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RESEARCH



# On the least common multiple of random $q$ -integers

Carlo Sanna\*

\*Correspondence:  
carlo.sanna.dev@gmail.com  
Department of Mathematical  
Sciences, Politecnico di Torino,  
Corso Duca degli Abruzzi 24,  
10129 Torino, Italy  
C. Sanna is a member of GNSAGA  
of INdAM and of CryptTO, the  
group of Cryptography and  
Number Theory of Politecnico di  
Torino

## Abstract

For every positive integer  $n$  and for every  $\alpha \in [0, 1]$ , let  $\mathcal{B}(n, \alpha)$  denote the probabilistic model in which a random set  $\mathcal{A} \subseteq \{1, \dots, n\}$  is constructed by picking independently each element of  $\{1, \dots, n\}$  with probability  $\alpha$ . Cilleruelo, Rué, Šarka, and Zumalacárregui proved an almost sure asymptotic formula for the logarithm of the least common multiple of the elements of  $\mathcal{A}$ . Let  $q$  be an indeterminate and let  $[k]_q := 1 + q + q^2 + \dots + q^{k-1} \in \mathbb{Z}[q]$  be the  $q$ -analogue of the positive integer  $k$ . We determine the expected value and the variance of  $X := \deg \text{lcm}([\mathcal{A}]_q)$ , where  $[\mathcal{A}]_q := \{[k]_q : k \in \mathcal{A}\}$ . Then we prove an almost sure asymptotic formula for  $X$ , which is a  $q$ -analogue of the result of Cilleruelo et al.

**Keywords:** Asymptotic formula, Least common multiple,  $q$ -analogue, Random set

**Mathematics Subject Classification:** Primary: 11N37, Secondary: 11B99

## 1 Introduction

A nice consequence of the Prime Number Theorem is the asymptotic formula

$$\log \text{lcm}(1, 2, \dots, n) \sim n, \quad \text{as } n \rightarrow +\infty, \quad (1)$$

where  $\text{lcm}$  denotes the least common multiple. Indeed, precise estimates for  $\log \text{lcm}(1, \dots, n)$  are equivalent to the Prime Number Theorem with an error term. Thus, a natural generalization is to study estimates for  $L_f(n) := \log \text{lcm}(f(1), \dots, f(n))$ , where  $f$  is a well-behaved function, for instance, a polynomial with integer coefficients. (We ignore terms equal to 0 in the  $\text{lcm}$  and we set  $\text{lcm} \emptyset := 1$ .) When  $f \in \mathbb{Z}[x]$  is a linear polynomial, the product of linear polynomials, or an irreducible quadratic polynomial, asymptotic formulas for  $L_f(n)$  were proved by Bateman et al. [3], Hong et al. [10], and Cilleruelo [6], respectively. In particular, for  $f(x) = x^2 + 1$ , Rué et al. [15] determined a precise error term for the asymptotic formula. When  $f$  is an irreducible polynomial of degree  $d \geq 3$ , Cilleruelo [6] conjectured that  $L_f(n) \sim (d - 1)n \log n$ , as  $n \rightarrow +\infty$ , but this is still an open problem. However, bounds for  $L_f(n)$  were proved by Maynard and Rudnick [13], and Sah [16]. Moreover, Rudnick and Zehavi [14] studied the growth of  $L_f(n)$  along a shifted family of polynomials.

Another direction of research consists in considering the least common multiple of random sets of positive integers. For every positive integer  $n$  and every  $\alpha \in [0, 1]$ , let

$\mathcal{B}(n, \alpha)$  denote the probabilistic model in which a random set  $\mathcal{A} \subseteq \{1, \dots, n\}$  is constructed by picking independently each element of  $\{1, \dots, n\}$  with probability  $\alpha$ . Cilleruelo et al. [9] studied the least common multiple of the elements of  $\mathcal{A}$  and proved the following result (see [1] for a more precise version, and [4, 5, 7, 8, 12, 17–19] for other results of a similar flavor).

**Theorem 1.1** *Let  $\mathcal{A}$  be a random set in  $\mathcal{B}(n, \alpha)$ . Then, as  $\alpha n \rightarrow +\infty$ , we have*

$$\log \text{lcm}(\mathcal{A}) \sim \frac{\alpha \log(1/\alpha)}{1 - \alpha} \cdot n,$$

with probability  $1 - o(1)$ , where the factor involving  $\alpha$  is meant to be equal to 1 for  $\alpha = 1$ .

*Remark 1.1* In the deterministic case  $\alpha = 1$ , we have  $\mathcal{A} = \{1, \dots, n\}$  (surely) and Theorem 1.1 corresponds to (1).

Let  $q$  be an indeterminate. The  $q$ -analog of a positive integer  $k$  is defined by

$$[k]_q := 1 + q + q^2 + \dots + q^{k-1} \in \mathbb{Z}[q].$$

The  $q$ -analogs of many other mathematical objects (factorial, binomial coefficients, hypergeometric series, derivative, integral...) have been extensively studied, especially in Analysis and Combinatorics [2, 11]. For every set  $\mathcal{S}$  of positive integers, let  $[\mathcal{S}]_q := \{[k]_q : k \in \mathcal{S}\}$ .

The aim of this paper is to study the least common multiple of the elements of  $[\mathcal{A}]_q$  for a random set  $\mathcal{A}$  in  $\mathcal{B}(n, \alpha)$ . Our main results are the following:

**Theorem 1.2** *Let  $\mathcal{A}$  be a random set in  $\mathcal{B}(n, \alpha)$  and put  $X := \deg \text{lcm}([\mathcal{A}]_q)$ . Then, for every integer  $n \geq 2$  and every  $\alpha \in [0, 1]$ , we have*

$$\mathbb{E}[X] = \frac{3}{\pi^2} \cdot \frac{\alpha \text{Li}_2(1 - \alpha)}{1 - \alpha} \cdot n^2 + O(\alpha n (\log n)^2), \tag{2}$$

where  $\text{Li}_2(z) := \sum_{k=1}^{\infty} z^k/k^2$  is the dilogarithm and the factor involving  $\alpha$  is meant to be equal to 1 when  $\alpha = 1$ . In particular,

$$\mathbb{E}[X] \sim \frac{3}{\pi^2} \cdot \frac{\alpha \text{Li}_2(1 - \alpha)}{1 - \alpha} \cdot n^2,$$

as  $n \rightarrow +\infty$ , uniformly for  $\alpha \in [0, 1]$ .

**Theorem 1.3** *Let  $\mathcal{A}$  be a random set in  $\mathcal{B}(n, \alpha)$  and put  $X := \deg \text{lcm}([\mathcal{A}]_q)$ . Then there exists a function  $v : (0, 1) \rightarrow \mathbb{R}^+$  such that, as  $\alpha n / ((\log n)^3 (\log \log n)^2) \rightarrow +\infty$ , we have*

$$\mathbb{V}[X] = (v(\alpha) + o(1)) n^3. \tag{3}$$

Moreover, the upper bound

$$\mathbb{V}[X] \ll \alpha n^3, \tag{4}$$

holds for every positive integer  $n$  and every  $\alpha \in [0, 1]$ .

As a consequence of Theorems 1.2 and 1.3, we obtain the following  $q$ -analog of Theorem 1.1.

**Theorem 1.4** *Let  $\mathcal{A}$  be a random set in  $\mathcal{B}(n, \alpha)$ . Then, as  $\alpha n \rightarrow +\infty$ , we have*

$$\deg \text{lcm}([\mathcal{A}]_q) \sim \frac{3}{\pi^2} \cdot \frac{\alpha \text{Li}_2(1-\alpha)}{1-\alpha} \cdot n^2,$$

*with probability  $1 - o(1)$ , where the factor involving  $\alpha$  is meant to be equal to 1 for  $\alpha = 1$ .*

*Remark 1.2* In the deterministic case  $\alpha = 1$ , we have (see Lemma 4.1 below)

$$\deg \text{lcm}([1, 2, \dots, n]_q) = \sum_{1 < d \leq n} \varphi(d),$$

and Theorem 1.4 corresponds to the well-known asymptotic formula  $\sum_{d \leq n} \varphi(d) \sim \frac{3}{\pi^2} n^2$  (Lemma 3.3 below) for the sum of the first values of the Euler function  $\varphi$ .

*Remark 1.3* In Theorem 1.4 the condition  $\alpha n \rightarrow +\infty$  is necessary. Indeed, if  $\alpha n \leq C$ , for some constant  $C > 0$ , then

$$\mathbb{P}[\mathcal{A} = \emptyset] = (1 - \alpha)^n \geq \left(1 - \frac{C}{n}\right)^n \rightarrow e^{-C}$$

as  $n \rightarrow +\infty$ , and so no (nontrivial) asymptotic formula for  $\deg \text{lcm}([\mathcal{A}]_q)$  can hold with probability  $1 - o(1)$ .

We conclude this section with some possible questions for further research on this topic. Alsmeyer, Kabluchko, and Marynych [1, Corollary 1.5] proved that, for fixed  $\alpha \in [0, 1]$  and for a random set  $\mathcal{A}$  in  $\mathcal{B}(n, \alpha)$ , an appropriate normalization of the random variable  $\log \text{lcm}(\mathcal{A})$  converges in distribution to a standard normal random variable, as  $n \rightarrow +\infty$ . In light of Theorems 1.2 and 1.3, it is then natural to ask whether the random variable

$$\frac{\deg \text{lcm}([\mathcal{A}]_q) - \frac{3}{\pi^2} \cdot \frac{\alpha \text{Li}_2(1-\alpha)}{1-\alpha} \cdot n^2}{\sqrt{v(\alpha)n^3}}$$

converges in distribution to a normal random variable, or to some other random variable.

Another problem could be considering polynomial values, similarly to the results done in the context of integers, and studying  $\text{lcm}([f(1)]_q, \dots, [f(n)]_q)$  for  $f \in \mathbb{Z}[x]$  or, more generally,  $\text{lcm}([f(k)]_q : k \in \mathcal{A})$  with  $\mathcal{A}$  a random set in  $\mathcal{B}(n, \alpha)$ .

## 2 Notation

We employ the Landau–Bachmann “Big Oh” and “little oh” notations  $O$  and  $o$ , as well as the associated Vinogradov symbol  $\ll$ , with their usual meanings. Any dependence of the implied constants is explicitly stated or indicated with subscripts. For real random variables  $X$  and  $Y$ , depending on some parameters, we say that “ $X \sim Y$  with probability  $1 - o(1)$ ”, as the parameters tend to some limit, if for every  $\varepsilon > 0$  we have  $\mathbb{P}[|X - Y| > \varepsilon|Y|] = o_\varepsilon(1)$ , as the parameters tend to the limit. We let  $(a, b)$  and  $[a, b]$  denote the greatest common divisor and the least common multiple, respectively, of two integers  $a$  and  $b$ . As usual, we write  $\varphi(n)$ ,  $\mu(n)$ ,  $\tau(n)$ , and  $\sigma(n)$ , for the Euler totient function, the Möbius function, the number of divisors, and the sum of divisors, of a positive integer  $n$ , respectively.

## 3 Preliminaries

In this section we collect some preliminary results needed in later arguments.

**Lemma 3.1** *We have*

$$\sum_{m \leq x} \tau(m) \ll x \log x,$$

for every  $x \geq 2$ .

*Proof* See, e.g., [20, Part I, Theorem 3.2]. □

**Lemma 3.2** *We have*

$$\sum_{[e_1, e_2] > x} \frac{1}{e_1 e_2 [e_1, e_2]} \ll \frac{\log x}{x},$$

for every  $x \geq 2$ .

*Proof* From Lemma 3.1 and partial summation, it follows that

$$\begin{aligned} \sum_{m > x} \frac{\tau(m)}{m^2} &= \left[ \frac{\sum_{m \leq t} \tau(m)}{t^2} \right]_{t=x}^{+\infty} + 2 \int_x^{+\infty} \frac{\sum_{m \leq t} \tau(m)}{t^3} dt \\ &\ll \int_x^{+\infty} \frac{\log t}{t^2} dt = \left[ -\frac{\log t + 1}{t} \right]_{t=x}^{+\infty} \ll \frac{\log x}{x}. \end{aligned}$$

Let  $e := (e_1, e_2)$  and  $e'_i := e_i/e$  for  $i = 1, 2$ . Then we have

$$\begin{aligned} \sum_{[e_1, e_2] > x} \frac{1}{e_1 e_2 [e_1, e_2]} &\leq \sum_{e \geq 1} \frac{1}{e^3} \sum_{e'_1 e'_2 > x/e} \frac{1}{(e'_1 e'_2)^2} = \sum_{e \geq 1} \frac{1}{e^3} \sum_{m > x/e} \frac{\tau(m)}{m^2} \\ &\ll \sum_{e \leq x/2} \frac{1}{e^3} \frac{\log(x/e)}{x/e} + \sum_{e > x/2} \frac{1}{e^3} \ll \frac{\log x}{x} + \frac{1}{x^2} \ll \frac{\log x}{x}, \end{aligned}$$

as desired. □

Let us define

$$\Phi(x) := \sum_{n \leq x} \varphi(n) \quad \text{and} \quad \Phi(a_1, a_2; x) := \sum_{n \leq x} \varphi(a_1 n) \varphi(a_2 n),$$

for every  $x \geq 1$  and for all positive integers  $a_1, a_2$ .

**Lemma 3.3** *We have*

$$\Phi(x) = \frac{3}{\pi^2} x^2 + O(x \log x),$$

for every  $x \geq 2$ .

*Proof* See, e.g., [20, Part I, Theorem 3.4]. □

**Lemma 3.4** *We have*

$$\Phi(a_1, a_2; x) = C_1(a_1, a_2) x^3 + O(\sigma(a_1) \sigma(a_2) x^2 (\log x)^2), \tag{5}$$

for every  $x \geq 2$ , where

$$C_1(a_1, a_2) := \frac{a_1 a_2}{3} \sum_{d_1, d_2 \geq 1} \frac{\mu(d_1) \mu(d_2)}{d_1 d_2 [d_1 / (a_1, d_1), d_2 / (a_2, d_2)]} \tag{6}$$

and the series is absolutely convergent.

*Proof* From the identity  $\varphi(n)/n = \sum_{d|n} \mu(d)/d$ , it follows that

$$\begin{aligned} \sum_{n \leq x} \frac{\varphi(a_1 n)}{a_1 n} \frac{\varphi(a_2 n)}{a_2 n} &= \sum_{n \leq x} \left( \sum_{d_1 | a_1 n} \frac{\mu(d_1)}{d_1} \sum_{d_2 | a_2 n} \frac{\mu(d_2)}{d_2} \right) \\ &= \sum_{\substack{d_1 \leq a_1 x \\ d_2 \leq a_2 x}} \frac{\mu(d_1)}{d_1} \frac{\mu(d_2)}{d_2} \#\{n \leq x : d_1 | a_1 n \text{ and } d_2 | a_2 n\} \\ &= \sum_{\left[ \frac{d_1}{(a_1, d_1)}, \frac{d_2}{(a_2, d_2)} \right] \leq x} \frac{\mu(d_1)}{d_1} \frac{\mu(d_2)}{d_2} \left( \frac{x}{\left[ d_1/(a_1, d_1), d_2/(a_2, d_2) \right]} + O(1) \right). \end{aligned}$$

Let  $c_i := (a_i, d_i)$  and  $e_i := d_i/c_i$ , for  $i = 1, 2$ . On the one hand, we have

$$E_1 := \sum_{\left[ \frac{d_1}{(a_1, d_1)}, \frac{d_2}{(a_2, d_2)} \right] \leq x} \frac{1}{d_1 d_2} \leq \sum_{c_1 | a_1} \frac{1}{c_1} \sum_{c_2 | a_2} \frac{1}{c_2} \sum_{e_1 \leq x} \frac{1}{e_1} \sum_{e_2 \leq x} \frac{1}{e_2} \ll \frac{\sigma(a_1) \sigma(a_2)}{a_1 a_2} (\log x)^2.$$

On the other hand, thanks to Lemma 3.2, we have

$$\begin{aligned} E_2 &:= \sum_{\left[ \frac{d_1}{(a_1, d_1)}, \frac{d_2}{(a_2, d_2)} \right] > x} \frac{1}{d_1 d_2 \left[ d_1/(a_1, d_1), d_2/(a_2, d_2) \right]} \\ &\leq \sum_{c_1 | a_1} \frac{1}{c_1} \sum_{c_2 | a_2} \frac{1}{c_2} \sum_{[e_1, e_2] > x} \frac{1}{e_1 e_2 [e_1, e_2]} \ll \frac{\sigma(a_1) \sigma(a_2) \log x}{a_1 a_2 x}, \end{aligned}$$

which, in particular, implies that the series

$$C_0(a_1, a_2) := \sum_{d_1, d_2 \geq 1} \frac{\mu(d_1) \mu(d_2)}{d_1 d_2 \left[ d_1/(a_1, d_1), d_2/(a_2, d_2) \right]}$$

is absolutely convergent. Therefore, we obtain

$$\begin{aligned} \sum_{n \leq x} \frac{\varphi(a_1 n)}{a_1 n} \frac{\varphi(a_2 n)}{a_2 n} &= (C_0(a_1, a_2) + O(E_2)) x + O(E_1) \\ &= C_0(a_1, a_2) x + O\left( \frac{\sigma(a_1) \sigma(a_2)}{a_1 a_2} (\log x)^2 \right). \end{aligned} \tag{7}$$

Now (5) follows from (7) by partial summation and since  $C_1(a_1, a_2) = \frac{a_1 a_2}{3} C_0(a_1, a_2)$ .  $\square$

*Remark 3.1* The obvious bound  $\varphi(m) \leq m$  yields  $C_1(a_1, a_2) \leq \frac{a_1 a_2}{3}$  (which is not so obvious from (6)).

We end this section with an easy observation that will be useful later.

*Remark 3.2* It holds  $1 - (1 - x)^k \leq kx$ , for all  $x \in [0, 1]$  and for all integers  $k \geq 0$ .

### 4 Proofs

Henceforth, let  $\mathcal{A}$  be a random set in  $\mathcal{B}(n, \alpha)$ , let  $[\mathcal{A}]_q$  be its  $q$ -analog, and put  $L := \text{lcm}([\mathcal{A}]_q)$  and  $X := \text{deg } L$ . For every positive integer  $d$ , let us define

$$I_{\mathcal{A}}(d) := \begin{cases} 1 & \text{if } d | k \text{ for some } k \in \mathcal{A}; \\ 0 & \text{otherwise.} \end{cases}$$

The following lemma gives a formula for  $X$  in terms of  $I_{\mathcal{A}}$  and the Euler function.

**Lemma 4.1** *We have*

$$X = \sum_{1 < d \leq n} \varphi(d) I_{\mathcal{A}}(d). \tag{8}$$

*Proof* For every positive integer  $k$ , it holds

$$[k]_q = \frac{q^k - 1}{q - 1} = \prod_{\substack{d|k \\ d > 1}} \Phi_d(q),$$

where  $\Phi_d(q)$  is the  $d$ th cyclotomic polynomials. Since, as it is well known, every cyclotomic polynomial is irreducible over  $\mathbb{Q}$ , it follows that  $L$  is the product of the polynomials  $\Phi_d(q)$  such that  $d > 1$  and  $d | k$  for some  $k \in \mathcal{A}$ . Finally, the equality  $\deg(\Phi_d(q)) = \varphi(d)$  and the definition of  $I_{\mathcal{A}}$  yield (8).  $\square$

Let  $\beta := 1 - \alpha$ . The next lemma provides two expected values involving  $I_{\mathcal{A}}$ .

**Lemma 4.2** *For all positive integers  $d, d_1, d_2$ , we have*

$$\mathbb{E}[I_{\mathcal{A}}(d)] = 1 - \beta^{\lfloor n/d \rfloor} \tag{9}$$

and

$$\mathbb{E}[I_{\mathcal{A}}(d_1)I_{\mathcal{A}}(d_2)] = 1 - \beta^{\lfloor n/d_1 \rfloor} - \beta^{\lfloor n/d_2 \rfloor} + \beta^{\lfloor n/d_1 \rfloor + \lfloor n/d_2 \rfloor - \lfloor n/[d_1, d_2] \rfloor}.$$

*Proof* On the one hand, by the definition of  $I_{\mathcal{A}}$ , we have

$$\mathbb{E}[I_{\mathcal{A}}(d)] = \mathbb{P}[\exists k \in \mathcal{A} : d | k] = 1 - \mathbb{P}\left[\bigwedge_{m \leq \lfloor n/d \rfloor} (dm \notin \mathcal{A})\right] = 1 - \beta^{\lfloor n/d \rfloor},$$

which is (9). On the other hand, by linearity of the expectation and by (9), we have

$$\begin{aligned} \mathbb{E}[I_{\mathcal{A}}(d_1)I_{\mathcal{A}}(d_2)] &= \mathbb{E}[I_{\mathcal{A}}(d_1) + I_{\mathcal{A}}(d_2) - 1 + (1 - I_{\mathcal{A}}(d_1))(1 - I_{\mathcal{A}}(d_2))] \\ &= \mathbb{E}[I_{\mathcal{A}}(d_1)] + \mathbb{E}[I_{\mathcal{A}}(d_2)] - 1 + \mathbb{E}[(1 - I_{\mathcal{A}}(d_1))(1 - I_{\mathcal{A}}(d_2))] \\ &= 1 - \beta^{\lfloor n/d_1 \rfloor} - \beta^{\lfloor n/d_2 \rfloor} + \mathbb{E}[(1 - I_{\mathcal{A}}(d_1))(1 - I_{\mathcal{A}}(d_2))], \end{aligned}$$

where the last expected value can be computed as

$$\begin{aligned} \mathbb{E}[(1 - I_{\mathcal{A}}(d_1))(1 - I_{\mathcal{A}}(d_2))] &= \mathbb{P}[\forall k \in \mathcal{A} : d_1 \nmid k \text{ and } d_2 \nmid k] \\ &= \mathbb{P}\left[\bigwedge_{\substack{k \leq n \\ d_1 | k \text{ or } d_2 | k}} (k \notin \mathcal{A})\right] = \beta^{\lfloor n/d_1 \rfloor + \lfloor n/d_2 \rfloor - \lfloor n/[d_1, d_2] \rfloor}, \end{aligned}$$

and second claim follows.  $\square$

We are ready to compute the expected value of  $X$ .

*Proof of Theorem 1.2* From Lemmas 4.1 and 4.2, it follows that

$$\mathbb{E}[X] = \sum_{1 < d \leq n} \varphi(d) \mathbb{E}[I_{\mathcal{A}}(d)] = \sum_{1 < d \leq n} \varphi(d)(1 - \beta^{\lfloor n/d \rfloor}). \tag{10}$$

Moreover, since  $\lfloor n/d \rfloor = j$  if and only if  $n/(j + 1) < d \leq n/j$ , we get that

$$\begin{aligned} \sum_{d \leq n} \varphi(d)(1 - \beta^{\lfloor n/d \rfloor}) &= \sum_{j \leq n} (1 - \beta^j) \sum_{n/(j+1) < d \leq n/j} \varphi(d) \\ &= \sum_{j \leq n} (1 - \beta^j) \left( \Phi\left(\frac{n}{j}\right) - \Phi\left(\frac{n}{j+1}\right) \right) \\ &= \alpha \sum_{j \leq n} \beta^{j-1} \Phi\left(\frac{n}{j}\right) \\ &= \frac{3}{\pi^2} \cdot \alpha \sum_{j \leq n} \frac{\beta^{j-1}}{j^2} \cdot n^2 + O\left(\alpha \sum_{j \leq n} \frac{n}{j} \log\left(\frac{n}{j}\right)\right) \\ &= \frac{3}{\pi^2} \cdot \frac{\alpha \operatorname{Li}_2(1 - \alpha)}{1 - \alpha} \cdot n^2 + O(\alpha n(\log n)^2), \end{aligned} \tag{11}$$

where we used Lemma 3.3. Putting together (10) and (11), and noting that, by Remark 3.2, the addend of (11) corresponding to  $d = 1$  is  $1 - \beta^n = O(\alpha n)$ , we get (2). The proof is complete.  $\square$

Now we consider the variance of  $X$ .

*Proof of Theorem 1.3* From Lemmas 4.1 and 4.2, it follows that

$$\begin{aligned} \mathbb{V}[X] &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\ &= \sum_{1 < d_1, d_2 \leq n} \varphi(d_1) \varphi(d_2) \left( \mathbb{E}[I_{\mathcal{A}}(d_1) I_{\mathcal{A}}(d_2)] - \mathbb{E}[I_{\mathcal{A}}(d_1)] \mathbb{E}[I_{\mathcal{A}}(d_2)] \right) \\ &= \sum_{1 < d_1, d_2 \leq n} \varphi(d_1) \varphi(d_2) \beta^{\lfloor n/d_1 \rfloor + \lfloor n/d_2 \rfloor - \lfloor n/[d_1, d_2] \rfloor} (1 - \beta^{\lfloor n/[d_1, d_2] \rfloor}). \end{aligned} \tag{12}$$

Let us define

$$V_n(\alpha) := \frac{1}{n^3} \sum_{d_1, d_2 \leq n} \varphi(d_1) \varphi(d_2) \beta^{\lfloor n/d_1 \rfloor + \lfloor n/d_2 \rfloor - \lfloor n/[d_1, d_2] \rfloor} (1 - \beta^{\lfloor n/[d_1, d_2] \rfloor}).$$

Clearly, we have

$$V_n(\alpha) - \frac{\mathbb{V}[X]}{n^3} \ll \frac{1}{n^3} \sum_{d \leq n} \varphi(d) \beta^n (1 - \beta^{\lfloor n/d \rfloor}) \leq \frac{1}{n^3} \sum_{d \leq n} d \ll \frac{1}{n}.$$

Hence, in order to prove (3), it suffices to show that  $V_n(\alpha) = v(\alpha) + o(1)$ .

For all vectors  $\mathbf{a} := (a_1, a_2)$  and  $\mathbf{j} := (j_1, j_2, j_3)$  with components in the set of positive integers, define the quantities

$$\rho_1(\mathbf{a}, \mathbf{j}) := \max\left(\frac{1}{a_1(j_1 + 1)}, \frac{1}{a_2(j_2 + 1)}, \frac{1}{a_1 a_2(j_3 + 1)}\right)$$

and

$$\rho_2(\mathbf{a}, \mathbf{j}) := \min\left(\frac{1}{a_1 j_1}, \frac{1}{a_2 j_2}, \frac{1}{a_1 a_2 j_3}\right).$$

Let  $d := (d_1, d_2)$  and  $a_i := d_i/d$  for  $i = 1, 2$ . Then the equalities

$$j_1 = \left\lfloor \frac{n}{d_1} \right\rfloor, \quad j_2 = \left\lfloor \frac{n}{d_2} \right\rfloor, \quad j_3 = \left\lfloor \frac{n}{[d_1, d_2]} \right\rfloor,$$

are equivalent to

$$j_1 \leq \frac{n}{a_1 d} < j_1 + 1, \quad j_2 \leq \frac{n}{a_2 d} < j_2 + 1, \quad j_3 \leq \frac{n}{a_1 a_2 d} < j_3 + 1,$$

which in turn are equivalent to

$$\frac{n}{a_1(j_1 + 1)} < d \leq \frac{n}{a_1 j_1}, \quad \frac{n}{a_2(j_2 + 1)} < d \leq \frac{n}{a_2 j_2}, \quad \frac{n}{a_1 a_2(j_3 + 1)} < d \leq \frac{n}{a_1 a_2 j_3},$$

that is,

$$\rho_1(\mathbf{a}, \mathbf{j}) n < d \leq \rho_2(\mathbf{a}, \mathbf{j}) n.$$

Therefore, letting

$$S_n := \{(\mathbf{a}, \mathbf{j}) \in \mathbb{N}^5 : (a_1, a_2) = 1, \exists d \in \mathbb{N} \text{ s.t. } \rho_1(\mathbf{a}, \mathbf{j}) n < d \leq \rho_2(\mathbf{a}, \mathbf{j}) n\}$$

and

$$S(\mathbf{a}, \mathbf{j}; n) := \frac{1}{n^3} \sum_{\rho_1(\mathbf{a}, \mathbf{j}) n < d \leq \rho_2(\mathbf{a}, \mathbf{j}) n} \varphi(a_1 d) \varphi(a_2 d),$$

we have

$$V_n(\alpha) = \sum_{(\mathbf{a}, \mathbf{j}) \in S_n} \beta^{j_1 + j_2 - j_3} (1 - \beta^{j_3}) S(\mathbf{a}, \mathbf{j}; n).$$

Now let us define

$$v(\alpha) := \sum_{(\mathbf{a}, \mathbf{j}) \in S_\infty} \beta^{j_1 + j_2 - j_3} (1 - \beta^{j_3}) D(\mathbf{a}, \mathbf{j}), \tag{13}$$

where

$$S_\infty := \bigcup_{m \geq 1} S_m = \{(\mathbf{a}, \mathbf{j}) \in \mathbb{N}^5 : (a_1, a_2) = 1, \rho_1(\mathbf{a}, \mathbf{j}) < \rho_2(\mathbf{a}, \mathbf{j})\}$$

and

$$D(\mathbf{a}, \mathbf{j}) := C_1(a_1, a_2) (\rho_2(\mathbf{a}, \mathbf{j})^3 - \rho_1(\mathbf{a}, \mathbf{j})^3),$$

recalling that  $C_1(a_1, a_2)$  is defined by (6). The convergence of series (13) follows easily from Remark 3.1,  $\rho_2(\mathbf{a}, \mathbf{j}) \leq 1/(a_1 a_2 j_3)$ , and the fact that  $\min(j_1, j_2) \geq j_3$  for all  $(\mathbf{a}, \mathbf{j}) \in S_\infty$ .

Thanks to Lemma 3.4, for each  $(\mathbf{a}, \mathbf{j}) \in S_n$  we have

$$S(\mathbf{a}, \mathbf{j}; n) = D(\mathbf{a}, \mathbf{j}) + O\left(\sigma(a_1) \sigma(a_2) \rho_2(\mathbf{a}, \mathbf{j})^2 \cdot \frac{(\log n)^2}{n}\right).$$

Consequently, we get that

$$V_n(\alpha) = v(\alpha) - \Sigma_1 + O\left(\Sigma_2 \cdot \frac{(\log n)^2}{n}\right), \tag{14}$$

where

$$\Sigma_1 := \sum_{(\mathbf{a}, \mathbf{j}) \in S_\infty \setminus S_n} \beta^{j_1 + j_2 - j_3} (1 - \beta^{j_3}) D(\mathbf{a}, \mathbf{j})$$

and

$$\Sigma_2 := \sum_{(\mathbf{a}, \mathbf{j}) \in \mathcal{S}_n} \beta^{j_1+j_2-j_3}(1 - \beta^{j_3}) \sigma(a_1) \sigma(a_2) \rho_2(\mathbf{a}, \mathbf{j})^2.$$

Now we have to bound both  $\Sigma_1$  and  $\Sigma_2$ .

If  $(\mathbf{a}, \mathbf{j}) \in \mathcal{S}_\infty \setminus \mathcal{S}_n$  then  $(\rho_2(\mathbf{a}, \mathbf{j}) - \rho_1(\mathbf{a}, \mathbf{j}))n < 1$  and consequently, also by Remark 3.1,

$$\begin{aligned} D(\mathbf{a}, \mathbf{j}) &\ll a_1 a_2 (\rho_2^3 - \rho_1^3) = a_1 a_2 (\rho_1^2 + \rho_1 \rho_2 + \rho_2^2) (\rho_2 - \rho_1) \ll \frac{a_1 a_2 \rho_2^2}{n} \\ &\leq \frac{1}{a_1 a_2 j_3^2 n}, \end{aligned} \tag{15}$$

where, for brevity, we wrote  $\rho_i := \rho_i(\mathbf{a}, \mathbf{j})$  for  $i = 1, 2$ .

If  $(\mathbf{a}, \mathbf{j}) \in \mathcal{S}_\infty$  then, as we already noticed,  $\min(j_1, j_2) \geq j_3$  and, moreover,

$$\frac{j_2}{j_3 + 1} < a_1 < \frac{j_2 + 1}{j_3} \quad \text{and} \quad \frac{j_1}{j_3 + 1} < a_2 < \frac{j_1 + 1}{j_3}.$$

Hence, we have

$$\begin{aligned} \sum_{(\mathbf{a}, \mathbf{j}) \in \mathcal{S}_\infty} \frac{\beta^{j_1+j_2-j_3}(1 - \beta^{j_3})}{a_1 a_2 j_3^2} &\leq \sum_{j_3 \geq 1} \frac{1 - \beta^{j_3}}{j_3^2} \sum_{j_1, j_2 \geq j_3} \beta^{j_1+j_2-j_3} \sum_{\substack{j_2/(j_3+1) < a_1 < (j_2+1)/j_3 \\ j_1/(j_3+1) < a_2 < (j_1+1)/j_3}} \frac{1}{a_1 a_2} \\ &\ll \sum_{j_3 \geq 1} \frac{1 - \beta^{j_3}}{j_3^2} \sum_{j_1, j_2 \geq j_3} \beta^{j_1+j_2-j_3} = \frac{1}{\alpha^2} \sum_{j \geq 1} \frac{(1 - \beta^j) \beta^j}{j^2} \\ &\leq \frac{1}{\alpha} \sum_{j \leq 1/\alpha} \frac{1}{j} + \frac{1}{\alpha^2} \sum_{j > 1/\alpha} \frac{1}{j^2} \ll \frac{\log(1/\alpha) + 1}{\alpha}, \end{aligned} \tag{16}$$

where we used the inequality  $1 - \beta^j \leq \alpha j$ , which follows from Remark 3.2.

On the one hand, from (15) and (16) it follows that

$$\Sigma_1 \ll \frac{\log(1/\alpha) + 1}{\alpha n} = o(1), \tag{17}$$

as  $\alpha n / ((\log n)^3 (\log \log n)^2) \rightarrow +\infty$  (actually,  $\alpha n / \log n \rightarrow +\infty$  is sufficient).

On the other hand, from  $\rho_2(\mathbf{a}, \mathbf{j}) \leq 1/(a_1 a_2 j_3)$ , (16), and the bound  $\sigma(m) \ll m \log \log m$  (see, e.g., [20, Part I, Theorem 5.7]) it follows that

$$\begin{aligned} \Sigma_2 &\leq \sum_{(\mathbf{a}, \mathbf{j}) \in \mathcal{S}_n} \frac{\beta^{j_1+j_2-j_3}(1 - \beta^{j_3})}{a_1 a_2 j_3^2} \cdot \frac{\sigma(a_1) \sigma(a_2)}{a_1 a_2} \ll \frac{(\log(1/\alpha) + 1)(\log \log n)^2}{\alpha} \\ &= o\left(\frac{n}{(\log n)^2}\right), \end{aligned} \tag{18}$$

as  $\alpha n / ((\log n)^3 (\log \log n)^2) \rightarrow +\infty$ .

At this point, putting together (14), (17), and (18), we obtain  $V_n(\alpha) = v(\alpha) + o(1)$ , as desired. The proof of (3) is complete.

It remains only to prove the upper bound (4). From (12) it follows that

$$\begin{aligned} \mathbb{V}[X] &\leq \sum_{[d_1, d_2] \leq n} \varphi(d_1) \varphi(d_2) \beta^{\lfloor n/d_1 \rfloor + \lfloor n/d_2 \rfloor - \lfloor n/[d_1, d_2] \rfloor} (1 - \beta^{\lfloor n/[d_1, d_2] \rfloor}) \\ &\leq \sum_{[d_1, d_2] \leq n} d_1 d_2 \cdot \frac{\alpha n}{[d_1, d_2]} = \alpha n \sum_{[d_1, d_2] \leq n} (d_1, d_2) \leq \alpha n \sum_{d \leq n} d \sum_{a_1 a_2 \leq n/d} 1 \\ &= \alpha n \sum_{d \leq n} d \sum_{m \leq n/d} \tau(m) \ll \alpha n^2 \sum_{d \leq n} \log\left(\frac{n}{d}\right) = \alpha n^2 (n \log n - \log(n!)) < \alpha n^3, \end{aligned}$$

where we used Remark 3.2, Lemma 3.1, and the bound  $n! > (n/e)^n$ . Thus (4) is proved.  $\square$

*Proof of Theorem 1.4* By Chebyshev's inequality, Theorems 1.2 and 1.3, we have

$$\mathbb{P}[|X - \mathbb{E}[X]| > \varepsilon \mathbb{E}[X]] \leq \frac{\mathbb{V}[X]}{(\varepsilon \mathbb{E}[X])^2} \ll \frac{\alpha n^3}{(\varepsilon \alpha n)^2} = \frac{1}{\varepsilon^2 \alpha n} = o_\varepsilon(1),$$

as  $\alpha n \rightarrow +\infty$ . Hence, using again Theorem 1.2, we get

$$X \sim \frac{3}{\pi^2} \cdot \frac{\alpha \operatorname{Li}_2(1 - \alpha)}{1 - \alpha} \cdot n^2,$$

with probability  $1 - o(1)$ , as  $\alpha n \rightarrow +\infty$ .  $\square$

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#### References

- Alsmeyer, G., Kabluchko, Z., Marynych, A.: Limit theorems for the least common multiple of a random set of integers. *Trans. Am. Math. Soc.* **372**(7), 4585–4603 (2019)
- Andrews, G.E.: *q-Series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra*, CBMS Regional Conference Series in Mathematics, vol. 66, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI (1986)
- Bateman, P., Kalb, J., Stenger, A.: Solution to problem 10797: a limit involving least common multiples. *Am. Math. Monthly* **109**(4), 393–394 (2002)
- Bostan, A., Marynych, A., Raschel, K.: On the least common multiple of several random integers. *J. Number Theory* **204**, 113–133 (2019)
- Buraczewski, D., Iksanov, A., Marynych, A.: A Brownian weak limit for the least common multiple of a random  $m$ -tuple of integers. <https://arxiv.org/abs/2004.05643>
- Cilleruelo, J.: The least common multiple of a quadratic sequence. *Compos. Math.* **147**(4), 1129–1150 (2011)
- Cilleruelo, J., Guijarro-Ordóñez, J.: Ratio sets of random sets. *Ramanujan J.* **43**(2), 327–345 (2017)
- Cilleruelo, J., Ramana, D.S., Ramaré, O.: Quotient and product sets of thin subsets of the positive integers. *Proc. Steklov Inst. Math.* **296**(1), 52–64 (2017)
- Cilleruelo, J., Rué, J., Šarka, P., Zumalacárregui, A.: The least common multiple of random sets of positive integers. *J. Number Theory* **144**, 92–104 (2014)
- Hong, S., Qian, G., Tan, Q.: The least common multiple of a sequence of products of linear polynomials. *Acta Math. Hungar.* **135**(1–2), 160–167 (2012)
- Kac, V., Cheung, P.: *Quantum Calculus*. Universitext. Springer, New York (2002)
- Mastrostefano, D.: On maximal product sets of random sets, (to appear on *Journal of Number Theory*) <https://arxiv.org/abs/2005.04663>
- Maynard, J., Rudnick, Z.: A lower bound on the least common multiple of polynomial sequences, (to appear on *Rivista di Matematica dell'Università di Parma*) <https://arxiv.org/abs/1910.13218>
- Rudnick, Z., Zehavi, S.: On Cilleruelo's conjecture for the least common multiple of polynomial sequences. <https://arxiv.org/abs/1902.01102>
- Rué, J., Šarka, P., Zumalacárregui, A.: On the error term of the logarithm of the lcm of a quadratic sequence. *J. Théor. Nombres Bordeaux* **25**(2), 457–470 (2013)
- Sah, A.: An improved bound on the least common multiple of polynomial sequences. *J. Théor. Nombres Bordeaux* **32**(3), 891–899 (2020)
- Sanna, C.: On the L.C.M. of shifted Fibonacci numbers <https://arxiv.org/abs/2007.13330>
- Sanna, C.: A note on product sets of random sets. *Acta Math. Hungar.* **162**(1), 76–83 (2020)
- Sanna, C.: On the L.C.M. of random terms of binary recurrence sequences. *J. Number Theory* **213**, 221–231 (2020)
- Tenenbaum, G.: *Introduction to Analytic and Probabilistic Number Theory*, third ed., Graduate Studies in Mathematics, vol. 163, American Mathematical Society, Providence, RI, Translated from the 2008 French edition by Patrick D. F. Ion (2015)

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