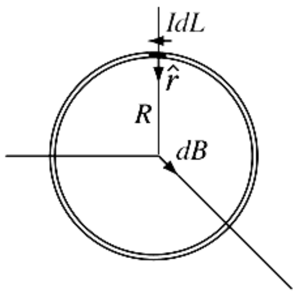
BC) $\boldsymbol{B}_{BC}=\frac{\mu_o}{4\pi}\int_B^C I_o \frac{d\boldsymbol{l}\times\boldsymbol{r}}{r^3}=-\frac{\mu_o}{4\pi} I_o \int_B^C \frac{dl}{r^3}\boldsymbol{k}$ since $d\boldsymbol{l}\times\boldsymbol{r}=-r\,dl\,\boldsymbol{k}$.

Being $dl=r\,d\theta$, we have $\boldsymbol{B}_{BC}=-\frac{\mu_o}{4\pi} I_o \int_0^{\pi/2} \frac{d\theta}{r}\boldsymbol{k}=-\frac{\mu_o}{4\pi} I_o \frac{\pi}{2r_2}\boldsymbol{k}=-\frac{\mu_o I_o}{8r_2}\boldsymbol{k}$

DA) It is the same as for BC but $d\boldsymbol{l}\times\boldsymbol{r}=r\,dl\,\boldsymbol{k}$, then: $\boldsymbol{B}_{DA}=\frac{\mu_o I_o}{8r_1}\boldsymbol{k}$.

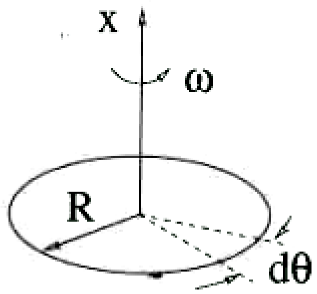
$$\boldsymbol{B}_{DA}=\frac{\mu_o I_o}{8}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\boldsymbol{k}$$

58) Evaluate the magnetic induction at the centre of a current loop.



$d\boldsymbol{B}=\frac{\mu_o I}{4\pi R^3} d\boldsymbol{L}\times\boldsymbol{R}=\frac{\mu_o I}{4\pi R^2} dL\,\boldsymbol{k}$ then $\boldsymbol{B}=\frac{\mu_o I}{4\pi R^2}\oint dL=\frac{\mu_o I}{2R}\boldsymbol{k}$

For a point on the z-axis: $\boldsymbol{B}=\frac{\mu_o I}{2}\frac{R^2}{(z^2+R^2)^{3/2}}\boldsymbol{k}$.



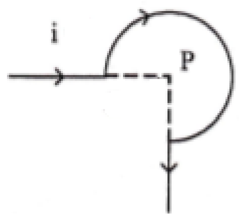
59) A charge q is uniformly distributed on a ring made of insulating material. The ring is rotating about its axis with a constant angular speed.

The density of charge is: $\lambda = q/(2\pi R)$. The arc is $ds = R\,d\theta$.

$$dq = \lambda\,ds = \frac{q}{2\pi R} R\,d\theta = \frac{q}{2\pi} d\theta \quad ; \quad I = \frac{dq}{dt} = \frac{q}{2\pi}\frac{d\theta}{dt} = \frac{q}{2\pi}\omega$$

The magnetic induction for the points of the axis is:

$$\boldsymbol{B} = \frac{\mu_o I}{2}\frac{R^2}{(z^2+R^2)^{3/2}}\,\boldsymbol{k} = \frac{\mu_o q\,\omega}{4\pi}\frac{R^2}{(z^2+R^2)^{3/2}}\,\boldsymbol{k}$$



60) The electric current i flows in the conductor wire. What is the magnetic induction at the centre of the circle (P)?

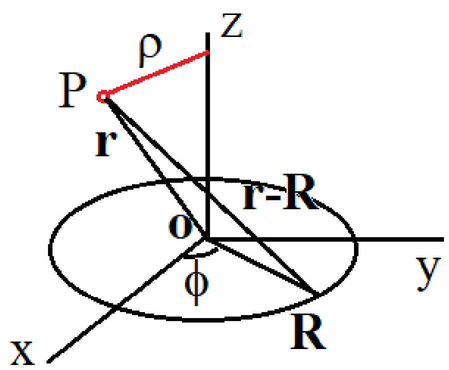
The magnetic field formula at the centre of the coil with a number of loops:

$$B = \frac{\mu_o N I}{2r}$$

In our case: N =3/4.

$$B = \frac{3}{4}\frac{\mu_o I}{2r}$$

61) Find the vector potential created by a current *I* flowing along a ring of radius *R*. Examine the special case when the observation point lies on the axis of the ring. [3].



Let us use a system of cylindrical coordinates.

$$A_\phi=\frac{\mu_o I}{4\pi}\oint\frac{d l_{\phi'}}{|\mathbf{r}-\mathbf{r}'|}=\frac{\mu_o I}{4\pi}\int_0^{2\pi}\frac{R\cos\phi' d\phi'}{(R^2+\rho^2+z^2-2R\rho\cos\phi')^{1/2}}$$

Let us use: $\phi'=\pi+2\theta$.

$$A_\phi=\frac{\mu_o R I}{\pi}\int_0^{\pi/2}\frac{(2\sin^2\theta-1)d\theta}{((R+\rho)^2+z^2-4R\rho\sin^2\theta)^{1/2}}$$

Let us define: $k^2=\frac{4R\rho}{(R+\rho)^2+z^2}$, then:

$$A_\phi=\frac{\mu_o I}{k\pi}\left(\frac{R}{\rho}\right)^{1/2}\left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right] ,$$

$K(k)=\int_0^{\pi/2}\frac{d\theta}{(1-k^2\sin^2\theta)^{1/2}}$ is an elliptic integral of the first kind, and

$E(k)=\int_0^{\pi/2}(1-k^2\sin^2\theta)^{1/2}d\theta$ is an elliptic integral of the second kind.

These two functions are related in the following manner:

$$\frac{dK}{dk}=\frac{E}{k(1-k^2)}-\frac{K}{k}$$

$$\frac{dE}{dk}=\frac{E}{k}-\frac{K}{k}$$

The magnetic field is given as:

$$H_\rho=\frac{I}{2\pi}\frac{z}{\rho[(R+\rho)^2+z^2]^{1/2}}\left[-K(k)+\frac{R^2+\rho^2+z^2}{(R-\rho)^2+z^2}E(k)\right]$$

$$H_z=\frac{I}{2\pi}\frac{1}{\rho[(R+\rho)^2+z^2]^{1/2}}\left[K(k)+\frac{R^2-\rho^2-z^2}{(R-\rho)^2+z^2}E(k)\right]$$

$$H_\phi=0$$

In the case that $\rho\rightarrow 0$, $H_\rho=0$, $H_\phi=0$, $H_z=\frac{R^2 I}{2(R^2+z^2)^{3/2}}$.

62) Find the magnetic field strength and the vector of magnetic induction created by a uniform magnetic ball. The radius of the ball is *R* and its magnetization vector $\boldsymbol{M}$ [3].

Outside the ball, we have that $\boldsymbol{B_2}=\mu_o \boldsymbol{H_2}$; $\text{curl}\,\boldsymbol{H_2}=0$: $\text{div}\,\boldsymbol{H_2}=0$. These relations mean that it exists a scalar so that $\Delta \varphi_m=0$, and that $\boldsymbol{H_2}=-\nabla \varphi_m$.

$$\varphi_m=\sum_{l=0}^{\infty} a_l \frac{P_l(\cos\theta)}{r^{l+1}}$$

Let us remember that, in spherical coordinates:

$$: \ \text{grad} f=\frac{\partial f}{\partial r}\boldsymbol{e_r}+\frac{1}{r}\frac{\partial f}{\partial \theta}\boldsymbol{e_\theta}+\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\boldsymbol{e_\phi} \ ,$$

Then:

$$\boldsymbol{H_2}=\sum_l \left[a_l \frac{(l+1)P_l(\cos\theta)}{r^{l+2}}\boldsymbol{e_r}-\frac{a_l}{r^{l+2}}\frac{dP_l}{d\theta}\boldsymbol{e_\theta}\right]$$

We have also that $\boldsymbol{M}=\chi \boldsymbol{H}$; $\boldsymbol{B}=\mu\mu_o \boldsymbol{H}$.

Inside the ball, fields $\boldsymbol{H_1}$, $\boldsymbol{B_1}$, $\boldsymbol{M}$ are parallel. Using the continuity of B_r and H_θ at $r=R$, we have:

$$B_1\cos\theta=\mu_o \sum_{l=0}^{\infty} \frac{(l+1)a_l P_l(\cos\theta)}{R^{l+2}}$$

$$H_1 \sin\theta = \mu_o \sum_{l=0}^{\infty} \frac{a_l}{R^{l+2}} \frac{d\,P_l(\cos\theta)}{d\theta}$$

According to boundary conditions, the nonzero coefficients are those with $l=1$.

$$\boldsymbol{H_1} = \frac{\boldsymbol{B_1}}{\mu_o} - \boldsymbol{M} \rightarrow B_1 = \mu_o \frac{2a_1}{R^3} \quad , \; M - \frac{B_1}{\mu_o} = \frac{a_1}{R^3}$$

then:

$$a_1 = \frac{1}{3} M R^3 \quad , \quad B_1 = \frac{2}{3} \mu_o M$$

Outside the ball:

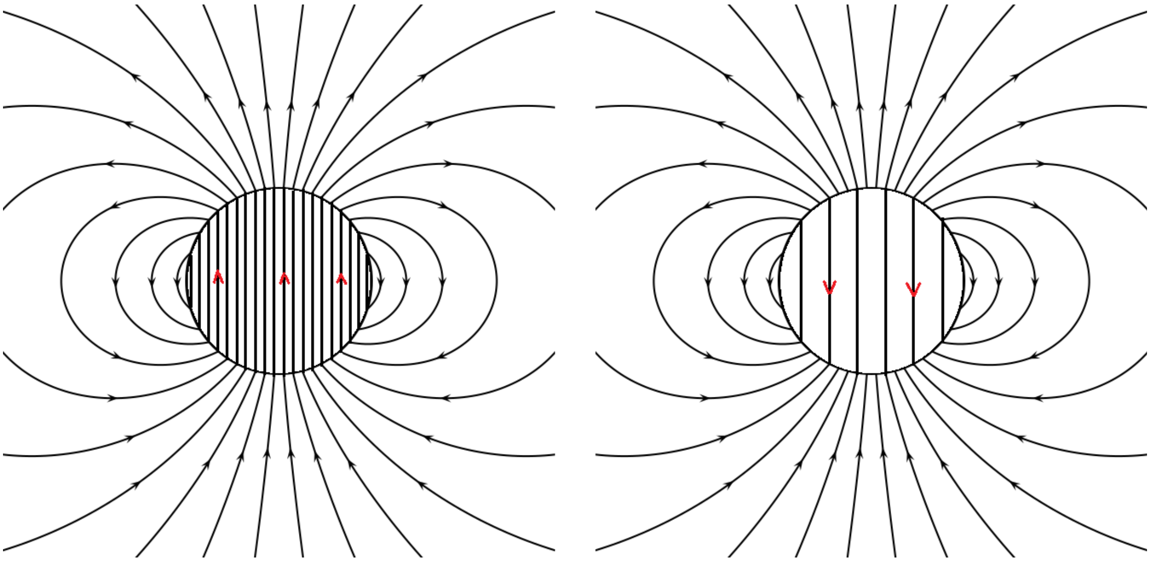
$$\boldsymbol{H_2} = \frac{R^3}{3}\left[3\boldsymbol{r}\frac{(\boldsymbol{M}\cdot\boldsymbol{r})}{r^5} - \frac{\boldsymbol{M}}{r^3}\right] \quad ; \quad \boldsymbol{B_2} = \mu_o \boldsymbol{H_2} \quad .$$

Therefore, outside the ball, the magnetic field is the field of a dipole having a magnetic moment:

$$m = \frac{4\pi}{3} R^3 M$$

Inside the ball: $\boldsymbol{B_1} = \frac{2}{3}\mu_o \boldsymbol{M}$; $\boldsymbol{H_1} = -\frac{1}{3}\boldsymbol{M}$.

In the following image, we can see $\boldsymbol{B}$, $\boldsymbol{H}$ inside the sphere. Note that the magnetic strength is a vector which is opposite to the magnetization.



The images are based on a drawing courtesy Geek3, for Wikipedia. Description tells "Drawing of a homogeneously magnetized spherical magnet with exactly computed magnetic field lines. A spherical magnet has the remarkable property that its field outside the magnet is identical to that of an ideal point-like dipole. Inside the magnetized volume, the field is exactly constant and aligned along the north-south axis." The field mentioned in the description is the vector magnetic induction (on the left). On the right the magnetic strength is illustrated.

63) Determine the magnetic field strength on the axis of a magnet of cylinder shape. The radius of the magnet is *R*. Its length is *d*. The magnetization is M_o [3].

We have again $\boldsymbol{H}=-\nabla\varphi_m$, where $\Delta\varphi_m=0$ in empty space.

Let us suppose $\boldsymbol{B}=\mu_o(\boldsymbol{H}+\boldsymbol{M})$. Since $\mathrm{div}\,\boldsymbol{B}=0$, $\mathrm{div}\,\boldsymbol{H}=-\mathrm{div}\,\boldsymbol{M}$.
We have the effective magnetic charge density:

$$\Delta\varphi_m=-\rho_m=-\mathrm{div}\,\boldsymbol{M}$$

If an interface is present: $\sigma_m=\boldsymbol{n}\cdot\boldsymbol{M}$, which is the effective magnetic surface change density. The magnetic potential is therefore:

$$\varphi_m(\boldsymbol{r})=-\frac{1}{4\pi}\int\frac{\mathrm{div}\,\boldsymbol{M}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}dV'-\frac{1}{4\pi}\oint\frac{M_n}{|\boldsymbol{r}-\boldsymbol{r}'|}dS'$$

$\boldsymbol{M}(\boldsymbol{r})$ is the magnetization vector of the magnet. S' is any closed surface that contains the magnet.
In our case, $\boldsymbol{M}=const$, $\mathrm{div}\,\boldsymbol{M}=0$. M_n is nonzero only on the basis of the magnet.

$$\varphi_m=\frac{M_o}{4\pi}\int\frac{dx\,dy}{[(z+d/2)^2+x^2+y^2]^{1/2}}-\frac{M_o}{4\pi}\int\frac{dx\,dy}{[(z-d/2)^2+x^2+y^2]^{1/2}}$$

$$\varphi_m=\frac{M_o}{2}\left[\sqrt{(z+d/2)^2+R^2}-|z+d/2|-\sqrt{(z-d/2)^2+R^2}+|z-d/2|\right]$$

Inside the magnet:

$$H_z=-\frac{\partial\varphi_m}{\partial z}=\frac{M_o}{2}\left(\frac{z+d/2}{[(z+d/2)^2+R^2]^{1/2}}+\frac{z-d/2}{[(z-d/2)^2+R^2]^{1/2}}-2\right)$$

The Poisson equation is $\Delta\varphi=f$, where Δ is the Laplace operator.
The Poisson equation is solved by the Green's function:

$$\varphi(\boldsymbol{r})=-\int\frac{f(\boldsymbol{r}')}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}dV'$$

Remember the potential given by a body distribution of charges.

---

64) A sphere of radius *R* rotates about the z-axis with angular velocity Ω . Its surface is electrically charged with a density σ_o . Find the vector potential and the magnetic field strength inside and outside the sphere [3].

The motion of the sphere generates a surface current. In spherical coordinates with the polar axis directed along the axis of rotation:

$$I_\phi=\sigma\,\Omega\,R\sin\theta$$

There are no currents either inside and outside the sphere. We can use a magnetic potential.

$$\boldsymbol{H}=-\nabla\psi_1 \quad r<R \qquad ; \qquad \boldsymbol{H}=-\nabla\psi_2 \quad r>R$$

$$\psi_1=\sum_{l=0}^{\infty}A_l r^l P_l(\cos\theta) \qquad ; \qquad \psi_1=\sum_{l=0}^{\infty}B_l r^{-(l+1)}P_l(\cos\theta)$$

Boundary conditions becomes:

$$\frac{\partial \psi_1}{\partial r} = \frac{\partial \psi_2}{\partial r} \quad ; \quad \frac{\partial \psi_1}{r \partial \theta} - \frac{\partial \psi_2}{r \partial \theta} = i_\phi$$

Therefore: $A_1 = \frac{2}{3} \sigma \Omega R$; $B_1 = \frac{1}{3} \sigma \Omega R^4$; $A_l = B_l = 0$, *for* $l \neq 1$.

Inside the sphere:

$$\boldsymbol{H} = \frac{2}{3} \sigma R \boldsymbol{\Omega}$$

The magnetic field has the direction of the z-axis, the axis of rotation.

Outside the sphere:

$$\boldsymbol{H} = \frac{3(\boldsymbol{m} \cdot \boldsymbol{r}) \boldsymbol{r}}{4 \pi r^5} - \frac{\boldsymbol{m}}{4 \pi r^3}$$

where $m = \frac{4 \pi}{3} R^4 \sigma \Omega$. Outside the sphere, the magnetic field is the field is the field of the magnetic moment $\boldsymbol{m}$.

Consider a specimen of magnetic material subjected to a constant magnetic field $\boldsymbol{H_o}$. A magnetization $\boldsymbol{M_o}$. parallel to $\boldsymbol{H_o}$, is created in the specimen. In addition to the constant magnetic field, a variable magnetic field $\boldsymbol{h}$, perpendicular to the constant field, and rotating with angular frequency ω . The amplitude of the field is $h \ll H_o$. Determine the additional magnetization and the condition of the resonance [3].

The equation governing the magnetic moment:

$$\frac{d\boldsymbol{M}}{dt}=\mu_o g[\boldsymbol{M}\times\boldsymbol{H}]$$

where the field is: $\boldsymbol{H}=\boldsymbol{H_o}+\boldsymbol{h}\,\exp(-i\omega t)$.

Let us look for the solution: $\boldsymbol{M}=\boldsymbol{M_o}+\boldsymbol{m}\,\exp(-i\omega t)$.

Since $h\ll H_o$, let us assume $m\ll M_o$.

$$-i\omega m_x=-\mu_o g M_o h_y+\mu_o g m_y H_o$$
$$-i\omega m_y=\mu_o g M_o h_x-\mu_o g m_x H_o$$
$$-i\omega m_z=0$$

$$m_x=\chi h_x-i\nu h_y \qquad m_x=\chi h_x-i\nu h_y \qquad m_z=0$$

In this solution:

$$\chi=g M_o\mu_o\frac{\omega_o}{\omega^2-\omega_o^2} \quad , \quad \nu=g M_o\mu_o\frac{\omega}{\omega^2-\omega_o^2} \quad ,$$

where : $\omega_o=\mu_o g H_o$. When ω approaches ω_o , we observe the resonance.

---

65) Find the self-inductance *L* per unit length of a transmission line that consists of two coaxial cylinders of radiuses R_1 and R_2 ($R_1<R_2$). The space between the conductors is filled with a substance having a permittivity μ [3].

The magnetic field between the coaxial conductors is $H=I/(2\pi r)$, where I is the current. The energy is:

$$W=\frac{1}{2}\int BH\,dV=\frac{\mu_o\mu I^2}{4\pi}\ \ln\frac{R_2}{R_1}$$

We have also:

$$W=\frac{1}{2}LI^2$$

Therefore:

$$L=\frac{\mu_o\mu}{2\pi}\ \ln\frac{R_2}{R_1}\quad .$$

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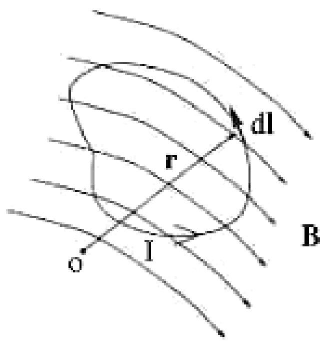
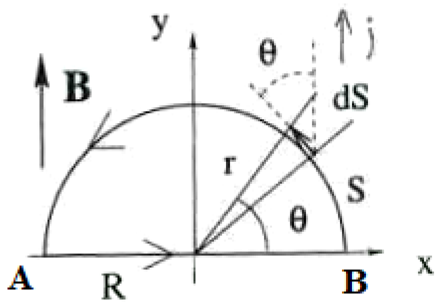


Let us consider a loop carrying a current *I*. The loop in in a magnetic field with induction $\boldsymbol{B}$. On the loop we have a force and a torque:

$$\boldsymbol{F}=I\oint d\boldsymbol{l}\times\boldsymbol{B}\ \ ;\ \ \boldsymbol{M_o}=I\oint \boldsymbol{r}\times d\boldsymbol{l}\times\boldsymbol{B}$$



66) Let us consider the semicircular loop with a current *I*, as in the figure. The loop is in the *xy*-plane. Find the force on the loop.

$$\boldsymbol{F}=\boldsymbol{F}_{\boldsymbol{AB}}+\boldsymbol{F}_{\boldsymbol{BA}}=I\int_A^B d\boldsymbol{l}\times\boldsymbol{B}+I\int_B^A d\boldsymbol{l}\times\boldsymbol{B}=I\int_A^B dx\,\boldsymbol{i}\times B\,\boldsymbol{j}+I\int_B^A ds\,\boldsymbol{t}\times B\,\boldsymbol{j}$$

where $\boldsymbol{t}$ is the unit vector tangent the curve. Then:

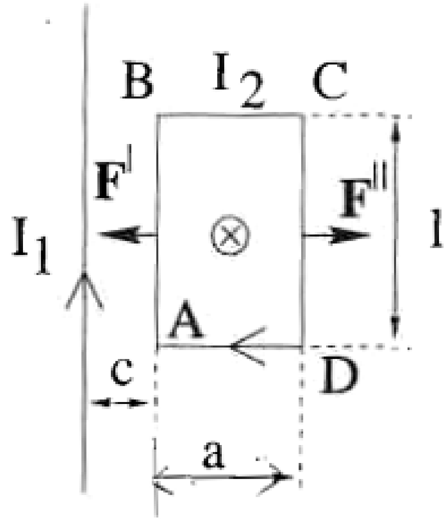
$$\boldsymbol{F}=\boldsymbol{k}\left(2\,IRB-IRB\int_0^\pi \sin\theta\,d\theta\right)=0$$

67) Calculate the force between two straight parallel wires of infinite length with electric current I_1 and I_2 if the distance between the wires is d. The permittivity of the medium between the conductors is μ [3].

$$F=\frac{\mu\,\mu_o}{2\pi}\frac{I_1 I_2}{d}$$

This is the modulus of the force vector. The force is attractive, when the currents are flowing in the same direction, repulsive when they are flowing opposite.

This problem is discussed in all the physics books. Suggested reading: https://courses.lumenlearning.com/physics/chapter/22-10-magnetic-force-between-two-parallel-conductors/



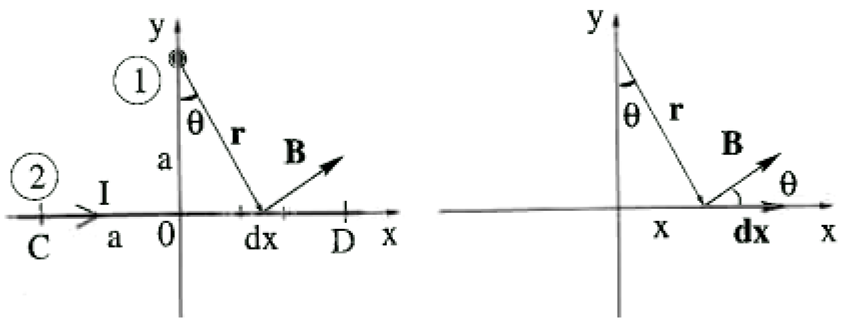
68) A straight wire is carrying current I_1 . It is in the same plane where there is the loop ABCDA in the following figure. The loop is carrying current I_2 . The sides of the loop, AB and CD, are parallel to the wire, and the sides DA and BC, perpendicular. The length are *l* and *a,* as given in the figure. The distance of AB from the wire is *c*. Find the force on the loop [10].

$$\frac{F'}{l}=\frac{\mu_o}{2\pi}\frac{I_1 I_2}{c}$$

$$\frac{F''}{l}=\frac{\mu_o}{2\pi}\frac{I_1 I_2}{c+a}$$

$$\frac{F'}{l}-\frac{F''}{l}=\frac{\mu_o}{2\pi}I_1 I_2\left(\frac{1}{c}-\frac{1}{c+a}\right)=\frac{\mu_o}{2\pi}I_1 I_2\frac{a}{c(c+a)}$$

The force on the loop is directed towards the wire.



69) Two straight wires 1 and 2 are placed as shown in the figure. They are carrying the same current I . Find the torque on the segment CD of wire 2 [10].

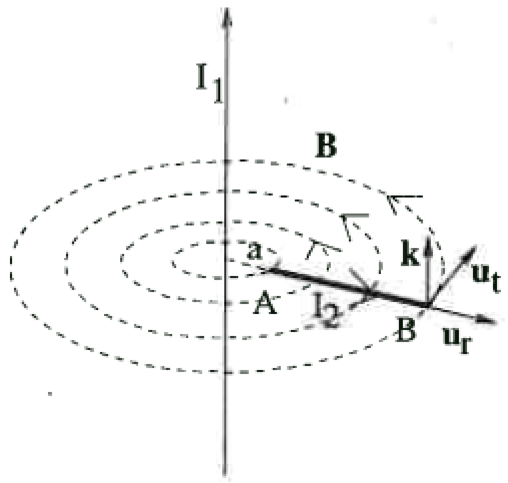
Wire 1 creates the field: $B=\frac{\mu_o I}{2\pi r}$.

$$d\boldsymbol{F}=I\,d\boldsymbol{x}\times\boldsymbol{B} \quad ; \quad dF=I\,B\,dx\sin\theta=\frac{\mu_o I^2}{2\pi}\frac{dx}{r}\sin\theta$$

$$d\tau_o=x\,dF=r\sin\theta\,dF=\mu_o I^2\,\frac{dx}{2\pi}\,\sin^2\theta=\mu_o I^2\,\frac{a\,d\theta}{2\pi}\,\tan^2\theta$$

$$\tau_o=\mu_o I^2\,\frac{a}{2\pi}\int_{-\pi/4}^{\pi/4}\tan^2\theta\,d\theta$$

The torque turns the wire 2 to have it parallel to 1.



70) A rod AB with length l is in the field of magnetic induction generated by a straight infinitely long wire carrying current I_1. The rod is perpendicular to the wire, as given by the figure. AB is carrying a current I_2. Find the force on AB.

$$\boldsymbol{B}=\frac{\mu_o I_1}{2\pi r}\,\boldsymbol{u_t} \quad ; \quad d\boldsymbol{F}=I_2\, d\boldsymbol{l}\times\boldsymbol{B} \quad ; \quad d\boldsymbol{l}=dr\,\boldsymbol{u_r} \quad ; \quad \boldsymbol{u_r}\times\boldsymbol{u_t}=\boldsymbol{k}$$

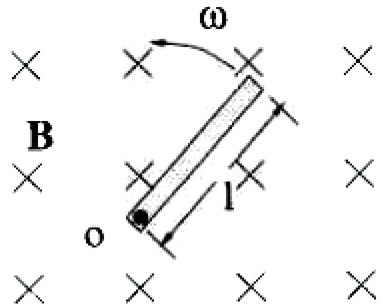
$$\boldsymbol{F}=\int_B^B d\boldsymbol{F}=\frac{\mu_o I_1 I_2}{2\pi}\int_a^{a+l}\frac{dr}{r}\,\boldsymbol{k}=\frac{\mu_o I_1 I_2}{2\pi}\ln\frac{a+l}{a}\,\boldsymbol{k}$$

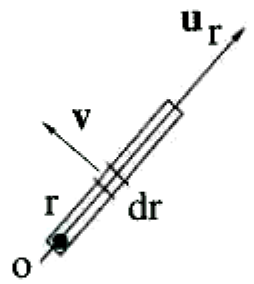
Quasi-stationary fields

Magnetic flux and Induced Emf

For a discussion, see for instance https://courses.lumenlearning.com/physics/chapter/23-

1-induced-emf-and-magnetic-flux/

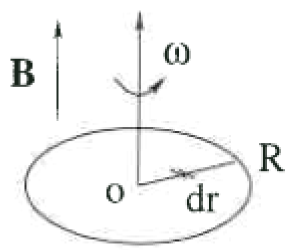




71) A rod made of conducting material and having a length l is rotating with angular constant velocity ω about O. Th field $\boldsymbol{B}$ is perpendicular to the plane where the rod is rotating. Find the induced tension in the rod.

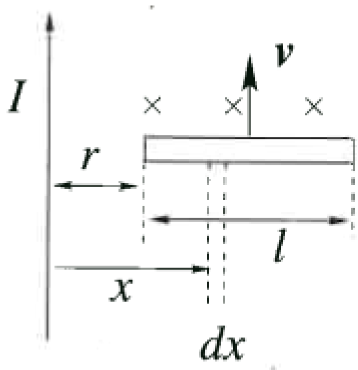
Each piece of the rod dr has a velocity $v=\omega r$. Each element gives a contribution:

$$d\varepsilon=\boldsymbol{v}\times\boldsymbol{B}\cdot\boldsymbol{u_r}\,dr=\omega B r\,dr \quad ; \quad \varepsilon=\int_0^l \omega B r\,dr=\frac{1}{2}B\omega l^2$$



72) Let us consider a disc made of conducting material. The disc rotates about its centre with ω , constant angular velocity. Find the tension between the rim of the disc and its centre [10].

$$\varepsilon = \int_0^R B\,\omega\,r\,dr = \frac{1}{2} B\,\omega\,R^2$$



73) A rod, having length *l*, moves with speed *v*, as shown in the figure. The velocity is parallel to the wire carrying the current *I*. The wire and the rod are in the same plane. Find the tension between the ends of the rod.

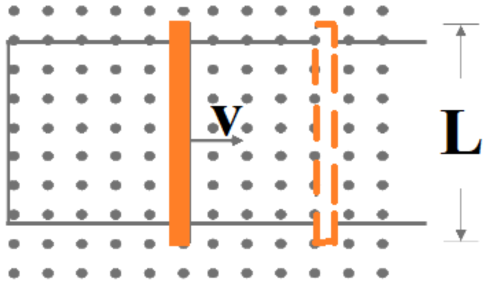
The field **B** is not uniform. The force of Lorentz on the charges is given by:

$$\boldsymbol{F} = q\,\boldsymbol{v}\times\boldsymbol{B} \quad ; \quad \boldsymbol{B}(x) = q\,\boldsymbol{v}\times\boldsymbol{B}(x)$$

$$F(x) = q\,v\,B(x) = q\,v\frac{\mu_o}{2\pi}\frac{I}{x} \quad ; \quad E(x) = v\,B(x) = v\frac{\mu_o}{2\pi}\frac{I}{x}$$

$$|\varepsilon| = \int_r^{r+l} E(x)\,dx = \frac{\mu_o v}{2\pi}\;I\;\ln\left(\frac{r+l}{r}\right)$$

Motional emf - It is well known that Faraday and Henry discovered that a current in a loop can be created by moving a magnet, and also by moving a loop through a magnetic field. This method of having a current is the *electromagnetic induction*. For instance, if we move a conducting rod perpendicularly to a magnetic field, a magnetic force on its charges sends the electrons to one end, creating an excess of positive charges at the other end. As a consequence, we have a polarization of charge giving *potential difference* or *emf* ε between the ends of the rod. If we have a loop linked to the rod, the induced ε produces a current I .

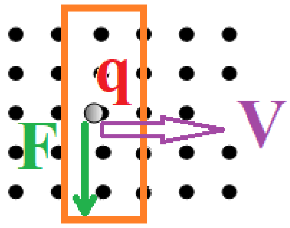


The field B is perpendicular to the page, towards the reader. The motional emf produced in the rod is $\varepsilon = vBL$. If the circuit has a resistance R, the induced current is:

$$I = \frac{\varepsilon}{R} = \frac{vBL}{R}$$

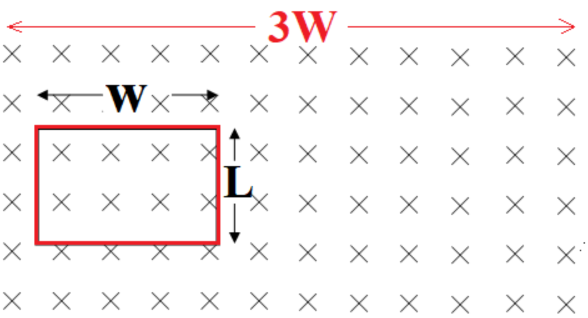
74) In the figure given above, the rod is moving to the right. Find the direction of the induced emf in the rod. That is, what is the positive end of the rod?

Consider for instance a positive charge inside the rod. As the charge moves to the right with the rod, we can determine the direction of the magnetic force on the charge.

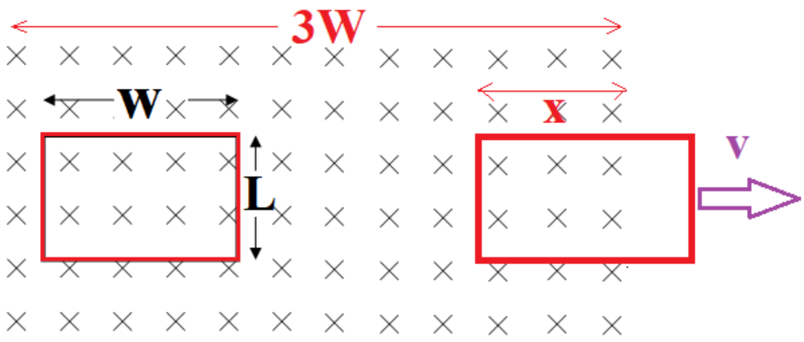


The force on a positive charge in the rod is downward toward the bottom of the page. Therefore, the lower end in the figure is positive.

75) A rectangular loop of height *L* and width *w is* in the plane of the figure. In the space there is a region of magnetic field wide *3w*. The magnetic field is directed into the page.



The loop is in the plane of the figure. Its surface is perpendicular to the magnetic field lines. The *magnetic flux* through the loop is the scalar product of the field ***B*** and the surface ***A*** through which it fluxes: $\Phi = \boldsymbol{B} \cdot \boldsymbol{A}$. If the loop is at rest in the field, the flux is equal to $BA = BLx$, where x is the part of w in the field.



But if the loop is moving in the magnetic field with speed *v*, the flux through the loop does not remain constant and changes with time. The *Faraday Law of induction* states that an emf ε will be induced in a loop of wire through which the flux is changing:

$$\varepsilon=-\frac{d\Phi}{dt}$$

The induced emf in the loop is caused by the *change in flux*.

The "minus" sign is given by the Lenz Law. This law states that the induced emf in the conductor is responsible of a change in flux, which opposes the change in flux that produced it. It means that the induced emf the induced change in flux is against the old change in flux. Actually, the reason of this law is in the conservation of energy.

The the emf induced in the loop as the loop is passing through the magnetic field is

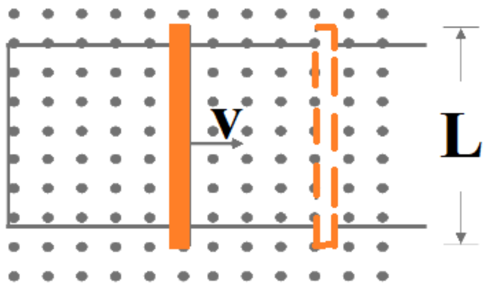
$$\varepsilon=-\frac{\mathrm{d}\Phi}{dt}=-\frac{d(BA)}{dt} \quad ; \quad \varepsilon=-B\frac{[d(Lx)]}{dt}=-BL\left(\frac{dx}{dt}\right)=-BLv$$

The current can be found by Ohm's law:

$$I=\frac{\varepsilon}{R}=-\frac{BLv}{R}$$

In the case depicted in the figure previously given, we have that the current is flowing in a direction when the loop is entering the field, and in the opposite direction, when the it is leaving the field. Note that when the loop is all inside or outside the field, the current is zero, because we have no emf.

76) A conducting rod of length 0.30 m and resistance 10.0 Ω moves with a speed of 2.0 m/s through a magnetic field of 0.20 T which is directed out of the page. Find emf.

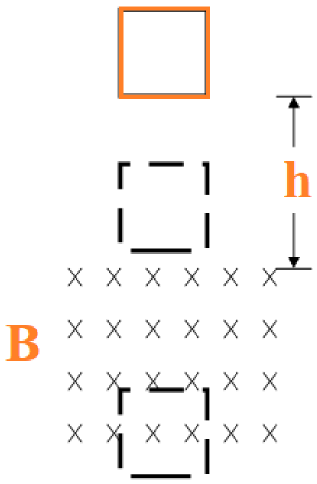


Neglecting the resistance of the other parts of the loop, evaluate the current and the dissipated power.

77) A square loop of side a = 0.4 m, mass m = 1.5 kg, and resistance 5.0 Ω falls from rest from a height h = 1.0 m toward a uniform magnetic field **B** which is directed into the page as shown by the figure.
Determine the speed of the loop just before it enters the magnetic field. As soon as the loop enters the magnetic field, an emf ε and a current I is induced in the loop.
Is the direction of the induced current in the loop clockwise or counter-clockwise?

Let us suppose that, when the loop enters the magnetic field, it falls through with a constant velocity.

Evaluate the magnetic induction B, required to keep the loop falling at a constant velocity.



Conservation of energy :

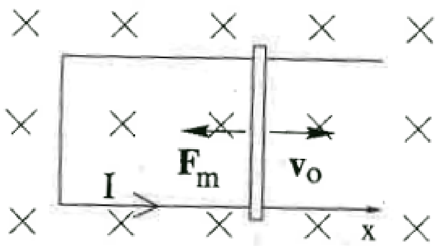
$$U_{in}=K_{fin}$$
$$mgh=\frac{1}{2}mv$$
$$v=\sqrt{2gh}=\sqrt{2(10\,m/s^2)(1.0\ m)}=4.5\ m/s$$

The current moves around the loop counter-clockwise.

To have the loop moving with constant speed, we need a magnetic force upward just balances the weight of the loop downward.

$$F_B=mg=(1.5\ kg)(10\ m/s^2)=15\ N$$

78) A rod of mass *m* and length *l* moves on two parallel wires without friction, as in the figure. The rod has an initial speed v_o . Find the speed of the rod as a function of time [10].



The magnetic force is: $F_m = -IlB$. From Newton's Law:

$$ma = m\frac{dv}{dt} = -IlB \quad ; \quad m\frac{dv}{dt} = -\frac{\varepsilon}{R} lB = -\frac{Blv}{R} lB \ .$$

R is the resistance of the loop.

$$m\frac{dv}{dt} = -\frac{B^2 l^2 v}{R} \rightarrow \int_{v_o}^{v} \frac{dv'}{v'} = -\int_{t_o}^{t} \left(\frac{B^2 l^2}{mR}\right) dt'$$

$$\ln\frac{v}{v_o} = -\frac{B^2 l^2}{mR} t \rightarrow v = v_o \exp\left(-t B^2 l^2 / mr\right)$$

79) A circular loop of wire has a radius of 0.025 m and a resistance of 3.0 Ω. It is placed in a 1.6 T magnetic field which is directed through the loop perpendicularly to its surface and then turned off uniformly, over a period of 0.10 s. What is the current in the wire during the time that the magnetic field changes from 1.6 T to zero?

$$|\varepsilon| = \frac{\Delta\Phi}{\Delta t} = \pi r^2 \frac{\Delta B}{\Delta t} = 0.031\, V$$

Felici's Law

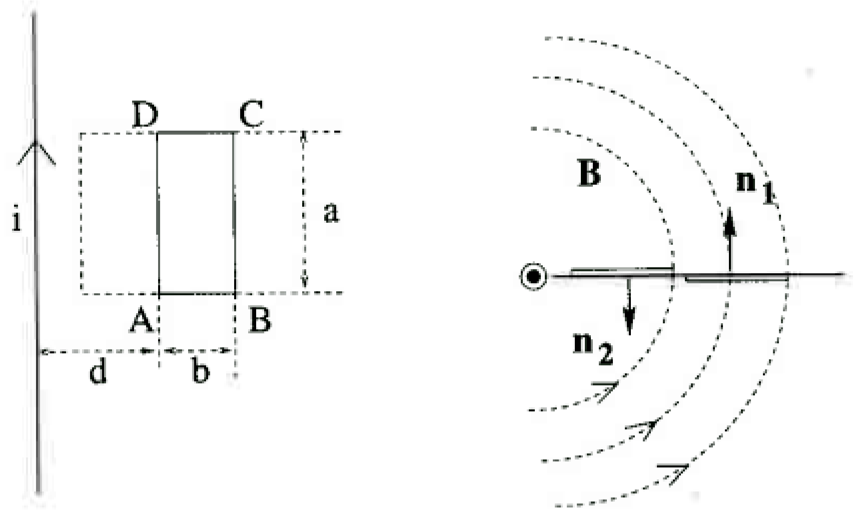
This law allows one to calculate the net charge through a circuit configuration in which there is a current induced by a variable magnetic field, such a conductor coil immersed in a variable magnetic field. By means of this law, we calculate net charge in a period using initial flux and final flux [10].

$$q(t) = \frac{1}{\mathrm{R}}[\Phi(0) - \Phi(t)]$$

In fact:

$$q(t) = \int_0^t i(\tau)d\tau = \frac{1}{\mathrm{R}}\int_0^t f_{em}(\tau)d\tau = \frac{1}{\mathrm{R}}[\Phi(0) - \Phi(t)]$$

When $f_{em}(t) = -\dfrac{d\Phi(t)}{dt}$



80) A loop ABCDA has a resistance R in in a plane where there is a very long straight wire carrying current *i*. The loop turns of 180° about the side AD, parallel to the wire, at a distance *d*. Find the charge flowing in the loop during the rotation.

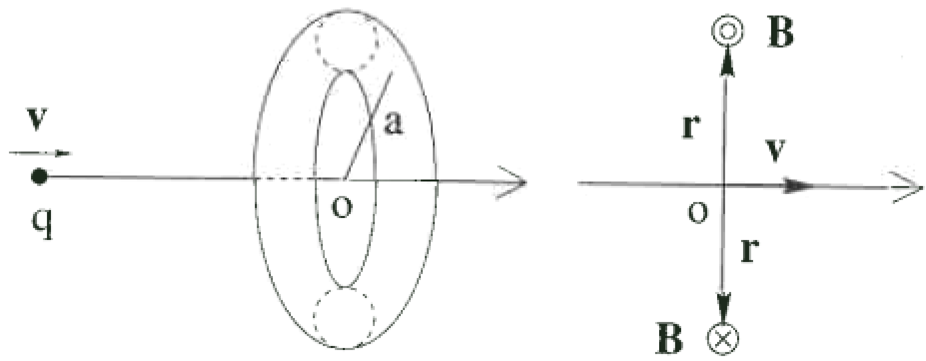
We have an initial and a final flux.

$$\Phi_1 = \frac{\mu_o i a}{2\pi}\ln\frac{d+b}{d}$$

$$\Phi_2 = -\frac{\mu_o i a}{2\pi}\ln\frac{d}{d-b}$$

Felici's Law tells:

$$|q| = \frac{|\Delta\Phi|}{R} = \frac{|\Phi_1 - \Phi_2|}{R} = \frac{\mu_o i a}{2\pi R} \ln \frac{d+b}{d-b}$$



81) In the figure it is shown a toroidal solenoid. Evaluate the total charge Q which moves in the solenoid, from the beginning of the motion to the time instance when the charge passes in O [10].

The field generated by the moving charge is: $\boldsymbol{B} = \frac{\mu_o}{4\pi} q \frac{\boldsymbol{v} \times \boldsymbol{r}}{r^3}$.

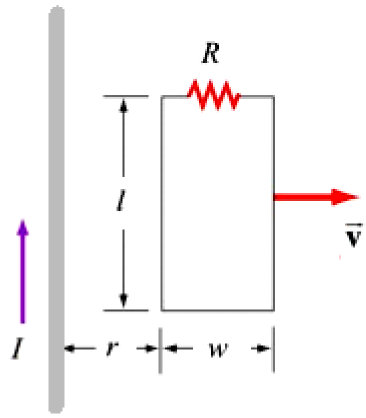
When the particle is far from the solenoid, the flux in it is zero. When the particle is passing in O, the flux through the N loops:

$$\Phi_{spira} = A \frac{\mu_o q v}{4\pi a^2}$$

Then, according to Felici's Law:

$$Q = \frac{\Phi_{tot}}{R} = N A \frac{\mu_o q v}{4\pi a^2 R}$$

82) A rectangular loop of sides with lengths l and w moves with a constant velocity away from an infinitely long straight wire carrying a current I in the plane of the loop, as shown in the figure. The total resistance of the loop is R.



Using Ampere's law, find the magnetic field at a distance s away from the straight current-carrying wire.

What is the magnetic flux through the rectangular loop at the instant when the lower side with length is at a distance r away from the straight current-carrying wire, as shown in the figure?

At the instant the lower side is a distance r from the wire, find the induced emf and the corresponding induced current in the rectangular loop. Which direction does the induced current flow?

$$\oint \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_o I \quad \text{, then} \quad B = \frac{\mu_o I}{2\pi s} \quad \text{into the page.}$$

$$\Phi = \int \boldsymbol{B} \cdot \boldsymbol{n}\, dS = \int_r^{r+w} \left(\frac{\mu_o I}{2\pi s} \right) l\, ds = \frac{\mu_o I l}{2\pi} \ln \frac{r+w}{r}$$

$$\epsilon=-\frac{d\Phi}{dt}=\frac{\mu_o I l}{2\pi}\frac{vw}{r(r+w)}$$

Poynting vector

The Poynting vector represents the energy flux (the energy transfer per unit area per unit time) of an electromagnetic field. The SI unit of the Poynting vector is the watt per square metre (W/m^2). It is named after its discoverer John Henry Poynting, who first derived it in 1884. Oliver Heaviside also discovered it independently in the more general form that recognises the freedom of adding the curl of an arbitrary vector field to the definition.

Let us remember some formulas previously given:

$$\text{rot}\,\text{grad}\, f=0 \quad (1) \quad ; \quad \text{div}\,\text{rot}\,\boldsymbol{A}=0 \quad (2)$$
$$\text{rot}\,\text{rot}\,\boldsymbol{A}=\text{grad}\,\text{div}\,\boldsymbol{A}-\Delta\boldsymbol{A} \quad (3) \quad ; \quad \text{div}(f\,\boldsymbol{A})=f\,\text{div}\,\boldsymbol{A}+\text{grad}\,f\cdot\boldsymbol{A} \quad (4)$$
$$\text{div}(\boldsymbol{A}\times\boldsymbol{B})=\boldsymbol{B}\cdot\text{rot}\,\boldsymbol{A}-\boldsymbol{A}\cdot\text{rot}\,\boldsymbol{B} \quad (5) \quad ; \quad \text{rot}(f\,\boldsymbol{A})=\text{grad}\,f\times\boldsymbol{A}+f\,\text{rot}\,\boldsymbol{A} \quad (6)$$

(5) is interesting for a discussion concerning the Poynting vector [2]. Let us assume $\boldsymbol{A}\times\boldsymbol{B}=\boldsymbol{S}=\frac{1}{\mu}\boldsymbol{E}\times\boldsymbol{B}$, where vector **E** is the electric field and **B** the magnetic induction.

$$\frac{1}{\mu}\text{div}(\boldsymbol{E}\times\boldsymbol{B})=\frac{1}{\mu}(\boldsymbol{B}\cdot\text{rot}\,\boldsymbol{E}-\boldsymbol{E}\cdot\text{rot}\,\boldsymbol{B}) \quad (7)$$

Due to Maxwell equations in the empty space:

$$\frac{1}{\mu}\text{div}(\boldsymbol{E}\times\boldsymbol{B})=-\frac{1}{\mu}\left(\boldsymbol{B}\cdot\frac{\partial\boldsymbol{B}}{\partial t}+\mu\epsilon\,\boldsymbol{E}\cdot\frac{\partial\boldsymbol{E}}{\partial t}\right)=-\frac{\partial}{\partial t}[u_B+u_E] \quad (8)$$

where $u = u_B + u_E$ is the energy per unit of volume. Therefore:

$$\mathrm{div}(\boldsymbol{S}) = -\frac{\partial u}{\partial t} \quad (9)$$

(9) is written in the case of an empty space, without densities of currents.
Let us consider ***B*** *uniform* in the volume we are considering. Let us suppose that ***B*** is uniform as the time goes on. The curl is zero. Using (7) and (8):

$$\frac{1}{\mu}\mathrm{div}(\boldsymbol{E}\times\boldsymbol{B}) = \frac{1}{\mu}(\boldsymbol{B}\cdot\mathrm{rot}\,\boldsymbol{E}) \;=\; -\frac{1}{\mu}\left(\boldsymbol{B}\cdot\frac{\partial\boldsymbol{B}}{\partial t}\right) = -\frac{\partial}{\partial t}[u_B] \quad (10)$$

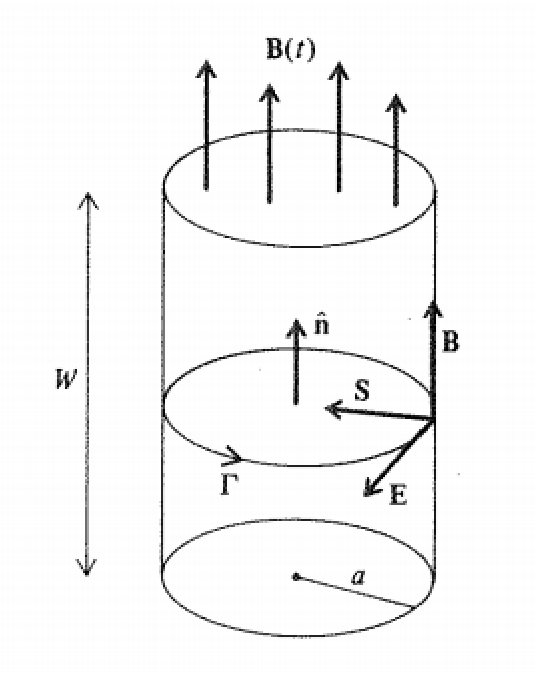
In (10), we do not find the derivative of the energy of the electric field. Let us use Maxwell. We have $\mathrm{rot}\,\boldsymbol{B} = \mu\epsilon\,\partial\boldsymbol{E}/\partial t$. If the magnetic field ***B*** is uniform, and it remains uniform during the time, its curl is always zero. The electric field must be constant in time, and the same for the energy of this field. That is, the electric field does not gain or lose energy. Since ***E*** is constant in time, $\mathrm{rot}\,\boldsymbol{E} = -\partial\boldsymbol{B}/\partial t$ gives that the variation of the magnetic induction is constant and therefore ***B*** is a linear function of time.
Now, let us consider a uniform electric field, constant in time. Its curl is zero. Using again $\mathrm{rot}\,\boldsymbol{E} = -\partial\boldsymbol{B}/\partial t$, we have that the magnetic induction is constant.
Therefore:

$$\frac{1}{\mu}\mathrm{div}(\boldsymbol{E}\times\boldsymbol{B}) = -\frac{1}{\mu}(\boldsymbol{E}\cdot\mathrm{rot}\,\boldsymbol{B}) \;=\; -\epsilon\left(\boldsymbol{E}\cdot\frac{\partial\boldsymbol{E}}{\partial t}\right) = -\frac{\partial}{\partial t}[u_E] \quad (11)$$

Since **B** is constant, Maxwell gives $\mathrm{rot}\,\boldsymbol{B} = \mu\epsilon\,\partial\boldsymbol{E}/\partial t$, an electric **E** which is uniform must be a linear function of time.
These relations explain some difficulties in solving exercises that we find in textbooks and on the web, and that are concerning volumes with uniform electric (in flat

capacitors) and magnetic (solenoids or air gap) fields, but that are changing over time. We must therefore pay close attention to the formulation of problems.

Let us see two examples.



"Una opportuna distribuzione di correnti provoca in una regione dello spazio avente forma cilindrica con raggio a e lunghezza W l'insorgere di un campo magnetico uniforme diretto come l'asse del cilindro e avente modulo variabile nel tempo. Si determini il campo elettrico sulla superficie laterale del cilindro supponendo che la densità di carica elettrica netta sia nulla in tutto lo spazio. Si mostri inoltre che la variazione nel tempo dell'energia del campo magnetico nella regione considerata è uguale al flusso del vettore di Poynting attraverso la sua superficie laterale." [13]. A certain distribution of currents causes, in a region of the space having a cylindrical shape with radius a and length W, the presence of a uniform magnetic field directed as the axis of the cylinder and having a modulus that varies over time. Determine the electric field on the lateral surface of the cylinder assuming that the net electric charge density is zero in all space. Moreover, show that the variation in time of the energy of the magnetic field in the considered region is equal to the flux of the Poynting vector through its lateral surface

Using Faraday-Lenz we have that the induced electric field on the surface of the cylinder is:

$$\boldsymbol{E}=-\frac{a}{2}\frac{\partial B}{\partial t}\boldsymbol{u}_{\phi} \qquad (12)$$

$\boldsymbol{u}_{\phi}$ is the unit vector tangent to curve Γ, as given in the figure.
The flux of the Poynting vector can be easily calculated as:

$$\Phi_{\boldsymbol{S}}=\pi\epsilon C^2 a^2 W B\frac{\partial B}{\partial t} \qquad (13)$$

If we evaluate the energy of the magnetic induction ***B*** inside the cylinder, we have $U_{\boldsymbol{B}}=\frac{\pi}{2}\epsilon C^2 a^2 W B^2$, and therefore (13).

However, according to our previous discussion, the statement of the problem is right only in the case that the uniform field **B**(t) is a linear function of time. In this manner, the derivative with respect to time of the electric field is zero and the energy of the electric field is constant. If the magnetic field were a non-linear function with respect to time, the problem would be not properly stated, in the case that we ask that it also be uniform. The flux of the Poynting vector could not be equal only to the variation of magnetic energy. It should also involve the variation of the energy linked to the electric field.

Let us consider another problem [14].

"Un condensatore piano con armature circolari, caricato alla d.d.p. V_o, viene lasciato scaricare attraverso un resistore di resistenza R_o. Calcolare il flusso totale di energia dall'interno all'esterno del condensatore durante la scarica". La soluzione recita: "Dobbiamo calcolare i campi **E** e **B** e da questi il vettore di Poynting **S** durante la scarica". A flat capacitor with circular armatures, charged to voltage V_o, can discharge through a resistance resistor R_o. Calculate the total energy taht flows from the inside to the outside of the capacitor during the discharge. The solution given by the boo [14] tells that We must calculate the fields **E** and **B** and, from them, the Poynting vector **S** during the discharge.

The proposed solution does not tell the field **E** inside the capacitor be uniform; however it must be so due to the modulus of the induced **B,** given at page 359, $B=\frac{\mu\epsilon R}{2}\frac{\partial E}{\partial t}$.

The solution given by the book tells that this is the modulus of the magnetic field on the lateral surface of the cylinder, representing the geometry of the capacitor, having a circular section of radius *R*. The modulus of the Poynting vector is:

$$S=\frac{EB}{\mu}=\epsilon\frac{R}{2}E\frac{\partial E}{\partial t} \quad (14)$$

The book tells that this modulus is equal to the variation of the energy linked to the presence of the electric field.

The problem must therefore be consídred with caution. In the discharge process of the capacitor, the variation of the electric field with respect to time is not linear. The derivative of the electric field with respect to time is not a constant and consequently the magnetic field is a function of time. If this field is a function of time, its energy also comes into play together with the energy due to the presence of the electric field. Since the variation of the energy is linked to the flow of the Poynting vector, (14) is valid only if the variation of the energy due to the magnetic field is zero, or if we consider it negligible. Moreover, even the assumption of an electric field in the capacitor as uniform is an approximation.

References

[1] Smirnov V. I., Corso di Matematica Superiore, Editori Riuniti, 1978, Volume 2, Pagina 391.

[2] Sparavigna, Amelia Carolina. (2019, December 18). La divergenza del prodotto vettoriale ed il vettore di Poynting. Zenodo. http://doi.org/10.5281/zenodo.3583296

[3] Grechko, L.G., Sugakov, V.I., Tomasevich, O.F., & Fedorchenko, A.M. (1977).

Problems in Theoretical Physics. Mir Publishers, Moscow.
[4] Barbero, G., Sparavigna, A., & Strigazzi, A. (1990). The structure of the distortion free-energy density in nematics: second-order elasticity and surface terms. Il Nuovo Cimento D, 12(9), 1259-1272.
[5] Feynman, R.P., Leighton, R.B., & Sands, M. (1975). The Feynman Lectures on Physics, Volume II, Part I. Inter European Editions.
[6] Kirk T. McDonald (2000). Complex Functions and Electrostatics. Publication Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544 (February 28, 2000).
[7] Jeans, James H. (1908). The Mathematical Theory of Electricity and Magnetism. Cambridge University Press.
[8] Tikhonov, Andrey N.; Samarskii, Alexander A. (1963). Equations of Mathematical Physics. New York: Dover Publications. p. 354. ISBN 0-486-66422-8.
[9] Lifšits Evgenij M., & Pitaevskij, Lev P. (1986). Elettrodinamica dei mezzi continui, Editori Riuniti.
[10] Sparavigna, A. (1997). Fisica II. Esercizi e Prove d'Esame. Società Editrice Esculapio, Bologna.
[11] Nino Zanghi. Introduzione ai Metodi Matematici della Fisica. Complementi. https://www.ge.infn.it/~zanghi/metodi/ZULUC09.pdf
[12] Nigro, M., & Voci, C. (1986). Problemi di Fisica Generale - Elettromagnetismo - Ottica. Edizioni Libreria Cortina. Padova.
[13] Luigi Quartapelle, Cristina Lenardi, Martino Travagnin. Fisica 2. Problemi d'esame svolti (1993-1995). Esculapio.
[14] Massimo Nigro e Cesare Voci. Fisica Generale, Elettromagnetismo Ottica, Edises 1986, pagina 359.