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# Aspects of Supegravity and String theory

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# Abstract

The aim of this thesis is to present the results obtained during my PhD, concerning String Field Theory and Supergravity.
In the first Part, we introduce the main mathematical tools needed to perform an analysis of the regularities properties of the Intertwining solution in Open Cubic Superstring Field Theory. This analysis leads to constraints on the explicit form of the tachyon vacuum solution, which, if satisfied, guarantee that the intertwining solution is well-defined as a string field and that it does not produce ambiguous terms when inserted in the equations of motion.
In the second Part of this thesis, we introduce the geometric formulation of Supergravity and its applications to condensed matter systems and holography. In the first case, we build an analogy between Supergravity and graphene-like materials, where Supersymmetry proves to be fundamental to describe the electronic properties of these materials. Motivated by this result, we then perform a holographic analysis of the asymptotic properties of $\mathcal{N} = 2$ pure AdS Supergravity in presence of a boundary and, by following the AdS/CFT prescriptions, we prove that the Ward identities of the dual theory are satisfied, which signals that there are no anomalies.

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# Contents

# 1 Introduction

Over the decades, String theory and Supergravity have drawn a lot of interest in the academic world as they arose as candidates for unifying all four fundamental particle interactions.
Strong nuclear, weak and electromagnetic interaction have been successfully described by the Standard Model and, from a theoretical point of view, they correspond to internal symmetries of the action, called *gauge* symmetries. Gravity is instead described by General Relativity, which is a geometrical theory of spacetime, whose action is invariant under diffeomorphisms. These transformations, unlike gauge symmetries, involve spacetime itself and for this reason are fundamentally different.
The profound distinction between these two kinds of symmetries is also reflected by the *hierarchy problem*, which is the name often used to refer to the immense difference in strength between gravity and the other interactions.
A theory capable of properly quantising gravity and describing all fundamental interactions is called a Theory of Everything (TOE): such a theory should also be able to solve the *hierarchy problem* and explain, through a Higgs mechanism, the breakdown of the original symmetry group at low energies.
One way of solving the *hierarchy problem* is by introducing a symmetry associating to each particle a partner obeying the opposite statistics. In this way, it can be shown that the mass of the Higgs field, whose value would otherwise be driven, by perturbative corrections, all the way to the Plank scale, does not receive contributions from such high energy scales. The mentioned symmetry, called Supersymmetry, is then believed to be a possible solution to this problem and a consistent phenomenological theory extending the Standard Model should is expected to feature it.
Both bosonic String theory and General Relativity can be made supersymmetric: one then talks about Superstring theory and Supergravity as the resulting theories.

Let us start with a general overview of these two theories, briefly describing their origins and main underlying ideas.
String theory is a theory of 1-dimensional laces, which was first proposed to explain the strong nuclear force, but Quantum Chromodynamics (QCD) proved to be better suited for that purpose. It was then realised that String theory was not meant to be discarded as the idea was far reaching: the spectrum of such theory, given by the vibrational modes of open and closed strings, contains indeed both the photon and the graviton.
The introduction of Supersymmetry brought even more interest in String theory, as it could now include fermions, which are a key part of the Standard Model.
As a compromise for these astounding properties, Superstring theory can only be formulated on a 10-dimensional target spacetime: the necessity of this condition can be shown in several ways. For example, one has to set to zero the mass of gauge fields which arise in the open string spectrum in the light-cone quantization. Another way of retrieving the same result is to notice that the worldsheet theory is a (Super-)Conformal Field Theory: one of the defining features of these theories is that the trace of the energy momentum tensor vanishes. At the quantum level, though, this trace is proportional to the total central charge of the CFT: the

vanishing of the trace then requires that

$$c^{\text{tot}} = c^X + c^\psi + c^{\text{ghosts}} = \frac{3}{2}D - 15 = 0 \implies D = 10,$$

where $X^\mu$ are the coordinates of the target spacetime in which the strings propagate, $\psi_\mu$ are their supersymmetric partners and the ghost fields are associated with the symmetries of the worldsheet theory.

One way of dealing with this unusual feature of String theory is by means of dimensional compactification: in this way, the extra dimensions are assumed to be compact and to form, for example, small circles (toroidal compactification) that can only be seen at high enough energies. This idea goes back to the early work of Kaluza and Klein and turns a feature, which could certainly appear as dangerous, in an advantageous property, since compactifications can be used to build a variety of phenomenological models in which spacetime is effectively 4-dimensional.

Another mathematical consequence of String theory are the Dirichlet-branes, in short D-branes, which are hypersurfaces of the target manifold described by the boundary conditions of Dirichlet type imposed on the end points of open strings. Apart from their mathematical origin, their physical importance has been noticed in virtue of the fact that they can lead to non-abelian gauge symmetries. Furthermore, they have been the key to understand the process of tachyon condensation, which explains the presence of a tachyonic particle in the spectrum of the open bosonic string.

The richness of this mathematical setup is shown by the fact that there are five different consistent Superstring theories, type IIB, type IIA, type I, Heterotic $SO(32)$ and Heterotic $E_8 \times E_8$. This feature has again turned into an interesting property, as they can all be linked by a series of dualities, consisting of the so-called S-duality and T-duality: the first one is a strong-weak duality obtained by inverting the coupling constant, whereas the second one is deeply linked to compactification and to the fact that strings are 1-dimensional objects that can wind around compact directions.

These different theories can actually be seen as describing particular limits of a 11-dimensional more general theory, the M-theory, whose high energy description is still unknown: in this framework, the mentioned dualities can be further generalised by the so called U-dualities, which consist in their combination.

One of the main conceptual criticisms against String theory has been that it is not manifestly background independent. This can be traced to the fact that the action, written in superconformal gauge, depends on an explicit choice of background metric

$$S = -\frac{1}{8\pi}\int \mathrm{d}\sigma \; G_{\mu\nu}(X)\left(\frac{2}{\alpha'}\partial_\alpha X^\mu \partial^\alpha X^\nu + 2\mathrm{i}\overline{\psi}^\mu \gamma^\alpha \partial_\alpha \psi^\nu\right)$$

Without entering into details, here $G_{\mu\nu}(X)$ is a non-dynamical background metric, which is usually taken to be the flat metric. From a relativistic point of view, a theory of gravity should in principle be background independent and progress in this direction has been achieved both for open and closed strings. In this thesis, we will further investigate this subject in the context of open strings.

As we argued, String theory contains a graviton in its spectrum: in order for such a theory do correctly describe gravity, this particle has indeed to satisfy the Einstein's equations, or their supersymmetric generalisation. This can be seen by requiring that the theory remains (super)conformal even a the quantum level. This condition is encoded in the vanishing of the beta function, which reproduces the gravity equations in the bosonic case and the equations of motion of Supergravity, in presence of Supersymmetry. Furthermore, the effective action in the background fields, having the vanishing of the beta function as equations of motion, is a Supergravity action.

It must be noticed, though, that Supergravity was first introduced in 1976 independently from String theory, as a supersymmetric generalization of Einstein's General Relativity. In addition to the background independence, Supergravity unfortunately inherits from General Relativity the fact that it is not renormalizable, because one would need to add to the starting $D = 4$ lagrangian an infinite number of counterterms to account for all the diverging diagrams.
At first, it was believed that Supersymmetry could make the theory finite by itself: however, in the simplest case, $\mathcal{N} = 1$ Supergravity with matter is divergent at one loop order. The introduction of more supersymmetries, which lead to the so-called Extended Supergravities, does improve the behaviour of the theory, which nonetheless remains divergent up until $\mathcal{N} = 8$. In the presence of the maximum number of supersymmetries the situation is still uncertain and the computation of scattering amplitudes at seveth loop order, which is the one where divergences could start showing up, is still under way. Despite this, even if this theory would prove to be finite, Extended Supergravities are not chiral theories, in the sense that in the same supermultiplet one has both left and right handed fields, thus belonging to the same representation of the R-Symmetry group, whereas this does not seem to be the case in the Standard Model.
All these elements brought to the conclusion that Supergravity should be interpreted as a low energy effective field theory of some more fundamental theory in an analogous way as the old Fermi theory was an effective theory for the weak interaction, which could be taken as valid only for low enough energies. As anticipated before, Superstring theory is a candidate for the needed fundamental theory, but in principle Supergravity could admit other UV completions. This interpretation does not downplay the role of Supergravity, as such an effective theory selects only the low lying modes out of the infinite tower of string states. This theory indeed should be understood as an intermediate step between energy levels available today and those needed to access stringy effects. Unfortunately, as of now, all phenomenological models derived from both theories lack of an experimental support.

A new and renovated interest in both Supergravity and Superstring theory has been brought by one of the most important developments in the last decades: the gauge-gravity duality. This is a correspondence between theories of quantum gravity in $D = d + 1$ dimensions and non-gravitational QFTs on the $d$ dimensional boundary. This idea was first proposed by Maldacena in 1997, who related type IIB Superstring theory on the 10-dimensional $AdS_5 \times S^5$ curved background to $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $SU(\mathcal{N})$, which is a Conformal Field Theory (CFT).
This correspondence, which for this reason is often called AdS/CFT duality, can be restated

as a duality between a gravity theory, like General Relativity, Supergravity or String theory and a gauge theory. By considering the low energy limit of String theory, this conjecture provides an important tool to investigate strong coupling regimes on the QFT side, which can not be reached by ordinary perturbation theory.
The AdS/CFT correspondence can in principle be expressed as an equality of path integrals [1, 2]

$$Z_G[\Phi_0] = Z_{CFT}[\mathcal{J}],$$

where $\Phi_0$ is the value of all the fields at the boundary of an asymptotically Anti de-Sitter space and $\mathcal{J}$ are the sources of the CFT. Furthermore, one must have a match between symmetries of the two theories: in particular global symmetries in the CFT side must correspond to local ones on the gravity counterpart. For example, if one has global Supersymmetry on one side, the other must have local Supersymmetry, which implies Supergravity.
Most crucially, if one wants to infer properties of a CFT and obtain physical and finite results from the gravity side, one has to consider finite gravity actions: this is achieved by means of the so called holographic renormalization, which is a way of removing IR divergences, corresponding to UV ones in the QFT side. This procedure consists in adding to the action suitable boundary terms, which therefore do not alter the equations of motion of the fields: they instead remove singularities coming from the presence of a boundary. We will further discuss this issue during this thesis and we will focus in particular on topological invariant terms.
The applications of this duality are countless and range from Black holes to Nuclear physics and even to Condensed matter physics. In particular, the possibility of studying properties of superconductors, superfluids or graphene-like systems near Dirac points through theories of gravity, which as particle theories are out of our experimental reach, is an idea as fascinating as powerful. Indeed, by relating the physics described by such seemingly different theories, one could for example experimentally test ideas and results of String theory. In this interpretation, without giving up on its privileged role as a fundamental theory of nature, String theory could also be considered as an insightful mathematical framework.

This thesis is organised as follows: in Part I we will briefly review the basic concepts and purposes of String Field Theory as well as its formalism, which heavily relies on 2-dimensional CFT results. We will then focus on Cubic Open String Field Theory and tachyon condensation. We will study analytic solutions of the cubic equations of motion, paying particular attention to the the intertwining solution. After describing the defining features of this solution in the bosonic case, we will study its properties in the superstring case. We will then understand under which conditions the solution is well defined as a string field: with this in mind, we will notice that by extending the set of fields used to describe the solution, we will improve its behaviour, thus allowing to study cases that were previously prohibited.
In Part II, we will start by reviewing the geometric approach to Supergravity: we will focus on mathematical and physical principles allowing the construction of lagrangians for fields in the gravitational multiplet. This framework will outline the two research paths that we will analyse in this thesis: in the first one we will study a particular vacuum theory with the aim of constructing a model describing relativistic spin 1/2 particles, which will be associated to

the wavefunction of charge carriers in of graphene-like materials. The second research line will instead be devoted to $\mathcal{N} = 2$ pure Supergravity with negative cosmological constant, in presence of a spacetime with a boundary. In particular, we will be interested in understanding if the addition of topological boundary terms to the starting bulk lagrangian yields a consistent boundary theory. This holographic analysis will be the first step to further explore and strengthen the link between Supergravity and graphene models.
At the end of both Parts, we will conclude by commenting the obtained results and by discussing the possible future developments that the research activity carried out during the PhD has outlined.

## Part I

# String Field Theory

String theory, which is usually described as a quantum mechanical theory, can be studied in a second quantised approach, in terms of oscillations of fields around a stable vacuum. This approach is therefore called String Field Theory (SFT) and can be applied to both closed and open strings: closed SFT describes fluctuations around a closed string background, without D-branes, whereas open SFT deals with fluctuations around a D-brane system living in a chosen closed string background.
From a historical point of view, such framework has provided, in the open bosonic string case, a better understanding of the tachyon mode appearing in the spectrum of the open string and of the non-manifest background independence of String theory.
Regarding the first problem, in a quantum field theory context, the presence of a tachyon is understood as a sign of instability of the vacuum: in this case it suggests that the D-brane the open strings are attached to is unstable. This problem has been first addressed by Ashoke Sen in [3], where the author conjectured that the tachyon potential possesses a minimum, called tachyon vacuum, with an associated energy density, measured with respect to the unstable vacuum, equal to minus the tension of the D-brane. Sen also conjectured that, in such minimum, the starting D-brane, together with open strings, should vanish and spacetime should be left in a closed string vacuum: the rolling of the theory from the unstable vacuum to the tachyon vacuum has been called tachyon condensation. The absence of physical states around this configuration means that all infinitesimal shifts around the tachyon vacuum are pure gauge.
Following this idea, lower dimensional D-branes should arise as classical solutions on the background of the tachyon vacuum.
The condensation process was first formulated in the context of bosonic String theory, where a tachyon naturally appears in the spectrum, but it can also be studied in Superstring theory, as tachyons arise, for example, from open strings stretched between D-branes and anti D-branes.

The conjecture and its implications have first been tackled from a perturbative point of view, by studying, with increasing precision, the difference in energy density between the unstable vacuum and the tachyon vacuum in the tachyon potential. In this analysis, the infinite expansion of the string field

$$\Phi = \int \mathrm{d}^D k \big(\phi(k) c_1 |0,k\rangle + \chi(k) c_0 |0,k\rangle + A_\mu(k) \alpha^\mu_{-1} c_1 |0,k\rangle + ...\big), \tag{1.1}$$

which would in principle contain the entirety of the string excitations, is truncated at a certain level: this operatively allowed to study the mentioned energy difference, computed from the Witten's cubic action of [4]. This procedure confirmed the conjecture with an incredible precision. For a review of this approach, see [5].

The analytical proofs of Sen's conjectures, in the bosonic case, have been provided in [6], [7] and [8] and are characterised by a different mathematical approach to this problem, relying on an algebra of string fields capable generating exact solutions of the equations of motion. We will briefly review this method in the following Section, as it will be the main tool that we will use in this part of the thesis.
The Superstring case is instead more delicate, as the construction of the theory requires more effort: in particular, the cubic action used in the bosonic case has to be replaced by a non-polynomial action, built of fields living in the Large Hilbert space [9]. Clearly, finding solutions of the equations of motion derived from this action is challenging and the standard analysis performed in the bosonic case is no longer sufficient.
However, it turns out that analytic solutions of the cubic equations of motion can be used to build solutions of the non-polynomial ones, as shown in [10]. For this reason, by following the work done in [11], we will provide a solution of the cubic equations of motion in the superstring case, together with a careful study of its potential divergences.
The solution extends the results of [7] and [12], to the superstring case and describes generic D-brane systems starting from a reference one, as we will see.
We will conclude with some comments on possible future developments.

# 2 Worldsheet and Conformal Field Theories

One of the main applications of Conformal Field Theory is to String theory: the string action, once the worldsheet metric has been fixed to the conformal gauge value, is indeed still invariant under a residual symmetry, generated by the so called conformal Killing vectors. This means that the gauge fixing choice is preserved under this set of diffeomorphisms and that the theory must be invariant under them.
Furthermore, the requirement of conformal invariance completely fixes the dimension of the background target space in which the string worldsheet is embedded, as we will see. For this reason, we will now review the main tools of 2d CFTs, which will play an important role when discussing the results presented in this thesis. The main references for this analysis are [13–17].

## 2.1 Bosonic string setting

The worldsheet is usually parametrised by the real coordinates $(\tau, \sigma)$, in such a way that $\tau \in (-\infty, +\infty)$ and $\sigma \in [0, \pi)$, $\sigma \in [0, 2\pi)$ for open and closed strings respectively. In these coordinates, the worldsheet is thought of as a strip, for the open strings and as a cylinder for closed ones. One then usually performs a Wick rotation $\tau \to -\mathrm{i}\tau$ from the Minkowskian signature to the Euclidean one and defines two complex coordinates

$$w = \tau - \mathrm{i}\sigma, \qquad \bar{w} = \tau + \mathrm{i}\sigma.$$

.

At last, the obtained coordinates are mapped to

$$z = e^{w}, \qquad \bar{z} = e^{\bar{w}}, \tag{2.1}$$

which describe the Upper Half Plane (UHP) in the open string case and the full complex plane in the closed one.

Equal $\tau$ lines, which describe equal time lines, are mapped, in the coordinates $z, \bar{z}$ to (semi)circles centered in the origin, which instead corresponds to past infinity. For this reason, time ordered product of fields is mapped into radial ordering, which is defined in the following way

$$R(\Phi_1(z)\Phi_2(w)) = \begin{cases} \Phi_1(z)\Phi_2(w) \text{ for } |z| > |w| \\ (-1)^{|\Phi_1||\Phi_2|}\Phi_2(w)\Phi_1(z) \text{ for } |w| > |z|, \end{cases}$$

where $\Phi_1$ and $\Phi_2$ are two generic fields, whose Grassmanality is denoted by $|\Phi_{1,2}|$.

In the following, any product of fields will be considered to be radially ordered and we will therefore drop the ordering symbol. As a consequence of radial ordering, the equal time commutator is then defined as

$$[\Phi_1(z), \Phi_2(w)]|_{|z|=|w|} = \lim_{\delta \to 0}\{\Phi_1(z)\Phi_2(w)|_{|z|=|w|+\delta} - \Phi_2(w)\Phi_1(z)|_{|z|=|w|-\delta}\}. \tag{2.2}$$

Let us now first focus on the closed string case, whereas the open string one will be dealt with later on.

In $d = 2$, conformal transformations, which are solutions to the conformal Killing vector equation, actually coincide with the holomorphic and anti-holomorphic coordinate transformations given by

$$z' = f(z), \qquad \bar{z}' = \bar{f}(\bar{z}).$$

The main mathematical objects studied in CFTs are the primary fields, which transform as tensors under conformal transformations

$$\Phi'(z', \bar{z}') = \left(\frac{\partial z'}{\partial z}\right)^{-h} \left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{-\bar{h}} \Phi(z, \bar{z}), \tag{2.3}$$

where $h, \bar{h}$ are called weights and the combinations $h+\bar{h}$ and $h-\bar{h}$ are respectively called scaling dimension and conformal spin of $\Phi$.
An holomorphic field can be expressed in the complex plane as a mode expansion

$$\Phi(z) = \sum_{n\in\mathbb{Z}} \Phi_n z^{-n-h}, \tag{2.4}$$

where the various modes can be obtained by Cauchy's residue theorem as

$$\Phi_n = \oint_{C_0} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \Phi(z) z^{n+h-1}, \tag{2.5}$$

$C_0$ being a circle around the origin.
Since the energy momentum tensor of a CFT is classically traceless and covariantly conserved, its only two components are holomorphic $T_{zz} \equiv T(z)$ and anti-holomorphic $T_{\bar{z}\bar{z}} \equiv \bar{T}(\bar{z})$: a generic infinitesimal conformal transformation $\delta z = \xi(z)$, $\delta\bar{z} = \bar{\xi}(\bar{z})$ then leads to the following conserved quantity

$$T_{\xi,\bar{\xi}} = \frac{1}{2\pi\mathrm{i}} \oint_{C_0} \left(\mathrm{d}z\, \xi(z) T(z) + \mathrm{d}\bar{z}\, \bar{\xi}(\bar{z}) \bar{T}(\bar{z})\right). \tag{2.6}$$

We therefore obtain an infinite number of conserved quantities, parametrised by the vectors $\xi, \bar{\xi}$: this is a reflection of the fact that, as we will mention, the conformal algebra in two dimensions is infinite dimensional.
The above quantity generates infinitesimal transformations of primary fields: for holomorphic fields we have

$$\delta_\xi \Phi(z) = -[T_\xi, \Phi(z)], \tag{2.7}$$

which can be compared with the defining relation (2.3). This allows to obtain the Operator Product Expansion (OPE), which in general describes the short distance behaviour of two fields, between the energy momentum tensor and the chosen primary field

$$T(z)\Phi(w) = \frac{h\Phi(w)}{(z-w)^2} + \frac{\partial\Phi(w)}{z-w} + \text{finite terms}. \tag{2.8}$$

We see that the weight of the primary operator always appears as the coefficient of the $(z-w)^{-2}$ term.
By examining the transformation law of the energy momentum tensor, one can inspect its OPE with itself:

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{finite terms}. \tag{2.9}$$

The first term appearing in the above formula depends on a number $c$ and in general prevents us from considering the energy momentum tensor as a primary field of conformal weight $h=2$. An analogous formula can be written for the anti-holomorphic sector.
The T-T OPE (2.9) can be equivalently restated in terms of the energy momentum modes

$$L_n = \oint \frac{\mathrm{d}z}{2\pi\mathrm{i}} z^{n+1} T(z)$$

as

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n}. \tag{2.10}$$

We then see that the $c$-term appearing in (2.9) generates the unique central extension of the classical Witt algebra. The algebra (2.10) is called Virasoro algebra and the number $c$ is referred to as the central charge.
The central charge measures how much of the conformal invariance is lost at the quantum level: a non-anomalous String theory should always have $c = 0$.
We observe that (2.10) allows for a closed finite subalgebra generated by $L_0, L_{\pm 1}$, which describe the infinitesimal transformations $\delta z = \alpha + \beta z + \gamma z^2$. These generators define the group of transformations $PSL(2, \mathbb{C}) = SL(2, \mathbb{C})/\mathbb{Z}_2$.

In a generic quantum field theory, the space of states and the set of local operators differ from each other, whereas in CFTs there exists an isomorphisms which relates them: this is the so called state-operator correspondence and can be concretely expressed for in-states as

$$|\Phi\rangle := \lim_{z,\bar{z}\to 0} \Phi(z, \bar{z})|0\rangle, \tag{2.11}$$

where $|0\rangle$ is the in-vacuum state and $\Phi$ is a primary field. Regularity of the constructed state at past infinity means that

$$\Phi_n|0\rangle = 0 \text{ for } n > -h, \tag{2.12}$$

as a consequence of (2.5). Out-states are defined as Hermitian conjugate of in-states and their regularity requires

$$\langle 0|\Phi_n = 0 \text{ for } n < h. \tag{2.13}$$

In particular, we can consider the Virasoro generators $L_m$, which have to satisfy

$$L_n|0\rangle = 0 \text{ for } n > -2 \quad \wedge \quad \langle 0|L_n = 0 \text{ for } n < 2.$$

We then see that the previously considered $PSL(2, \mathbb{C})$ algebra annihilates both the in- and the out-vacuum, which is therefore called $SL(2, \mathbb{C})$ invariant vacuum.
As we will see in the next sections, when considering correlation functions, one can define another notion of out-state, the so called BPZ conjugate, which reads

$$(|\Phi\rangle)_{\text{BPZ}} := \langle 0|I \circ \Phi(0), \tag{2.14}$$

where $I(z) = -\frac{1}{z}$.
The state-operator correspondence relates unitary representations of the Virasoro algebra to primary fields: indeed, it can be shown that the state $|\Phi\rangle$ is a highest weight representation satisfying

$$L_0|\Phi\rangle = h|\Phi\rangle, \tag{2.15}$$

where $h$ is the conformal weight of the holomorphic field $\Phi(z)$. Since $L_0$ is associated to dilatations in the complex plane, which are time translations on the cylinder and therefore coincides with the hamiltonian, the conformal weight can be understood as the energy of the state, being the eigenvalue of such operator.

Let us now consider the open string case: as we have said, the worldsheet is mapped to the UHP, where the real axis corresponds to the edges of the strip. One usually imposes a physical condition on the energy momentum tensor of the theory

$$T(z) = \overline{T}(\bar{z})\Big|_{z=\bar{z}} \tag{2.16}$$

indicating that there is no flow of momentum across the boundary. This suggests us to perform an analytic continuation of energy momentum tensor to the lower half plane, a procedure called doubling trick. We therefore identify $T(z') = \bar{T}(z)$ for $Im(z') < 0$, yielding a single holomorphic energy momentum tensor on the whole complex plane.
This condition breaks the conformal symmetry from two independent Virasoro algebras to a single (diagonal) Virasoro algebra. whose generators are defined in the usual way as

$$L_n = \oint_{C_0} \frac{\mathrm{d}z}{2\pi \mathrm{i}} z^{n+1} T(z).$$

Open strings are characterised by the possibility of having different boundary conditions on the two endpoints: for this reason, the theory describing them is actually a Boundary CFT (BCFT). The latter deals with boundary condition changing operators (bcco), which, inserted at the origin of the UHP, allow to change boundary conditions between the negative and positive real axis, which correspond, in the strip frame, to the worldlines drawn by the two endpoints.

### 2.1.1 Free CFTs and BRST symmetry

We discuss now two examples of free CFTs, needed for the analysis of bosonic String theory: the scalar fields $X^\mu$ and the bc-system. Since the closed string CFT actually splits into two isomorphic sectors, left-moving (holomorphic) and a right-moving (anti-holomorphic), whereas the open string BCFT has only one, we will mainly focus on the main properties of the holomorphic sector.

The Polyakov action in the conformal gauge for strings propagating in flat Minkowski space reads

$$S^X = \frac{1}{2\pi\alpha'} \int \mathrm{d}^2 z \, \partial X^\mu(z,\bar{z}) \bar{\partial} X_\mu(z,\bar{z}), \qquad \mu = 0, 1, \ldots, D-1, \tag{2.17}$$

where, in virtue of the equations of motion, $X^\mu(z,\bar{z}) = X^\mu(z) + \bar{X}(\bar{z})$. The holomorphic energy momentum tensor reads

$$T^X(z) = -\partial X^\mu \partial X_\mu, \tag{2.18}$$

where we have set $\alpha' = 1$. In order for it to be finite, the conformal normal ordering $: \ldots :$ defined for generic fields as

$$: A(z)B(z) := \lim_{w \to z} A(w)B(z) - \text{poles} = \oint \frac{dw}{2\pi \mathrm{i}} \frac{A(w)B(z)}{w-z}$$

is intended. The T-T OPE agrees with (2.9) and reads

$$T^X(z)T^X(w) = \frac{D/2}{(z-w)^4} + \frac{2\,T^X(w)}{(z-w)^2} + \frac{\partial T^X(w)}{z-w} + \ldots$$

and allows to retrieve $c^X = D$. The free bosons $X^\mu$ are not primary fields, as they do not transform according (2.8), but the current $J^\mu = \mathrm{i}\sqrt{2}\,\partial X^\mu$ is a primary field of conformal weight $h = 1$.

The action describing the (holomorphic part of the) bc-system is instead given by

$$S^{\mathrm{bc}} = \frac{1}{2\pi} \int \mathrm{d}^2 z\, b \bar{\partial} c, \tag{2.19}$$

where the OPE between the two anticommuting fields is given by

$$b(z)c(w) = c(z)b(w) = \frac{1}{z-w} + \ldots. \tag{2.20}$$

The action is conformally invariant when the fields transform as primaries with

$$h_b = \lambda, \qquad h_c = 1 - \lambda,$$

for any real number $\lambda$. The energy momentum tensor for the bc-system is given by

$$T^{\mathrm{bc}}(x) = (1-\lambda)\partial b c - \lambda b \partial c, \tag{2.21}$$

with associated central charge $c^{\mathrm{bc}} = -3(2\lambda - 1)^2 + 1$. Furthermore, the bc-system is invariant under a $U(1)$ symmetry $\delta b = -\mathrm{i}\epsilon b$, $\delta c = \mathrm{i}\epsilon c$, which generates a current $j^{\mathrm{bc}}(z) = -bc\,(z)$ satisfying

$$T^{\mathrm{bc}}(z) j^{\mathrm{bc}}(w) = \frac{1-2\lambda}{(z-w)^3} + \frac{j^{\mathrm{bc}}(w)}{(z-w)^2} + \frac{\partial j^{\mathrm{bc}}(w)}{z-w} + \ldots. \tag{2.22}$$

This last formula shows that the ghost current does not transform as a primary field, unless $\lambda = \frac{1}{2}$. One can then define a conserved charge

$$Q^{\mathrm{bc}} = \oint_{C_0} \frac{\mathrm{d}z}{2\pi \mathrm{i}}\, j^{\mathrm{bc}}(z),$$

which associates a number to each field: in particular $Q^{\mathrm{bc}} c(z) = +1$, $Q^{\mathrm{bc}} b(z) = -1$.
The $\lambda = 2$ bc-system arises when gauge-fixing the worldsheet diffeomorphisms of the Polyakov string and the two fields are therefore called ghosts.

The combination of the matter CFT with the ghost $\lambda = 2$ CFT is a theory with central charge $c^{\text{tot}} = D - 26$: this number appears in the expression of the vacuum expectation value of the trace of the energy momentum tensor and it is referred to as the Weyl anomaly. Its vanishing allows to preserve the conformal symmetry of String theory at the quantum level and fixes the dimension of the target spacetime at $D = 26$.
The $\lambda = 2$ ghost fields can be expanded according to (2.4) as

$$c(z) = \sum_{n\in\mathbb{Z}} c_n z^{-n-1}, \qquad b(z) = \sum_{n\in\mathbb{Z}} b_n z^{-n-2}, \tag{2.23}$$

whose modes satisfy (2.12)

$$c_n|0\rangle = 0 \quad \forall n \geq 2, \qquad b_n|0\rangle = 0 \quad \forall n \geq -1. \tag{2.24}$$

After gauge fixing of the symmetries, the combined theory still preserves a part of the original gauge symmetry. This residual redundancy is called BRST symmetry and generates a current in the form

$$j_B(z) = c\left(T^X + \frac{1}{2}T^{\text{bc}}\right)(z) + \frac{3}{2}\partial^2 c(z), \tag{2.25}$$

which can be used to define a BRST charge

$$Q_B = \oint_{C_0} \frac{\mathrm{d}z}{2\pi \mathrm{i}} j_B(z) \implies Q^2 = 0.$$

The latter actually satisfies the nilpotency condition $Q_B^2 = 0$ at the critical dimension $D = 26$.
The BRST charge satisfies the following commutation relations

$$\begin{aligned} [Q_B, X^\mu(z)] = c\partial X^\mu(z), \quad [Q_B, T^X(z) + T^{\text{bc}}(z)] = \frac{D-26}{12}\partial^3 c(z), \\ [Q_B, c(z)] = c\partial c(z), \quad [Q_B, b(z)] = T^X(z) + T^{\text{bc}}(z), \end{aligned} \tag{2.26}$$

where the bracket $[\cdot,\cdot]$ is the graded commutator. Physical states are BRST invariant states, which are not exact, namely

$$Q_B|\Phi\rangle = 0, \qquad |\Phi\rangle \neq Q_B|\chi\rangle \tag{2.27}$$

This means that there is a correspondence between physical states and the cohomology of the BRST operator: any physical quantity should not depend on the representative we choose from each class.
Finally, due to the fact that the mode $c_0$ commutes with the hamiltonian $L_0$ and that $[c_0, b_0] = 1$, the vacuum is actually degenerate and does not coincide with the usual $SL(2,\mathbb{C})$ vacuum. This is also a consequence of the $U(1)$ anomaly of the ghost current (2.22), in the $\lambda = 2$ case. Indeed it can be proved that physical states are built from $|\downarrow\rangle := c_1|0\rangle$ and that the only non-vanishing correlator is $\langle 0|c_{-1}c_0c_1|0\rangle$, with ghost number three.

We conclude this discussion, by noticing that the bc-system in the $\lambda = \frac{1}{2}$ case can be reformulated by calling $b \to \psi$ and $c \to \bar{\psi}$. By then redefining the fields as

$$\psi = \frac{1}{\sqrt{2}}(\psi_1 + \mathrm{i}\psi_2), \qquad \bar{\psi} = \frac{1}{\sqrt{2}}(\psi_1 - \mathrm{i}\psi_2),$$

one obtains two $\psi$-theories with central charge $c = \frac{1}{2}$. This latter case will gain relevance in the upcoming discussion on superstring theory.

## 2.2 The RNS formalism for Superstring theory

From a phenomenological point of view, bosonic String theory is unsatisfactory as it cannot describe fermions, which can instead be considered if one introduces Supersymmetry. The correct mathematical framework needed to study this theory is the one of Super Conformal Field Theories (SCFTs): superconformal symmetry appears as a consequence of the symmetries of the fermionic string in the superconformal gauge. We now start reviewing the basic properties of $\mathcal{N} = 1$ SCFT.

One first defines coordinates $\boldsymbol{z} = (z, \theta)$, $\bar{\boldsymbol{z}} = (\bar{z}, \bar{\theta})$ on a superspace, where $\theta$, $\bar{\theta}$ are fermionic quantities satisfying $\theta^2 = \bar{\theta}^2 = 0$. A field is called holomorphic chiral superconformal primary if it is written as

$$\boldsymbol{\Phi}(\boldsymbol{z}) = \Phi_0(z) + \theta\Phi_1(z) \iff \bar{D}\boldsymbol{\Phi} = 0 \tag{2.28}$$

and if it transforms under superconformal transformations $\boldsymbol{z}' = \boldsymbol{z}'(z'(z,\theta), \theta'(z,\theta))$ as

$$\boldsymbol{\Phi}(z) = (D\theta')^{2h}\boldsymbol{\Phi}'(\boldsymbol{z}'). \tag{2.29}$$

Here the covariant derivative in superspace is defined as $D = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}$.
One accordingly defines a superfield containing both the energy momentum tensor and the supercurrent $T_F(z)$, generating infinitesimal superconformal transformations

$$\boldsymbol{T}(\boldsymbol{z}) = T_F(z) + \theta T(z), \tag{2.30}$$

whose OPE with the $\Phi_0$ component in (2.28) reads

$$\begin{aligned} T(z)\Phi_0(w) &= \frac{h\Phi_0(w)}{(z-w)^2} + \frac{\partial\Phi_0(w)}{z-w} + \ldots, \\ T_F(z)\Phi_0(w) &= \frac{\Phi_1(w)}{z-w}. \end{aligned} \tag{2.31}$$

The components of the super energy momentum tensor satisfy the following relations

$$\begin{aligned} T(z)T(w) &= \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots, \\ T(z)T_F(w) &= \frac{\frac{3}{2}T(w)}{(z-w)^2} + \frac{\partial T_F(w)}{z-w} + \ldots, \end{aligned}$$

$$T_F(z)T_F(w) = \frac{c/6}{(z-w)^3} + \frac{T(w)}{z-w} \tag{2.32}$$

and can be expressed in terms of their modes as

$$\begin{aligned} T(z) &= \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \\ T_F(z) &= \sum_{r\in\mathbb{Z}+a} G_r z^{-r-\frac{3}{2}}. \end{aligned} \tag{2.33}$$

The parameter $a$ has been introduced here to distinguish between two cases: integer modings $a = 0$ correspond to half integer powers, whereas half integers modings $a = \frac{1}{2}$ correspond to integer powers.
These two cases respectively correspond to the Ramond(R) and Neveu-Schwarz (NS) boundary conditions that one can impose for the fermions appearing in the supersymmetric generalisation of the Polyakov action

$$S^{(x,\psi)} = \frac{1}{4\pi}\int \mathrm{d}^2 z \left(\frac{2}{\alpha'}\partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial}\psi_\mu + \tilde{\psi}^\mu \partial \tilde{\psi}_\mu\right), \tag{2.34}$$

where the fermions $\psi$, $\tilde{\psi}$ are described by the $\lambda = \frac{1}{2}$ bc-system shown at the end of the previous Subsection.
For the closed string, both right moving and left moving fermions have to be provided with (possibly different) boundary conditions, whereas in the open string case they can be combined into a single holomorphic spinor satisfying one of the two boundary conditions. In the latter case, the doubling trick procedure has also to be applied on the energy momentum tensor, as in the bosonic theory and on the supercurrent $T_F(z)$.
From (2.32) one can derive the algebra that the modes satisfy

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n}, \\ [L_m, G_r] &= \left(\frac{1}{2}m - r\right) G_{m+r}, \\ [G_r, G_s] &= 2L_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r,-s}. \end{aligned} \tag{2.35}$$

Only in the NS case, where $a$ is half integer, there exists a finite dimensional subalgebra generated by $L_0$, $L_{\pm 1}$, $G_{\pm\frac{1}{2}}$. The latter actually generalises the $SL(2,\mathbb{C})$ algebra to the superalgebra of $OSp(1,2)$. An $OSp(1,2)$ invariant vacuum can be then defined only in the NS case and the regularity of both components of the super energy momentum tensor implies

$$L_n|0\rangle = 0 \quad \forall n \geq -2, \qquad G_r|0\rangle = 0 \quad \forall r \geq -\frac{3}{2}. \tag{2.36}$$

At last, let us observe that the global supersymmetry algebra on the complex plane

$$G^2_{-\frac{1}{2}} = L_{-1}. \tag{2.37}$$

can be obtained from the last relation in (2.35).

### 2.2.1 Free SCFTs and BRST symmetry

We now discuss, as we did for the bosonic string, free SCFTs examples and their relation to the BRST quantisation of the Superstring Theory.

Let us start from the supersymmetric generalisation of the Polyakov action, (2.34): the explicit expression of its super energy momentum tensor reads

$$T^{(X,\psi)} = -\partial X^{\mu}\partial X_{\mu}(z) - \frac{1}{2}\psi^{\mu}\partial\psi_{\mu}(z), \tag{2.38}$$

whose central charge is given by $c^{(X,\psi)} = \frac{3}{2}D$. The process of gauge fixing of the redundancies of the starting action introduces ghost fields: in particular, bosonic(fermionic) symmetries induce fermionic(bosonic) ghosts: besides the known bc-system, one has to introduce two bosonic superghosts $\beta$ and $\gamma$. These fields can all be arranged in superfields as

$$\boldsymbol{B}(\boldsymbol{z}) = \beta(z) + \theta b(z), \qquad \boldsymbol{C}(\boldsymbol{z}) = c(z) + \theta\gamma(z) \tag{2.39}$$

and appear in the total ghost action as

$$S^{\text{ghost}} = \frac{1}{2\pi}\int \mathrm{d}z\, \left(b\bar{\partial}c + \beta\bar{\partial}\gamma\right). \tag{2.40}$$

The $\beta\gamma$-system in the generic case $h_{\beta} = \lambda$, $h_{\gamma} = 1-\lambda$ has a central charge $c^{\beta\gamma} = 3(2\lambda-1)^2-1$. In the gauge fixing process, these bosonic ghosts appear with $\lambda = \frac{3}{2}$ and the total central charge is $c^{\text{ghost}} = c^{\text{bc}} + c^{\beta\gamma} = -15$. To preserve superconformal invariance at the quantum level, we find that the critical dimension of the superstring is

$$c^{(X,\psi)} + c^{\text{ghost}} = 0 \implies D = 10. \tag{2.41}$$

However, as a consequence of the action being linear in the derivatives and due to the bosonic nature of the superghost fields, it turns out that the energy of the vacuum is unbounded from below. In order to avoid this problem, it is more convenient to “bosonise” the $\beta\gamma$ system: this procedure consists in introducing new fields allowing to reproduce the same properties and behaviour of the chosen ghost CFT.
In order to do so, let us consider a $\lambda = 1$ bc-system, whose fields are known in the literature as $\eta$ and $\xi$ and whose conformal weights are $h_{\eta} = 1$ and $h_{\xi} = 0$. Moreover, let us introduce a scalar field $\phi$, whose action is given by

$$S^{\phi} = -\frac{1}{8\pi}\int \mathrm{d}^2z\,\sqrt{h}\left(h^{\alpha\beta}\partial_{\alpha}\phi\partial_{\beta}\phi + QR\phi\right). \tag{2.42}$$

The number $Q$ is a positive background charge and can be determined by matching the properties of the resulting CFT with the $\beta\gamma$-system: indeed

$$c^{(\beta\gamma)} = c^{(\eta\xi)} + c^{\phi} \implies 11 = -2 + c^{\phi} \implies c^{\phi} = 13 \tag{2.43}$$

The explicit expression of the energy momentum tensor in in the superconformal gauge picks up a contribution proportional to $Q$ and reads

$$T^{\phi} = -\frac{1}{2}\partial\phi\partial\phi - \frac{Q}{2}\partial^2\phi. \tag{2.44}$$

The central charge associated to this system can be computed from the OPE $\partial\phi(z)\partial\phi(w) = \frac{-1}{(z-w)^2} + \ldots$ and reads

$$c^{\phi} = 1 + 3Q^2 = 13 \implies Q = 2.$$

In this way, the bosonic ghosts $\beta$ and $\gamma$ can be then related to the newly introduced fields as

$$\beta(z) = e^{-\phi(z)}\partial\xi(z), \qquad \gamma(z) = e^{\phi(z)}\eta(z), \tag{2.45}$$

where it is now trivial to show that the two ghosts satisfy the correct OPE relations.
Notice that the scalar field $\xi$ appears in (6.33) only through its derivative: this implies that an arbitrary OPE of superghost will never generate $\xi$, and therefore the zero mode $\xi_0$, but only its derivatives. Since operators are in one to one correspondence with states, we can distinguish between a Large Hilbert space containing states generated by $\xi$ and a Small Hilbert space excluding them.
At last, one can introduce the following two currents:

$$j^{\eta\xi}(z) = -\xi\eta(z), \qquad j^{p} = \partial\phi(z) + \xi\eta(z). \tag{2.46}$$

The first one is the standard current associated to a bc-system and its charge associates to each field the corresponding super ghost number

$$Q^{\eta\xi}\beta(z) = -\beta(z), \quad Q^{\eta\xi}\gamma(z) = \gamma(z), \quad Q^{\eta\xi}\eta(z) = \eta(z), \quad Q^{\eta\xi}\xi(z) = \xi(z), \tag{2.47}$$

whereas the second current introduces a new quantum number, called picture number, in the following way

$$Q^{p}\beta(z) = Q^{p}\gamma(z) = 0, \quad Q^{p}\eta(z) = -\eta(z), \quad Q^{p}\xi(z) = \xi(z), \quad Q^{p}e^{q\phi}(z) = qe^{q\phi(z)}. \tag{2.48}$$

In Superstring theory, every state has different equivalent representations in terms of vertex operators with different picture numbers. However, similarly to the fact that the anomaly of the bosonic ghost current meant that non-vanishing correlators had to contain three ghosts $c$, here both currents in (2.46) are anomalous at the quantum level. Therefore, it turns out that non vanishing correlators in Superstring theory must have ghost number 3 and picture number $-2$ in order for them not to be zero.

We now review the main results concerning BRST quantisation in the supersymmetric case: as said for the bosonic theory, after gauge fixing, the action preserves some residual symmetries, which induce a charge

$$Q_B = \oint \frac{\mathrm{d}z}{2\pi\mathrm{i}}\Big(\Big(cT^{(X,\psi)} + \frac{1}{2}cT^{\text{ghost}}\Big) + \Big(\gamma T_F^{(X,\psi)} + \frac{1}{2}\gamma T_F^{\text{ghost}}\Big)\Big), \tag{2.49}$$

where the quantities appearing in the above formula are given by

$$\begin{aligned} T^{\text{ghost}} &= -(\partial b)c - 2b\partial c - \frac{1}{2}(\partial\beta)\gamma - \frac{3}{2}\beta\partial\gamma, \\ T_F^{\text{ghost}} &= (\partial\beta)c + \frac{3}{2}\beta\partial c - 2b\gamma. \end{aligned} \tag{2.50}$$

The obtained charge is nilpotent in the critical dimension $D = 10$, has picture number zero and satisfies the following identities

$$\begin{aligned} &[Q_B, c(w)] = c\partial c(w) - \gamma^2(w), \quad [Q_B, b(w)] = T^{(X,\psi)}(w) + T^{\text{ghost}}(w), \\ &[Q_B, \gamma(w)] = c\partial\gamma(w) - \frac{1}{2}\gamma\partial c(w), \quad [Q_B, \gamma^2(w)] = c\partial\gamma\gamma(w) + \gamma c\partial\gamma(w) - \gamma^2\partial c(w). \end{aligned} \tag{2.51}$$

Furthermore, as discussed previously, physical states belong to the cohomology of the BRST charge. These results will be of central importance for the upcoming discussion, where we will consider the cubic formulation of Open String Field Theory (OSFT) and its relation to the background independence problem of the open string.

# 3 Selected topics on Cubic Open String Field Theory

In order to study fluctuations of a D-brane, it is useful to group all the excitations of an open string attached to the given D-brane in a single object, the string field. As a consequence of the state-operator correspondence, string fields are elements of a state space $\mathcal{H}$ generated by the worldsheet BCFT defining the D-brane. We are now interested in reviewing the different approaches developed for studying the dynamics of this object.
Let us first start with the bosonic case: in 1986, Witten formulated an axiomatic approach to OSFT [4], based on a graded vector space

$$\mathcal{A} = \bigoplus_{q=-\infty}^{+\infty} \mathcal{A}^q, \tag{3.1}$$

whose grading $q$ is identified with the ghost number of the string fields. This vector space is equipped with a nilpotent operator, coinciding with the BRST charge,

$$\begin{aligned} &Q_B : \mathcal{A}^q \to \mathcal{A}^{q+1}, \\ &Q_B^2 = 0 \end{aligned} \tag{3.2}$$

and with an associative, non-commutative product, called Witten's star product

$$\begin{aligned} &\star : \mathcal{A}^p \otimes \mathcal{A}^q \to \mathcal{A}^{p+q}, \\ &(a \star b) \star c = a \star (b \star c), \end{aligned} \tag{3.3}$$

for $a, b, c \in \mathcal{A}$. The latter operation has to be compatible with the structure introduced previously: this means that the BRST charge has to satisfy the Leibnitz rule

$$Q_B(a \star b) = Q_B(a) \star b + (-1)^{gh(a)} a \star Q_B(b). \tag{3.4}$$

As a final operation, Witten proposed a linear map

$$\int : \mathcal{A} \to \mathbb{C}, \tag{3.5}$$

called integration, with the following properties

$$\begin{aligned} \int Q_B a &= 0, \\ \int a \star b &= (-1)^{gh(a)gh(b)} \int b \star a, \end{aligned} \tag{3.6}$$

$\forall a, b \in \mathcal{A}$. The star product is interpreted as the gluing of the right half piece of a first string to the left half of a second one, effectively yielding a third open string. The integration operation is instead thought of as the gluing of the left and right halves of a single string.
All these operations have an equivalent formulation in the CFT language: the Witten's integral, for example, becomes a CFT correlator built out of the BPZ inner product, which, as we said, is non-zero provided that there are three insertions of ghost $c$. Therefore the integrand must be at ghost number three as well.
By making use of these definitions, Witten studied an action for a dynamical string field $\Phi$ with ghost number one, capable of generalising the physical condition (2.27) to the interacting case. The action resembles a Chern-Simons one and reads

$$S = -\frac{1}{g_0^2}\left(\frac{1}{2}\int \Phi \star Q_B \Phi + \frac{1}{3}\int \Phi \star \Phi \star \Phi\right), \tag{3.7}$$

where $g_0$ is the open string coupling, related to the closed one by $g_0^2 = g_{\text{closed}}$. This action is invariant under an infinitesimal non-abelian gauge symmetry given by

$$\delta\Phi = Q_B \Lambda + (\Phi \star \Lambda - \Lambda \star \Phi), \tag{3.8}$$

with ghost number zero gauge parameter $\Lambda$ and the equations of motion are

$$Q_B \Phi + \Phi \star \Phi = 0. \tag{3.9}$$

The cubic action (3.7) is then well defined from the point of view of the axioms stated above, as the total ghost number of the integrand is indeed three. As stated above, this construction can be obtained by analogy with non-abelian Chern-Simons theories, where the string field $\Phi$ and the ghost number are identified with the gauge field and the rank of a differential form respectively.
From this point of view, the cubic term in the action, which is responsible for interactions, is actually the only non-linear extension compatible with the non-abelian gauge symmetry (3.8).

The cubic action we just introduced cannot be straightforwardly upgraded to the supersymmetric case: if we focus on the NS sector, the natural picture number of the dynamical string field is $-1$ and the Witten's integral would have the correct total ghost number, but the wrong total picture number, $-3$. One way to avoid this problem is to choose instead a dynamical string field at picture number 0 and to consider a double-step picture changing operator $Y_{-2}(z) := Y_{-1}(z)Y_{-1}(\bar{z})$: this allows to obtain the Modified Cubic action [18, 19], which reads

$$S = -\frac{1}{g_0^2}\left(\frac{1}{2}\int Y_{-2}\Psi \star Q_B\Psi + \frac{1}{3}\int Y_{-2}\Psi \star \Psi \star \Psi\right), \tag{3.10}$$

where the operators $Y_{-1}(z) = c\partial\xi e^{-2\phi}(z)$ are inserted at the string midpoint. However, the main problem of this approach comes from the equations of motion, which read

$$Y_{-2}(Q_B\Psi + \Psi \star \Psi) = 0. \tag{3.11}$$

Since the double-step picture changing operator $Y_{-2}(z)$ has a non-trivial kernel, one could in principle obtain, in the linearised theory, solutions which are not in the cohomology of the BRST operator. As of today, it is still not clear if this approach truly leads to inconsistencies.

A very different approach for the OSFT in the NS sector has been proposed by Berkovits in 1995 [9]: the dynamical string field described with this formulation belongs to the Large Hilbert space, it is Grassmann even and with picture number 0. The action takes the form of a Wess-Zumino-Witten model

$$S = \frac{1}{2}\int \left[\left(e^{-\Phi}Q_Be^{\Phi}\right)\left(e^{-\Phi}\eta_0e^{\Phi}\right) - \int_0^1 dt\ \left(e^{-t\Phi}\partial_te^{t\Phi}\right)\left\{\left(e^{-t\Phi}Q_Be^{t\Phi}\right), \left(e^{-t\Phi}\eta_0e^{t\Phi}\right)\right\}\right], \tag{3.12}$$

where the integral is defined by the Witten's gluing prescription of the strings and where we omitted the $\star$ in all products of string fields.
The action enjoys a non-linear gauge invariance under the infinitesimal transformation

$$\delta e^{\Phi} = (Q_B\Lambda_1)e^{\Phi} + e^{\Phi}(\eta_0\Lambda_2),$$

where $(\Lambda_1, \Lambda_2)$ are two gauge parameters of ghost number $-1$ and its equations of motion read

$$\eta_0(e^{-\Phi}Q_Be^{\Phi}) = 0. \tag{3.13}$$

We can have a grasp of the meaning of these equations of motion if we look at the linearised theory: they read $Q_B(\eta_0\Phi) = 0$, from which we see that the zero mode of $\eta$ maps the dynamical string field into the Small Hilbert space. The equations of motion then reduce to the known physical condition.

The equations of motion (3.13) are in principle very hard to solve, because they are non-polynomial. However, they can be formally solved by ignoring the $Y_{-2}$ operator in (3.11), as shown in [10]: this simplification effectively allows to reduce to the previously considered equations of motion (3.9), where now $\Psi$ is a picture number 0 dynamical string field. Indeed, let $\Psi_*$ be a solution of such equations: if we impose the relation

$$\Psi_* = e^{-\Phi}Q_Be^{\Phi}, \tag{3.14}$$

we see that the cubic equations of motion are trivially satisfied. The above equation relates solutions of the Berkovits theory to solutions of the cubic approach. From the latter point of view, the solution is written as a pure gauge transformation with gauge parameter $e^{\Phi}$: however, this field and its inverse are not defined as string fields in the space of states of the Chern-Simons-like action (3.10), which ultimately means that the solution $\Psi_*$ can be of physical interest. The very same argument has been used to understand the physical implications of the tachyon vacuum solution that we will describe in the next Subsection.
The identification in (3.14) can be rewritten as

$$Q_{0\Psi_*}e^{\Phi} = 0, \tag{3.15}$$

in terms of a modified BRST operator

$$Q_{AB}C = Q_BC + AC - (-1)^{gh(C)}CB. \tag{3.16}$$

This operator is actually nilpotent if the subscripts are solutions of the cubic equations of motion. In this case we then have a solution

$$e^{\Phi} = Q_{0\Psi_*}\beta. \tag{3.17}$$

The real challenge, once we have a solution of the equations of motion, is to find a string field $\beta$ making $e^{\Phi}$ invertible. Nonetheless, finding solutions of the cubic equations of motion does provide a first step towards solving the Berkovits' ones. For this reason, in this thesis we will be interested in these kind of solutions and in particular in their regularity properties.

## 3.1 Analytic solutions of the cubic equations of motion

As we have mentioned, the Witten's cubic approach has an equivalent formulation in terms of CFT tools: indeed, the action (3.7) can be rewritten as

$$S(\Phi) = -\frac{1}{g_0^2}\left[\frac{1}{2}\langle\Phi, Q_B\Phi\rangle + \frac{1}{3}\langle\Phi, \Phi\star\Phi\rangle\right], \tag{3.18}$$

where $\langle\cdot,\cdot\rangle$ is the BPZ bilinear inner product, defined as

$$\langle\phi_1,\phi_2\rangle = \langle I\circ\phi_1(0)\phi_2(0)\rangle_{\text{UHP}}. \tag{3.19}$$

If $\xi$ is the original coordinate frame on the upper half plane, we consider the following map

$$z = f(\xi) = \frac{2}{\pi}\text{arctan}\xi = \frac{\mathrm{i}}{\pi}\ln\left(\frac{\mathrm{i}-\xi}{\mathrm{i}+\xi}\right), \tag{3.20}$$

which defines a new coordinate frame, called sliver frame, on the UHP. This definition maps the right half of the string ($\xi = e^{\mathrm{i}\theta}$ with $\theta \in [0,\pi]$) into a semi infinite vertical line with $Re z = -\frac{1}{2}$ and positive imaginary part. A detailed representation of this transformation is shown in Figure 1 consistently with the conventions of [20].
If one considers the insertion of the identity operator in the origin of the $\xi$ coordinates, one

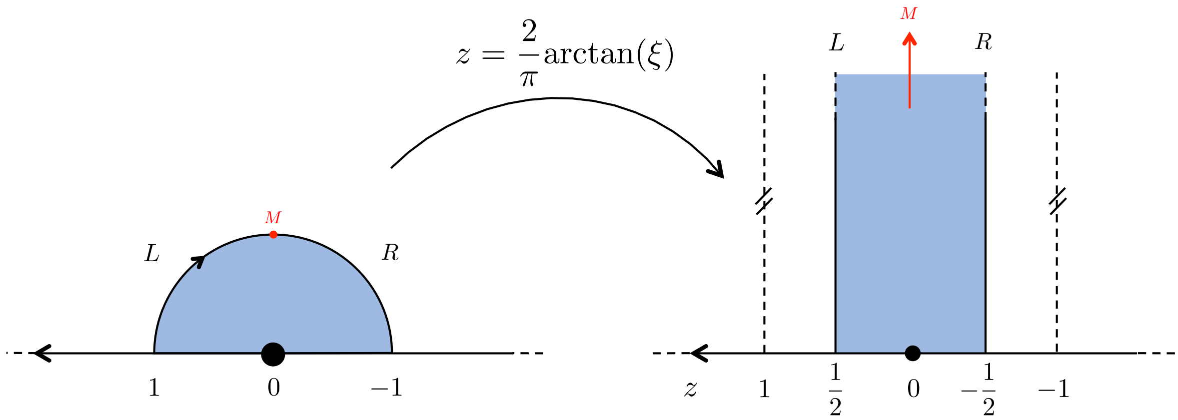


Figure 1: On the left the half unit disk of the upper half plane, on the right the semi-infinite strip in the sliver frame.

obtains in the sliver frame a semi infinite strip of width 1. This state can be multiplied by itself to obtain strips of arbitrary natural width. These states can be further generalised to strips with real non negative width and it can be shown that they actually form and algebra, called wedge algebra, under the star product

$$\Omega^{\alpha} \star \Omega^{\beta} = \Omega^{\alpha+\beta}, \tag{3.21}$$

where $\Omega^{\alpha}$ is a wedge state of width $\alpha > 0$. These states are actually the key to analytically solve the equations of motion: the solutions are indeed obtained by considering operator insertions inside wedge states, which have been well examined and reviewed in [6, 21–23].
In particular let us consider the following GSO(+) string fields

$$\begin{aligned} K: \quad & \langle \phi, K \rangle = \langle f \circ \phi(0) \mathcal{K} \rangle_{C_1}, \qquad B: \quad \langle \phi, B \rangle = \langle f \circ \phi(0) \mathcal{B} \rangle_{C_1}, \\ c: \quad & \langle \phi, c \rangle = \left\langle f \circ \phi(0)\; c\left(-\frac{1}{2}\right) \right\rangle_{C_1}, \qquad \gamma^2: \quad \langle \phi, \gamma^2 \rangle = \left\langle f \circ \phi(0)\; \gamma^2\left(-\frac{1}{2}\right) \right\rangle_{C_1}. \end{aligned} \tag{3.22}$$

These states are defined as inner products on a cylinder $C_1$ with a generic test state $\phi$, as it is usually done in SFT and the operator insertions for the $K$, $B$ string fields are defined as

$$\mathcal{K} = \int_{V_\gamma} \frac{\mathrm{d}z}{2\pi \mathrm{i}} T(z), \qquad \mathcal{B} = \int_{V_\gamma} \frac{\mathrm{d}z}{2\pi \mathrm{i}} b(z), \tag{3.23}$$

where $V_\gamma$ is a vertical infinite line with $Rez = \gamma$, with $-\alpha - \frac{1}{2} < \gamma < -\frac{1}{2}$, in the direction from $-\mathrm{i}\infty$ to $+\mathrm{i}\infty$.
These states are represented as in Figure 2.
Due to the properties of the insertions and thanks to the wedge state algebra, the obtained string fields satisfy the following differential relations[1]

$$QB = 0, \qquad QK = 0, \qquad Qc = c\partial c - \gamma^2, \qquad Q\gamma^2 = cK\gamma^2 - \gamma^2 Kc \tag{3.24}$$

[1]In the following we will omit the subscript to the BRST charge to ease the notation.

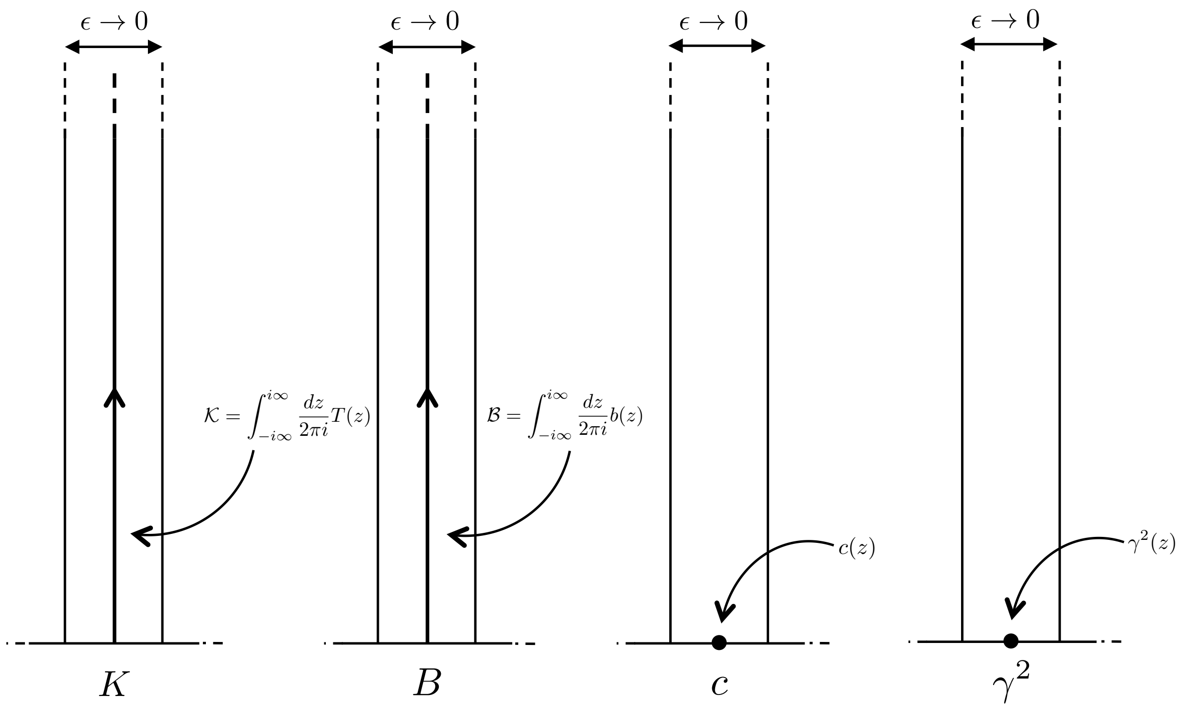


Figure 2: The string fields are obtained as wedge states of infinitesimal width with insertions of operators.

and the algebraic ones

$$[K, B] = 0, \quad [B, c] = 1, \quad [B, \gamma^2] = 0, \quad [c, \gamma^2] = 0, \quad B^2 = c^2 = 0. \tag{3.25}$$

Among all these string fields, $K$ has a very important role, as it generates wedge states, which, as said, are of key importance for these solutions. This relation is encoded in the fact that

$$\Omega^\alpha = e^{-\alpha K}.$$

A generic function of $F(K)$ will have to be expressed as a sum of exponentials in the following way

$$F(K) = \int_0^\infty \mathrm{d}t\, f(t) e^{-tK}. \tag{3.26}$$

Another property of the string field $K$ is given by its role in partial derivatives of string fields, because they can be expressed as $\partial \cdot = [K, \cdot\,]$.
We now focus on specific solutions and their meaning: before checking the explicit expressions, we observe that one can always reduce to bosonic solutions by setting all $\gamma^2$-terms to zero.

**The tachyon vacuum**

The first solution of the equations of motion (3.9) that we are going to analyse is the tachyon vacuum solution. This specific solution can be written in the Okawa form [22] as

$$\Psi_{\text{tv}} = \sqrt{F(K)}\left(c\frac{K}{1-F(K)}Bc + B\gamma^2\right)\sqrt{F(K)}, \tag{3.27}$$

where $F(K)$ satisfies the following conditions

$$F(0)=1, \quad F'(0)<0, \quad F(\infty)=0, \quad F(K)<1.$$

By using (3.24) and (3.25), it is trivial to show that the above expression satisfies the equations of motion. This solution does also satisfy Sen's conjecture: the associated energy is indeed $E = -S(\Psi) = -\frac{1}{2\pi^2}$ and equals minus the energy density of the collapsing D-brane. Furthermore, one can define a well-behaving string field, called homotopy string field

$$H(K) = \frac{1-F(K)}{K}, \tag{3.28}$$

satisfying $QH + [\Psi_{\text{tv}}, H] = \mathcal{I}$, where $\mathcal{I}$ is the identity string field, an empty infinitesimal width strip. This guarantees that the cohomology of the shifted BRST operator $Q_{\text{tv}} = Q + [\Psi_{\text{tv}}, \cdot]$ is empty and ultimately that there are not open string states in the tachyon vacuum.
A very special case is obtained by equating

$$H(K) = F(K) \implies F(K) = \frac{1}{1+K}. \tag{3.29}$$

The above choice is called simple solution and will play an important role in the upcoming discussion.
In the bosonic case, where the superghost terms are absent, the solution (3.27) is well understood as describing the condensation of the tachyonic mode of the open string [6]. The superstring case is more subtle, as this solution is only constructed out of GSO(+) string fields and seems to exist even on a stable BPS D-brane. To clarify this point, we remind that, since our formulation comes from the Modified cubic approach [23], the dynamical string field has picture number zero. In the natural $-1$ picture there is only one tachyonic state, which is correctly mapped away as it belongs to the GSO($-$) sector; however, due to the fact that the picture lowering operators $Y_{-1}$ are not invertible, there are actually states in the 0 picture that cannot be obtained from the $-1$ level, as shown in [24]. Among these states, which are called auxiliary states, there is the level $L_0 = -1$ state $ce^{ik\cdot X}(0)|0\rangle$, which is usually called tachyon for his similarity with the bosonic case. The latter state belongs to the GSO(+) and the solution (3.27) that we have studied exactly describes its condensation.
The tachyon vacuum we just reviewed is a fundamental ingredient for the solution we are going to discuss now.

### The Intertwining solution

In the original paper [7], the authors studied a set of bosonic solutions of (3.9) showing that open strings attached to a starting D-brane system can rearrange themselves to create

a new one, sharing the same closed string background. The idea is that the starting D-brane can be annihilated by a process of tachyon condensation and from the latter a new D-brane, described by a possibly different BCFT, can emerge.
These intertwining solutions clearly address the open string background independence: the original solution dealt with time independent backgrounds, but the construction has been further generalised in [12], to include any kind of open string background.
This problem can be formulated in SFT by saying that two theories formulated on different backgrounds are related by field redefinition. One can in particular postulate that the field redefinition takes the form

$$\Psi^{(0)} = \Psi_* + f(\Psi^{(*)}), \tag{3.30}$$

where

- $\Psi^{(0)} \in \mathcal{H}_0$ is the dynamical field of the reference D-brane,
- $\Psi^{(*)} \in \mathcal{H}_*$ is the dynamical field of the target D-brane,
- $\Psi_* \in \mathcal{H}_0$ is a classical solution of the reference string field theory,
- $f : \mathcal{H}_0 \to \mathcal{H}_*$ is an invertible linear transformation.

The explicit form of the intertwining solution in the cubic approach reads

$$\Psi_* = \Psi_{\rm tv} - \Sigma \Psi_{\rm tv} \bar{\Sigma} \tag{3.31}$$

and the equations of motion are satisfied provided that

$$Q_{\rm tv}\Sigma = Q_{\rm tv}\bar{\Sigma} = 0, \qquad \bar{\Sigma}\Sigma = 1.$$

The first tachyon vacuum appearing in (3.31) belongs to the $\mathcal{H}_0$ Hilbert space, whereas the one between the two $\Sigma$ fields belongs to $\mathcal{H}_*$.
Both formulations of the intertwining solution, [7] and [12], rely on boundary condition changing operators: in this thesis we will focus on time independent backgrounds, whose simpler, but more restrictive, construction avoids some of the complications of the flag states introduced in [12]. Indeed, in the particular case of static backgrounds, the intertwining string fields $(\Sigma, \overline{\Sigma})$ can be expressed as

$$\Sigma = Q_{\rm tv}(\sqrt{H}\sigma B\sqrt{H}), \qquad \overline{\Sigma} = Q_{\rm tv}(\sqrt{H}B\overline{\sigma}\sqrt{H}),$$

with $(\sigma, \overline{\sigma})$ being infinitesimal width strip with insertions of weight zero (super)conformal primaries. These operators can be defined in terms of matter primaries of generic weight $h$ $(\sigma^{(h)}, \overline{\sigma}^{(h)})$ as

$$\begin{aligned} \sigma(x) &= e^{i\sqrt{h}X^0}\sigma^{(h)}(x), \\ \overline{\sigma}(x) &= e^{-i\sqrt{h}X^0}\overline{\sigma}^{(h)}(x), \end{aligned} \qquad \text{such that} \qquad \begin{cases} \lim_{x\to 0} \quad \overline{\sigma}(x)\sigma(0) = 1, \\ \lim_{x\to 0} \quad \sigma(x)\overline{\sigma}(0) = \frac{g_*}{g_0} \end{cases}.$$

Here $g_{*,0} = \langle 1 \rangle_{BCFT_{*,0}}$ are the disk partition functions in the respective BCFTs. The disk partition functions will have different values if the D-brane configurations will have different energies: this means that in general $\lim_{x\to 0} \sigma(x)\overline{\sigma}(0) \neq 1$.
As an example of such operators, we can consider the translation of a D-brane in a certain spacetime direction ($X^1$) over a distance ($d$). In this case the boundary condition changing operators are written as

$$\sigma(x) = e^{\mathrm{i}d(X^0+\tilde{X}^1)}(x),$$
$$\overline{\sigma}(x) = e^{-\mathrm{i}d(X^0+\tilde{X}^1)}(x),$$

where $\tilde{X}^1 = X^1(z) - \bar{X}^1(\bar{z})|_{z=\bar{z}=x}$.
Another more interesting example is given by the creation of D-branes of codimension ($2n$): $Dp$-$D(p\pm 2n)$. In these cases, the boundary condition changing operators are given in terms of twist fields ($\Delta, \overline{\Delta}$), which are needed to implement the change in the $X^\mu$ boundary conditions and of (bosonised) spin fields, which instead account for the spinors $\psi$

$$\sigma(x) = e^{\mathrm{i}\sqrt{\frac{n}{4}}X^0}\Delta\ e^{\frac{i}{2}\sum_{i=1}^n H_i}(x),$$
$$\overline{\sigma}(x) = e^{-\mathrm{i}\sqrt{\frac{n}{4}}X^0}\overline{\Delta}\ e^{-\frac{i}{2}\sum_{i=1}^n H_i}(x).$$

The string fields obtained from the insertions of such operators can be added to the $KBc\gamma^2$ algebra and satisfy, besides the already mentioned $\overline{\sigma}\sigma = 1$, $\sigma\overline{\sigma} \neq 1$, the following derivation relations

$$Q\sigma = c\partial\sigma + \gamma\delta\sigma, \qquad Q\bar{\sigma} = c\partial\bar{\sigma} + \gamma\delta\bar{\sigma} \tag{3.32}$$

and algebraic properties

$$\begin{gathered}[B,\sigma] = [B,\bar{\sigma}] = 0, \quad [c,\sigma] = [c,\bar{\sigma}] = 0, \quad [\gamma^2,\sigma] = [\gamma^2,\bar{\sigma}] = 0, \\ [c,\partial\sigma] = 0, \qquad [c,\partial\bar{\sigma}] = 0.\end{gathered} \tag{3.33}$$

The derivation relations ultimately follow from the expression of the BRST charge (2.49) and from (2.31), where $\delta\sigma$ is the supersymmetric partner of $\sigma$.

The explicit expression of the intertwining solution can be written in terms of the tachyon vacuum (3.27) as

$$\begin{aligned}\Psi_* = &\sqrt{F}\left(c\frac{B}{H}c + B\gamma^2\right)\sqrt{F} - \sqrt{H}\sigma\sqrt{\frac{F}{H}}\left(c\frac{B}{H}c + B\gamma^2\right)\sqrt{\frac{F}{H}}\bar{\sigma}\sqrt{H} + \sqrt{H}Q\sigma BFQ\bar{\sigma}\sqrt{H} \\ &- \left(\sqrt{H}Q\sigma B\sqrt{\frac{F}{H}}c\sqrt{\frac{F}{H}}\bar{\sigma}\sqrt{H} + \text{conj.}\right) - \left(\sqrt{H}\left[\sqrt{\frac{F}{H}}c\sqrt{\frac{F}{H}},\sigma\right]B\sqrt{\frac{F}{H}}c\sqrt{\frac{F}{H}}\bar{\sigma}\sqrt{H} + \text{conj.}\right) \\ &- \left(\sqrt{H}Q\sigma BF\left[\bar{\sigma},\sqrt{\frac{F}{H}}c\sqrt{\frac{F}{H}}\right]\sqrt{H} + \text{conj.}\right) - \sqrt{H}\left[\sqrt{\frac{F}{H}}c\sqrt{\frac{F}{H}},\sigma\right]BF\left[\bar{\sigma},\sqrt{\frac{F}{H}}c\sqrt{\frac{F}{H}}\right]\sqrt{H},\end{aligned} \tag{3.34}$$

where "conj" indicates the reality conjugate of the term on the left. This operation is performed by reading the corresponding term from right to left and by considering

$$\begin{aligned}
&K^\ddagger = K, \quad B^\ddagger = B, \quad c^\ddagger = c;\\
&(\gamma^2)^\ddagger = \gamma^2;\\
&\sigma^\ddagger = \overline{\sigma}, \quad \overline{\sigma}^\ddagger = \sigma;\\
&(Q\sigma)^\ddagger = -Q\overline{\sigma}, \quad (Q\overline{\sigma})^\ddagger = -Q\sigma.
\end{aligned} \tag{3.35}$$

As an example, we can take

$$\sqrt{H}Q\sigma B\sqrt{\frac{F}{H}}c\sqrt{\frac{F}{H}}\overline{\sigma}\sqrt{H} \quad \xrightarrow{\text{"conj"}} \quad -\sqrt{H}\sigma\sqrt{\frac{F}{H}}c\sqrt{\frac{F}{H}}BQ\overline{\sigma}\sqrt{H}.$$

If one plugs in the explicit expression of $Q\sigma$, $Q\overline{\sigma}$, the solution becomes

$$\begin{aligned}
\Psi_* = &\sqrt{F}\left(c\frac{B}{H}c + B\gamma^2\right)\sqrt{F} - \sqrt{H}c\frac{1}{H}\sigma BF\overline{\sigma}\frac{1}{H}c\sqrt{H} - \sqrt{H}\sigma\sqrt{\frac{F}{H}}B\gamma^2\sqrt{\frac{F}{H}}\overline{\sigma}\sqrt{H}\\
&+\sqrt{H}\gamma\delta\sigma BF\gamma\delta\overline{\sigma}\sqrt{H} - \left(\sqrt{H}\gamma\delta\sigma BF\overline{\sigma}\frac{1}{H}c\sqrt{H} + \text{conj.}\right) - \sqrt{H}\sigma\left[\sqrt{\frac{F}{H}}, c\right]\frac{B}{H}\left[c, \sqrt{\frac{F}{H}}\right]\overline{\sigma}\sqrt{H}\\
&+\left(\sqrt{H}c\frac{B}{H}\sigma\sqrt{\frac{F}{H}}\left[\sqrt{\frac{F}{H}}, c\right]\overline{\sigma}\sqrt{H} + \text{conj.}\right) - \left(\sqrt{H}\gamma\delta\sigma B\sqrt{\frac{F}{H}}\left[c, \sqrt{\frac{F}{H}}\right]\overline{\sigma}\sqrt{H} + \text{conj.}\right)\\
&+\left(\sqrt{H}c\frac{B}{H}\sigma F\left[\overline{\sigma}, \sqrt{\frac{F}{H}}\left[\sqrt{\frac{F}{H}}, c\right]\right]\sqrt{H}+\text{conj.}\right)+\left(\sqrt{H}\gamma\delta\sigma BF\left[\overline{\sigma}, \sqrt{\frac{F}{H}}\left[\sqrt{\frac{F}{H}}, c\right]\right]\sqrt{H}+\text{conj.}\right)\\
&+\left(\sqrt{H}\sigma\left[\sqrt{\frac{F}{H}}, c\right]B\sqrt{\frac{F}{H}}\left[\overline{\sigma}, \sqrt{\frac{F}{H}}\left[\sqrt{\frac{F}{H}}, c\right]\right]\sqrt{H} + \text{conj.}\right)\\
&-\sqrt{H}\left[\left[c, \sqrt{\frac{F}{H}}\right]\sqrt{\frac{F}{H}}, \sigma\right]BF\left[\overline{\sigma}, \sqrt{\frac{F}{H}}\left[\sqrt{\frac{F}{H}}, c\right]\right]\sqrt{H},
\end{aligned} \tag{3.36}$$

which, in the bosonic simple solution case reduces to the know expression [7].
The solution, written here in terms of a generic function $F(K)$, must be well defined as a string field and it cannot not lead to ambiguities when inserted in the equations of motion. In the latter case, we observe that, as a consequence of the properties of the boundary condition changing operators, depending on the choice of $F(K)$, one may have associativity anomalies of the star product. In particular, one needs to avoid expressions containing the triple product $\sigma\overline{\sigma}\sigma$, which leads to inconsistencies depending on how we perform the computation

$$\frac{g_*}{g_0}\sigma = (\sigma\overline{\sigma})\sigma \neq \sigma(\overline{\sigma}\sigma) = \sigma.$$

For this reason, in the following section, we will focus on criteria capable of selecting consistent choices of the function $F(K)$, which ultimately means selecting consistent tachyon vacuum solutions.

# 4 Regularity conditions of the Intertwining solution

In this Section we motivate sufficient conditions on the starting tachyon vacuum to ensure that the solution itself is well-defined and that ambiguous products do not appear in the equations of motion. This analysis is motivated by the fact that in the simple solution case, the superstring solution (3.36) is ill-defined as it presents the collision of two boundary condition changing operators, which causes problems in the equations of motion.

In order to do that, let us pick a representative class of tachyon vacuum solutions

$$F(K) = \left(1 - \frac{1}{\nu}K\right)^{\nu} = \frac{(-\nu)^{-\nu}}{\Gamma(-\nu)} \int_0^{\infty} \mathrm{d}t\, t^{-\nu-1} e^{\nu t} \Omega^t, , \tag{4.1}$$

parametrised by a negative number $\nu < 0$. This number represents the leading level in the $K \to \infty$ limit of $F(K)$ and measures how identity-like a string field is. From a rigorous point of view, this analysis is performed by making use of the dual $\mathcal{L}^-$ level expansion [25]: indeed, the eigenvalues of the operator $\frac{1}{2}\mathcal{L}^- = \frac{1}{2}(\mathcal{L}_0 - \mathcal{L}_0^*)$, where $\mathcal{L}_0$ is the scaling generator in the sliver frame and $\mathcal{L}_0^*$ is its BPZ conjugate, are $\nu$, when acting on functions of $K$ and the conformal weight of the insertions, when acting on wedge states.
From (4.1) one sees that the contributions from states close to the identity string field, which correspond to the $t \to 0$ limit, become increasingly suppressed as $\nu \to -\infty$, as claimed.
As special cases of the above formula, we see that $\nu = -1$ corresponds to the already discussed simple solution, whereas $\nu = -\infty$ corresponds to Schnabl's solution [6].
Being completely determined in terms of $F(K)$, the homotopy string field can also then be written as a sum of wedge states

$$H(K) = \int_0^{\infty} \mathrm{d}t\, \frac{\Gamma(-\nu, -\nu t)}{\Gamma(-\nu)} \Omega^t. \tag{4.2}$$

Since we want to consider both bosonic and supersymmetric bounds on the level $\nu$ and since the bounds we derive in the two cases are different, we will specify it by using $\nu_{\text{boson}}$ and $\nu_{\text{super}}$. In light of the above discussion regarding the simple solution in the superstring case, we already know that

$$\nu_{\text{super}} < -1. \tag{4.3}$$

The question is whether this bound is sufficient, or if it should be further strengthened.
In order to fully address these potential problems, we make the following assumptions

$$\sigma(s)\overline{\sigma}(0) = \text{regular}, \qquad \sigma(s)\delta\overline{\sigma}(0) = \text{regular}.$$

The first condition is satisfied by construction and the second one has been explicitly checked in all the examples previously mentioned. As a consequence of these assumptions, one has that

$$\begin{aligned} \sigma(s)\partial\overline{\sigma}(0) &\sim \text{ less singular than simple pole,} \\ \partial\sigma(s)\partial\overline{\sigma}(0) &\sim \text{ less singular than double pole,} \end{aligned}$$

$$\delta\sigma(s)\delta\overline{\sigma}(0) \sim \text{ less singular than simple pole,}$$
$$\partial\sigma(s)\delta\overline{\sigma}(0) \sim \text{ less singular than simple pole.}$$

We can then formulate the following rule.

**Claim 1.** *Let $\mathcal{O}_1$ represent $\sigma, \partial\sigma$ or $\delta\sigma$ and $\mathcal{O}_2$ represent $\overline{\sigma}, \partial\overline{\sigma}$ or $\delta\overline{\sigma}$. Then the state*

$$\mathcal{O}_1 G(K)\mathcal{O}_2 \tag{4.4}$$

*suffers from no OPE divergence provided that its leading level in the dual $\mathcal{L}^-$ level expansion is less than or equal to* 0 *if the state is GSO even, and less than or equal to* 1/2 *if the state is GSO odd.*

To understand this claim, let us consider the following example

$$\mathcal{O}_1 F(K)\mathcal{O}_2 = \int_0^\infty \mathrm{d}t f(t) O_1 \Omega^t \mathcal{O}_2.$$

We are interested in the limit $t \to 0$, where the OPE between the two operators becomes relevant. It is clear that

$$\mathcal{O}_1(t)\mathcal{O}_2(0) \sim t^{h-h_1-h_2}\phi^{(h)}(0),$$

for a generic field $\phi^{(h)}$. Furthermore, by making use of the Schwinger integral, the leading component of $f(t)$ is $t^{-\nu-1}$. This means that in order for the integral to be finite in 0, we need to have

$$-\nu - 1 + h - h_1 - h_2 > -1 \overset{h\geq 0}{\Longrightarrow} \nu + h_1 + h_2 \leq 0.$$

Since this claim contains only two operators, it can be used to check if the solution, as a string field, contains divergences.

In the first way of expressing the solution (3.34), we notice in particular two terms

$$\begin{gathered}\sqrt{H}Q\sigma BFQ\overline{\sigma}\sqrt{H},\\ \sqrt{H}Q\sigma B\sqrt{\frac{F}{H}}c\sqrt{\frac{F}{H}}\overline{\sigma}\sqrt{H}.\end{gathered} \tag{4.5}$$

Since ghosts can always be ignored in this analysis, as we are considering OPE divergences coming from collision of two boundary condition changing operators, the matter sector component of these states reduces to

$$\begin{aligned}\partial\sigma F\partial\overline{\sigma} \quad &\sim \quad \partial\sigma K^\nu \partial\overline{\sigma},\\ \partial\sigma \frac{F}{H}\overline{\sigma} \quad &\sim \quad \partial\sigma K^{\nu+1}\overline{\sigma}, \quad K \to \infty.\end{aligned}$$

These expressions do not suffer from OPE divergences if

$$\nu \leq -2 \quad \text{(no OPE divergences in (3.34))}. \tag{4.6}$$

All remaining terms in the solution do not alter this bound, which then means that $F(K)$ must fall off as $K^{-2}$ or faster to be certain that OPE divergences are absent from (3.34), in either bosonic or supersymmetric cases. However, the original solution [7] was written in the simple case and it is finite. This is due to the fact that the explicit form of the BRST variation of the boundary condition changing operators cancels these OPE divergences. Indeed, by repeating the same analysis on the solution (3.36), we see that it is much safer

$$\nu_{\rm boson} \leq 0 \quad \text{or} \quad \nu_{\rm super} \leq -1 \quad \text{(no OPE divergences in (3.36)).} \tag{4.7}$$

Now let us turn to issues concerning three boundary condition changing operators. These do not affect the solution $\Psi_*$ by itself as a state, but they are related to the validity of the equations of motion and in particular the quadratic term $\Psi_*^2$. For this reason, we formulate the following claim.

**Claim 2.** *Let $\mathcal{O}_1, \mathcal{O}_2$ and $\mathcal{O}_3$ represent three primary operators and consider the state*

$$\mathcal{O}_1 G_1(K) \mathcal{O}_2 G_2(K) \mathcal{O}_3. \tag{4.8}$$

*Simultanous collision of all three operators do not render this state undefined provided that its leading level in the dual $\mathcal{L}^-$ level expansion is less than $h$, where $h$ is the lowest dimension of a primary operator which has nonvanishing contraction with the state.*

To better understand this claim, we contract the above expression with a test state $\Omega\mathcal{O}\Omega^\infty$, where $\mathcal{O}$ is a primary operator. Its precise form is not crucial, as we are interested in the short distance behaviour when $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ collide. Furthermore, we chose a test state containing the sliver state $\Omega^\infty$ to be able to perform the computation directly on the UHP, without needing a conformal transformation from the Cylinder to the UHP. The obtained expression reads

$$\mathrm{Tr}\Big[\Omega^\infty \mathcal{O}_1 G_1(K) \mathcal{O}_2 G_2(K) \mathcal{O}_3 \Omega \mathcal{O}\Big] = \int_0^\infty \mathrm{d}t_1 \mathrm{d}t_2 \, g_1(t_1) g_2(t_2) \Big\langle \mathcal{O}_1(t_1{+}t_2) \mathcal{O}_2(t_2) \mathcal{O}_3(0) \mathcal{O}(-1) \Big\rangle_{\rm UHP},$$

where $g_1$ and $g_2$ are the inverse Laplace transforms of $G_1$ and $G_2$. We now perform a change of integration variables

$$L = t_1 + t_2, \qquad \theta = \frac{t_2}{t_1 + t_2}, \tag{4.9}$$

together with a conformal transformation on the 4-point function $f(z) = \frac{z}{z+1} \frac{L+1}{L}$, in such a way that $\mathcal{O}_1$ is inserted at 1, $\mathcal{O}_3$ is inserted at 0, and $\mathcal{O}$ is inserted at infinity. This yields the following expression

$$\begin{aligned} &\mathrm{Tr}\Big[\Omega^\infty \mathcal{O}_1 G_1(K) \mathcal{O}_2 G_2(K) \mathcal{O}_3 \Omega \mathcal{O}\Big] = \\ &= \int_0^\infty \mathrm{d}L \int_0^1 \mathrm{d}\theta \, L g_1(L(1-\theta)) g_2(L\theta) \left(\frac{1}{L}\frac{1}{L+1}\right)^{h_1} \left(\frac{1}{L}\frac{L+1}{(L\theta+1)^2}\right)^{h_2} \left(\frac{L+1}{L}\right)^{h_3} \left(\frac{L}{L+1}\right)^{h} \\ &\quad \times \left\langle \mathcal{O}_1(1) \mathcal{O}_2\left(\frac{L+1}{L\theta+1}\theta\right) \mathcal{O}_3(0) I \circ \mathcal{O}(0) \right\rangle_{\rm UHP}. \end{aligned} \tag{4.10}$$

We are interested in the behaviour of the integrand towards $L = 0$, which is when $\mathcal{O}_1, \mathcal{O}_2$ and $\mathcal{O}_3$ collide. For small $L$ we have

$$g_1(L(1-\theta)) \sim L^{-\nu_1-1}(1-\theta)^{-\nu_1-1}, \quad g_2(L\theta) \sim L^{-\nu_2-1}\theta^{-\nu_2-1}, \tag{4.11}$$

where $\nu_1, \nu_2$ are the leading levels of the dual $\mathcal{L}^-$ expansion of $G_1$ and $G_2$. This means that the whole integrand can be approximately expressed as

$$L^{-\nu_1-\nu_2-h_1-h_2-h_3+h-1}(1-\theta)^{-\nu_1-1}\theta^{-\nu_2-1}\Big\langle \mathcal{O}_1(1)\mathcal{O}_2(\theta)\mathcal{O}_3(0) I \circ \mathcal{O}(0)\Big\rangle_{\text{UHP}}. \tag{4.12}$$

The $\theta$-integration will be finite assuming that the OPE between $\mathcal{O}_2$ and $\mathcal{O}_1$, and between $\mathcal{O}_2$ and $\mathcal{O}_3$, is sufficiently regular; whether this is the case is equivalent to the question of whether the states $\mathcal{O}_1 G_1(K)\mathcal{O}_2$ and $\mathcal{O}_2 G_2(K)\mathcal{O}_3$ are separately finite, which is not our present concern. Our interest is the convergence of the integration over $L$ towards $L = 0$, which will be unproblematic provided that

$$\nu_1 + \nu_2 + h_1 + h_2 + h_3 < h, \tag{4.13}$$

as claimed.
Before analysing the solution, let us consider a particular string field: $\sigma\overline{\sigma}\sigma$. We already know that this state is ambiguous due to the associativity anomaly and claim 2 does not seem to apply as it assumes that the operators are separated by wedge states with strictly negative $\mathcal{L}^-$ level. However, this can be easily dealt with by rewriting it as

$$\begin{aligned}\sigma\overline{\sigma}\sigma \quad = \quad & -\partial\sigma\frac{1}{1+K}\overline{\sigma}\frac{1}{1+K}\partial\sigma + (1+K)\sigma\frac{1}{1+K}\overline{\sigma}\frac{1}{1+K}\partial\sigma \\ & -\partial\sigma\frac{1}{1+K}\overline{\sigma}\frac{1}{1+K}\sigma(1+K) + (1+K)\sigma\frac{1}{1+K}\overline{\sigma}\frac{1}{1+K}\sigma(1+K).\end{aligned}$$

The argument below claim 2 now applies to all terms and only the first one can be problematic, as it seems to diverge as $L \to 0$, where for matter sector operators, the lowest conformal dimension of a probe state is $h = 0$. However, the $\theta$-integration exactly vanishes and the final result is a product $0 \times \infty$, which is clearly ambiguous. Therefore, the first term is exactly where the ambiguity of $\sigma\overline{\sigma}\sigma$ is hidden and this discussion further strengthens claim 2.

Coming back to the solution, we now analyse potential inconsistencies in the $\Psi_*^2$ term of the equations of motion. For the superstring solution (3.36), the strongest bound is given by the product of

$$\begin{gathered}\sqrt{H}\sigma\sqrt{\frac{F}{H}}B\gamma^2\sqrt{\frac{F}{H}}\overline{\sigma}\sqrt{H}, \\ \sqrt{H}\gamma\delta\sigma BF\gamma\delta\overline{\sigma}\sqrt{H}\end{gathered} \tag{4.14}$$

with

$$\sqrt{H}c\frac{1}{H}\sigma BF\overline{\sigma}\frac{1}{H}c\sqrt{H}. \tag{4.15}$$

These products respectively read

$$\sqrt{H}\sigma\sqrt{\frac{F}{H}}B\gamma^2\sqrt{\frac{F}{H}}\overline{\sigma}\sigma F\overline{\sigma}\frac{1}{H}c\sqrt{H},$$
$$\sqrt{H}\gamma\delta\sigma BF\gamma\delta\overline{\sigma}\sigma F\overline{\sigma}\frac{1}{H}c\sqrt{H}.$$

One can again strip off the ghosts and obtain the following expressions

$$\begin{aligned}\sigma\frac{F}{H}\overline{\sigma}\sigma &\sim \sigma K^{\nu+1}\overline{\sigma}\sigma,\\ \delta\sigma F\delta\overline{\sigma}\sigma &\sim \delta\sigma K^{\nu}\delta\overline{\sigma}\sigma, \qquad (K\to\infty),\end{aligned}$$

which in virtue of claim 2 impose the following bound

$$\nu_{\text{super}} < -1. \tag{4.16}$$

This means that the solution (3.36) is free from ambiguous terms provided that $\nu_{\text{super}} < -1$ or $\nu_{\text{bosonic}} < 0$, whereas the stronger bound $\nu \le -2$ is needed for (3.34), for both bosonic and fermionc string.

To clarify the obtained results, regarding both claims, we end with an observation regarding the commutators appearing in the solution (3.36): let $\sqrt{f/h}(t)$ represent the inverse Laplace transform of $\sqrt{F/H}$ and write

$$\begin{aligned}\left[c,\sqrt{\frac{F}{H}}\right] &= \int_0^\infty \mathrm{d}t\,\sqrt{f/h}(t)[c,\Omega^t]\\ &= \int_0^\infty \mathrm{d}t\int_0^1 \mathrm{d}\theta\left(t\sqrt{f/h}(t)\right)\Omega^{t(1-\theta)}\partial c\Omega^{t\theta}.\end{aligned} \tag{4.17}$$

Note the extra factor of $t$ which appears in the integrand. This means, from the point of view of separation of the matter sector boundary condition changing operators, the commutator with $c$ can be seen as equivalent to

$$\left[c,\sqrt{\frac{F}{H}}\right] \to -\frac{\mathrm{d}}{\mathrm{d}K}\sqrt{\frac{F}{H}}. \tag{4.18}$$

As an example of this procedure, let us consider the term

$$\sqrt{H}\sigma\left[\sqrt{\frac{F}{H}},c\right]\frac{B}{H}\left[c,\sqrt{\frac{F}{H}}\right]\overline{\sigma}\sqrt{H}, \tag{4.19}$$

which from our point of view can be rewritten as

$$\sigma\left(\frac{\mathrm{d}}{\mathrm{d}K}\sqrt{\frac{F}{H}}\right)^2\frac{1}{H}\overline{\sigma} \sim \sigma K^{\nu}\overline{\sigma}, \qquad (K\to\infty). \tag{4.20}$$

This is the reason why terms involving commutators of $c$ with $\sqrt{F/H}$ do not require a stronger bound than $\nu_{\text{super}} < -1$ or $\nu_{\text{bosonic}} < 0$.

## 4.1 Discussion on the obtained results and future perspectives

We conclude Part I of this thesis focused on SFT, by commenting on the obtained results. In the previous Section, we tried to understand how to generalise the bosonic intertwining solution first introduced in [7] to the superstring case. The naive attempt to keep working in the simple solution case proved to be inconsistent, because the new fermionic terms introduced ambiguities in the solution.

We then focused on quantitatively measure if and how much one has to move away from the simple solution case, by considering the most general function $F(K)$ and by studying its properties.

To do that, we introduced the level expansion $\nu$ and we formulated two rules allowing to study the behaviour of the solution as a string field and when inserted into the equations of motion. The best behaving solution required an upper bound $\nu_{\text{super}} < -1$ to avoid all kind of ambiguities.

This analysis, together with the discussion in [11] concerning the inner transformation $\phi$ of the intertwining solution, which maps the solution for a given $F(K)$ to the same solution written in terms of another function $\phi \circ F(K)$, confirms that the simple solution case is the only upper bound on the superstring case solution.

Another approach to takle this problem could be to extend the possible range of string fields by introducing the supersymmetric partner of $K$, a string field made of the insertion of the fermion current $T_F(z)$ [26].

By paying the price of enlarging the algebra, it may be possible to cancel ambiguous terms and to reach the simple solution case $\nu_{\text{super}} = -1$. This is an interesting perspective that will be analysed in the future.

In both cases, the interest in a superstring generalisation of the intertwining solution, derived in the context of the Modified cubic approach, lies in the fact that it naturally leads to the WZW-Berkovits' approach, because, as we have shown, it can be considered a first step towards finding solutions of the non-polynomial equations of motion.

# Part II
# Supergravity

## 5 Geometrical approach

Supergravity is a direct generalization of General Relativity, differing from the latter for the presence of local Supersymmetry. Both gravity theories can be described in a formal way by making use of the powerful tools of Differential geometry: in this section we will give a formulation of the so called *Geometrical approach* to gravity and Supergravity [27–30] by making use of Lie groups, Lie algebras and Principal fiber bundles. In particular, in what follows we will restrict to Lie (super)groups whose Lie algebra can be split as

$$\mathcal{G} = \mathcal{K} \oplus \mathcal{H} = \mathcal{I} \oplus \mathcal{O} \oplus \mathcal{H}. \tag{5.1}$$

Here $\mathcal{I}$ and $\mathcal{O}$ are the subalgebras of translations and supersymmetric transformations respectively, whereas $\mathcal{H}$ is a gauge algebra containing the so-called Lorentz algebra $so(1, D-1)$. For physical applications, we will be interested in the following algebra structure

$$\begin{aligned}
[\mathcal{H},\mathcal{H}] \subset \mathcal{H}, \quad [\mathcal{H},\mathcal{I}] \subset \mathcal{I}, \quad [\mathcal{H},\mathcal{O}] \subset \mathcal{O}, \\
[\mathcal{I},\mathcal{I}] \subset \mathcal{H}, \quad [\mathcal{I},\mathcal{O}] \subset \mathcal{O}, \quad \{\mathcal{O},\mathcal{O}\} \subset \mathcal{I} \oplus \mathcal{H},
\end{aligned} \tag{5.2}$$

which determines the role of Lorentz symmetry, translations and Supersymmetry transformations.

First of all, let us notice that Lie groups can only describe vacuum configurations, where field dynamics is absent. Lie groups are indeed "rigid", since their (pseudo-)Riemannian geometry is completely fixed in terms of the structure constants $C^C_{AB}$ describing the Lie algebra $\mathcal{G}$: indeed, in the case of semisimple Lie groups, the Killing form actually becomes a metric, defined by

$$g_{AB} = C^F_{AE} C^E_{BF},$$

where $A, B, \ldots = 1, \ \ldots, \dim G$ label the generators of the Lie algebra $\{T_A\}$.

For vacuum solutions, one identifies the forms dual to the generators of the subalgebras $\mathcal{I}$ to the vielbein of the vacuum space.

Introducing dynamics, namely moving to a general situation in which the structure of space-time/superspace is not fixed, requires a more complicated mathematical structure, the one of Principal Bundles. See [31–33] for a review of the main ideas and results used in the following.

**Definition 5.1.** *A Principal Bundle is a quadruple $(P, \mathcal{M}, \pi, G)$, where $P$ is the total space, $\mathcal{M}$ is the base manifold, the typical fiber $F$ is diffeomorphic to the Lie group $G$ and $\pi : P \to \mathcal{M}$ is a surjective map.*

In particular, $\pi^{-1}(x)$ is diffeomorphic to $G$, $\forall x \in \mathcal{M}$ and each point $p \in P$ is locally expressed as a couple $(x, g)$ through maps

$$t_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G,$$

called local trivializations, where $\{U_\alpha\}$ is an atlas on $\mathcal{M}$ defining local coordinates. In this geometric construction, $\mathcal{M}$ will be a (super-)manifold representing spacetime or its supersymmetric generalization, superspace.
The base space is immersed in the Principal fiber bundle in such a way that it is not possible, in general, to single it out: this formalism is the key to introduce dynamics and can actually be used to also describe vacuum configurations, if one considers a single point as the base manifold.
In this framework, connections are fundamental concepts for both maths and physics: from the former point of view, they are a choice of an equivariant horizontal space $H \subset TP$, satisfying $H_{\tilde{R}_g p} = \tilde{R}_{g*} H_p$, where $\tilde{R}_g : P \to P$ is the right action of $g \in G$ on the Principal bundle, whereas in physics they correspond to gauge fields and allow to consider covariant derivatives in the associated bundles, where other physical fields like scalars and spinors are defined.
As we will shortly see, connections allow to decompose the tangent space $TP$ into horizontal and vertical spaces, the latter denoted by $VP$ and containing all vector satisfying $\pi_*(v) = 0$. This means that each vector $v \in TP$ can be rewritten as a sum

$$v = v_V + v_H.$$

Moreover, there is a canonical isomorphism relating the vertical space $V_p P$ to the Lie algebra $\mathcal{G}$: indeed, for any $\hat{v} \in \mathcal{G}$, corresponding to the group element $g$, meaning that $\hat{v} = \frac{\mathrm{d}g(s)}{\mathrm{d}s}\Big|_{s=0}$, one can build the associated *fundamental vector field*

$$v(p) = \left.\frac{\mathrm{d}(\tilde{R}_{g(s)} p)}{\mathrm{d}s}\right|_{s=0}.$$

This is actually an isomorphism of vector spaces as they share the same dimension.
Connections can be defined in multiple ways, all reproducing the same features, but in this formulation we will be interested in the following one.

**Definition 5.2.** *A connection on a Principal Bundle* $(P, \mathcal{M}, \pi, G)$ *is a map*

$$\boldsymbol{\omega} : TP \to \mathcal{G},$$

*satisfying*

- $\boldsymbol{\omega}(v) = \hat{v}$, *where* $v \in VP$ *and* $\hat{v} \in \mathcal{G}$ *are related by the canonical isomorphism between* $VP$ *and* $\mathcal{G}$,
- $\boldsymbol{\omega}$ *depends differentiably on* $p \in P$,
- $R_g^* \boldsymbol{\omega}_{\boldsymbol{p}}(v) = \boldsymbol{\omega}_{\tilde{R}_g p}(\tilde{R}_{g*} v) = Ad_{g^{-1}}(\boldsymbol{\omega}_p(v))$,

*where* $Ad_g(\gamma) = (L_g R_g^{-1})_*(\gamma)$ *is the adjoint representation of* $g \in G$ *on* $\gamma \in \mathcal{G}$.

We remark that the adjoint representation can be written in the usual way $Ad_g(\gamma) = g\gamma g^{-1}$ only in the case of the General Linear group or one of its subgroups.
With this definition, the horizontal subspace is obtained as

$$H_p = \{v \in T_pP : \boldsymbol{\omega}_p(v) = 0\} \tag{5.3}$$

and satisfies the needed equivariance property. This means that the tangent vector space $TP$ is accordingly split as

$$TP = VP \oplus H = \mathcal{G} \oplus H$$

and one can then choose a basis for such vector space. If $\{t_A\}$ are the fundamental vector fields associated to the Lie algebra generators $\{T_A\}$, one can then choose

$$TP = \text{span}\{t_A, u_M\},$$

where $\{u_M\}$ is a basis for the horizontal space defined by the connection, with $M = 1, \cdots, \dim \mathcal{M}$.
In order to construct a basis for the cotangent space $T^*P$, let us notice that, by definition of connection, we have that

$$\boldsymbol{\omega}(t_A) = T_A.$$

This result can also be proved in the following way: take a local section $\sigma : x \in U_\alpha \to p_0 \in P$ and define $\boldsymbol{A} \in \Omega^1(U_\alpha) \otimes \mathcal{G}$ as $\boldsymbol{A} = \sigma^*\boldsymbol{\omega}$. The connection in $p_0 = \sigma(x)$ can be defined as

$$\bar{\boldsymbol{\omega}}_{p_0}(v) = \boldsymbol{A}(\pi_* v) + \hat{v}_V$$

and can be extended to a generic $p = \tilde{R}_g p_0 \in P$ through the adjoint representation

$$\boldsymbol{\omega}_p(u) = Ad(g^{-1})(\bar{\boldsymbol{\omega}}_{p_0}(\tilde{R}_{g*}^{-1} u)), \qquad \forall u \in T_pP.$$

In order to apply the obtained connection to the basis of the vertical subspace $\{t_A\}|_p$, we first compute

$$\boldsymbol{\omega}_{p_0}(\tilde{R}_{g*}^{-1} t_A|_p) = \boldsymbol{A}(\pi_* \tilde{R}_{g*}^{-1} t_A|_p) + \widehat{\tilde{R}_{g*}^{-1} t_A|_p} = Ad(g)(T_A),$$

where we used that $\pi_* \tilde{R}_{g*}^{-1} t_A = 0$. We finally have

$$\boldsymbol{\omega}_p(t_A) = Ad(g^{-1})\bar{\boldsymbol{\omega}}_{p_0}(\tilde{R}_{g*}^{-1} t_A) = Ad(g^{-1})(Ad(g)(T_A)) = T_A.$$

Since this whole evaluation is really independent of the chosen local section, we can conclude that our claim holds. As a consequence, since $\boldsymbol{\omega}$ is an element of $\Omega^1(P) \otimes \mathcal{G}$, it can be rewritten as

$$\boldsymbol{\omega} = \omega^A \otimes T_A. \tag{5.4}$$

In virtue of the property shown above, we must have $\omega^A(t_B) = \delta^A_B$. Moreover, if the horizontal vector $u_M$ is written as

$$u_M = \frac{\partial}{\partial X^M} + u^A_M t_A$$

where $X^M = (x^\mu, \theta^\alpha)$ are the coordinates on superspace, then it is also true that $\boldsymbol{\omega}^A(u_M) = 0$. Therefore, one has to set $-u^A_M = \boldsymbol{\omega}^A(\frac{\partial}{\partial X^M}) \equiv \omega^A_M$.
The final step needed to determine a basis of $T^*P$ is to consider the one-form

$$U^M = \mathrm{d}X^M + U^M_A \boldsymbol{\omega}^A,$$

which behaves in the following way

$$U^M(t_A) = U^M_B \delta^B_A = U^M_A, \qquad U^M(u_N) = \delta^M_N.$$

We see that by choosing $U^M_A = 0$, thus setting $U^M = \mathrm{d}X^M$, we obtain a basis of the cotangent space

$$T^*P = \mathrm{span}\{\boldsymbol{\omega}^A, \mathrm{d}X^M\} \equiv \mathrm{span}\{\mu^{\mathcal{A}}\}, \qquad \mathcal{A} = \{A, M\}$$

obeying at the following rules

$$\begin{aligned} \boldsymbol{\omega}^A(u_N) = 0, &\qquad \mathrm{d}X^M(u_N) = \delta^M_N, \\ \boldsymbol{\omega}^A(t_B) = \delta^A_B, &\qquad \mathrm{d}X^M(t_B) = 0. \end{aligned} \tag{5.5}$$

The connection can be used to define an important physical object, the curvature. To this end, we first introduce the exterior covariant derivative of a r-form $\phi$ on $P$ as

$$D\phi(v_{(1)}, \cdots, v_{(r+1)}) = \mathrm{d}_P\phi(v_{H(1)}, \cdots, v_{H(r+1)}).$$

The curvature is then defined in the following way.

**Definition 5.3.** *The curvature $\boldsymbol{\Omega}$ of a connection $\boldsymbol{\omega}$ is defined as the covariant exterior derivative of the connection*

$$\boldsymbol{\Omega} = D\boldsymbol{\omega}.$$

The curvature 2-form satisfies the Cartan structural equation

$$\boldsymbol{\Omega} = \mathrm{d}_P\boldsymbol{\omega} + [\boldsymbol{\omega}, \boldsymbol{\omega}],$$

which can be rewritten in the more common way by making use of (5.4)

$$\boldsymbol{\Omega}^A \otimes T_A = \left(\mathrm{d}_P\boldsymbol{\omega}^A + \frac{1}{2}C^A_{BC}\boldsymbol{\omega}^B \wedge \boldsymbol{\omega}^C\right) \otimes T_A, \tag{5.6}$$

where we exploited the bracket definiton $[\boldsymbol{\omega}, \boldsymbol{\omega}] = (\boldsymbol{\omega}^A \wedge \boldsymbol{\omega}^A) \otimes [T_A, T_B]$.
The curvature two-form satisfies a set of identities, called Bianchi identities, which are a consequence of the fact that the differential is nilpotent $\mathrm{d}^2_P = 0$: they read

$$D\boldsymbol{\Omega} = 0 \iff \nabla\boldsymbol{\Omega}^A = \mathrm{d}_P\boldsymbol{\Omega}^A + C^A_{BC}\boldsymbol{\omega}^B \wedge \boldsymbol{\Omega}^C = 0. \tag{5.7}$$

With all these tools at our disposal, we now define the transformation laws of the fields $\boldsymbol{\omega}^A$ under an infinitesimal vertical automorphism generated by $\epsilon = \epsilon^A t_A$. Vertical automorphisms, which are active transformations that do change the point on the fiber bundle, can also be rewritten and interpreted in terms of passive transformations, by considering changes of trivializations. In the former case, they act on the right with $\tilde{R}_g$, whereas in the latter one they are written in terms of left group multiplication. One usually refers to them as *gauge transformations.* We then have

$$\begin{aligned}
\delta\boldsymbol{\omega}^A &\equiv \mathcal{L}_\epsilon \boldsymbol{\omega}^A \\
&= (i_\epsilon \mathrm{d}_P + \mathrm{d}_P i_\epsilon)\boldsymbol{\omega}^A = \mathrm{d}_P\epsilon^A + i_\epsilon\left(\mathrm{d}_P\boldsymbol{\omega}^A \pm \frac{1}{2}C^A_{BC}\boldsymbol{\omega}^B \wedge \boldsymbol{\omega}^B\right) \\
&= (\mathrm{d}_P\epsilon^A + C^A_{BC}\boldsymbol{\omega}^B\epsilon^C) + i_\epsilon\boldsymbol{\Omega}^A \equiv \nabla\epsilon^A + i_\epsilon\boldsymbol{\Omega}^A.
\end{aligned} \tag{5.8}$$

Since $\boldsymbol{\Omega}^A$ is a two-form on $P$, it can be expanded along the basis as

$$\boldsymbol{\Omega}^A = \Omega^A_{\mathcal{BC}}\mu^{\mathcal{B}} \wedge \mu^{\mathcal{C}}$$

and in order to obtain the usual gauge transformations, one has to impose the following conditions.

*Physical Requirement* 5.1. The curvature must satisfy the horizontality condition

$$i_{t_B}\boldsymbol{\Omega}^A = 0,$$

which effectively requires it to be expanded as $\boldsymbol{\Omega}^A = \Omega^A_{MN}\mathrm{d}X^M \wedge \mathrm{d}X^N$.

This last expression would be enough for usual gauge theories, where it is known that the infinitesimal transformations of the gauge fields are given by the covariant derivative of the gauge parameter. General Relativity and Supergravity are not ordinary gauge theories, as this condition is not enough to reproduce the known transformations rules of the fields. These theories, which are called *natural* theories, as opposed to *gauge-natural* ones, are characterised by the strong importance assumed by the tangent space $T\mathcal{M}$ of the base manifold. In order to correctly reproduce the physical transformation laws of the fields for these theories, one has to introduce the following mechanism.

**Principal bundle breakdown**: consider the following linear map

$$\theta : \mathcal{T}_p \subset V_pP \to T_{\pi(p)}\mathcal{M}$$

where $\mathcal{T}_p$ is generated by the fundamental vector fields associated to the subalgebra of translations and supersymmetric transformations $\mathcal{K}$ in (5.1). In particular, if $\mathcal{P}_a$ are associated to translations and $Q_\alpha$ to supersymmetric transformations, we define

$$\begin{aligned}
\theta(\mathcal{P}_a) &= V_a^M\frac{\partial}{\partial X^M} = V_a^\mu\partial_\mu + V_a^\alpha\frac{\partial}{\partial\theta^\alpha}, \\
\theta(Q_\alpha) &= \Psi_\alpha^M\frac{\partial}{\partial X^M} = \Psi_\alpha^\mu\partial_\mu + \Psi_\alpha^\beta\frac{\partial}{\partial\theta^\beta}.
\end{aligned} \tag{5.9}$$

The relation that we are defining is actually an isomorphism of vector spaces, since $\dim \mathcal{T}_p = \dim T\mathcal{M}$ and since we are relating the bases of the two vector spaces. The same identification can be performed for cotangent spaces: one indeed takes the map

$$\theta^* = \mathcal{T}^*_p \to T^*_{\pi(p)}\mathcal{M}$$

relating $\boldsymbol{V}^a$ and $\boldsymbol{\Psi}^\alpha$, which are the one-forms $\boldsymbol{\omega}^A$ associated again to translations and supersymmetric transformations, to $\mathrm{d}X^M = \{\mathrm{d}x^\mu, \mathrm{d}\theta^\alpha\}$.
This is the crucial step that allows to relate a subset of the gauge transformations to diffeomorphisms in superspace. Indeed, an infinitesimal vertical automorphism on the fiber of the principal bundle along the direction $\mathcal{P}_a$ is now associated to a transformation on spacetime itself. The name *Principal bundle breakdown* exactly refers to this fact: this linear map destroys the verticality of some directions, by "pulling down" some fibers and projecting them on superspace.
One could have clearly started by considering a Principal fiber bundle where only the true gauge group is taken as the typical fiber, but the power of this construction lies in the expression (5.8), which allows to easily compute the transformation laws of all physical fields. In particular, the identification we just defined reduces the dimension of the cotangent space $T^*P$: its basis has to be accordingly modified, as $\boldsymbol{V}^a$ and $\boldsymbol{\Psi}^\alpha$ are no longer linearly independent from $\mathrm{d}x^\mu$ and $\mathrm{d}\theta^\alpha$.
To obtain the physical transformation laws of General Relativity and Supergravity, one first applies the Principal bundle breakdown to (5.8) by considering

$$\epsilon = \epsilon^a \theta(\mathcal{P}_a) + \epsilon^\alpha \theta(Q_\alpha) + \epsilon^{\hat{A}} t_{\hat{A}}$$
$$\boldsymbol{\Omega}^A = \Omega^A_{\hat{\mathcal{B}}\hat{\mathcal{C}}}\, \mu^{\hat{\mathcal{B}}} \wedge \mu^{\hat{\mathcal{C}}}, \qquad \mu^{\hat{\mathcal{A}}} = \{\mathrm{d}X^M, \boldsymbol{\omega}^{\hat{A}}\},$$

where in this context hatted indices exclude those directions generated by translations and supersymmetric transformations and then imposes Physical Requirement 5.1.

As an example, let us compute the transformation laws of the spin connection $\omega^{ab}$ in General Relativity, where all fermions are switched off: we start with (5.8) and we obtain

$$\begin{aligned}
\delta\omega^{ab} &= \nabla\epsilon^{ab} + i_\epsilon \boldsymbol{\Omega}^{ab} \\
&= \nabla\epsilon^{ab} + \epsilon^{ef}\Omega^{ab}_{\hat{B}\hat{C}}\mu^{\hat{B}}(t_{ef})\mu^{\hat{C}} + 2\epsilon^e \Omega^{ab}_{\hat{B}\hat{C}}\mu^{\hat{B}}(\theta(\mathcal{P}_e))\mu^{\hat{C}} \\
&= \nabla\epsilon^{ab} + 2\epsilon^{cd}\Omega^{ab}_{cd\ \hat{C}}\mu^{\hat{C}} + 2\epsilon^e \Omega^{ab}_{\mu\ \hat{C}} dx^\mu(\theta(\mathcal{P}_e))\mu^{\hat{C}} \\
&= \nabla\epsilon^{ab} + 2\epsilon^e \Omega^{ab}_{\mu\ \hat{C}} dx^\mu(V^\nu_e \partial_\nu)\mu^{\hat{C}} \\
&= \nabla\epsilon^{ab} + 2\epsilon^e \Omega^{ab}_{e\hat{C}}\mu^{\hat{C}} \\
&= \nabla\epsilon^{ab} + 2\epsilon^e \Omega^{ab}_{ef} V^f .
\end{aligned}$$

In order to obtain this result, which is indeed the known one, in the second line we expanded the curvature only on $\hat{A}$ directions, as mentioned, whereas in the fourth and sixth lines we used the Physical Requirement 5.1. In the fourth line we also used the Principal bundle breakdown.

In virtue of this mechanism, the geometric approach to Supergravity treats Supersymmetries as diffeomorphisms on a base manifold having both bosonic and fermionic directions. This proves to be extremely useful to construct Lagrangians, to perform computations and to derive transformation rules of fields, but it introduces degrees of freedom, which are absent in the standard spacetime formulation. Indeed, physical actions will be formulated on a bosonic

submanifold M of $\mathcal{M}$ determined by the condition $\theta^\alpha = \mathrm{d}\theta^\alpha = 0$.
The procedure of extending the spacetime information to the whole superspace without introducing new physical content is called *rheonomic extension mapping*: this can be achieved by requiring that a field $\boldsymbol{\omega}^A(x, \theta, y)$, where $y^{\hat{A}}$ are the coordinates on the group, is uniquely determined in terms of spacetime quantities.
To understand this idea, let us take a specific combination of $\theta(\mathcal{P}_a)$ and $\theta(Q_\alpha)$ having only supersymmetric component

$$\epsilon = \epsilon^\alpha \frac{\partial}{\partial \theta^\alpha}.$$

We see that the field evaluated at $(x, \delta\theta, y)$ can be written as an infinitesimal transformation along $\epsilon$ as

$$\begin{aligned}
\boldsymbol{\omega}^A(x, \delta\theta, y) &= \boldsymbol{\omega}^A(x, 0, y) + \nabla\epsilon^A(x, 0, y) + 2\epsilon^\alpha \boldsymbol{\Omega}^A_{\hat{\mathcal{B}}\hat{\mathcal{C}}}(x, 0, y)\ \mu^{\hat{\mathcal{B}}} \left(\frac{\partial}{\partial\theta^\alpha}\right) \mu^{\hat{\mathcal{C}}}(x, 0, y) \\
&= \boldsymbol{\omega}^A(x, 0, y) + \nabla\epsilon^A(x, 0, y) + 2\epsilon^\alpha \boldsymbol{\Omega}^A_{\alpha\hat{\mathcal{C}}}(x, 0, y)\ \mu^{\hat{\mathcal{C}}}(x, 0, y) + \\
&+ 2\epsilon^\alpha \boldsymbol{\Omega}^A_{\hat{B}\hat{\mathcal{C}}}(x, 0, y)\ \mu^{\hat{B}} \left(\frac{\partial}{\partial\theta^\alpha}\right) \mu^{\hat{\mathcal{C}}}(x, 0, y) \\
&= \boldsymbol{\omega}^A(x, 0, y) + \nabla\epsilon^A(x, 0, y) + 2\epsilon^\alpha \boldsymbol{\Omega}^A_{\alpha M}(x, 0, y)\ \mathrm{d}X^M(x, 0, y),
\end{aligned}$$

where in the last line we applied the Physical Requirement 5.1. As we mentioned before, all terms in the last expression must be functions of spacetime only: we are then led to the following requirement.

*Physical Requirement* 5.2. The components of the curvature along supersymmetric directions must be expressed in terms of spacetime ones

$$\Omega^A_{\alpha M} = C^{A|\mu\nu}_{\alpha M|B} \Omega^B_{\mu\nu}, \tag{5.10}$$

where $C^{A|\mu\nu}_{\alpha M|B}$ are constants.

In this way, it is possible to reconstruct the whole superspace, by starting from a bosonic submanifold.
Before starting to talk about lagrangians and their building rules, let us comment on the relation between Physical Requirement 5.2 and the Bianchi identities, which hold whenever $\mathrm{d}_P^2 = 0$. The compatibility of (5.10), which are conditions on the curvature, with the Bianchi identities imposes conditions on the inner components $\Omega^A_{\mu\nu}$, which are none other than the equations of motion.

The formalism we described so far works wonderfully for the construction of lagrangians for pure Supergravities, where the only physical fields appearing in the theory are the one-forms $\boldsymbol{\omega}^A$. Let us now review this procedure by describing the following building rules:

a) **Geometricity**: the lagrangian must be a $\dim(M)$-form constructed out of forms and differential of forms. Since the Hodge operator of a form in spacetime is not well defined

in superspace, one is allowed to use 0-form fields whose equations of motion effectively reproduce the Hodge star.
The action is then obtained by integrating the lagrangian on the hypersurface $M$

$$S = \int_{M \subset P} L.$$

For this reason, the fields appearing in the lagrangian must be forms defined on spacetime/superspace: fields associated to the $\mathcal{K}$ subalgebra are already projected down to the base manifold, thanks to the Principal bundle breakdown $V^a \equiv \theta^*(\boldsymbol{V}^a)$ and $\Psi^\alpha \equiv \theta^*(\boldsymbol{\Psi}^\alpha)$, whereas fields associated to $\mathcal{H}$ can be pulled back through a local canonical section $\sigma$, as $\sigma^* \boldsymbol{\omega}^{\hat{A}}$.

b) **Gauge invariance**: the lagrangian must be $\mathcal{H}$-invariant.

c) **Homogeneous scaling law**: the decomposition of the algebra (5.1) allows for rescalings of the generators, which are then transferred to the fields appearing in the lagrangian. Each term in the latter must scale homogeneously under this scaling law: this means that each term must scale as $[w]^{\dim M - 2}$, the scale weight of the Einstein term.

d) **Vacuum**: the equations of motion derived from the lagrangian must admit the solution $\boldsymbol{\Omega}^A = 0$, corresponding to the flat connection condition.

e) **Physical requirements 5.1 and 5.2**: the curvature components must satisfy

$$\Omega^A_{\hat{B}\hat{C}} = 0, \qquad \Omega^A_{\alpha M} = C^{A|\mu\nu}_{\alpha M|B} \Omega^B_{\mu\nu},$$

which are both needed to ensure that transformation rules of the fields are well defined and that the lagrangian does not contain too much information.

Now that the basic concepts of the geometric approach to Supergravity have been laid down, we will focus on applications. The first one will be concerned with the construction of a possibly predictive model of a 2-dimensional lattice material starting from a suitable vacuum theory, whereas the second application will make use of the full power of this approach, as we will deal with a dynamical theory, $\mathcal{N} = 2$ pure Supergravity with negative cosmological constant, for manifolds with a boundary.
In this second application, we will also be concerned with studying the asymptotic properties of the obtained theory.

# 6 $\mathcal{N}$-extended AdS$_4$ vacuum theory: a model for graphene-like materials

As a first application of the powerful approach formulated in the previous Section, we now want to build a model capable of describing the electronic properties of $(2+1)$-dimensional materials in a top-down approach [34].

Gravity can surprisingly be used to build an analogy relating spacetime to condensed matter systems living on a lattice. In this context, geometric quantities are linked to properties and defects of the lattice [35]: for example, the curvature is associated to disclinations via the Frank vector, whereas torsion is related to dislocations through the Burgers vector.
The introduction of fermionic matter on the gravity side is expected to further extend this analogy to include the description of the electronic properties of these materials. However, the gravitini, appearing in a pure Supergravity theory, are spin 3/2 massless particles and have no counterpart on the condensed matter side: a substantial improvement in this direction has been achieved in [36] and goes by the name of "Unconventional Supersymmetry". Indeed, the authors managed to express the gravitini appearing in the Supergravity theory in terms of a spin 1/2 field $\chi^A$. Our purpose in this section will then be to construct a consistent theory for these spinors, which, as we will argue, will have to obey a Dirac equation, while simultaneously reproducing as many features of the lattices of these materials as possible.

To this end, having in mind possible applications of holography, where, as we will discuss in the next Section, one performs appropriate asymptotic limits to study the properties of the boundary theory, let us consider an AdS$_4$ vacuum theory generated by $G = OSp(\mathcal{N}|4)$: as said, in these cases everything is determined in terms of the chosen Lie group and its algebra, as the only fields in the theory are those appearing in the Maurer-Cartan form $\boldsymbol{\omega}$. The fact that we decided to use the same symbol $\boldsymbol{\omega}$ to indicate both the connection on the Principal fiber bundle and the Maurer-Cartan form is a consequence of the fact that the latter can be considered a connection on a Principal fiber bundle in which the base manifold is a single point.
The curvature associated to the chosen connection is flat, which means that

$$\boldsymbol{\Omega} = \mathrm{d}_P\boldsymbol{\omega} + [\boldsymbol{\omega}, \boldsymbol{\omega}] = 0.$$

Notice that, for our purposes, it may be possible to consider more generic configurations: for example, one could take fluctuations around the AdS$_4$ vacuum, preserving the full symmetry at radial infinity, where scalars and spin 1/2 fields, which would in general appear in the gravitational multiplet, would be required to be frozen at their boundary value. We will keep a conservative approach and consider only the vacuum configuration.

The $OSp(\mathcal{N}|4)$ Lie algebra is described by the following relations

$$\begin{aligned}
[L_{\mathcal{AB}}, L_{\mathcal{CD}}] &= \eta_{\mathcal{AD}} L_{\mathcal{BC}} - \eta_{\mathcal{AC}} L_{\mathcal{BD}} + \eta_{\mathcal{BC}} L_{\mathcal{AD}} - \eta_{\mathcal{BD}} L_{\mathcal{AC}}, \\
[T_{AB}, T_{CD}] &= \delta_{AD} T_{BC} - \delta_{AC} T_{BD} + \delta_{BC} T_{AD} - \delta_{BD} T_{AC}, \\
[L_{\mathcal{AB}}, Q^\alpha_A] &= -\frac{1}{2}(\tilde{\Gamma}_{\mathcal{AB}})^\alpha{}_\beta Q^\beta_A, \\
[T_{AB}, Q^\alpha_C] &= 2\delta^D{}_{[A}\delta_{B]C} Q^\alpha_D, \\
\{Q^\alpha_A, Q^\beta_B\} &= \frac{1}{2\ell}(\tilde{\Gamma}^{\mathcal{EF}} C_5)^{\alpha\beta}\delta_{AB} L_{\mathcal{EF}} - \frac{1}{\ell} C_5^{\alpha\beta} T_{AB}.
\end{aligned} \tag{6.1}$$

Here, the first two properties describe the bosonic subalgebra of the subgroup $\mathrm{O}(\mathcal{N}) \times SO(2,3)$

where[2]

$$\begin{aligned}\mathcal{A},\mathcal{B},\ldots &= 0,1,2,3,4\\ A,B,\ldots &= 1,\ldots,\mathcal{N}\,,\\ \eta_{\mathcal{AB}} &= \mathrm{diag}(+,-,-,-,+)\,,\end{aligned} \tag{6.2}$$

whereas the remaining ones extend the mentioned subalgebra to a supersymmetric one. In particular, the fermionic generators $Q^\alpha_A$ are Majorana spinors transforming in the fundamental representation of $SO(\mathcal{N})$ and in the spinorial representation of the Lorentz group, with $\alpha,\beta,\ \ldots = 0,1,2,3,4$. We refer to the Appendix B for properties of gamma matrices and spinors.
This algebra can be recasted in the (5.1) form by writing it in a manifestly covariant way with respect to the $D=4$ Lorentz group: this is achieved by defining $\mathcal{A}=(a,4)$, where $a=0,1,2,3$ and $L_{a4} := \ell\,\mathcal{P}_a$.
The chosen Lie algebra can also be expressed in terms of the Maurer-Cartan form

$$\boldsymbol{\omega} = \frac{1}{2}\hat{\omega}^{\mathcal{AB}}\,L_{\mathcal{AB}} + \frac{1}{2}A^{CD}\,T_{CD} + \overline{\Psi}^A_\alpha\,Q^\alpha_A, \tag{6.3}$$

where one identifies, as explained previously, $\hat{\omega}^{a4} := \ell^{-1}\,V^a$ with the vielbein of the vacuum space AdS$_4$ and $\ell$ with its radius. The structure equations are then obtained as

$$\begin{aligned}&\mathrm{d}\hat{\omega}^{ab} + \hat{\omega}^a{}_c\wedge\hat{\omega}^{cb} - \frac{1}{\ell^2}V^a\wedge V^b - \frac{1}{2\ell}\left(\overline{\Psi}^A\wedge\Gamma^{ab}\Psi_A\right) = 0,\\ &\mathrm{d}V^a + \hat{\omega}^a{}_b\wedge V^b - \frac{\mathrm{i}}{2}\left(\overline{\Psi}^A\wedge\Gamma^a\Psi_A\right) = 0,\\ &\mathrm{d}A^{CD} + A^C{}_B\wedge A^{BD} + \frac{1}{\ell}\left(\overline{\Psi}^C\wedge\Psi^D\right) = 0,\\ &\mathrm{d}\Psi^A + \frac{1}{4}\,\hat{\omega}^{ab}\wedge\Gamma_{ab}\Psi^A + \frac{i}{2\ell}\,V^a\wedge\Gamma_a\Psi^A + A^{AB}\wedge\Psi_B = 0,\end{aligned} \tag{6.4}$$

where spinor indices have been omitted. In order to reproduce the known results of [37] and to generalise the construction of [36], we now consider a local coordinate patch, in which the boundary of AdS$_4$ is locally AdS$_3$: this means that

$$\hat{g} = -\frac{\ell^2}{z^2}\mathrm{d}z^2 + \frac{\ell^2}{z^2}g_{AdS_3},$$

where $g_{AdS_3}$ is the metric of AdS$_3$. Let us then study the equations (6.4) at the radial boundary: for this reason, it is convenient to further split the rigid index $a$ into $a=(i,3)$, where $i=0,1,2$ labels the boundary dreibein and $a=3$ labels the vierbein along the radial direction. The superalgebra is further decomposed in terms of $SO(1,1)\times SO(1,2)\subset SO(2,3)$, having $L_{34}$ and $L_{ij}$ as their generators, respectively. By defining the following quantities

$$V^i_\pm := \frac{1}{2}\left(\ell\hat{\omega}^{i3}\pm V^i\right) \tag{6.5}$$

[2] Notice that we are going to relabel indices already used in the previous section, which however will not be needed anymore, so this shouldn't be source of confusion.

and by decomposing the gravitini in their chiral components with the $\Gamma^3$ matrix

$$\Psi^A = \Psi^A_+ + \Psi^A_- \,, \qquad \Gamma^3 \Psi^A_\pm = \pm \mathrm{i}\, \Psi^A_\pm \,, \tag{6.6}$$

one can effectively write the algebra in a way in which the $SO(1,1)$-grading is manifest. With these definitions, The structure equations become

$$\begin{aligned}
&\mathrm{d}\hat{\omega}^{ij} + \hat{\omega}^i{}_k \wedge \hat{\omega}^{kj} + \frac{4}{\ell^2}\, V^{[i}_+ \wedge V^{j]}_- - \frac{1}{\ell}\left(\overline{\Psi}^A_+ \wedge \Gamma^{ij}\Psi_{A-}\right) = 0\,,\\
&\mathrm{d}V^i_\pm + \hat{\omega}^i{}_j \wedge V^j_\pm \mp \frac{1}{\ell}\, V^i_\pm \wedge V^3 \mp \frac{\mathrm{i}}{2}\left(\overline{\Psi}^A_\pm \wedge \Gamma^i \Psi_{A\pm}\right) = 0\,,\\
&\mathrm{d}V^3 - \frac{1}{\ell}\,(V^i_+ + V^i_-) \wedge V_i + \overline{\Psi}^A_- \wedge \Psi_{A+} = 0\,,\\
&\mathrm{d}A^{CD} + A^C{}_M \wedge A^{MD} + \frac{2}{\ell}\left(\overline{\Psi}^{[C}_+ \wedge \Psi^{D]}_-\right) = 0\,,\\
&\mathrm{d}\Psi^{M\beta}_\pm + \frac{1}{4}\,\hat{\omega}^{ij} \wedge \left(\Gamma_{ij}\Psi^M_\pm\right)^\beta \pm \frac{\mathrm{i}}{\ell}\, V^i_\pm \wedge \left(\Gamma_i \Psi^M_\mp\right)^\beta \pm \frac{1}{2\ell}\, V^3 \wedge \Psi^{M\beta}_\pm + \delta^M{}_{[C}\delta_{D]B}\, A^{CD} \wedge \Psi^{B\beta}_\pm = 0\,.
\end{aligned} \tag{6.7}$$

Since we are working in a vacuum configuration, these Maurer-Cartan equations hold everywhere, both in the bulk and at the boundary. In particular, one can perform a specific asymptotic limit and study these equations projected on the cotangent space to the boundary. To this end, let us consider the following scaling

$$\begin{aligned}
&V^i_+(x,z) = \left(\frac{\ell}{z}\right) E^i(x) + \mathcal{O}\left(\frac{z}{\ell}\right)\,, \qquad V^i_-(x,z) = -\frac{1}{4}\left(\frac{\ell}{r}\right) E^i(x) + O\left(\frac{\ell^2}{r^2}\right)\,,\\
&\hat{\omega}^{ij}(x,z) = \omega^{ij}_\mu(x)\, dx^\mu + \ldots\,, \qquad \hat{A}^{AB}(x,z) = A^{AB}_\mu(x)\, dx^\mu + \ldots\,,
\end{aligned} \tag{6.8}$$

where $x$ are the local coordinates on $\mathrm{AdS}_3$. As for the spinors, we require them to behave in the following way

$$\begin{aligned}
&\Psi^A_{+\mu}(x,z)\, dx^\mu = \sqrt{\frac{\ell}{z}}\begin{pmatrix}\psi^A(x)\\ \mathbf{0}\end{pmatrix} + \mathcal{O}\left(\sqrt{\frac{z}{\ell}}\right),\\
&\Psi^A_{-\mu}(x,z)\, dx^\mu = \frac{1}{2}\sqrt{\frac{z}{\ell}}\begin{pmatrix}\mathbf{0}\\ k^{AB}\psi_B(x)\end{pmatrix} + \mathcal{O}\left(\left(\frac{\ell}{r}\right)^{\frac{3}{2}}\right),
\end{aligned} \tag{6.9}$$

where $k_{AB}$ is a symmetric metric satisfying $k^{AC}\, k_{CB} = \delta_{AB}$. If we restrict ourselves to the cotangent space to the boundary, the third equation in (7.132) vanishes automatically, because $V^3 = \mathrm{d}V^3 = 0$ follows from the radial foliation that we are considering, $\hat{\omega}^3{}_i \wedge E^i = 0$ due to the general properties of the extrinsic curvature and the bilinear $\overline{\psi}^A \wedge \psi_A$ is automatically zero.

The remaining equations are rewritten as

$$
\begin{aligned}
&\mathrm{d}\omega^{ij}+\omega^{i}{}_{k}\wedge\omega^{kj}-\frac{1}{\ell^{2}}\,E^{i}\wedge E^{j}-\frac{1}{2\ell}\,\left(\overline{\psi}^{A}\wedge\gamma^{ij}k_{AB}\psi^{B}\right)=0\,,\\
&\mathrm{d}E^{i}+\omega^{i}{}_{j}\wedge E^{j}-\frac{\mathrm{i}}{2}\,\left(\overline{\psi}^{A}\wedge\gamma^{i}\psi_{A}\right)=0\,,\\
&\mathrm{d}A^{CD}+A^{C}_{M}\wedge A^{MD}+\frac{1}{\ell}\,\overline{\psi}^{[C}\wedge k^{D]B}\psi_{B}=0\,,\\
&\mathrm{d}\psi^{A}+\frac{1}{4}\,\omega^{ij}\;\wedge\gamma_{ij}\,\psi^{A}+\frac{\mathrm{i}}{2\ell}\,E^{i}\wedge\gamma_{i}\,k^{AB}\psi_{B}+A^{AB}\wedge\psi_{B}=0\,.
\end{aligned}
\tag{6.10}
$$

Let us notice that the matrix $k_{AB}$ introduced in the asymptotic behaviour of the gravitini allows to break the full R-symmetry group: indeed, one can always bring such metric into a diagonal form

$$
k_{AB}=\begin{pmatrix}\mathbf{1}_{p\times p} & \mathbf{0}_{p\times q}\\ \mathbf{0}_{q\times p} & -\mathbf{1}_{q\times q}\end{pmatrix}
\tag{6.11}
$$

through an $O(\mathcal{N})/(O(p)\times O(q))$ rotation, where $p+q=\mathcal{N}$. By doing so, the R-symmetry is broken to $O(\mathcal{N})\to O(p)\times O(q)$, which means that a generic $O(\mathcal{N})$ index splits into $A=(a_1,a_2)$, where $a_1=1,\ldots,p$ and $a_2=p+1,\ldots,\mathcal{N}$.
The condition (6.11) has strong consequences on the Maurer-Cartan equations for the boundary gauge field $A^{BC}$: the gauge fields $A^{a_1a_2}$ associated to the coset of rotations used to set the $k_{AB}$ matrix in a block diagonal form decouple from spinors

$$
\mathrm{d}A^{a_1b_2}+A^{a_1}{}_{c_1}\wedge A^{c_1b_2}+A^{a_1}{}_{c_2}\wedge A^{c_2b_2}=0
$$

and must be set to zero in order to obtain consistent gravitini equations. The other structure equations can be reproduced as equations of motion of the following lagrangian

$$
\begin{aligned}
L=&-\frac{1}{2}R^{ij}\wedge E^{k}\epsilon_{ijk}+\frac{1}{6L^{2}}\epsilon^{ijk}E_{i}\wedge E_{j}\wedge E_{k}-\overline{\Psi}^{a_1}\wedge\mathcal{D}[\omega,A]\Psi_{a_1}-\overline{\Psi}^{a_2}\wedge\mathcal{D}[\omega,A]\Psi_{a_2}\\
&-\frac{\mathrm{i}}{2L}\overline{\Psi}^{a_1}\wedge E^{i}\wedge(\gamma_i\Psi_{a_1})+\frac{\mathrm{i}}{2L}\overline{\Psi}^{a_2}\wedge E^{i}\wedge(\gamma_i\Psi_{a_2})+\\
&-\frac{L}{2}\left(A^{a_1b_1}\wedge\mathrm{d}A_{b_1a_1}+\frac{2}{3}A^{a_1b_1}\wedge A^{b_1c_1}\wedge A^{c_1a_1}\right)+\\
&+\frac{L}{2}\left(A^{a_2b_2}\wedge\mathrm{d}A_{b_2a_2}+\frac{2}{3}A^{a_2b_2}\wedge A^{b_2c_2}\wedge A^{c_2a_2}\right),
\end{aligned}
\tag{6.12}
$$

where $R^{ij}=\mathrm{d}\omega^{ij}+\omega^{i}{}_{k}\omega^{kj}$ and $\mathcal{D}[\omega,A]\psi^{A}=\mathrm{d}\psi^{A}+\frac{1}{4}\,\omega^{ij}\;\wedge\gamma_{ij}\,\psi^{A}+A^{AB}\wedge\psi_{B}$.
It is crucial to notice, though, that one can rearrange the fields in this theory in such a way to describe the superalgebra

$$
OSp(p|2)_{+}\times OSp(q|2)_{-}.
$$

The subscripts "$\pm$" used here and in the following do not refer to the $SO(1,1)$-grading, but they will be used to ease the notation and because they will play an important role in the

physical interpretation of this model in terms of graphene-like materials.
By defining the following quantities

$$
\begin{aligned}
&\Omega^i_{(\pm)} := \omega^i \pm \frac{E^i}{\ell}\,, \quad \psi_+ := (\psi^{a_1})\,, \quad \psi_- := (\psi^{a_2})\,, \quad A_+ := (A^{a_1 b_1})\,, \quad A_- := (A^{a_2 b_2})\,,\\
&\mathcal{D}[\Omega_+,\, A_+]\psi_+ := \left(\mathrm{d}\psi^{a_1} + \frac{\mathrm{i}}{2}\Omega^i_+ \wedge \gamma_i \psi^{a_1} + A^{a_1 b_1} \wedge \psi^\beta_{b_1}\right)\,,\\
&\mathcal{D}[\Omega_-,\, A_-]\psi_- := \left(\mathrm{d}\psi^{a_2} + \frac{\mathrm{i}}{2}\Omega^i_- \wedge \gamma_i \psi^{a_2} + A^{a_2 b_2} \wedge \psi_{b_2}\right)\,,
\end{aligned}
$$

where $\omega^i := \frac{1}{2}\,\epsilon^{ijk}\,\omega_{jk}$, one obtains a compact form of the Maurer-Cartan equations

$$R_\pm{}^i := \mathrm{d}\Omega^i_\pm - \frac{1}{2}\epsilon^{ijk}\Omega_{\pm\, j} \wedge \Omega_{\pm\, k} = \pm\frac{\mathrm{i}}{\ell}\left(\overline{\psi}_\pm \wedge \gamma^i \psi_\pm\right)\,; \tag{6.13a}$$

$$\mathcal{D}[\Omega_\pm,\, A_\pm]\psi_\pm = 0\,, \tag{6.13b}$$

$$\mathcal{F}^{a_1 b_1} := \mathrm{d}A^{a_1 b_1} + A^{a_1}{}_{c_1} \wedge A^{c_1 b_1} = -\frac{1}{\ell}\left(\overline{\psi}^{a_1} \wedge \psi^{b_1}\right)\,, \tag{6.13c}$$

$$\mathcal{F}^{a_2 b_2} := \mathrm{d}A^{a_2 b_2} + A^{a_2}{}_{c_2} \wedge A^{c_2 b_2} = \frac{1}{\ell}\left(\overline{\psi}^{a_2} \wedge \psi^{b_2}\right)\,. \tag{6.13d}$$

Let us notice that the definitions used above ignore the upper/lower position of the R-symmetry indices: this is possible because such indices transform in the fundamental of $SO(\mathcal{N})$. Moreover, the definition and properties of the 3-dimensional Levi-Civita symbol are listed in Appendix A.
The action (6.12) can be then rewritten in a compact form

$$
\begin{aligned}
&L = L_{(+)} - L_{(-)} - \frac{1}{2}\,\mathrm{d}(\Omega_{+k} \wedge \Omega^k_-)\,,\\
&L_{(\pm)} := \frac{1}{2}\left(\Omega_{\pm\, i}\mathrm{d}\Omega^i_\pm - \frac{1}{3}\,\epsilon_{ijk}\,\Omega^i_\pm \wedge \Omega^j_\pm \wedge \Omega^k_\pm\right) + \mathrm{Tr}\left(A_\pm \wedge \mathrm{d}A_\pm + \frac{2}{3}\,A_\pm \wedge A_\pm \wedge A_\pm\right) \pm\\
&\qquad \pm \frac{2}{\ell}\overline{\psi}_\pm \wedge \mathcal{D}[\Omega_\pm,\, A_\pm]\psi_\pm\,,
\end{aligned} \tag{6.14}
$$

which exactly reproduces the result obtained in [37]: the action having the Maurer-Cartan equations of the supergroup $OSp(p|2)_+ \times OSp(q|2)_-$ as equations of motion can be written as a difference of Chern-Simons lagrangians plus a boundary term. This exact term is actually a Gibbons-Hawking-York term [38, 39], as it can be rewritten as

$$-\frac{\ell}{2}\,\mathrm{d}(\Omega_{+k} \wedge \Omega^k_-) = \mathrm{d}(\omega_i \wedge E^i) = -\mathrm{d}(e_2\, K\, \mathrm{d}x^2),$$

where $e_2$ is the determinant of the vielbein at the boundary of AdS$_3$, $E^2|_{T\partial \mathrm{AdS}_3} = 0$ and $K$ is the trace of the extrinsic curvature $\omega^{s2} = K^s{}_t E^t$, with $s, t = 0, 1$. [3]
We now discuss an interesting property which emerges from this formulation of the theory.

[3] Let us remark that the coordinate patch chosen in the 4-dimensional analysis describes only a part of the full boundary. It then comes with no surprise that such boundary, AdS$_3$ can itself have a boundary.

**Generalised Parity symmetry**

Let us now consider a reflection along the y-axis, described by $t \to t,\ x \to -x,\ y \to y$, combined with a particular transformation acting on spinors and gauge fields.
Such transformation acts on vielbein and spin connection in the following way[4]

$$E^i \to \mathcal{O}_y{}^i{}_j\, E^j\,, \qquad \omega^i \to -\mathcal{O}_y{}^i{}_j\, \omega^j\,, \qquad \Longrightarrow\ \Omega^i_\pm \to -\mathcal{O}_y{}^i{}_j\, \Omega^j_\mp, \tag{6.15}$$

where $\mathcal{O}_y = \mathrm{diag}(+1, -1, +1)$ and on the remaining fields as

$$\psi_\pm \to \sigma^1\, \psi_\mp\,, \qquad A_\pm \to A_\mp\,. \tag{6.16}$$

This particular set of transformations maps the $OSp(p|2)_+ \times OSp(q|2)_-$ model into a $OSp(q|2)_+ \times OSp(p|2)_-$ one: in general this is not a symmetry of the theory, but it becomes one in the particular case $p = q$. Indeed, by making use of the following identities,

$$\begin{aligned} \overline{\psi}_\pm \wedge \gamma^i \psi_\pm \;&\to\; \mathcal{O}_y{}^i{}_j\, \overline{\psi}_\mp \wedge \gamma^j \psi_\mp\,, \\ \Omega_\pm\, \gamma^i \psi_\pm \;&\to\; \Omega_\mp\, \sigma^1 \gamma^i \psi_\mp\,, \\ \overline{\psi}_\pm\, \mathcal{D}[\Omega_\pm\,,\, A_\pm]\, \psi_\pm \;&\to\; -\overline{\psi}_\mp\, \mathcal{D}[\Omega_\mp\,,\, A_\mp]\, \psi_\mp\,, \end{aligned} \tag{6.17}$$

one can prove that the equations of motion of the lagrangian, i.e. the Maurer-Cartan equations, are just shuffled between themselves, whereas the lagrangian $L \to -L$, as $L_{(\pm)} \to L_{(\mp)}$.
Let us end this small paragraph with a comment on the role of this discrete symmetry: as we said in the beginning of this section, our purpose is to model electronic properties of 2-dimensional materials. The fact that this theory enjoys this symmetry is somewhat surprising, as it allows to describe the parity properties of bipartite lattices, as we will argue. The case $p = q$ will then play an important role in what follows.
Notice that this property is a novelty even from the $\mathcal{N} = 2$ point of view explored in [40], where, in our current notation, the authors chose $p = 0,\ q = 2$.

**Decomposition of the boundary gravitini**

The key distinguishing feature of the Unconventional Supersymmetry [36] is given by the following decomposition

$$\psi_\pm = \mathrm{i}\gamma_i e^i \chi_\pm, \tag{6.18}$$

in terms of a spin $1/2$ particle and a bosonic driebein $e^i$ describing an $\mathrm{AdS}_3'$ space. Here unprimed and primed quantities respectively refer to the distinction between the starting target space and a new purely bosonic spacetime that we are introducing, which will be called world volume.
From this point of view, the isometry group $\mathrm{SL}(2,\mathbb{R})'_+ \times \mathrm{SL}(2,\mathbb{R})'_-$ of the tangent space to the world-volume geometry and the bosonic subgroup $\mathrm{SL}(2,\mathbb{R})_+ \times \mathrm{SL}(2,\mathbb{R})_-$ of the gauge group

[4] Recall that the vielbein is a Lorentz vector, whereas $\omega^i$ is a pseudo-vector.

are in principle unrelated. A link between these two groups can be created if we provide enough information: indeed one can first identify the index $i$ of $e^i$ and $E^i$ and then relate the connection $\Omega'^{i}_{\pm} = \omega'^{\,i} \pm e^i/\ell'$, written here in terms of a torsionless connection $\omega'^i$, with the one of $\mathrm{SL}(2,\mathbb{R})_{\pm}$.This last step will be dealt with in the next paragraph.
Furthermore, gamma matrices should in principle be considered as intertwiners between primed and unprimed groups, as the spinors $\chi_{\pm}$ transform in the $\left(\mathbf{\frac{1}{2}},\mathbf{0}\right)$ or $\left(\mathbf{0},\mathbf{\frac{1}{2}}\right)$ representation of $\mathrm{SL}(2,\mathbb{R})'_{+} \times \mathrm{SL}(2,\mathbb{R})'_{-}$. This is another reason to lean towards the identification between these groups, as it allows to avoid this complication.

Let us first analyse the effect of this decomposition on the bosonic structure equations in (6.13):

$$
\begin{aligned}
R^i_{\pm} &= \pm\frac{1}{\ell}\,\overline{\chi}_{\pm}\chi_{\pm}\epsilon^{ijk}\,e_j\wedge e_k\,,\\
\mathcal{D}[\Omega_{\pm}]E^i &= \mp\frac{1}{\ell}\,\epsilon^{ijk}\,E_j\wedge E_k + \frac{1}{2}\left(\overline{\chi}_{+}\chi_{+} + \overline{\chi}_{-}\chi_{-}\right)\epsilon^{ijk}\,e_j\wedge e_k\,,\\
\mathcal{F}^{a_1b_1} &= -\frac{\mathrm{i}}{\ell}\left(\overline{\chi}^{a_1}\gamma^i\chi^{b_1}\right)\,\epsilon_{ijk}e^j\wedge e^k\,,\qquad \mathcal{F}^{a_2b_2} = \frac{\mathrm{i}}{\ell}\left(\overline{\chi}^{a_2}\gamma^i\chi^{b_2}\right)\,\epsilon_{ijk}e^j\wedge e^k\,,
\end{aligned}
\tag{6.19}
$$

where $\chi_{+} := (\chi_{a_1})$, $\chi_{-} := (\chi_{a_2})$. Since we are identifying the indices $i$, one can compute the covariant derivative of the world volume vielbein with respect to the target space connections: this can be split into

$$
\mathcal{D}[\Omega_{\pm}]e^i = \beta_{\pm}e^i + \tau_{\pm}\epsilon^{ijk}e_j\wedge e_k\,,
\tag{6.20}
$$

where $\beta_{\pm}$ and $\tau_{\pm}$ are 1- and 0-forms, respectively.
Notice that the decomposition (6.18) introduces a local scale invariance called Nieh-Yan-Weyl (NYW) symmetry, which leaves the gravitino and therefore all target space quantities invariant

$$
e^i \to \lambda(x)\,e^i\,,\qquad \chi_{\pm} \to \frac{1}{\lambda(x)}\,\chi_{\pm}\,,\qquad \lambda \neq 0\,.
\tag{6.21}
$$

The expression (6.20) retains its form under a NYW rescaling provided that

$$
\beta_{\pm} \to \beta_{\pm} + \frac{d\lambda}{\lambda}\,,\qquad \tau_{\pm} \to \frac{1}{\lambda}\,\tau_{\pm}\,.
\tag{6.22}
$$

The transformation law of $\beta_{\pm}$ suggest that it is a connection under this local scale transformation.
To justify the name "world volume" given to the $\mathrm{AdS}'_3$ space, we impose a relation between the two vielbeins in our theory, $E^i = f^i(e)$ and we choose in particular

$$
E^i = f(x)e^i.
\tag{6.23}
$$

Let us now explore the implications of this simple ansatz: once the relation between the target space and world volume has been defined, the covariant derivative (6.20), expressed in terms of $\omega^i$, becomes

$$
\mathcal{D}[\Omega_{\pm}]e^i = \mathcal{D}[\omega]e^i \mp \frac{f}{\ell}\epsilon^{ijk}e_j\wedge e_k
$$

$$= \beta e^i + \tau \epsilon^{ijk} e_j \wedge e_k \mp \frac{f}{\ell} \epsilon^{ijk} e_j \wedge e_k,$$

where in the last line we used $\mathcal{D}[\omega]e^i = \beta \wedge e^i + \tau\, \epsilon^{ijk} e_j \wedge e_k$. We are then led to the following identifications

$$\beta_+ = \beta_- = \beta\,, \qquad \tau_+ + \frac{f}{\ell} = \tau_- - \frac{f}{\ell} = \tau\,. \tag{6.24}$$

The exterior derivative of (6.20) leads to constraints on $\beta_\pm$ and $\tau_\pm$, which are a consequence of the fact that $\epsilon^{ijk} R_{\pm j} e_k = 0$

$$\mathrm{d}\beta_\pm = 0, \qquad \mathrm{d}\tau_\pm + \beta_\pm \tau_\pm = 0. \tag{6.25}$$

This means that either $\tau_\pm = 0$ and $\beta_+ = \beta_-$ is a generic closed form, or that $\tau_\pm \neq 0$ and $\beta_\pm = -\frac{\mathrm{d}\tau_\pm}{\tau_\pm} = -\mathrm{d}\ln|\tau_\pm|$. Notice that this last relation can be interpreted by saying that $\beta_\pm$ can be produced by the scale transformation $e^i \to (\tau_\pm)^{-1} e^i$. Therefore, at least locally, $\beta_\pm = \beta$ can be gauged away to zero and correspondingly $\tau_\pm$ can be set equal to constants. In particular, one can fix one of them to a chosen value, while the other one will be consequently obtained from (6.24). This can be globally achieved only in absence of topological obstructions.

Relations involving the function $f(x)$ can instead be obtained by implementing (6.23) in the second equation appearing in (6.19). By comparing it with (6.20), one obtains

$$\mathrm{d}f + \beta f = 0, \tag{6.26}$$

$$f\,\tau = \frac{1}{2} \left( \overline{\chi}_+ \chi_+ + \overline{\chi}_- \chi_- \right), \tag{6.27}$$

where the first equation can be solved by choosing $f = \alpha_\pm\, \tau_\pm$, where $\alpha_\pm$ are dimensionful constants. This result can be used in the computation of the exterior derivative of $R^i_\pm$, which yields

$$\mathrm{d}(\overline{\chi}_\pm \chi_\pm) = -2\,\beta\, \overline{\chi}_\pm \chi_\pm$$

and since $\beta$ can be set to zero in a local patch, both bilinears $\overline{\chi}_\pm \chi_\pm$ must be constants. In this specific case, they can then be related by a generic constant $k$

$$\overline{\chi}_+ \chi_+ = k\, \overline{\chi}_- \chi_-\ . \tag{6.28}$$

The whole discussion on the bosonic structure equations can be summed up in the following table

$$
\begin{array}{ccl}
\mathcal{D}[\Omega_\pm]e^i & \xrightarrow{E^i=fe^i} & \begin{cases} \beta_+ = \beta_- = \beta \\ \tau_+ + \frac{f}{\ell} = \tau_- - \frac{f}{\ell} = \tau \end{cases} \\[2ex]
\mathcal{D}[\Omega_\pm]e^i & \xrightarrow{\mathrm{d},\ \tau_\pm\neq 0} & \begin{cases} \mathrm{d}\beta_\pm = 0 \\ \beta_\pm = -\mathrm{d}\ln|\tau_\pm| \end{cases} \\[2ex]
\mathcal{D}[\Omega_\pm]E^i & \xrightarrow{E^i=fe^i} & \begin{cases} \mathrm{d}f + \beta f = 0, \\ f\,\tau = \frac{1}{2}\left(\overline{\chi}_+\chi_+ + \overline{\chi}_-\chi_-\right) \end{cases} \\[2ex]
R^i_\pm & \xrightarrow{\mathrm{d},\ \ \beta=0} & \overline{\chi}_+\chi_+ = k\,\overline{\chi}_-\chi_- .
\end{array}
$$

Moving on to the fermionic structure equations, let us define a covariant derivative, in the generic $\beta$ case, including also the local NYW symmetry

$$\hat{\mathcal{D}} = \mathcal{D} + w\,\beta \implies \hat{\mathcal{D}}[\Omega_\pm,\, A_\pm]\,\chi_\pm = \mathcal{D}[\Omega_\pm,\, A_\pm]\,\chi_\pm + \beta\,, \tag{6.29}$$

where $w$ is the NYW weight of the field ($-1$ for $e^i$ and $+1$ for $\chi_\pm$). This derivative can be then used to express (6.13b) as

$$\gamma_{[i}\hat{\mathcal{D}}_{j]}[\Omega_\pm,\, A_\pm]\,\chi_\pm = \tau_\pm\,\epsilon_{ijk}\,\gamma^k\chi_\pm\,. \tag{6.30}$$

This last equation actually contains key information, which can be retrieved by taking its projections, i.e. by multiplying it by $\gamma^{ij}$ and $\gamma^i$. In the first case, one obtains two massive Dirac equations

$$\hat{\not{\mathcal{D}}}[\Omega_\pm,\, A_\pm]\,\chi_\pm = -3\,\mathrm{i}\,\tau_\pm\,\chi_\pm\,, \tag{6.31}$$

while in the second case we have

$$\hat{\mathcal{D}}_i[\Omega_\pm,\, A_\pm]\,\chi_\pm = -\mathrm{i}\,\tau_\pm\,\gamma_i\chi_\pm\,. \tag{6.32}$$

By inserting the decomposition (6.18) into the lagrangian (6.14), we see that the full set of structure equations can be then derived as its equations of motion. Indeed we have

$$
\begin{aligned}
S = \int \Bigg[ & \frac{1}{2}\left(\Omega_{+i}\wedge \mathrm{d}\Omega^i_+ - \frac{1}{3}\epsilon^{ijk}\Omega_{+i}\wedge\Omega_{+j}\wedge\Omega_{+k}\right) - \frac{1}{2}\left(\Omega_{-i}\wedge \mathrm{d}\Omega^i_- - \frac{1}{3}\epsilon^{ijk}\Omega_{-i}\wedge\Omega_{-j}\wedge\Omega_{-k}\right) + \\
& + \left(A^{a_1b_1}\wedge \mathrm{d}A_{b_1a_1} + \frac{2}{3}A^{a_1b_1}\wedge A^{b_1c_1}\wedge A^{c_1a_1}\right) - \left(A^{a_2b_2}\wedge \mathrm{d}A_{b_2a_2} + \frac{2}{3}A^{a_2b_2}\wedge A^{b_2c_2}\wedge A^{c_2a_2}\right) - \\
& - \frac{2\mathrm{i}}{\ell}\epsilon^{ijk}\overline{\chi}^{a_1}\left\{\gamma_k\hat{\mathcal{D}}[\Omega_+, A^{a_1b_1}]\chi_{a_1} + \mathrm{i}\tau_+\chi_{a_1}e_k\right\}\wedge e_i\wedge e_j - \\
& - \frac{2\mathrm{i}}{\ell}\epsilon^{ijk}\overline{\chi}^{a_2}\left\{\gamma_k\hat{\mathcal{D}}[\Omega_-, A^{a_2b_2}]\chi_{a_2} + \mathrm{i}\tau_-\chi_{a_2}e_k\right\}\wedge e_i\wedge e_j - \frac{1}{2}\,\mathrm{d}(\Omega_{+k}\wedge\Omega^k_-)\Bigg]\,.
\end{aligned}
$$

Most notably, the introduction of the Unconventional Supersymmetry has led us to a theory with spin 1/2 particles following massive Dirac equations. As we will see, this is the key development towards the extension of the analogy between 2-dimensional graphene-like materials and gravitational theories with the inclusion of fermionic matter.

#### A relation between connections of target space and world volume

As said in the previous paragraph, to completely identify the two groups under consideration, one also has to establish a link between their connections. Since the world volume is a bosonic space, one can consider a torsionless spin connection $\omega'^i$ such that $\mathcal{D}[\omega']e^i = 0$. Expressing such connection in terms of a target space one as

$$\omega'^i = \Omega^i_+ + \tau_+ \, e^i = \Omega^i_- + \tau_- \, e^i \, . \tag{6.33}$$

means introducing the desired relation between the two groups. This particular choice descends from the fact that $\mathcal{D}[\omega']e^i = 0$ is indeed zero if $\beta = 0$. We will therefore work under this assumption. Furthermore, this identification of connections allows to interpret (6.20) as the world volume torsion tensor.
Once the relation between the groups has been accomplished, one can compute the Dirac equation in terms of the torsionless connection, which becomes

$$\not{\mathcal{D}}[\omega', \, A_\pm]\chi_\pm = -\frac{3}{2}\, \mathrm{i}\, \tau_\pm \, \chi_\pm \, , \tag{6.34}$$

where the mass term is given by

$$m_\pm = \frac{3}{2}\, \tau_\pm \tag{6.35}$$

and the Riemann tensor associated to such connection

$$R^i[\omega'] = \frac{1}{2}\left(\frac{f^2}{\ell^2} + \tau^2 + \frac{k_{AB}\overline{\chi}^A\chi^B}{\ell}\right)\epsilon^{ijk}e_j \wedge e_k \, . \tag{6.36}$$

We notice that, since we are working in the $\beta = 0$ case, the terms in the parenthesis are constants and define an effective cosmological constant, which also receives contributions from fermions. In particular, one can choose the residual NYW symmetry to identify the radii of the two $AdS_3$ spaces. Notice that, in this case, a generic identification of torsionful connections

$$\Omega^i_{(\xi)} \equiv \omega^i + \frac{\xi}{\ell}\, E^i, \qquad \Omega'^i_{(\xi)} \equiv \omega'^i + \frac{\xi}{\ell}\, e^i,$$

parametrised by a constant $\xi$, is compatible with (6.33) provided that

$$\tau = \frac{\xi}{\ell}(f-1)\, , \tag{6.37}$$

holds. This condition can be further rewritten as

$$\xi \, f\,(f-1) = \frac{\ell}{2}\,(\overline{\chi}_+\chi_+ + \overline{\chi}_-\chi_-)\, . \tag{6.38}$$

When $\xi = 0$, we have $\omega'^i = \omega^i$, which means that $\omega^i$ has to be torsionless: by looking at (6.19), we see that this is possible only if

$$\overline{\chi}_+ \chi_+ = -\overline{\chi}_- \chi_- ,$$

which is the exact same condition that we obtain from (6.38). This relation can be also achieved when $f = 0,\ f = 1$: the first case is a singular one, as the target space vielbein would be zero, whereas the second case identifies the two vielbeins. Since $E^i$ is a supersymmetric vielbein, it comes with no surprise that the fermionic contribution to the torsion is required to vanish.
Finally, when $\xi = \pm 1$, we have $\Omega'^i_\pm = \Omega^i_\pm$: it turns out that by choosing $f = 1/2$ one can require either $\tau_+$ or $\tau_-$ to vanish, or equivalently $\chi_+$ or $\chi_-$ to be massless.
This whole set of possibilities will be explored in the next paragraph, where we will try to make contact with graphene-like materials and their properties.

#### Graphene-like materials and interpretation

In this paragraph, we will first state some of the properties that these materials share and we will then interpret them in terms of the results that we have found up until now.
Graphene-like materials are characterised by a honeycomb 2-dimensional lattice, made of two inequivalent sublattices, where A and B sites live, as it is shown in Figure 3. For example, pure graphene is composed of six carbon atoms, but in other similar materials different elements are also allowed, like in boron nitride.
In this analysis we will work in a regime where wave functions have long wavelength compared to the characteristic lattice length: in this way, the charge carriers will feel the lattice as a continuum and defects of the former as resulting from curvature or torsion of the latter.
In pure, isolated graphene the First Brillouin Zone (FBZ) is again a hexagonal lattice in momentum space, rotated by $\pi/2$ with respect to the original lattice, in which the inequivalent sites are called valleys $\boldsymbol{K},\ \boldsymbol{K}'$.

A reflection with respect to the y-axis exchanges both the A and B sites and the $\boldsymbol{K}$ and $\boldsymbol{K}'$ valleys provided that we combine it with a time-reversal transformation. This indeed allows to obtain the transformation $k_x \to k_x\,,\ k_y \to -k_y$ on a momentum vector.
Near these points, also called Dirac points, the pseudoparticles describing the charge carriers behave in a relativistic way: this can be inferred from the linear dispersion relation between the energy and the quasi-momentum. In these valleys, the conduction and valence band of graphene, which correspond to the positive and negative eigenvalues of the Hamiltonian, touch each other, forming the so called Dirac cones. This means that the charge carriers possess an additional pseudo-spin number, associated to the two valleys.
In virtue of their relativistic behaviour, the pseudoparticles obey the Dirac equation, which is massless if the graphene layer is pure and isolated. However, mass terms, which create a gap between conduction and valence bands even at the Dirac points, can be introduced in two different ways, as shown by Semenoff [41] and Haldane [42]. The Semenoff mass term is introduced by considering an on-site staggered potential, which clearly breaks the symmetry

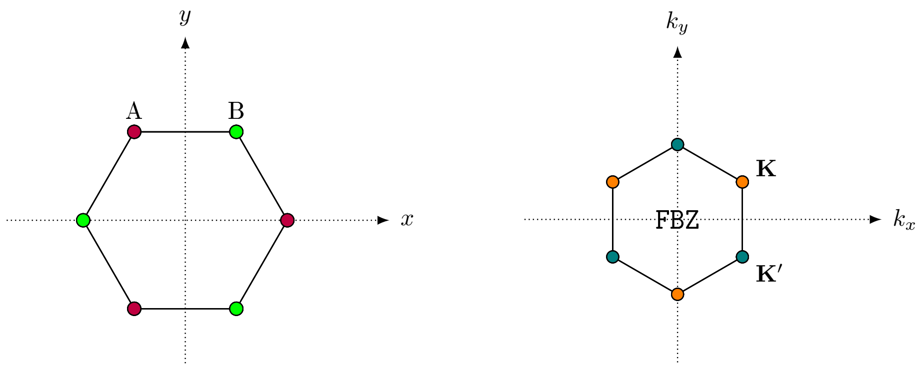


Figure 3: The honeycomb lattice and the FBZ are characterised by two inequivalent sites, belonging to two different sublattices.

under reflections, whereas the Haldane mass is generated by including periodic local magnetic flux densities, with zero net flux over the lattice cell. The mass terms in the two Dirac points are then obtained as

$$m_{\mathbf{K}} = M - 3\sqrt{3}\, t_2 \sin\varphi \,, \qquad m_{\mathbf{K}'} = M + 3\sqrt{3}\, t_2 \sin\varphi \,, \tag{6.39}$$

where $M$ is the Semenoff staggered potential, $t_2$ is the next-to-nearest hopping amplitude and $\varphi$ is an Aharonov-Bohm phase due to the local magnetic fluxes. The Semenoff mass term, as said, is parity odd, while the Haldane one is parity even and this property will allow an important identification in terms of gravity quantities.
The single-electron wave function around one valley is usually described as a two-component complex spinor $\zeta$

$$\zeta = \begin{pmatrix} \sqrt{n_A}\, e^{i\alpha_A} \\ \sqrt{n_B}\, e^{i\alpha_B} \end{pmatrix} \,, \tag{6.40}$$

associated to the gamma matrix basis

$$\underline{\gamma}^0 = -\sigma^3 \,, \qquad \underline{\gamma}^1 = -i\,\sigma^2 \,, \qquad \underline{\gamma}^2 = i\,\sigma^1 \,. \tag{6.41}$$

Here $n_A,\ n_B$ are the probability densities for the electron in the $\pi$-orbitals, whereas $\alpha_A,\ \alpha_B$ are their wave function phases. One in particular has that

$$n_A + n_B = \zeta^\dagger \zeta \,, \qquad n_B - n_A = \bar{\zeta}\zeta \,. \tag{6.42}$$

The first quantity is the total electron probability density associated to a single valley, whereas the second one describes the asymmetry between the probability densities.

In order to make contact with this description of graphene-like materials, we shall restrict ourselves in our Supergravity model, to the case in which Supersymmetry is defined by even integers $p$ and $q$, since this allows to arrange the real spinors $\chi_\pm$ into $p/2$ and $q/2$ Dirac spinors.

Furthermore, the reflection symmetry of the valleys is reminiscent of the parity symmetry of the Supergravity model in the case $p = q$: for this reason, we will take the simplest choice $p = q = 2$.
In this case the real spinors $\chi_+ = \chi^{a_1}$ and $\chi_- = \chi^{a_2}$, with $a_1 = 1, 2$, $a_2 = \dot{1}, \dot{2}$, can be rearranged to define complex spinors

$$\tilde{\chi}_+ = \chi^1 + \mathrm{i}\chi^2, \qquad \tilde{\chi}_- = \chi^{\dot{1}} + \mathrm{i}\chi^{\dot{2}}, \tag{6.43}$$

which will be associated to each valley through

$$\tilde{\chi}_+ = \sqrt{\frac{\ell}{2}}\, U\, \zeta_{\boldsymbol{K}}\,, \qquad \tilde{\chi}_- = \sqrt{\frac{\ell}{2}}\, U\, \zeta_{\boldsymbol{K'}}. \tag{6.44}$$

Here the matrix

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -\mathrm{i} & \mathrm{i} \end{pmatrix}$$

relates the spinor basis used for $\chi$ to the one defined above for $\zeta$ through $U^\dagger \gamma^i U = \underline{\gamma}^i$. By then defining the 4-dimensional spinors

$$\chi = \tilde{\chi}_+ \oplus \tilde{\chi}_-, \qquad Z = \zeta_{\boldsymbol{K}} \oplus \zeta_{\boldsymbol{K'}},$$

one obtains that

$$\overline{\chi}\chi = \overline{\chi}_+\chi_+ + \overline{\chi}_-\chi_- = \frac{\ell}{2}(\overline{Z}Z) = \frac{\ell}{2}(\overline{\zeta}_{\boldsymbol{K}}\zeta_{\boldsymbol{K}} + \overline{\zeta}_{\boldsymbol{K'}}\zeta_{\boldsymbol{K'}}) = \frac{\ell}{2}(n_B^{\boldsymbol{K}} - n_A^{\boldsymbol{K}} + n_B^{\boldsymbol{K'}} - n_A^{\boldsymbol{K'}})$$

This relation, together with (6.38) allows to obtain a lower bound on the total probability densities of one of the two sublattices

$$\begin{aligned} \xi > 0\,, \quad n_A^{\boldsymbol{K}} + n_A^{\boldsymbol{K'}} &= -\frac{4\xi}{\ell^2}\, f(f-1) + n_B^{\boldsymbol{K}} + n_B^{\boldsymbol{K'}} \geq -\frac{4\xi}{\ell^2}\, f(f-1) \geq \frac{\xi}{\ell^2}\,, \\ \xi < 0\,, \quad n_B^{\boldsymbol{K}} + n_B^{\boldsymbol{K'}} &= \frac{4\xi}{\ell^2}\, f(f-1) + n_A^{\boldsymbol{K}} + n_A^{\boldsymbol{K'}} \geq \frac{4\xi}{\ell^2}\, f(f-1) \geq \frac{|\xi|}{\ell^2}\,. \end{aligned} \tag{6.45}$$

In the $\xi = 0$ case, one has $\overline{\chi}_+\chi_+ = -\overline{\chi}_-\chi_-$, which implies that the total probability density are the same in the two sublattices $n_A^{\boldsymbol{K}} + n_A^{\boldsymbol{K'}} = n_B^{\boldsymbol{K}} + n_B^{\boldsymbol{K'}}$.
At last, this discussion allows to finally understand the role of the labels $\pm$ in the Supergravity theory: they have to be associated to the valleys $\boldsymbol{K},\ \boldsymbol{K'}$. Moreover, in the $\beta = 0$ case, which is the one we decided to consider, $\overline{\chi}_\pm\chi_\pm$ are constants, which means that the difference $n_B - n_A$ in the probability densities of each valley is a constant index, which still has to be explored.
In light of this new understanding, let us discuss the role of the parity symmetry in the Supergravity theory: since $E^i = f\, e^i$, the action of the $\mathcal{O}_y$-parity on $E^i$ naturally extends to $e^i$:

$$e^i \;\rightarrow\; \mathcal{O}_Y{}^i{}_j\, e^j\,, \tag{6.46}$$

provided that $f$ is invariant. This requires the spinors $\chi_\pm$ to transform as follows

$$\chi_\pm \;\rightarrow -\sigma^1 \chi_\mp\,. \tag{6.47}$$

Under these transformations, the + sector is mapped into the − sector provided that

$$\beta \to \beta\,, \qquad \tau_\pm \to -\tau_\mp\,.$$

However, a specific choice of $\tau_\pm$, which characterises the world-volume background, remains invariant provided that

$$\tau_+ = -\tau_- \ . \tag{6.48}$$

In our conventions and with $\hbar = v_{\rm F} = 1$, the description of the relativistic properties of the charge carriers in a curved background, with a coupling to a gauge potential is given by

$$\begin{aligned}
\mathbf{K}: &\qquad \mathrm{i}\,\mathcal{D}_t[\omega',\,A_{\mathbf{K}}]\,\chi_{\mathbf{K}}(x) = \left(-\mathrm{i}\alpha^1\,\mathcal{D}_x[\omega',\,A_{\mathbf{K}}] - \mathrm{i}\alpha^2\,\mathcal{D}_y[\omega',\,A_{\mathbf{K}}] + m_{\mathbf{K}}\,\gamma^0\right)\chi_{\mathbf{K}}(x)\,,\\
\mathbf{K}': &\qquad \mathrm{i}\,\mathcal{D}_t[\omega',\,A_{\mathbf{K}'}]\,\chi_{\mathbf{K}'}(x) = \left(-\mathrm{i}\alpha^1\,\mathcal{D}_x[\omega',\,A_{\mathbf{K}'}] + \mathrm{i}\alpha^2\,\mathcal{D}_y[\omega',\,A_{\mathbf{K}'}] + m_{\mathbf{K}'}\,\gamma^0\right)\chi_{\mathbf{K}'}(x)\,,
\end{aligned} \tag{6.49}$$

where $\alpha^\ell \equiv \gamma^0\,\gamma^\ell$ $(\ell = 1, 2)$. These Dirac equations are invariant under the parity symmetry discussed above and the spinors appearing here are identified to the ones appearing in our model through

$$\chi_{\mathbf{K}}(x^\mu) = \chi_+(x^\mu)\,, \qquad \chi_{\mathbf{K}'}(x^0,\,x^1,\,x^2) = \sigma^1\chi_-(-x^0,\,x^1,\,x^2)\,. \tag{6.50}$$

From this last equation, one sees that, as said, the parity symmetry has to be combined with a time reversal transformation. Furthermore, one has to consistently relate the gauge fields

$$A_{\mathbf{K}} = A_+\,, \qquad A_{\mathbf{K}'} = A_- \tag{6.51}$$

and the masses

$$m_{\mathbf{K}} = m_+ = \frac{3}{2}\,\tau_+\,, \qquad m_{\mathbf{K}'} = m_- = \frac{3}{2}\,\tau_-\,. \tag{6.52}$$

Interestingly, these masses can be associated to the ones of the macroscopic model. Indeed, by rewriting $\tau_\pm$ as

$$\tau_\pm \equiv \frac{1}{2}\,(\tau_+ + \tau_-) \pm \frac{1}{2}\,(\tau_+ - \tau_-) \;=\; \tau \mp 2\,\frac{f}{\ell}\,, \tag{6.53}$$

where the second equal sign is a consequence of (6.24), we see that, in virtue of (6.48), the first term is parity odd, while the second is parity even. This behaviour is also shared by the Semenoff and Haldane masses, respectively. One can then conclude that

$$M = \frac{3}{2}\tau\,, \qquad \sqrt{3}\,t_2\sin(\varphi) = \frac{f}{\ell}\,. \tag{6.54}$$

As a final comment, let us notice that, from a phenomenological point of view, the charge carriers in the honeycomb lattice are "relativistic" in a peculiar way: indeed the Fermi velocity $v_{\rm F}$ plays the role of the speed of light in the Dirac equation, whereas the characteristic speed appearing in (6.49) and (6.34) is defined by the only vielbein appearing there, $e^i$ and coincides with the speed of light.
This problem can be dealt with in several ways: the first one would be to start with both world volume and target space having as a numerical value of the speed of light the one of

the Fermi velocity. Another possibility would be to have a target space with a standard light speed and a world volume with a Fermi velocity: in this case one would need an ansatz more generic than (6.23). Finally, both target space and worldvolume could have the same standard value for the light speed: in this case one would need a mechanism to transform the vielbein $e^i$ appearing in the Dirac equations of our model to effectively modify the light cone of the theory and reproduce the "correct" Dirac equations. This third path will be the one analysed in the following Subsection, in the pure GR case.

## 6.1 Dirac equations and metric structure

We now want to study a mechanism capable of changing the speed of light appearing in a Dirac equation. This effectively means changing the light cone of the theory, which in turns implies modifying the spacetime metric [43].
Let us notice that the obtained Dirac equations (6.34) are written in terms of a bosonic vielbein, so it will be sufficient to understand the desired mechanism in a standard General Relativity context. For the sake of simplicity, we will neglect the minimal coupling to gauge fields and we will not necessarily restrict to $AdS_3$. Indeed the construction that we are going to illustrate is general and could have applications that exceed our graphene model.

From a mathematical point of view [44], the standard approach to the Dirac equations is given in terms of spin structures and spin manifolds, the latter being spacetimes allowing the construction of principal bundles with $G = Spin_e(r, s)$ as a Lie group.

**Definition 6.1.** *A spin structure over* $(M, g)$ *is a pair* $(P, \Lambda)$*, where* $P$ *is a spin bundle and* $\Lambda : P \to SO_e(M, g)$ *is a vertical principal morphism with respect to the covering map* $l : Spin_e(r, s) \to SO_e(r, s)$.

This is equivalent to require the commutativity of the following diagrams

$$\begin{array}{ccc} P & \xrightarrow{\Lambda} & SO_e(M,g) \\ {\scriptstyle p}\big\downarrow{\scriptstyle Spin_e(r,s)} & & {\scriptstyle \hat{\pi}}\big\downarrow{\scriptstyle SO_e(r,s)} \\ M & = & M \end{array} \qquad \begin{array}{ccc} P & \xrightarrow{\Lambda} & SO_e(M,g) \\ \big\downarrow{\scriptstyle R_g} & & \big\downarrow{\scriptstyle R_{l(g)}} \\ P & \xrightarrow{\Lambda} & SO_e(M,g) \end{array}$$

Here $SO_e(M, g) \subset L(M)$, where $L(M)$ is the frame bundle, denotes the subbundle of oriented orthonormal frames with respect to a specific metric $g$ and $(r, s)$ denotes its signature. Spinors are then introduced by considering a representation $\hat{\lambda} : Cl(r, s) \times V \to V$ of the Clifford algebra on a (usually complex) vector space of dimension $k = dim(V)$, which restricts to a representation $\lambda : Spin_e(r, s) \times V \to V$ of the (connected component to the identity of the) spin group. Spinors are then sections of the associated vector bundle

$$S(P) = P \times_\lambda V, \tag{6.55}$$

called spinor bundle. The last ingredient needed to formulate the Dirac equations is a principal connection on the spin bundle $\hat{H}_p \in T_pP$, which allows to compute covariant derivatives on the associated bundle. In this context, automorphisms of the spin bundle will induce

transformations preserving the metric structure of the theory.
However, fixing a metric is not preferable for two reasons: from a general relativistic point of view, one should avoid fixing a background metric and for our purposes, we need to be able to find a mechanism capable of changing the metric structure. For this reason, we now introduce the spin frames.

**Definition 6.2.** *A spin frame over the spin manifold $M$ is a pair $(P, e)$, where $P$ is a spin bundle and $e : P \to L(M)$ is a vertical principal morphism with respect to the map $i \circ l$ : $Spin_e(r, s) \to GL(m, \mathbb{R})$.*

In other words, we must require the commutativity of the following diagrams

$$\begin{tikzcd} P \arrow[r, "e"] \arrow[d, "p"', "Spin_e(r,s)"] & L(M) \arrow[d, "\pi"', "GL(m,\mathbb{R})"] \\ M \arrow[r, equal] & M \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, "e"] \arrow[d, "R_g"] & L(M) \arrow[d, "R_{i\circ l(g)}"] \\ P \arrow[r, "e"] & L(M) \end{tikzcd}$$

As it is clear from the definition, no metric structure is required on the base manifold $M$. We will now see how the metric structure arises from this definition.
Let $U_\alpha$ be an open cover of $M$ and let $\overset{(\alpha)}{\sigma} : U_\alpha \subset M \to P$ be a family of local sections of $P$, defined on each open of $M$. These local sections are enough to completely determine the spin frame on the entire fiber, as the image can be extended by equivariance.
Furthermore, local sections on the spin bundle induces local sections on the frame bundle as

$$e(\overset{(\alpha)}{\sigma}) =: (x, \overset{(\alpha)}{e}{}_a), \tag{6.56}$$

where $a = 1, ..., \dim M$ and $\overset{(\alpha)}{e}{}_a \in T_xM$. Since the frame bundle always allows the natural section $(x, \partial_\mu)$, induced by the local coordinates $x^\mu$ on $U_\alpha \subset M$, where $\mu = 1, ..., \dim m$, one can express the induced local section with respect to such basis

$$(x, \overset{(\alpha)}{e}{}_a) = (x, \partial_\mu)\overset{(\alpha)}{e}{}^\mu_a.$$

The metric structure emerges as a consequence of these frames in the following way.

**Definition 6.3.** *Let $(P, e)$ be a spin frame and $p \in P$ a point in the spin bundle mapped into $e(p) = (x, \epsilon_a) \in e(P) \hookrightarrow L(M)$. We define the induced metric as*

$$g(\epsilon_a, \epsilon_b) := \eta_{ab}. \tag{6.57}$$

This intrinsic definition can be locally restated as

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu, \qquad g_{\mu\nu} := \epsilon^a_\mu \eta_{ab} \epsilon^b_\nu, \tag{6.58}$$

We now prove that the conditions for the existence of spin frames coincide with the ones for spin structures.

**Theorem 6.1.** *A spin frame $(P, e)$ over the manifold $M$ exists if and only if there exists a spin structure over $(M, g)$ for some metric $g$ on $M$.*

*Proof.* Given a spin frame $(P, e)$ over $M$, one can consider the image $e(P) = \{(x, e_a) : e_a \text{ is a basis of } T_xM \wedge g(e_a, e_b) = \eta_{ab}\}$, where we crucially remark that the metric appearing here is the one induced by the spin frame. One can then define a spin structure as

$$\begin{tikzcd} P \arrow[r, "\hat{e}"] \arrow[d, "p"'] & e(P) \arrow[d, "\pi|_{e(P)}"'] \\ M \arrow[r, equal] & M \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, "\hat{e}"] \arrow[d, "R_g"] & e(P) \arrow[d, "R_{i\circ l(g)}|_{SO_e(r,s)} = R_{l(g)}"] \\ P \arrow[r, "\hat{e}"] & e(P) \end{tikzcd}$$

On the contrary, given a spin structure $(P, \Lambda)$ over $(M, g)$ one can define a spin frame $(P, e = \hat{\imath} \circ \Lambda)$: this is indeed a spin frame since

$$e \circ R_g = \hat{\imath} \circ \Lambda \circ R_g = \hat{\imath} \circ R_{l(g)} \circ \Lambda = R_{i\circ l(g)} \circ \hat{\imath} \circ \Lambda = R_{i\circ l(g)} \circ e.$$

□

This can be restated by saying that the following diagrams commute

$$\begin{tikzcd} P \arrow[r, "\Lambda"] \arrow[d, "p"'] & SO_e(M,g) \arrow[r, "\hat{\imath}"] \arrow[d, "\hat{\pi}"'] & L(M) \arrow[d, "\pi"'] \\ M \arrow[r, equal] & M \arrow[r, equal] & M \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, "\Lambda"] \arrow[d, "R_g"] & SO_e(M,g) \arrow[r, hook, "\hat{\imath}"] \arrow[d, "R_{l(g)}"] & L(M) \arrow[d, "R_{i\circ l(g)}"] \\ P \arrow[r, "\Lambda"] & SO_e(M,g) \arrow[r, hook, "\hat{\imath}"] & L(M) \end{tikzcd}$$

To reinforce the necessity of using spin frames, even if existence conditions between spin structures and spin frames are equivalent, let us consider a one parameter family of spin structures $(P, \Lambda_t)$, where $t \in \mathbb{R}$. They are maps from the spin bundle to a shared given orthonormal frame bundle, each one differing by an orthogonal transformation.
A one parameter family of spin frames is instead a couple $(P, e_t)$: the images of these maps are different orthonormal frame bundles differing one another for the arising metric $g_t$. All these different bundles are related by a new set of transformations, that we define below.

**Definition 6.4.** *Let $(P, e)$ be a spin frame and $\Phi : L(M) \to L(M)$ a vertical principal automorphism of the frame bundle. We define $(P, \tilde{e})$ as the transformed spin frame that makes the following diagram commute.*

$$\begin{tikzcd} & & & L(M) \arrow[dd, "\pi"] \\ P \arrow[rrru, "\tilde{e}"] \arrow[rr, "e"] \arrow[dd, "p"] & & L(M) \arrow[ru, "\Phi"] \arrow[dd, "\pi"] & \\ & & & M \\ M \arrow[rr, equal] & & M \arrow[ru, equal] & \end{tikzcd}$$

*We will refer to the automorphism $\Phi$ as a spin frame transformation.*

To be concrete, let us remember that the action of such vertical automorphism is again completely determined if we know how it acts on local sections of the frame bundle. Indeed, consider the section induced by the spin frame $(P, e)$: one then has

$$(x, \overset{(\alpha)}{\tilde{e}}_a) = \tilde{e}(\overset{(\alpha)}{\sigma}) = \Phi(e(\overset{(\alpha)}{\sigma})) = \Phi(x, \overset{(\alpha)}{e}_a) = (x, \overset{(\alpha)}{e}_b)\phi^b_a(x),$$

where $\phi^b_a$ is a $GL(m, \mathbb{R})$-matrix. The expression we have given here in terms of a right principal action on local sections can be translated into a left action on the local expression of a generic point $e(p) = (x, \epsilon_a) = (x, \overset{(\alpha)}{e}_b)\epsilon^b_a$: indeed

$$(x, \overset{(\alpha)}{e}_c)\tilde{\epsilon}^c_a = \tilde{e}(p) = \Phi(e(p)) = \Phi((x, \overset{(\alpha)}{e}_b)\epsilon^b_a) = \Phi(x, \overset{(\alpha)}{e}_b)\epsilon^b_a = (x, \overset{(\alpha)}{e}_c)\phi^c_b(x)\epsilon^b_a,$$

is locally expressed as

$$\Phi : \begin{cases} x' = x \\ \tilde{\epsilon}^c_a = \phi^c_b(x)\epsilon^b_a. \end{cases} \tag{6.59}$$

One can alternatively choose the natural section $(x, \partial_\mu)$ and repeat this computation, which yields

$$\Phi : \begin{cases} x' = x \\ \tilde{\epsilon}^\mu_a = \phi^\mu_\nu(x)\epsilon^\nu_a, \end{cases} \tag{6.60}$$

where again $\phi^\mu_\nu$ is a $GL(m, \mathbb{R})-$matrix.

These spin frame transformations, which are locally described by matrices $\phi^b_a$, $\phi^\mu_\nu$, are the key to effectively change the metric structure of the theory. Let us remark that we are not changing the local expression of the coefficients $g_\mu\nu$, but we are really changing the metric structure, as it is formally proved in [43]. The key idea is that $\Phi$ is a vertical automorphism that does not change coordinates on the base manifold. This means that in every local patch the local expression of the two metrics differs.
The new metric is indeed obtained from (6.57) and (6.58) as

$$\tilde{g} = \bar{\phi}^\rho_\mu(x)g_{\rho\sigma}\bar{\phi}^\sigma_\nu(x))dx^\mu \otimes dx^\nu,$$

where $\bar{\phi}^\mu_\nu$ is the inverse of $\phi^\mu_\nu$.
As said in the beginning of this Subsection, Dirac equations are formulated in terms of a connection on the spin bundle. The same horizontal tangent subspace defined in (5.3) can be also described in terms of horizontal lifts $\hat{\omega} : TM \to \hat{H} \subset TP$ by

$$\hat{H}_p = \left\{\hat{\omega}(v) = v^\mu(\partial_\mu|_p - \omega^{ab}{}_\mu(x)\sigma_{ab}|_p), \forall v \in T_xM, \pi(p) = x\right\}, \tag{6.61}$$

where $\sigma_{ab}$ are a set of vertical right invariant vector fields on $P$ satisfying $\sigma_{(ab)} = 0$. From this point of view, it is crucial to notice that this connection is independent from spin frame transformations, which act on $L(M)$ and that the induced connections on the frame bundle

$H_{e(p)} := T_p e(\hat{H}_p)$ and $H_{\tilde{e}(p)} := T_p \tilde{e}(\hat{H}_p)$ are expressed in terms of the same coefficients $\omega^{ab}{}_\mu$, when written in the two induced trivializations. Indeed

$$e(p) = (x, \overset{(\alpha)}{e}{}_b)\epsilon^b_a : \qquad T_p e(\hat{\omega}(v)|_p) = v^\mu(\partial_\mu|_{e(p)} - \omega^{ab}{}_\mu(x)\rho_{ab}|_{e(p)}),$$
$$\tilde{e}(p) = (x, \overset{(\alpha)}{\tilde{e}}{}_b)\tilde{\epsilon}'^b_a : \qquad T_p \tilde{e}(\hat{\omega}(v)|_p) = v^\mu(\partial_\mu|_{\tilde{e}(p)} - \omega^{ab}{}_\mu(x)\tilde{\rho}'_{ab}|_{\tilde{e}(p)}),$$

where $\rho_{ab} = \epsilon^c_d \partial^d_{[a}\eta_{b]c}$ and $\tilde{\rho}'_{ab} = \tilde{\epsilon}'^c_d \tilde{\partial}'^d_{[a}\eta_{b]c}$ are the vertical vector fields on $T_{e(p)}e(P)$ and $T\tilde{e}(P)$ respectively.
However, by starting from connections on the frame bundle and pulling them back onto the spin bundle, one can obtain an explicit expression of $\omega^{ab}{}_\mu$ in terms of the induced metric structures, to express the Dirac equation in terms of the metric structure of the theory.
Let us then start from a connection on the frame bundle $H \in TL(M)$: in the natural trivialization the horizontal lift can be written as

$$\omega(v)|_{(x,\epsilon_a)} = v^\mu \left(\partial_\mu|_{(x,\epsilon_a)} - \omega^\alpha_{\beta\mu}(x)\rho^\beta_\alpha|_{(x,\epsilon_a)}\right), \tag{6.62}$$

where $\rho^\beta_\alpha|_{(x,\epsilon_a)} = \epsilon^\beta_a \partial^a_\alpha$ are the right invariant vector fields on $L(M)$ and $\omega^\alpha_{\beta\mu}$ are the coefficients which characterise the chosen connection.

**Lemma 6.2.** *Let $H$ be a connection on $L(M)$ and $(P, e)$ a spin frame. $H$ is tangent to $e(P) = SO_e(M, g)$ in a point $(x, \epsilon_a)$ (and hence in all of its points) if and only if*

$$\omega^{(ab)}_\mu := \eta^{c(a}\omega^{b)}{}_{c\mu} = 0,$$

*where the $(ab)$ indicates that the indices are symmetrised and*

$$\omega^b{}_{c\mu}(x) = e^b_\alpha(x)\left(\omega^\alpha_{\beta\mu}(x)e^\beta_c(x) + \partial_\mu e^\alpha_c(x)\right). \tag{6.63}$$

*Proof.* In order to prove this statement, let us express the horizontal lift of a vector on $M$ in the trivialization induced by the spin frame $(P, e)$. By doing so, one obtains

$$\omega(v)|_{(x,\epsilon_a)} = v^\mu(\partial_\mu|_{(x,\epsilon_a)} - \omega^a{}_{b\mu}(x)\rho^b_a|_{(x,\epsilon_a)}),$$

where $\omega^a{}_{b\mu}(x)$ is exactly given by (6.63) and $\rho^b_a = e^b_\beta \rho^\beta_\alpha e^\alpha_a$. In this derivation, we suppressed the index $\alpha$ labelling the local section $\overset{(\alpha)}{\sigma}$ on $P$ to ease the reading.
Since the right invariant vector fields on $SO_e(M, g)$ are

$$\rho_{[ab]} = \eta_{c[a}\rho^c_{b]}, \tag{6.64}$$

where $[ab]$ indicates skew symmetric indices, the lifted vectors become

$$\begin{aligned}\omega(v)|_{(x,\epsilon_a)} &= v^\mu\left(\partial_\mu|_{(x,\epsilon_a)} - \omega^{ab}{}_\mu(x)\rho_{ab}|_{(x,\epsilon_a)}\right)\\ &= v^\mu\left(\partial_\mu|_{(x,\epsilon_a)} - \omega^{ab}{}_\mu(x)(\rho_{(ab)} + \rho_{[ab]})|_{(x,\epsilon_a)}\right)\end{aligned}$$

$$= v^\mu \left( \partial_\mu|_{(x,\epsilon_a)} - \omega^{[ab]}{}_\mu(x)\rho_{[ab]}|_{(x,\epsilon_a)} \right),$$

where in the last step we made use of the hypothesis. This is indeed an horizontal vector on $SO_e(M,g)$.
On the contrary, take a horizontal vector of $L(M)$

$$\begin{aligned}\omega(v)|_{(x,\epsilon_a)} &= v^\mu \left( \partial_\mu|_{(x,\epsilon_a)} - \omega^{ab}{}_\mu(x)\rho_{ab}|_{(x,\epsilon_a)} \right) \\ &= v^\mu \left( \partial_\mu|_{(x,\epsilon_a)} - \omega^{(ab)}{}_\mu(x)\rho_{(ab)}|_{(x,\epsilon_a)} - \omega^{[ab]}{}_\mu(x)\rho_{[ab]}|_{(x,\epsilon_a)} \right).\end{aligned}$$

If we want it to be tangent to $SO_e(M,g)$, we need to require the coefficient of $\rho_{(ab)}$, which are not right invariant vector fields on $SO_e(M,g)$, to vanish. We then get our thesis $\omega^{(ab)}{}_\mu(x) = 0$. □

This lemma can be then used to prove the following lemma, which constrains the vertical part of the chosen connection to be in a specific form, if we want to project it on $e(P)$.

**Theorem 6.3.** *Let $H$ be a connection of $L(M)$, $v \in TM$ and $\{g\}$ be the horizontal lift induced by Levi-Civita connection of the metric $g$. $H$ then projects on $e(P) = SO_e(M,g)$ if and only if*

$$\omega(v)|_{(x,\epsilon_a)} = (\{g\}(v) + K(v))|_{(x,\tilde{\epsilon}_a)}, \tag{6.65}$$

*where $(x,\epsilon_a) \in e(P)$ and $K(v)$ is defined as*

$$\begin{aligned} K : TM &\to V(L(M)) \\ v &\mapsto g^{\alpha\gamma} K_{\gamma\beta\mu} v^\mu \rho^\beta_\alpha, \end{aligned}$$

*in terms of the contorsion tensor whose coefficients are given by*

$$K_{\gamma\beta\mu} = \frac{1}{2}\Big( g(\partial_\beta, T(\partial_\mu,\partial_\gamma)) + g(\partial_\mu, T(\partial_\beta,\partial_\gamma)) + g(\partial_\gamma, T(\partial_\beta,\partial_\mu)) \Big).$$

*Proof.* The first part of theproof of this theorem consists on showing that, under our assumptions,

$$\omega^\alpha_{\beta\mu} = \{g\}^\alpha_{\beta\mu} + g^{\alpha\gamma}K_{\gamma\beta\mu}, \tag{6.66}$$

where $\{g\}^\alpha_{\beta\mu}$ are the Christoffel symbols of the induced metric $g$. If the connection is projectable, then the previous lemma requires $\omega^{(ab)}_\mu = 0$: this in turn implies that

$$\begin{aligned}\omega^{ab}{}_\mu + \omega^{ba}{}_\mu &= e^a_\alpha \omega^\alpha_{\beta\mu} e^{\beta b} + e^b_\alpha \omega^\alpha_{\beta\mu} e^{\beta a} + e^a_\alpha \partial_\mu e^{\alpha b} + e^b_\alpha \partial_\mu e^{\alpha a} \\ &= e^a_\alpha \omega^\alpha_{\beta\mu} e^{\beta b} + e^b_\alpha \omega^\alpha_{\beta\mu} e^{\beta a} + e^a_\alpha \partial_\mu e^{\alpha b} + e^b_\alpha \partial_\mu (g^{\alpha\lambda} e^a_\lambda) = 0.\end{aligned}$$

The previous expression can be easily rewritten as

$$\partial_\mu g_{\rho\sigma} = g_{\sigma\lambda}\omega^\lambda_{\rho\mu} + g_{\rho\alpha}\omega^\alpha_{\sigma\mu}.$$

We now cyclicly permute the indices and appropriately sum and subtract the obtained relations: by doing so, we get

$$-\partial_\rho g_{\sigma\mu} + \partial_\mu g_{\rho\sigma} + \partial_\sigma g_{\mu\rho} = g_{\mu\lambda} T^\lambda_{\rho\sigma} + g_{\sigma\lambda} T^\lambda_{\rho\mu} + g_{\rho\lambda} T^\lambda_{\mu\sigma} + 2 g_{\rho\lambda} \omega^\lambda_{\sigma\mu},$$

where $T^\lambda_{\alpha\beta} := \omega^\lambda_{\alpha\beta} - \omega^\lambda_{\beta\alpha}$ is the torsion of the connection. It is skew symmetric in its lower indices by construction. Then one easily gets

$$\begin{aligned}\omega^\gamma_{\sigma\mu} &= \{g\}^\gamma_{\sigma\mu} + \frac{1}{2} g^{\gamma\rho} (g_{\mu\lambda} T^\lambda_{\sigma\rho} + g_{\sigma\lambda} T^\lambda_{\mu\rho} + g_{\rho\lambda} T^\lambda_{\sigma\mu}) \\ &= \{g\}^\gamma_{\sigma\mu} + \frac{1}{2} g^{\gamma\rho} (T_{\mu\sigma\rho} + T_{\sigma\mu\rho} + T_{\rho\sigma\mu}),\end{aligned}$$

which is our thesis.
Notice that, in virtue of the properties of the torsion, the contorsion tensor is skew symmetric in its first two indices, meaning

$$K_{(\alpha\beta)\gamma} = 0. \tag{6.67}$$

Finally, if (6.65) is true, we have

$$\omega^{ab}{}_\mu = e^a_\alpha (\{g\}^\alpha_{\beta\mu} e^{\beta b} + \partial_\mu e^{\alpha b}) + e^{a\gamma} K_{\gamma\beta\mu} e^{\beta b} \implies \omega^{(ab)}{}_\mu = 0,$$

where the first term can be easily shown to be skew symmetric by explicitly writing the Christoffel symbols, while the second is a consequence of (6.67). The connection is indeed projectable. □

We then see that the space projectable connections is modelled on *contorsion-type tensors*, i.e. it is a submodule of dimension $\frac{m^2}{2}(m-1)$ in the module of connections, which has instead dimension $m^3$.
Once a connection is projected on the subbundle $e(P)$, it is straightforward to pull it back to the spin bundle through the spin frame $(P, e)$: this procedure defines a connection $\hat{H}$ whose horizontal vectors are given by

$$\hat{\omega}(v)|_p = v^\mu (\partial_\mu|_p - \omega^{ab}{}_\mu(x) \sigma_{ab}|_p), \tag{6.68}$$

where the coefficients $\omega^{ab}_\mu(x)$ are

$$\omega^{ab}{}_\mu(x) = e^a_\alpha(x) \left( (\{g\}^\alpha_{\beta\mu} + g^{\alpha\gamma} K_{\gamma\beta\mu}) e^\beta_c(x) + \partial_\mu e^\alpha_c(x) \right) \eta^{cb}. \tag{6.69}$$

as a consequence of Theorem 6.3. As promised, the coefficients $\omega^{ab}{}_\mu$ of the connection on the spin bundle are now expressed in terms of the induced metric structure.

Now suppose we have another connection $\tilde{H} \in L(M)$ and another spin frame $(P, \tilde{e})$: in virtue of Theorem 6.3, one can project it on $\tilde{e}(P)$ and pull it back on the spin bundle. A vector on the base manifold is then lifted as

$$\begin{cases} \hat{\tilde{\omega}}(v)|_p = v^\mu (\partial_\mu|_p - \Omega^{ab}{}_\mu(x) \sigma_{ab}|_p), \\ \Omega^{ab}{}_\mu(x) = \tilde{e}^a_\alpha(x) \left( (\{\tilde{g}\}^\alpha_{\beta\mu} + \tilde{g}^{\alpha\gamma} \tilde{K}_{\gamma\beta\mu}) \tilde{e}^\beta_c(x) + \partial_\mu \tilde{e}^\alpha_c(x) \right) \eta^{cb}, \end{cases} \tag{6.70}$$

where $\tilde{g}$ is the new induced metric and $\tilde{K}_{\gamma\beta\mu}$ is the contorsion of the new connection, completely unconstrained.
If the two connections on the frame bundle are unrelated, the connections pulled back onto the spin bundle will be different. However, this situation changes when we consider spin frame transformations: take indeed the connection $T\Phi(H)$, where $H$ is a projectable one on $e(P)$. We now prove that the contorsion $\tilde{K}_{\alpha\beta\gamma}$ of the obtained connection has to satisfy a certain property in order for $T\Phi(H)$ to be projectable on $\tilde{e}(P)$.
Indeed, the relation between the coefficient characterising the connection $T\Phi(H)$ and those of $H$ can be obtained by computing the push forward of horizontally lifter vectors. One obtains that

$$\tilde{\omega}^{\alpha}_{\beta\mu}(x) = \phi^{\alpha}_{\gamma}(x)\left(\omega^{\gamma}_{\delta\mu}(x)\bar{\phi}^{\delta}_{\beta}(x) + \partial_{\mu}\bar{\phi}^{\gamma}_{\beta}(x)\right). \tag{6.71}$$

This can be rearranged by writing

$$k^{\alpha}_{\beta\mu} := \tilde{\omega}^{\alpha}_{\beta\mu} - \omega^{\alpha}_{\beta\mu} = \phi^{\alpha}_{\gamma}\overset{\omega}{\nabla}_{\mu}\bar{\phi}^{\gamma}_{\beta},$$

where $\overset{\omega}{\nabla}_{\mu}$ indicated the usual covariant derivative with respect to $\omega^{\alpha}_{\beta\mu}$.
Finally, one also needs an expression for the difference between the Christoffel symbols of the two induced metrics: using that $\overset{\{g\}}{\nabla}_{\mu} g_{\alpha\beta} = 0$ and $\overset{\{\tilde{g}\}}{\nabla}_{\mu}\tilde{g}_{\alpha\beta} = 0$, one obtains

$$h^{\alpha}_{\beta\mu} := \{\tilde{g}\}^{\alpha}_{\beta\mu} - \{g\}^{\alpha}_{\beta\mu} = \phi^{\alpha}_{\gamma}\overset{\{g\}}{\nabla}_{(\beta}\bar{\phi}^{\gamma}_{\mu)} + \phi^{\alpha}_{\gamma}g^{\gamma\delta}\phi^{\lambda}_{\delta}g_{\rho\sigma}\bar{\phi}^{\rho}_{(\beta}\overset{\{g\}}{\nabla}_{\mu)}\bar{\phi}^{\sigma}_{\lambda} - \phi^{\alpha}_{\gamma}g^{\gamma\delta}\phi^{\lambda}_{\delta}g_{\rho\sigma}\overset{\{g\}}{\nabla}_{\lambda}\bar{\phi}^{\rho}_{(\beta}\bar{\phi}^{\sigma}_{\mu)}.$$

The following theorem then holds

**Theorem 6.4.** *Let $H$ be a connection on the frame bundle, projectable on the subbundle $e(P)$ and take $T\Phi(H)$. The latter connection is projectable on $\tilde{e}(P)$ if and only if its contorsion satisfies*

$$\tilde{K}(v)|_{(x,\tilde{\epsilon}_a)} = (K(v) + k(v) - h(v))|_{(x,\tilde{\epsilon}_a)}., \tag{6.72}$$

*where $k(v)|_{(x,\tilde{\epsilon}_a)} = v^{\mu}k^{\alpha}_{\beta\mu}\tilde{\rho}^{\beta}_{\alpha}$ and $h(v)|_{(x,\tilde{\epsilon}_a)} = v^{\mu}h^{\alpha}_{\beta\mu}\tilde{\rho}^{\beta}_{\alpha}$*

*Proof.* The proof of this statement is now straightforward, in light of previous results. Indeed

$$\begin{aligned}\tilde{\omega}(v)|_{(x,\tilde{\epsilon}_a)} &= (\omega(v) + k(v))|_{(x,\tilde{\epsilon}_a)} = (\{g\}(v) + K(v) + k(v))|_{(x,\tilde{\epsilon}_a)} \\ &= (\{\tilde{g}\}(v) + \tilde{K}(v))|_{(x,\tilde{\epsilon}_a)},\end{aligned}$$

where in the second line we used the fact that $T\Phi(H)$ is projectable. From this we get

$$\tilde{K}(v)|_{(x,\tilde{\epsilon}_a)} = (K(v) + k(v) - h(v))|_{(x,\tilde{\epsilon}_a)}.$$

Vice versa, recast (6.72) in the following way

$$\begin{aligned}&\tilde{K}(v)|_{(x,\tilde{\epsilon}_a)} = (K(v) + k(v) - \{\tilde{g}\}(v) + \{g\}(v))|_{(x,\tilde{\epsilon}_a)} \\ \implies &(\{\tilde{g}\}(v) + \tilde{K}(v))|_{(x,\tilde{\epsilon}_a)} = (\{g\}(v) + K(v) + k(v))|_{(x,\tilde{\epsilon}_a)} = (\omega(v) + k(v))|_{(x,\tilde{\epsilon}_a)} = \tilde{\omega}(v)|_{(x,\tilde{\epsilon}_a)}.\end{aligned}$$

We then see that $T\Phi(H)$ is projectable on $\tilde{e}(P)$. $\square$

This statement can clearly be written in a more explicit way as

$$\tilde{K}_{\rho\beta\mu} = g_{\alpha\gamma}\bar{\phi}^{\alpha}_{\rho}\overset{\{g\}}{\nabla}_{[\mu}\bar{\phi}^{\gamma}_{\beta]} - g_{\alpha\gamma}\bar{\phi}^{\alpha}_{(\beta}\overset{\{g\}}{\nabla}_{\mu)}\bar{\phi}^{\gamma}_{\rho} + g_{\alpha\gamma}\overset{\{g\}}{\nabla}_{\rho}\bar{\phi}^{\alpha}_{(\beta}\bar{\phi}^{\gamma}_{\mu)} + \bar{\phi}^{\sigma}_{\rho}K_{\sigma\eta\mu}\bar{\phi}^{\eta}_{\beta}, \tag{6.73}$$

which is indeed skew-symmetric in its first two indices, as can be seen by recasting it as

$$\tilde{K}_{\rho\beta\mu} = \phi^{\sigma}_{\lambda}\overset{\{g\}}{\nabla}_{\mu}\bar{\phi}^{\lambda}_{[\beta}\tilde{g}_{\rho]\sigma} - \phi^{\sigma}_{\lambda}\tilde{g}_{\sigma[\rho}\overset{\{g\}}{\nabla}_{\beta]}\bar{\phi}^{\lambda}_{\mu} + \tilde{g}_{\mu\sigma}\phi^{\sigma}_{\lambda}\overset{\{g\}}{\nabla}_{[\rho}\bar{\phi}^{\lambda}_{\beta]} + \bar{\phi}^{\sigma}_{[\rho}\bar{\phi}^{\lambda}_{\beta]}K_{\sigma\lambda\mu}.$$

As a consequence of Theorem 6.4, one has that the pullback of $H$ and $T\Phi(H)$ on the spin bundle coincides, as expected. This can be seen by computing the coefficients $\omega^{ab}{}_{\mu}$ and by noticing that

$$\omega^{ab}{}_{\mu} = e^{a}_{\alpha}\left([\{g\}^{\alpha}_{\beta\mu} + g^{\alpha\gamma}K_{\gamma\beta\mu}]e^{\beta}_{c} + \partial_{\mu}e^{\alpha}_{c}\right)\eta^{cb} = \tilde{e}^{a}_{\alpha}\left([\{\tilde{g}\}^{\alpha}_{\beta\mu} + \tilde{g}^{\alpha\gamma}\tilde{K}_{\gamma\beta\mu}(\tilde{e},\phi,K)]\tilde{e}^{\beta}_{c} + \partial_{\mu}\tilde{e}^{\alpha}_{c}\right)\eta^{cb},$$

which implies that the lifted vectors on $P$ are

$$\hat{\omega}(v)|_{p} = v^{\mu}(\partial_{\mu}|_{p} - \omega^{ab}{}_{\mu}\sigma_{ab}|_{p}). \tag{6.74}$$

This conclusion was clearly expected and coincides with (6.61): however, as we argued previously, by starting from frame bundle connections, we managed to obtain two expressions of $\omega^{ab}{}_{\mu}$ in terms of the metrics induced by the two spin frames. This will indeed prove to be useful to formulate the Dirac equations in terms of the two different light cone structures.

Indeed, consider the triple $(e, \hat{H}, \hat{\lambda})$: if the covariant derivative on the associated spinor bundle is given by $\mathcal{D}_{\mu}[\hat{\omega}] = \partial_{\mu} + \frac{1}{4}\hat{\omega}^{ab}{}_{\mu}\gamma_{ab}$, the Dirac equation is then given by

$$\mathrm{i}e^{\mu}_{a}\gamma^{a}\mathcal{D}_{\mu}[\hat{\omega}]\psi + \mu\psi = \mathrm{i}e^{\mu}_{a}\gamma^{a}\mathcal{D}_{\mu}[\overset{\circ}{\omega}]\psi + \mu\psi - \frac{\mathrm{i}}{4}e^{\mu}_{a}K^{bc}{}_{\mu}\gamma^{a}[\gamma_{b},\gamma_{c}]\psi = 0, \tag{6.75}$$

where the gamma matrices are those associated to the Clifford generators $\gamma^{a} := \eta^{ab}\hat{\lambda}(\mathbf{e}_{b})$, $\overset{\circ}{\omega}{}^{ab}{}_{\mu}$ is the torsionless connection generated by the Christoffel symbols of the induced metric $g$ and $\mu$ is a generic mass term.
A new spin frame, obtained from the first one from a spin frame transformation generates then a new Dirac equation $(\tilde{e}, \hat{H}, \hat{\lambda})$

$$\mathrm{i}\tilde{e}^{\mu}_{a}\gamma^{a}\mathcal{D}_{\mu}[\hat{\omega}]\psi + \mu\psi = \mathrm{i}\tilde{e}^{\mu}_{a}\gamma^{a}\mathcal{D}_{\mu}[\overset{\circ}{\tilde{\omega}}]\psi + \mu\psi - \frac{\mathrm{i}}{4}\tilde{e}^{\mu}_{a}\tilde{K}^{bc}{}_{\mu}\gamma^{a}[\gamma_{b},\gamma_{c}]\psi = 0, \tag{6.76}$$

where $\overset{\circ}{\tilde{\omega}}$ is the torsionless connection generated by the Christoffel symbols of the new metric $\tilde{g}$ and $\tilde{K}_{\alpha\beta\gamma}$ is given by (6.73).
We have then developed a mechanism for changing the metric structure of a certain Dirac equation, by extending the range of possible transformations acting on the theory. The modified Dirac equation is then written in terms of a new spin frame, inducing a new metric structure: the price paid for this transformation is absorbed into the contorsion, which is now a function of the spin frame transformation itself.

Let us notice that even if we start from a torsionless connection $K_{\alpha\beta\gamma} = 0$, after we change spin frame, we generally end up with a term, $\tilde{K}_{\alpha\beta\gamma} \neq 0$, which describes the effect of an interaction with the spacetime torsion.
It is then a matter of choosing the appropriate matrix $\phi^{\mu}_{\nu}$, depending on which metric structure one wants to obtain.
Let us now apply these results to the graphene model constructed previously.

### Application to the graphene model

Let us compare the Dirac equations (6.34) with the ones in (6.75): we see that the torsionless connection $\omega'$ has to coincide with the torsionless connection $\overset{\circ}{\omega}$. Furthermore, the spinors $\psi$ must be identified with $\chi_{\pm}$ and the mass term $\mu$ has to be zero. In the graphene model, the torsion, with respect to the torsionful connection $\Omega_{\pm}$ is given by $T^i_{\pm} = \tau_{\pm}\epsilon^{ijk} e_j \wedge e_k$, in the $\beta = 0$ case. Then

$$T^i_{\pm\mu\nu} = 2\tau_{\pm}\epsilon^{ijk} e_{j\mu} e_{k\nu}$$

and the contorsion tensor is given by

$$\begin{aligned} K_{\pm\lambda\mu\nu} &= \tau_{\pm}(\epsilon^{ijk} e_{\lambda i} e_{\mu j} e_{k\nu} + \epsilon^{ijk} e_{\mu i} e_{\lambda j} e_{k\nu} + \epsilon^{ijk} e_{\nu i} e_{\mu j} e_{k\lambda}) = \\ &= -\tau_{\pm} e'_{(3)} \epsilon_{\lambda\mu\nu}, \end{aligned} \tag{6.77}$$

where $e'_{(3)}$ is the determinant of the metric induced by $e^i_{\mu}$. This allows to compute the source term appearing in the Dirac equation as

$$\begin{aligned} \frac{\mathrm{i}}{4} e^{\nu}_{i} K^{jk}_{\pm\nu} \gamma^i \gamma_{jk} &= \frac{\mathrm{i}}{4} K_{\pm\lambda\mu\nu} \gamma^{\nu} \gamma^{\lambda\mu} = -\frac{\mathrm{i}\tau_{\pm}}{4} e'_{(3)} \epsilon_{\lambda\mu\nu} \gamma^{\nu\lambda\mu} = -\frac{\mathrm{i}\tau_{\pm}}{4} e'_{(3)} \epsilon_{\lambda\mu\nu} \mathrm{i} \epsilon^{\nu\lambda\mu} \frac{1}{e'_{(3)}} \\ &= \frac{3}{2}\tau_{\pm}, \end{aligned}$$

which exactly reproduces the results of (6.34). We see that in the graphene model the interaction with the torsion actually becomes a mass term, as it does not depend on fields or on the spacetime point.
To understand if this this mechanism can be applied to our model, we need to check if the transformed Dirac equation preserves the parity symmetry of the honeycomb lattice. In particular, the torsion interaction terms have to behave like $\tau_{\pm}$ under reflections:

$$\begin{aligned} -\frac{\mathrm{i}}{4} \tilde{e}^{\mu}_{i} \tilde{e}^{\rho}_{j} \tilde{e}^{\beta}_{k} \tilde{K}_{\rho\beta\mu} \gamma^i \gamma^{jk} &= -\frac{\mathrm{i}}{4} e^{\lambda}_{i} e^{\sigma}_{j} e^{\tau}_{k} \phi^{\mu}_{\lambda} \phi^{\rho}_{\sigma} \phi^{\beta}_{\tau} \Big( \phi^{\eta}_{\xi} \overset{\{g\}}{\nabla}_{\mu} \bar{\phi}^{\xi}_{[\beta} \tilde{g}_{\rho]\eta} - \phi^{\eta}_{\xi} \tilde{g}_{\eta[\rho} \overset{\{g\}}{\nabla}_{\beta]} \bar{\phi}^{\xi}_{\mu} + \\ &\quad + \tilde{g}_{\mu\eta} \phi^{\eta}_{\xi} \overset{\{g\}}{\nabla}_{[\rho} \bar{\phi}^{\xi}_{\beta]} + \bar{\phi}^{\eta}_{[\rho} \bar{\phi}^{\xi}_{\beta]} K_{\eta\xi\mu} \Big) \gamma^i \gamma^{jk} \\ &= -\frac{\mathrm{i}}{4} e^{\lambda}_{i} e^{\sigma}_{j} e^{\tau}_{k} \Big( g_{\sigma\xi} \overset{\{g\}}{\nabla}_{\mu} \bar{\phi}^{\xi}_{\beta} \phi^{\mu}_{[\lambda} \phi^{\beta}_{\tau]} - g_{\tau\xi} \overset{\{g\}}{\nabla}_{\mu} \bar{\phi}^{\xi}_{\beta} \phi^{\mu}_{[\lambda} \phi^{\beta}_{\sigma]} + \end{aligned}$$

$$+ g_{\lambda\xi} \overset{\{g\}}{\nabla}_{\mu} \bar{\phi}^{\xi}_{\beta} \phi^{\mu}_{[\sigma} \phi^{\beta}_{\tau]} + \phi^{\mu}_{\lambda} K_{\sigma\tau\mu} \Big) \gamma^{i} \gamma^{jk}$$

$$= U + V, \tag{6.78}$$

where $U$ is the sum of the first three terms and V is the last one. Let us first analyse the latter, which can be rewritten as

$$V = -\frac{1}{2} \tau_{\pm} \phi^{\mu}_{\mu} - \frac{1}{2} \tau_{\pm} \phi^{\mu}_{\lambda} e^{\lambda}_{a} e^{b}_{\mu} \gamma^{a}{}_{b}, \tag{6.79}$$

by making use of (6.77). From this last expression, we see that this term indeed behaves like $\tau_{\pm}$ under the reflection symmetry and does not spoil the parity symmetry of the graphene.
The remaining $U$ term also preserves the mentioned symmetry, as it contains three gamma matrices.
As we noticed previously, in the starting Dirac equation the interaction term is actually a constant mass term, as $\tau_{\pm}$ does not depend on $x$, if $\beta = 0$: this in not the case for the transformed equations, as both $U$ and $V$ depend on fields and ultimately on $x$. There are then two possibilities: the first one would be to relate these terms to defects of the graphene lattice, as it is known that torsion is indeed linked to dislocations. In this way, one would just choose a spin frame transformation, thus determining a new metric structure and this same choice would also manifest itself in the type of spacetime torsion interaction. The second possibility would instead be to impose conditions on the spin frame transformation, in order to make the $U$ and $V$ terms spacetime independent quantities: this would allow to interpret them again as mass terms.
We end now with some comments and discussions on perspectives and future applications of the obtained results.

## 6.2 Comments and discussion

The obtained model for graphene-like materials is a framework to investigate a huge range of possible applications: indeed along this construction, we have made several choices, which can be relaxed, if necessary. For example, right at the start, we decided to work in a vacuum configuration, where scalars and vectors are frozen to the value they possess at the boundary: the introduction of scalars could lead to complications, as one would need to consider the construction used for gauged Supergravities, but it could also help to describe new phenomenological effects.
More concretely, notice that have set $\beta = 0$ in any point of the manifold: there could be topological global obstructions to this condition. This idea is closely linked to the one of domain walls, as one could have configurations in which the quantity $\overline{\chi}_{\pm} \chi_{\pm}$ has different constant values in any single patch.
Furthermore, if the condition $\beta = 0$ is not valid for any point of spacetime, the mass terms in the Dirac equation actually become spacetime dependent and have to be interpreted as source terms. This theme also appears after we change the light cone structure of the Dirac equation and could lead to the description of impurities and defects in the 2-dimensional materials.

The first version of this graphene model has been achieved in [40], in a $p = 0$, $q = 2$ case: from a phenomenological point of view this setting is less appealing than the one chosen here $p = q = 2$, as one loses the parity symmetry between $\mathbf{K}$ and $\mathbf{K}'$ valleys. However, from a Supergravity point of view it allows to work in less restrictive frameworks, as one can also consider dynamical configurations, like asymptotically AdS spaces.
Going back to the key steps needed to obtain our model, we see that it is sufficient that the Maurer-Cartan equations hold at least at the boundary and this proved to be the case in the $\mathcal{N} = 2$ pure Supergravity studied in [46]: in that paper the authors studied how to regularise the action in presence of manifold with a boundary and found that the needed counterterm require the Maurer-Cartan equations to be asymptotically satisfied.
This framework allows to study the theory from a holographic point of view: the idea is to understand if it is possible to retrieve the graphene model from a formal AdS/CFT perspective. The first step in this direction will be performed in the next Section, where we will analyse if the regularised theory leads to a consistent boundary theory.
The obtained model should also be considered as a preliminary result in view of the more interesting $p = q = 2$ case, which would however require a regularised $\mathcal{N} = 4$ pure Supergravity action, in presence of a cosmological constant, which has not been achieved yet, due to complications related to the presence of scalars and spin 1/2 spinors in the gravitational multiplet.

# 7 Holographic analysis of pure $\mathcal{N} = 2$ AdS$_4$ Supergravity

In this Section, we will perform a holographic analysis [45] of the results obtained in [46]: this proves to be useful and interesting for two reasons. The first one regards the way the $\mathcal{N} = 2$ theory has been regularised: as we will see, the authors managed to add specific boundary terms with the aim of restoring the supersymmetry invariance of the action, broken by the presence of a boundary. In doing so, the obtained action has been rewritten in a MacDowell-Mansouri form [47].
The nature of the added counterterms and the obtained action have then been compared to the results achieved in [48], in a pure AdS gravity context, where the authors managed to regularise the divergences of the action, while rewriting the action again in a MacDowell-Mansouri form.
From this point of view, it is then natural to ask if the obtained Supergravity theory is finite and if it yields a consistent AdS/CFT duality, which requires to understand and study the boundary theory, its symmetries, the currents and their Ward identities.
The second reason for our interest is linked, as mentioned at the end of the previous Section, to the graphene model: studying the asymptotic limit of such a Supergravity theory would help understanding if implementing the Unconventional Supersymmetry would allow to holographically obtain the model for two-dimensional materials.
We first start with a brief introduction on the AdS/CFT correspondence and the topological renormalization method in the context of pure GR. We will then perform a preliminary analysis in absence of fermions, whose contributions will be introduced later on.

## 7.1 Essential notions on the gauge/gravity duality and holography

In its original formulation, the gauge/gravity duality was introduced by Maldacena in [49] as a relation between a $\mathcal{N} = 4$ super Yang-Mills theory and a type IIB Superstring theory on a $AdS_5 \times S^5$ background, where the chosen compact manifold is necessary to reach the critical dimension of String theory.
In the limit in which the classical effective low-energy Supergravity description of the gravity side can be trusted, the corresponding regime of the dual theory is strongly coupled, which would be impossible to study with perturbative methods.
The holographic correspondence has been extended to more general backgrounds of the form $\mathrm{AdS}_D \times \mathcal{M}_{\mathrm{int}}$, possibly with less supersymmetry, which can be embedded in other string theories or M-theory, such as the maximally supersymmetric $\mathrm{AdS}_4 \times \mathrm{S}^7$ and $\mathrm{AdS}_7 \times \mathrm{S}^4$ solutions of $D = 11$ Supergravity and variants thereof. A valuable approach to the study of holography on a background of the form $\mathrm{AdS}_D \times \mathcal{M}_{\mathrm{int}}$ is to restrict to an effective $D$-dimensional low-energy Supergravity originating from superstring/M-theory compactified on the internal manifold $\mathcal{M}_{\mathrm{int}}$. The specific choice of $\mathcal{M}_{\mathrm{int}}$ determines the amount of Supersymmetry preserved by the theory and the general features of the effective theory.
This discussion can be summed up by stating that the AdS/CFT conjecture is a holographic relation between the $\mathrm{AdS}_D$ (Super)gravity theory and a $d = (D-1)$-dimensional (super)conformal field theory at the boundary of the AdS geometry.
Most interestingly, the duality has been extended, on the gravity side, from global AdS to backgrounds which have an AAdS geometry, reproducing the renormalization group flow of the dual theory to an infrared (IR) conformal fixed point, the energy scale being fixed by the radial coordinate on the $D$-dimensional background. Indeed, the essential ingredient for this correspondence is the conformal structure of the boundary of AAdS spaces.

The relation between the gravitational theory and the CFT is implemented by identifying the partition functions of the two theories in the following way [1]

$$Z_{\mathrm{gravity}}[\Phi_{i(0)}] = Z_{\mathrm{CFT}}[\mathcal{J} = \Phi_{i(0)}], \tag{7.1}$$

where $\Phi_{i(0)}$ are the boundary values of the gravity fields $\Phi_i$, the former being identified with the external sources in the CFT side. The generic partition function of a $d$-dimensional CFT is given by

$$Z_{\mathrm{CFT}}[\mathcal{J}] = \mathrm{e}^{\mathrm{i}W[\mathcal{J}]} = \int \mathcal{D}\phi \, \mathrm{e}^{\mathrm{i}I[\phi] + \mathrm{i}\int_{\partial\mathcal{M}} d^dx \, O(\phi)\cdot\mathcal{J}} , \tag{7.2}$$

where $W[\mathcal{J}]$ is the quantum effective action, $\mathcal{J}$ are the sources for the operators $\mathcal{O}(\phi)$ and $I[\phi]$ is an already renormalised action, written in terms of some fields $\phi$.
We therefore see that the boundary conditions on the gravity theory are mapped to the sources on the CFT side. Furthermore, one also has to relate the symmetries of the two theories: local symmetries on the gravity side correspond to global ones on the quantum field theory one.
The exact equivalence between the two partition functions means that, at least in principle, if the gravity partition function is known, one can perform computations on the other side of

the duality: for example, n-point functions for the conformal field theory can be computed as

$$\langle O(x_1)\cdots O(x_n)\rangle_{\rm CFT} = Z^{-1}_{\rm CFT}[0]\, \frac{\delta^n Z_{\rm CFT}[\mathcal{J}]}{\mathrm{i}\delta\mathcal{J}(x_1)\cdots \mathrm{i}\delta\mathcal{J}(x_n)}\bigg|_{\mathcal{J}=0}\,. \tag{7.3}$$

However, performing computations involving the String theory path integral can lead to complications, which can be avoided at low energies, where such theory can be well approximated by General Relativity and Supergravity: in this case, the path integral picks up the on-shell contribution through the saddle point approximation. The defining relation (7.1) can then be rewritten as

$$W[\mathcal{J}\equiv\Phi_{i(0)}] \simeq I_{\rm on-shell}[\Phi_{i(0)}]\,. \tag{7.4}$$

In this thesis we will work from a gravity perspective: since the CFT action in (7.2) has been properly renormalised from a UV point of view, divergences appearing in the action on gravity side, due to the presence of a boundary, have to be consistently removed. This process is called Holographic renormalization [50] and consists of adding suitable exact terms to the starting action to take care of its divergent IR behaviour.

Let us now focus on the specific case of $D=4$ AdS gravity: the local counterterms needed to regularise the IR behaviour of the gravity action have been initially found in [51], but a better understanding of their origin has been achieved in [48], where the authors proved that the regularisation of the $D=4$ AdS gravity action was due to the addition of topological invariant counterterms.
The authors indeed showed that adding the Gauss-Bonnet term to the usual Hilbert-Einsten action[5]

$$\begin{aligned} I &= I_{HE}+I_{GB} = \\ &= -\frac{1}{16\pi G}\int_M d^4x\,\sqrt{\hat g}\left((\hat{\mathcal{R}}(\hat g)+2\Lambda)-\frac{\ell^2}{4}\left(\hat{\mathcal{R}}_{\hat\mu\hat\nu\hat\alpha\hat\beta}(\hat g)\hat{\mathcal{R}}^{\hat\mu\hat\nu\hat\alpha\hat\beta}(\hat g)-4\hat{\mathcal{R}}_{\hat\mu\hat\nu}(\hat g)\hat{\mathcal{R}}^{\hat\mu\hat\nu}(\hat g)+\hat{\mathcal{R}}^2(\hat g)\right)\right), \end{aligned} \tag{7.5}$$

where $\Lambda$ is the cosmological constant and $\hat g_{\mu\nu}$ is the bulk metric, allows to retrieve the known results. The full action can be rewritten in a MacDowell-Mansouri form [47] as

$$I = \frac{\ell^2}{64\pi G}\int_M d^4x\,\sqrt{\hat g}\,W_{\hat\mu\hat\nu\hat\alpha\hat\beta}W^{\hat\mu\hat\nu\hat\alpha\hat\beta}, \tag{7.6}$$

where $W^{\hat\alpha\hat\beta}_{\hat\mu\hat\nu} = \hat{\mathcal{R}}^{\hat\alpha\hat\beta}_{\hat\mu\hat\nu}(\hat g) - \frac{1}{\ell^2}\delta^{[\hat\alpha\hat\beta]}_{[\hat\mu\hat\nu]}$ is the Weyl tensor, the curvature of the AdS group. Furthermore, the added boundary term can be rewritten as

$$I_{GB} = 32\pi^2\chi(M) + \int_{\partial M} d^3x\, B_3, \tag{7.7}$$

[5] In this Section, hatted objects will refer to bulk quantities, whereas unhatted ones will indicate boundary contributions.

where $\chi(M)$ is the Euler characteristic and $B_3$ is called Second Chern Form.
By making use of the Gaussian normal (radial) coordinates, described in Appendix C, the latter contribution can be expressed as

$$B_3 = -4\sqrt{h}\, \delta^{[\lambda\mu\nu]}_{[\rho\sigma\tau]} K^\lambda_\rho \left( \frac{1}{2} \mathcal{R}^{\mu\nu}_{\sigma\tau}(h) + \frac{1}{3} K^\mu_\sigma K^\nu_\tau \right) , \tag{7.8}$$

in terms of the extrinsic curvature $K_{\mu\nu} = \frac{1}{2N} \partial_r h_{\mu\nu}$.
Finally, to make contact with the standard analysis of [51], by adding and subtracting the Gibbons-Hawking-York term, one rewrites the starting action as

$$I = I_{HE} + \frac{1}{8\pi G} \int_{\partial M} d^3x \, \sqrt{h} K + \int_{\partial M} d^3x \, L_{\text{ct}}, \tag{7.9}$$

where the Euler characteristic term, being a constant, has been omitted. From an asymptotical point of view, $L_{\text{ct}}$ contains a single non vanishing contribution, which can be written as

$$L_{\text{ct}} = \frac{1}{8\pi G} \sqrt{h} \left( \frac{2}{\ell} - \frac{\ell}{2} \mathcal{R}(h) \right) . \tag{7.10}$$

This last expression exactly coincides with the one appearing in [51] for the $\text{AdS}_4$ case and correctly cancels the divergences of the bulk action.
We then see that by adding a single topological invariant term, we end up with a regularised gravity theory, in which the Dirichlet problem is also well-defined.

Inspired by this analysis, the authors of [46] managed to provide the $\mathcal{N} = 2$ pure $\text{AdS}_4$ Supergravity action in presence of a boundary with local counterterms both preserving Supersymmetry and allowing the full action to be written in a MacDowell-Mansouri form. In this thesis, we will focus on proving that the derived boundary theory is consistent and we will be interested in matching the local symmetries of the gravity theory with the global ones of the CFT, which in our approach match the residual asymptotic symmetries at radial infinity. At last, we will check that the obtained boundary theory reproduces the correct Ward identities on the QFT side, which will be obtained by taking the variation of (7.4), namely $\delta W[\mathcal{J} \equiv \Phi_{i(0)}] \simeq \delta I_{\text{on-shell}}[\Phi_{i(0)}]$.

## 7.2 Asymptotic symmetries in Einstein $\text{AdS}_4$ gravity

The analysis of asymptotic AdS spaces in $D = d+1$ dimensions is usually performed in terms of local coordinates $x^{\hat{\mu}} = (x^\mu, x^d)$, where $x^\mu$ ($\mu = 0, \dots d-1$) describe the boundary and $z = x^d$ is the radial coordinate, which reaches the AdS boundary at $z = 0$.
One can choose a proper patch near $z = 0$ in such a way that the bulk metric can be written in the Fefferman-Graham (FG) form

$$ds^2 = \hat{g}_{\hat{\mu}\hat{\nu}} \, \mathrm{d}x^{\hat{\mu}} \mathrm{d}x^{\hat{\nu}} = \frac{\ell^2}{z^2} \left( - \mathrm{d}z^2 + g_{\mu\nu}(x, z) \, \mathrm{d}x^\mu \mathrm{d}x^\nu \right) , \tag{7.11}$$

where $g_{\mu\nu}$ is a regular metric on the boundary admitting the following power-expansion in the radial coordinate $z$

$$g_{\mu\nu} = g_{(0)\mu\nu}(x) + \frac{z^2}{\ell^2}\, g_{(2)\mu\nu}(x) + \cdots . \tag{7.12}$$

Subleading components of the metric appear only in even powers, up until the order $z^{d-1}$ and their precise expression in terms of $g_{(0)\mu\nu}$ can be obtained by solving order by order the Einstein equations. As an example, $g_{(2)\mu\nu}$ is proportional to the Schouten tensor

$$g_{(2)\mu\nu} = \ell^2 \mathcal{S}_{\mu\nu} = \ell^2 \left( \mathring{\mathcal{R}}_{\mu\nu} - \frac{1}{2(d-1)}\, g_{(0)\mu\nu}\, \mathring{\mathcal{R}} \right) , \tag{7.13}$$

where $\mathring{\mathcal{R}}^{\mu}{}_{\nu\lambda\sigma}(g_{(0)})$ is the boundary Riemann curvature and $\mathring{\mathcal{R}}_{\mu\nu}$ and $\mathring{\mathcal{R}}$ are the corresponding Ricci tensor and Ricci scalar, respectively.
Moreover, when $D$ is odd, a logarithmic term $z^d \log z$ appears in the asymptotic expansion of the metric and the mode $g_{(d)\mu\nu}$ cannot be resolved from the equations of motion, as it is proportional to the holographic stress tensor of the theory [50, 52]. The FG form of the metric is obtained by suitably gauge-fixing the spacetime coordinate frame and it is preserved by a specific set of transformations [53], which include the Penrose-Brown-Henneaux (PBH) transformations [54, 55] and the asymptotic symmetries, whose parameters take value on the boundary. More concretely,

$$\begin{aligned} \delta\hat{g}_{zz} &= 0 \quad \Rightarrow \quad \hat{\xi}^z = z\sigma(x)\,, \\ \delta\hat{g}_{\mu z} &= 0 \quad \Rightarrow \quad \hat{\xi}^\mu = \xi^\mu(x) + \frac{z^2}{2\ell}\, g^{\mu\nu}_{(0)}\partial_\nu\sigma + \mathcal{O}(z^4)\,, \end{aligned} \tag{7.14}$$

where $\xi^\mu(x)$ and $\sigma(x)$ are arbitrary local parameters on the boundary. These transformations reduce instead to Weyl transformations when acting on the leading order of the boundary metric:

$$\delta g_{(0)\mu\nu} = \pounds_\xi g_{(0)\mu\nu} - 2\sigma\, g_{(0)\mu\nu}\,, \tag{7.15}$$

where the first term is due to boundary diffeomorphisms, whereas the second one is generated by radial ones. The asymptotic symmetries are of particular interest for our analysis, as they produce conservation laws which are then mapped into the holographic Ward identities of the boundary CFT.

### Holographic gauge-fixing in first order formalism

When treating the $D=4$ gravity theory in first order formalism, besides the known general coordinate transformations, which define local translations, generated by the parameters $p^a = \hat{\xi}^{\hat{\mu}} V^a_{\hat{\mu}}$, one introduces a local Lorentz invariance, described by the $j^{ab} = -j^{ba}$ parameters. The AdS gravity in first order formalism is invariant under the general transformations

$$\begin{aligned} \delta V^a &= \hat{\mathcal{D}}p^a - j^{ab}V_b + i_p\hat{T}^a\,, \\ \delta\hat{\omega}^{ab} &= \hat{\mathcal{D}}j^{ab} + \frac{2}{\ell^2}\, p^{[a}V^{b]} + i_p\hat{R}^{ab}, \end{aligned} \tag{7.16}$$

where $\hat{\mathcal{D}}(\hat{\omega})$ is the Lorentz-covariant derivative and $\hat{T}^a = \hat{\mathcal{D}}V^a$ is the torsion 2-form. We also introduced the AdS curvature $\hat{R}^{ab} = \hat{\mathcal{R}}^{ab}(\hat{\omega}) - \frac{1}{\ell^2}V^aV^b = \frac{1}{2}\hat{R}^{ab}_{\hat{\mu}\hat{\nu}}\mathrm{d}x^{\hat{\mu}}\mathrm{d}x^{\hat{\nu}}$, written here in terms of the Lorentz curvature $\hat{\mathcal{R}}^{ab}(\hat{\omega})$, whose contraction is given by $i_p\hat{R}^{ab} = p^cV^{\hat{\nu}}_c\hat{R}^{ab}_{\hat{\nu}\hat{\mu}}\mathrm{d}x^{\hat{\mu}}$. When fermions are absent, one can always choose a torsionless connection, which then implies that $i_p\hat{T}^a = 0$. Contractions of curvatures are in general present when one studies spacetimes which are not exactly globally AdS space.

First order formalism requires to fix some of the degrees of freedom of the vielbein and of the spin connection: in order to do so, we have 10 local parameters $(p^a, j^{ab})$ at our disposal to choose. This holographic gauge-fixing has to be performed on the radial components of the fields, as the radial evolution of gravity considers them as Lagrange multipliers, similarly as the lapse and shift functions in the Arnowitt-Deser-Misner (ADM) formulation of gravity [56]. This choice must provide the radial expansion of both fields and parameters and must allow for residual transformations inducing boundary Weyl dilatations. Furthermore, they have to induce transformation laws of the boundary fields leading to conservations laws.
The immediate choice $V^a_z = 0$, $\hat{\omega}^{ab}_z = 0$ produces inconsistencies, as the vielbein becomes non-invertible. A suitable gauge fixing for spacetime diffeomorphisms $p^a$ and Lorentz transformations $j^{ab}$ is

$$V^a_z = \frac{\ell}{z}\,\delta^a_3, \qquad \hat{\omega}^{ab}_z = 0\,. \tag{7.17}$$

These conditions are in principle sufficient to determine local symmetries: however, in order to reproduce the FG form of the metric (7.11), we choose an adapted frame, where the boundary is orthogonal to the radial coordinate, meaning that

$$V^3_\mu = 0\,. \tag{7.18}$$

With these choices, the vielbein of AAdS spaces becomes

$$V^i_{\ \mu} = \frac{\ell}{z}\,\hat{E}^i_{\ \mu}(x,z)\,, \tag{7.19}$$

where $\hat{E}^i_{\ \mu}$ is finite at the boundary $z = 0$. This means that is can be expanded in a power series as

$$\begin{aligned}\hat{E}^i_{\ \mu} &= E^i_{\ \mu} + \frac{z^2}{\ell^2}\,S^i_{\ \mu} + \frac{z^3}{\ell^3}\,\tau^i_{\ \mu} + \mathcal{O}(z^4)\,,\\ \hat{E}^\mu_{\ i} &= E^\mu_{\ i} - \frac{z^2}{\ell^2}\,S^\mu_{\ i} - \frac{z^3}{\ell^3}\,\tau_i^{\ \mu} + \mathcal{O}(z^4)\,,\end{aligned} \tag{7.20}$$

where we defined $E^\mu_{\ i}$ as the inverse vielbein.
Strictly speaking, the inverse vielbein $(E^{-1})^\mu_{\ i} \equiv E^\mu_{\ i}$ has the property $E^\mu{}_i = g^{\mu\nu}_{(0)}\eta_{ij}E^j{}_\nu = E_i{}^\mu$ following from the invertibility and symmetry of the metric. It implies that one can overlook the order of the indices in the vielbein and its inverse. Even if in principle the same argument holds for the bulk vielbein $V^i_{\ \mu}$ and its inverse $V^\mu_{\ i}$, this is not true for the higher-order terms in the expansion that are not necessarily invertible. Therefore, we will be able to overlook the order of indices for the boundary vielbein only.

These two tensors, $\hat{E}^i{}_\mu$ and $E^\mu{}_i$, project the indices between the boundary spacetime and its tangent space and satisfy

$$e = \det[V^a_{\hat{\mu}}] = \frac{\ell^4}{z^4}\,\hat{e}_3\,, \quad \hat{e}_3 = \det[\hat{E}^i{}_\mu]\,, \quad e_3 \equiv \det[E^i_\mu]\,. \tag{7.21}$$

Let us notice that, as a consequence of the expansion (7.12), we see that $g_{(1)\mu\nu} = 0$, which implies that linear terms in $z$ are absent in the expansion of $\hat{E}^i{}_\mu$. Furthermore, both $S^{ij} = S^i{}_\mu E^{\mu j}$ and $\tau^{ij} = \tau^i{}_\mu E^{\mu j}$ can be made symmetric by making use of the residual Lorentz transformations generated by the parameter

$$j^{ij} = \theta^{ij} + \frac{z}{\ell}\,j^{ij}_{(1)} + \frac{z^2}{\ell^2}\,j^{ij}_{(2)} + \frac{z^3}{\ell^3}\,j^{ij}_{(3)} + \mathcal{O}(z^4)\,. \tag{7.22}$$

The absence of linear terms in $\hat{E}^i{}_\mu$ implies that $j^{ij}_{(1)} = 0$, whereas from

$$\delta_j S^i{}_\mu = -\theta^{ij} S_{j\mu} - j^{ij}_{(2)} E_{j\mu}\,, \qquad \delta_j \tau^i{}_\mu = -\theta^{ij}\tau_{j\mu} - j^{ij}_{(3)} E_{j\mu}\,. \tag{7.23}$$

we see that the antisymmetric part is actually independent of $\theta^{ij}(x)$. By then setting $j^{ij}_{(2)} = j^{ij}_{(3)} = 0$, we can consistently choose

$$S^{[ij]} = 0\,, \qquad \tau^{[ij]} = 0\,. \tag{7.24}$$

This result actually extends to all coefficients in the expansion of $V^i{}_\mu$, which can be taken to be symmetric $E^{[ij]}_{(n)} \equiv E^{\mu[j}E^{i]}_{(n)\mu} = 0$ provided that $j^{ij}_{(n+1)} = 0$ and $n \geq 1$. In the end, one is left with a Lorentz parameter $j^{ij} = \theta^{ij}(x)$, which has a single term in the asymptotic expansion, representing the asymptotic parameter corresponding a holographic symmetry, as we will see.
These results and choices are consistent with the FG frame and with the known coefficients of the metric

$$\begin{aligned} g_{(0)\mu\nu} &= E_{i\nu}E^i{}_\mu\,, \\ g_{(2)\mu\nu} &= 2S_{\mu\nu} = \ell^2\mathcal{S}_{\mu\nu}\,, \\ g_{(3)\mu\nu} &= 2\tau_{\mu\nu}\,. \end{aligned} \tag{7.25}$$

provided that we identify $E^i_\mu$ as the vielbein at the conformal boundary, $S^i{}_\mu = \frac{\ell^2}{2}\,\mathcal{S}^i{}_\mu$ as proportional to the Schouten tensor and $\tau^i{}_\mu$ as the holographic stress tensor.
Moreover, once the expression of the vielbein is known, in absence of Supersymmetry, the expansion of the spin connection is completely determined from $\hat{\mathcal{D}}V^a = 0$. Indeed from

$$\hat{\omega}^{ab}_{\hat{\mu}} = V^{\hat{\nu}b}\left(-\partial_{\hat{\mu}}V^a_{\ \hat{\nu}} + \hat{\Gamma}^{\hat{\lambda}}_{\hat{\nu}\hat{\mu}}V^a_{\ \hat{\lambda}}\right)\,, \tag{7.26}$$

where $\hat{\Gamma}^{\hat{\lambda}}_{\hat{\nu}\hat{\mu}}$ is the affine Levi-Civita connection in the bulk, one can prove that $\hat{\omega}^{ab}_z = 0$, as discussed previously and

$$\hat{\omega}^{ij}_\mu \;=\; \hat{E}^{\nu j}\left(-\partial_\mu\hat{E}^i{}_\nu + \Gamma^\lambda_{\nu\mu}\hat{E}^i{}_\lambda\right) = \mathring{\omega}^{ij}_\mu(x,z)\,,$$

$$\hat{\omega}^{i3}_\mu \;=\; \frac{1}{z}\,\hat{E}^i{}_\mu - \frac{1}{2}\,k_{\mu\nu}\hat{E}^{\nu i}\,. \tag{7.27}$$

Here $\mathring{\omega}^{ij}_\mu(x,0) = \mathring{\omega}^{ij}_\mu(E)$ is the torsionless spin connection on the boundary, $\Gamma^\lambda_{\nu\mu} = \Gamma^\lambda_{\nu\mu}(g)$ is the Levi-Civita connection at the boundary and

$$k_{\mu\nu} \equiv \partial_z g_{\mu\nu} = \mathcal{O}(z)\,, \qquad \partial_z g^{\mu\nu} = -k^{\mu\nu}\,. \tag{7.28}$$

More explicitly, by expanding $\hat{E}^i{}_\mu$

$$\begin{aligned}
\hat{\omega}^{ij}_\mu \;&=\; \mathring{\omega}^{ij}_\mu(x,z) = \mathring{\omega}^{ij}_\mu(x) + \frac{z^2}{\ell^2}\,\omega^{ij}_{(2)\mu}(S,E) + \frac{z^3}{\ell^3}\,\omega^{ij}_{(3)\mu}(\tau,E) + \mathcal{O}(z^4)\,,\\
\hat{\omega}^{i3}_\mu \;&=\; \frac{1}{z}\,E^i{}_\mu - \frac{z}{\ell^2}\,\tilde{S}^i{}_\mu - \frac{2z^2}{\ell^2}\,\tilde{\tau}^i{}_\mu + \mathcal{O}(z^3)\,,
\end{aligned} \tag{7.29}$$

where

$$\tilde{S}^i{}_\mu \equiv S_\mu{}^i = S^i{}_\mu\,, \qquad \tilde{\tau}^i{}_\mu \equiv \frac{1}{4}\,\left(\tau^i{}_\mu + 3\tau_\mu{}^i\right) = \tau^i{}_\mu\,. \tag{7.30}$$

As an important remark, we see that in pure AdS gravity, the tensors $\tilde{S}^i{}_\mu$ and $\tilde{\tau}^i{}_\mu$ can be chosen symmetric and equal to $S^i{}_\mu$ and $\tau^i{}_\mu$: the same will not be true in presence of fermions, where we will see that the group theoretic definition of the boundary Schouten tensor is actually $\mathcal{S}^i{}_\mu = \frac{1}{\ell^2}\,(S^i{}_\mu + \tilde{S}^i{}_\mu)$, which correctly reduces to $\frac{2}{\ell^2}\,S^i{}_\mu$ when (7.30) holds. To add further details on the spin connection asymptotic expansion, from the vanishing of the torsion one obtains

$$E_j \wedge \omega^{ij}_{(2)} = \mathring{\mathcal{D}}S^i\,, \qquad E_j \wedge \omega^{ij}_{(3)} = \mathring{\mathcal{D}}\tau^i\,, \tag{7.31}$$

which can both be solved in terms of $\omega^{ij}_{(2)}$, $\omega^{ij}_{(3)}$, where $\mathring{\mathcal{D}}$ denotes the covariant derivative with respect to the connection $\mathring{\omega}^{ij}_\mu(E)$.

To be ready for understanding the transformation laws (7.16), one also needs the asymptotic expansion of the AdS curvature

$$\begin{aligned}
\hat{R}^{i3}_{\mu\nu} &= -z\,\mathcal{C}^i{}_{\mu\nu} + \mathcal{O}(z^2)\,, & \hat{R}^{i3}_{\mu z} &= \frac{3z}{\ell^3}\,\tau^i{}_\mu + \mathcal{O}(z^2)\,,\\
\hat{R}^{ij}_{\mu\nu} &= W^{ij}_{\mu\nu} - \frac{12z}{\ell^3}\,E^{[i}_{\ [\mu}\tau^{j]}_{\ \nu]} + \mathcal{O}(z^2)\,, & \hat{R}^{ij}_{\mu z} &= -\frac{2z}{\ell^2}\,\omega^{ij}_{(2)\mu} - \frac{3z^2}{\ell^3}\,\omega^{ij}_{(3)\mu} + \mathcal{O}(z^3)\,,
\end{aligned} \tag{7.32}$$

where $\mathcal{C}^i = \frac{1}{2}\,\mathcal{C}^i{}_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \mathring{\mathcal{D}}\mathcal{S}^i$ is three-dimensional Cotton tensor and

$$W^{ij} = \mathring{\mathcal{R}}^{ij} - 2E^{[i} \wedge \mathring{\mathcal{S}}^{j]} = 0\,. \tag{7.33}$$

is Weyl tensor, which vanishes in three dimensions. For this reason, the three dimensional Bianchi identity yields

$$E^{[i} \wedge \mathcal{C}^{j]} = 0\,, \tag{7.34}$$

giving that the Cotton tensor is traceless, $\mathcal{C}^i{}_{ij} = 0$.
Another consequence of the vanishing of the Weyl tensor in three dimensions is that we get $\hat{R}^{ab}\Big|_{z=0} = 0$: the fact that both tangent and radial components of the curvature vanish in the pure GR case will have to be relaxed in the Supergravity case, as we shall see in the next Subsection.

### Residual symmetries

The chosen gauge-fixing must be preserved by the transformations of the fields (7.16): one then has to impose that the variation of such gauge fixing remains constant and this constrains the form of the parameters. In the GR case, we have

$$
\begin{aligned}
0 &= \delta V^3_{\ z} = \partial_z p^3 \,, && (7.35)\\
0 &= \delta V^i_{\ z} = \partial_z p^i + \frac{\ell}{z}\, j^{i3} \,, && (7.36)\\
0 &= \delta V^3_{\ \mu} = \partial_\mu p^3 - \hat{\omega}^{i3}_\mu p_i + j^{i3} V_{i\mu} \,, && (7.37)\\
0 &= \delta \hat{\omega}^{i3}_z = \frac{1}{\ell z}\, p^i + \partial_z j^{i3} + i_p \hat{R}^{i3}_z \,, && (7.38)\\
0 &= \delta \hat{\omega}^{ij}_z = \partial_z j^{ij} + i_p \hat{R}^{ij}_z \,, && (7.39)
\end{aligned}
$$

where the contractions of the AdS curvature is given by

$$
\begin{aligned}
i_p \hat{R}^{i3}_z &= p^j \left( \frac{3z^2}{\ell^4}\, \tau^i_{\ j} + \mathcal{O}(z^3) \right) \,, \\
i_p \hat{R}^{i3}_\mu &= -p^3 \left( \frac{3z^2}{\ell^4}\, \tau^i_{\ \mu} + \mathcal{O}(z^3) \right) + p^j \left( \frac{z^2}{\ell}\, E^\nu_{\ j} \mathcal{C}^i_{\ \mu\nu} + \mathcal{O}(z^3) \right) \,, \\
i_p \hat{R}^{ij}_z &= p^k \left( -\frac{2z^2}{\ell^3}\, E^\mu_{\ k} \omega^{ij}_{(2)\mu} - \frac{3z^3}{\ell^4}\, E^\mu_{\ k} \omega^{ij}_{(3)\mu} + \mathcal{O}(z^4) \right) \,. \qquad (7.40)
\end{aligned}
$$

By plugging in these expressions, we get

$$
\begin{aligned}
0 &= \partial_z p^3 \,, && (7.41)\\
0 &= \partial_z j^{i3} + \frac{1}{\ell z}\, p^i + \frac{3z^2}{\ell^4}\, p^j \left( \tau^i_{\ j} + \mathcal{O}(z) \right) \,, && (7.42)\\
0 &= \partial_z p^i + \frac{\ell}{z}\, j^{i3} \,, && (7.43)\\
0 &= \partial_\mu p^3 - \hat{\omega}^{i3}_\mu p_i + j^{i3} V_{i\mu} \,, && (7.44)\\
0 &= \partial_z j^{ij} + p^k \left( -\frac{2z^2}{\ell^3}\, E^\mu_{\ k} \omega^{ij}_{(2)\mu} - \frac{3z^3}{\ell^4}\, E^\mu_{\ k} \omega^{ij}_{(3)\mu} + \mathcal{O}(z^4) \right) \,, && (7.45)
\end{aligned}
$$

which are equations for the parameters of the theory. In particular, (7.41) can be solved as

$$
p^3 = -\ell \sigma(x) \,, \qquad (7.46)
$$

with the boundary parameter $\sigma(x)$ introduced as an integration constant, whereas the next two ones, (7.42) and (7.43), can be decoupled by eliminating $j^{i3}$ and finding the differential equation in $p^i$, whose solution reads

$$
\begin{aligned}
p^i &= \frac{\ell}{z}\, \xi^i + \frac{z}{\ell}\, b^i + \frac{z^2}{\ell^2}\, \xi^j \tau^i_{\ j} + \mathcal{O}(z^3) \,, \\
j^{i3} &= \frac{1}{z}\, \xi^i - \frac{z}{\ell^2}\, b^i - \frac{2z^2}{\ell^3}\, \xi^j \tau^i_{\ j} + \mathcal{O}(z^3) \,. \qquad (7.47)
\end{aligned}
$$

The parameters $\xi^i(x)$ and $b^i(x)$ are new integration constants. At last, (7.45) can in principle be solved by

$$j^{ij} = \theta^{ij} + \frac{z^2}{\ell^2}\,\xi^\mu \omega^{ij}_{(2)\mu} + \frac{z^3}{\ell^3}\,\xi^\mu \omega^{ij}_{(3)\mu} + \mathcal{O}(z^4)\,. \tag{7.48}$$

However, we already have an expression for such parameter, $j^{ij} = \theta^{ij}$, meaning that one has to accordingly set

$$\omega^{ij}_{(2)\mu} = 0\,, \qquad \omega^{ij}_{(3)\mu} = 0\,. \tag{7.49}$$

Finally, one also has to make sure that the condition (7.44) remains satisfied after a transformation: this means that

$$0 = \delta V^3_{\ \mu} = -\ell\partial_\mu\sigma + \frac{2}{\ell}\,\xi_i S^i_{\ \mu} - \frac{2}{\ell}\,b^i E_{i\mu} + \mathcal{O}(z^2)\,, \tag{7.50}$$

has to be solved: the condition at leading order can be satisfied provided that the parameter $b^i$ is not independent, i.e.

$$b_i = -\frac{\ell^2}{2}\,E^\mu_{\ i}\partial_\mu\sigma + S^j_{\ i}\xi_j\,. \tag{7.51}$$

To sum up the analysis performed up until now, the expansion of the parameters is given by

$$\begin{aligned}
p^3 &= -\ell\sigma(x)\,,\\
p^i &= \frac{\ell}{z}\,\xi^i(x) + \frac{z}{\ell}\,b^i + \frac{z^2}{\ell^2}\,\xi^j\tau^i_{\ j} + \mathcal{O}(z^3)\,,\\
j^{i3} &= \frac{1}{z}\,\xi^i(x) - \frac{z}{\ell^2}\,b^i - \frac{2z^2}{\ell^3}\,\xi^j\tau^i_{\ j} + \mathcal{O}(z^3)\,,\\
j^{ij} &= \theta^{ij}(x)\,,
\end{aligned} \tag{7.52}$$

where the independent boundary parameters $\sigma(x)$, $\xi^i(x)$, $\theta^{ij}(x)$ are associated to dilatations, diffeomorphisms and Lorentz transformations, respectively. We can here state the obtained variation of the relevant fields appearing in the asymptotic expansion of vielbein and spin connection:

$$\begin{aligned}
\delta E^i_{\ \mu} &= \mathring{\mathcal{D}}_\mu\xi^i + \sigma E^i_{\ \mu} - \theta^{ij}E_{j\mu}\,,\\
\delta S^i_{\ \mu} &= \mathring{\mathcal{D}}_\mu b^i - \sigma S^i_{\ \mu} - \theta^{ij}S_{j\mu}\,,\\
\delta \tau^i_{\ \mu} &= \mathring{\mathcal{D}}_\mu\left(\xi^j\tau^i_{\ j}\right) - 2\sigma\tau^i_{\ \mu} - \theta^{ij}\tau_{j\mu}\\
\delta\mathring{\omega}^{ij}_\mu &= \mathring{\mathcal{D}}_\mu\theta^{ij} - 2E^{\nu[i}E^{j]}_{\ \ \mu}\partial_\nu\sigma + \frac{4}{\ell^2}\left(-\xi_k E^{[i}_{\ \mu}S^{j]k} + \xi^{[i}S^{j]}_{\ \ \mu}\right)\,.
\end{aligned} \tag{7.53}$$

To check that the obtained residual symmetries match the usual PBH transformations (7.15) in metric formalism, one can try and understand if one obtains the correct transformation rule for the coefficient of the metric $g_{(d)\mu\nu}$. As an example, it is known that the third coefficient, namely $g_{(3)\mu\nu}$, being proportional to the holographic stress tensor, transforms homogeneously and this is exactly what one gets using (7.53),

$$\delta g_{(3)\mu\nu} = \pounds_\xi g_{(3)\mu\nu} - \sigma g_{(3)\mu\nu}\,, \tag{7.54}$$

where one has to use $\mathring{\mathcal{D}}_{[\mu}E^i_{\ \nu]} = 0$ and $\mathring{\mathcal{D}}_{[\mu}\tau^i_{\ \nu]} = 0$. Here the last identity is the consequence of (7.31) and (7.49).

### Conservation law for conformal symmetry

In the AdS/CFT framework, the leading order fields $E^i_\mu$, $\mathring{\omega}^{ij}_\mu$ appearing in Riemann-Cartan AdS gravity remain arbitrary functions on the three-dimensional boundary and act as sources in the dual field theory.
The quantum effective action $W$ in first formalism is then given by

$$W[E,\omega] = -\mathrm{i} \ln Z[E,\omega]\,, \tag{7.55}$$

where the (external) gravitational sources $E^i_\mu$ and $\omega^{ij}_\mu$ are coupled to the energy-momentum tensor $J^\mu_{\ i}$ and the spin current $J^\mu_{\ ij}$. The variation of $W$ yields then

$$\delta W = \int \left( \delta E^i \wedge J_i + \frac{1}{2}\, \delta\omega^{ij} \wedge J_{ij} \right) \,, \tag{7.56}$$

where the 2-form currents $J = \frac{1}{2}\, J_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ are the Hodge dual of the usual Noether currents 1-forms ${}^*J = J_\mu\, \mathrm{d}x^\mu$

$$J^\mu = \frac{1}{2e_3}\, \epsilon^{\mu\nu\lambda} J_{\nu\lambda}\,. \tag{7.57}$$

Since the spin connection is not an independent source, we must have both in General Relativity and Supergravity that $J_{ij} = 0$. Plugging this result in (7.56) and using the expression of the variation of the vielbein, one obtains the following expression

$$0 = \delta W = \int \left[ -\xi^i \mathring{\mathcal{D}} J_i + \left( \sigma E^i - \theta^{ij} E_j \right) \wedge J_i \right] \,, \tag{7.58}$$

from which we can read the *classical conservation laws* of conformal symmetry in $d = 3$

$$\begin{aligned}
\xi^i :& \quad 0 = \mathring{\mathcal{D}} J_i\,, && \text{(conserved } J_{\mu\nu}) \\
\sigma :& \quad 0 = E^i \wedge J_i\,, && \text{(traceless } J_{\mu\nu}) \\
\theta^{ij} :& \quad 0 = E_i \wedge J_j - E_j \wedge J_i\,. && \text{(symmetric } J_{\mu\nu})
\end{aligned} \tag{7.59}$$

Note that we have the full Weyl symmetry on the boundary expressed in terms of the Belinfante-Rosenfeld tensor $J^\mu_{\ i}$, which can be written as $J_{\mu\nu} = -(3/\ell)\, \tau_{\mu\nu}$ by using the equations of motion. Notice that its tracelessness is not modified at the quantum level, because there is no conformal anomaly in three dimensions.

At last, let us comment on the transformation laws of the Schouten tensor: in AdS gravity, this expression can be computed both from the transformation rules of the vielbein and spin connection and the results must coincide. The first expression is given in (7.16), whereas the second one reads $\delta S^i = \mathring{\mathcal{D}} b^i - \sigma S^i - \theta^{ij} S_j + \frac{\ell^2}{2}\, i_\xi \mathcal{C}^i$: they differ for a term proportional to the contraction of the Cotton tensor, which appears rom the contraction of the curvature in (7.53).
However, we have chosen a particular gauge fixing, which makes $S_{ij}$ symmetric and, at the same time, implies $\omega^{ij}_{(2)\mu} = 0$: from (7.31) we then see that this requirement implies that the

Cotton tensor vanishes $\mathcal{C}^i{}_{\mu\nu} = 0$.
Our gauge fixing in fact restricts the asymptotic behaviour of spacetime to the one with conformally flat asymptotic boundaries: in the Supergravity case, we will relax this conformally flat boundary condition, because including fermions will naturally imply that $\tilde{S}_{ij} \neq S_{ij}$.

We could now compare the obtained conservation laws to the ones one would get by treating $\omega^{ij}$ and $S^i$ as independent fields and by taking $V^3_\mu \sim B_\mu \neq 0$. One would have to add in the variation of the action (7.56) the special conformal current $J_{(K)i}$ and the dilatation current $J_{(D)}$ via the respective couplings $\delta S^i \wedge J_{(K)i}$ and $\delta B \wedge J_{(D)}$ and $b^i$ would have to be treated as an independent parameter. They would read

$$\begin{aligned}
\xi^i \quad &: \quad \mathcal{D}J_i = B \wedge J_i + \frac{2}{\ell^2} S^j \wedge J_{ij} + \frac{2}{\ell} S_i \wedge J_{(D)} \,, \\
\sigma \quad &: \quad \ell dJ_{(D)} = -E^i \wedge J_i + S^i \wedge J_{(K)i} \,, \\
\theta^{ij} \quad &: \quad \mathcal{D}J_{ij} = 2E_{[i} \wedge J_{j]} + 2S_{[i} \wedge J_{(K)j]} \,, \\
b^i \quad &: \quad \mathcal{D}J_{(K)i} = \frac{2}{\ell^2} E^j \wedge J_{ij} - \frac{2}{\ell} E_i \wedge J_{(D)} - B \wedge J_{(K)i} \,,
\end{aligned} \tag{7.60}$$

where $\mathcal{D}$ is the covariant derivative with respect to the Lorentz connection $\omega^{ij} = \mathring{\omega}^{ij} - 2B^{[i} \wedge E^{j]}$. These expressions clearly reduce to (7.59), provided that $J_{ij} = 0$, $J_{(K)i} = 0$ and $J_{(D)} = 0$.
This discussion simply shows that the full conformal structure is already encoded in (7.59), even if some fields are composite. In other words, if some fields happen to be expressed in terms of other boundary fields, their corresponding currents become zero and their associated symmetries may be realised non-linearly, as it happens for $b^i$. A similar situation will be discussed in Subsection 7.7 for the superconformal group.
In the following Sections, we will extend the above analysis to the supersymmetric case.

### 7.3 Supergravity setting

In the geometric approach to the $\mathcal{N} = 2$ Supergravity in four-dimensional spacetime, one considers the Lie supergroup $OSp(2|4)$, whose algebra behaves like (5.2). The bosonic gauge algebra is given by the product $so(2,3) \times so(2)$, where the first factor corresponds to the isometry group of AdS$_4$ and the second one describes the R-symmetry of the theory. The physical fields, as described in Section 5, are the vielbein $V^a_{\hat{\mu}}$, the gravitino $\Psi_{\hat{\mu}A}$, the $SO(1,3)$ spin connection $\hat{\omega}^{ab}_{\hat{\mu}}$ and the graviphoton $\hat{A}_{\hat{\mu}}$.
Here the indices $A, B \ldots = 1, 2$ are in the fundamental representation of $so(2)$ and, as mentioned previously, hatted quantities refer to bulk objects.
Since we are now studying a dynamical theory, one introduces a Principal bundle with $OSp(2|4)$ as the fiber: the curvatures, which are defined as in (5.6), read[6]

$$\hat{\mathbf{R}}^{ab} \quad = \quad \hat{\mathcal{R}}^{ab} - \frac{1}{\ell^2} V^a V^b - \frac{1}{2\ell} \delta^{AB} \overline{\Psi}_A \Gamma^{ab} \Psi_B \,,$$

[6]Let us notice that, in comparison with the previous Section, the gauge field has been here rescaled by a factor $\frac{1}{\ell}$.

$$
\begin{aligned}
\hat{\mathbf{R}}^a &= \hat{\mathcal{D}} V^a - \frac{\mathrm{i}}{2}\,\overline{\Psi}^A \Gamma^a \Psi_A\,, \\
\hat{\boldsymbol{\rho}}_A &= \mathrm{d}\Psi_A + \frac{1}{4}\,\Gamma_{ab}\,\hat{\omega}^{ab} \wedge \Psi_A - \frac{1}{2\ell}\,\hat{A}\epsilon_{AB} \wedge \Psi^B - \frac{\mathrm{i}}{2\ell}\,\delta_{AB}\Gamma_a \Psi^B V^a\,, \\
\hat{\mathbf{F}} &= \mathrm{d}\hat{A} - \overline{\Psi}^A \wedge \Psi^B \epsilon_{AB}\,.
\end{aligned}
\tag{7.61}
$$

We refer, as done previously, to Appendix B for properties of spinors and gamma matrices. The defined curvatures satisfy the Bianchi "identities", which, as said, contain the information of the equations of motion

$$
\begin{aligned}
\hat{\mathcal{D}}\hat{\boldsymbol{R}}^{ab} &= \frac{2}{\ell^2} V^{[a}\hat{\boldsymbol{R}}^{b]} + \frac{1}{\ell}\overline{\Psi}^A \Gamma^{ab}\hat{\boldsymbol{\rho}}_A\,, \\
\hat{\mathcal{D}}\hat{\boldsymbol{R}}^{a} &= \hat{\boldsymbol{R}}^a{}_b V^b + \mathrm{i}\overline{\Psi}^A \Gamma^a \hat{\boldsymbol{\rho}}_A\,, \\
\hat{\mathcal{D}}\hat{\boldsymbol{\rho}}^A &= \frac{1}{2\ell}\hat{A}\epsilon^{AB}\hat{\boldsymbol{\rho}}_B - \frac{\mathrm{i}}{2\ell}\Gamma_a V^a \hat{\boldsymbol{\rho}}^A + \frac{1}{4}\hat{\boldsymbol{R}}_{ab}\Gamma^{ab}\Psi_A - \frac{1}{2\ell}\hat{\boldsymbol{F}}\epsilon^{AB}\Psi_B + \frac{\mathrm{i}}{2\ell}\Gamma_a \Psi^A \hat{\boldsymbol{R}}^a\,, \\
\mathrm{d}F &= 2\epsilon^{AB}\overline{\Psi}_A \hat{\boldsymbol{\rho}}_B\,.
\end{aligned}
\tag{7.62}
$$

These equations allow to completely determine the expansion of the $OSp(2|4)$ curvatures in Superspace: indeed, as a consequence of 5.1 and 5.2, the quantities in (7.61) must be expanded in Superspace only and the coefficients along supersymmetric directions must be proportional to the coefficients along the bosonic directions. By taking a generic ansatz and by plugging it into the Bianchi identities, one obtains the following rheonomic parametrization

$$
\begin{aligned}
\hat{\mathbf{R}}^a &= 0\,, \\
\hat{\mathbf{F}} &= \tilde{F}_{ab} V^a V^b\,, \\
\hat{\boldsymbol{\rho}}^A &= \tilde{\rho}^A_{ab} V^a V^b - \frac{\mathrm{i}}{2}\,\Gamma^a \Psi^B V^b \tilde{F}_{ab}\epsilon^{AB} - \frac{1}{2}\Gamma_5 \Gamma^a \Psi^B V^b\, {}^*\tilde{F}_{ab}\epsilon^{AB}\,, \\
\hat{\mathbf{R}}^{ab} &= \tilde{R}^{ab}_{cd} V^c V^d - \overline{\Theta}^{ab}_{A|c}\Psi_A V^c - \frac{1}{2}\,\overline{\Psi}_A \Psi_B \epsilon_{AB}\tilde{F}^{ab} - \frac{\mathrm{i}}{2}\overline{\Psi}_A \Gamma_5 \Psi_B \epsilon_{AB}\, {}^*\tilde{F}^{ab}\,.
\end{aligned}
\tag{7.63}
$$

The Lagrangian for a manifold without boundary can be obtained by following the building rules in Section 5. The result of such procedure is [7]

$$
\begin{aligned}
L^{\mathrm{bulk}} =& \frac{1}{4}\hat{\mathcal{R}}^{ab} V^c V^d \epsilon_{abcd} + \overline{\Psi}^A \Gamma_a \Gamma_5 \hat{\rho}_A V^a + \frac{\mathrm{i}}{2}\left(\hat{F} + \frac{1}{2}\,\overline{\Psi}^A \Psi^B \epsilon_{AB}\right)\overline{\Psi}^C \Gamma_5 \Psi^D \epsilon_{CD} \\
& - \frac{\mathrm{i}}{2\ell}\,\overline{\Psi}^A \Gamma_{ab}\Gamma_5 \Psi_A V^a V^b - \frac{1}{8\ell^2}\,V^a V^b V^c V^d \epsilon_{abcd} \\
& + \frac{1}{4}\left(\tilde{F}^{cd} V^a V^b \hat{F} - \frac{1}{12}\,\tilde{F}_{lm}\tilde{F}^{lm} V^a V^b V^c V^d\right)\epsilon_{abcd}\,.
\end{aligned}
\tag{7.64}
$$

The zero-form $\tilde{F}^{ab}$ appearing both in (7.63) and in the lagrangian above has been introduced to effectively perform the Hodge operator of the field strength $\hat{F}$ in superspace, as its equations of motion exactly yield $\tilde{F}^{ab} = F^{ab}$.
Let us notice that the very same lagrangian can be obtained by studying the effective theory

[7] In this Section we will often omit wedge symbols between forms in order to ease the reading.

around a vacuum of the generic $\mathcal{N}=2$ Supergravity lagrangian coupled to vector multiplets and hypermultiplets [57]. This vacuum, which is indeed an AdS vacuum, is obtained as a maximum of the scalar potential $V(\phi)$: most notably, it admits a consistent truncation of the theory, in which all massive fields can be set to zero. This indeed allows to retrieve the pure Supergravity case and in particular the lagrangian (7.64)
Since the lagrangian has to be integrated on the base manifold, considered as embedded into the Principal bundle, namely

$$S=\int_{\mathcal{M}\subset P} L^{\text{bulk}},$$

the fields appearing here must be forms in Superspace

$$\begin{aligned} V^a(x,\theta) &= V^a_{\hat\mu}(x,\theta)\mathrm{d}x^{\hat\mu}+V^a_{\alpha A}(x,\theta)\mathrm{d}\theta^{\alpha A}\,,\\ \hat\omega^{ab}(x,\theta) &= \hat\omega^{ab}_{\hat\mu}(x,\theta)\mathrm{d}x^{\hat\mu}+\hat\omega^{ab}_{\alpha A}(x,\theta)\mathrm{d}\theta^{\alpha A}\,,\\ \Psi^A_\alpha(x,\theta) &= \Psi^A_{\alpha\hat\mu}(x,\theta)\mathrm{d}x^{\hat\mu}+\Psi^A_{\alpha|\beta B}(x,\theta)\mathrm{d}\theta^{\beta B}\,,\\ \hat A(x,\theta) &= \hat A_{\hat\mu}(x,\theta)\mathrm{d}x^{\hat\mu}+\hat A_{\alpha A}(x,\theta)\mathrm{d}\theta^{\alpha A}\,. \end{aligned} \tag{7.65}$$

Moreover, one can always restrict to a bosonic hypersurface $M\subset\mathcal{M}$: this procedure allows to restrict the fields to spacetime quantities

$$\begin{aligned} V^a(x) &= V^a(x,\theta)|_{\theta=\mathrm{d}\theta=0}=V^a_{\hat\mu}(x,0)\mathrm{d}x^{\hat\mu}\,,\\ \hat\omega^{ab}(x) &= \hat\omega^{ab}(x,\theta)|_{\theta=\mathrm{d}\theta=0}=\hat\omega^{ab}_{\hat\mu}(x,0)\mathrm{d}x^{\hat\mu}\,,\\ \Psi^A(x) &= \Psi^A(x,\theta)|_{\theta=\mathrm{d}\theta=0}=\Psi^A_{\hat\mu}(x,0)\mathrm{d}x^{\hat\mu}\,,\\ \hat A(x) &= \hat A(x,\theta)|_{\theta=\mathrm{d}\theta=0}=\hat A_{\hat\mu}(x,0)\mathrm{d}x^{\hat\mu}\,. \end{aligned} \tag{7.66}$$

If the spacetime manifold has a boundary, one is forced to add exact terms to preserve Supersymmetry invariance: let us review this procedure, by following the analysis performed in [46].
When the lagrangian is integrated on the bosonic manifold $M$, considered as a submanifold of the whole superspace $\mathcal{M}$, the supersymmetric variation of the action is given by

$$\delta_\epsilon S=\int_M \mathcal{L}_\epsilon L=\int_M (i_\epsilon \mathrm{d}L+\mathrm{d}i_\epsilon L),$$

which implies

$$i_\epsilon \mathrm{d}L=0,\qquad i_\epsilon L|_{\partial M}=0. \tag{7.67}$$

The only possible total derivative terms compatible with the symmetries of the theory are

$$L^{\text{boundary}}=-\frac{\ell^2}{8}\left(\hat{\mathcal{R}}^{ab}\hat{\mathcal{R}}^{cd}\epsilon_{abcd}+\frac{8\mathrm{i}}{\ell}\,\bar{\hat\rho}^A\Gamma_5\hat\rho_A-\frac{2\mathrm{i}}{\ell}\,\hat{\mathcal{R}}^{ab}\overline{\Psi}^A\Gamma_{ab}\Gamma_5\Psi_A+\frac{4\mathrm{i}}{\ell^2}\,\mathrm{d}\hat A\,\overline{\Psi}^A\Gamma_5\Psi^B\epsilon_{AB}\right), \tag{7.68}$$

which allow the total lagrangian $L=L^{\text{bulk}}+L^{\text{boundary}}$ to satisfy the desired conditions (7.67), provided that

$$\hat{\mathbf{R}}^{ab}|_{\partial\mathcal{M}}=0\,,\quad \hat{\boldsymbol{\rho}}_A|_{\partial\mathcal{M}}=0\,,\quad \hat{\mathbf{F}}|_{\partial\mathcal{M}}=0\,. \tag{7.69}$$

Let us understand the origin of these constraints: to this end, first rewrite the full lagrangian in terms of the $OSp(2|4)$ curvatures in a MacDowell-Mansouri form

$$L = -\frac{\ell^2}{8}\hat{\boldsymbol{R}}^{ab}\wedge\hat{\boldsymbol{R}}^{cd}\epsilon_{abcd} - \mathrm{i}\ell\hat{\overline{\boldsymbol{\rho}}}^A\Gamma_5\wedge\hat{\boldsymbol{\rho}}_A + \frac{1}{4}\left(\tilde{F}^{cd}V^aV^b\hat{F} - \frac{1}{12}\tilde{F}_{lm}\tilde{F}^{lm}V^aV^bV^cV^d\right)\epsilon_{abcd}.$$

This expression, since we are now in spacetime, can be rewritten as

$$\mathcal{L}^{\text{spacetime}} = -\frac{\ell^2}{8}\hat{\boldsymbol{R}}^{ab}\wedge\hat{\boldsymbol{R}}^{cd}\epsilon_{abcd} - \mathrm{i}\ell\hat{\overline{\boldsymbol{\rho}}}^A\Gamma_5\wedge\hat{\boldsymbol{\rho}}_A + \frac{1}{4}\hat{\boldsymbol{F}}\wedge{}^*\hat{\boldsymbol{F}}, \tag{7.70}$$

where the Hodge dual in spacetime is defined in the standard way as

$${}^*\hat{\mathbf{F}} = \frac{1}{2}{}^*\hat{\mathbf{F}}_{\hat{\mu}\hat{\nu}}\,\mathrm{d}x^{\hat{\mu}}\wedge\mathrm{d}x^{\hat{\nu}} = \frac{e}{4}\epsilon_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}\,\hat{\mathbf{F}}^{\hat{\rho}\hat{\sigma}}\,\mathrm{d}x^{\hat{\mu}}\wedge\mathrm{d}x^{\hat{\nu}}\,.$$

By taking the variation of (7.70) one obtains a bulk term, giving the equations of motion, which we will discuss below and a boundary term, which reads

$$\int_{\partial M}\left(\frac{\ell^2}{4}\delta\hat{\omega}^{ab}\wedge\boldsymbol{R}^{cd}\epsilon_{abcd} - 2\mathrm{i}\ell\delta\overline{\Psi}^A\Gamma_5\wedge\hat{\boldsymbol{\rho}}_A + \frac{1}{2}\delta\hat{A}\wedge *\hat{\boldsymbol{F}}\right)\Bigg|_{\partial M},$$

from which we conclude that (7.69) has to hold, if the variation of the fields at the boundary is different from zero.
Let us notice that boundary terms (7.68) are the supersymmetric generalisation of the ones appearing in (7.5) and actually reduce to the Gauss-Bonnet term, when fermions are switched off. Moreover, the obtained action (7.70) is the direct generalisation of (7.6).

The equations of motion can be computed both from the bulk lagrangian and from (7.70): the two results clearly coincide, as the two lagrangian differ for exact terms. In the following analysis we will make use of both ways of computing the equations of motion, so we will state here the result of both methods. By using the bulk lagrangian, one obtains

$$\begin{aligned}
\delta\hat{\omega}^{ab}:&\quad V^c\hat{\boldsymbol{R}}^d\epsilon_{abcd} = 0 \quad\Rightarrow\quad \hat{\boldsymbol{R}}^a = 0\,,\\
\delta V^a:&\quad \frac{1}{2}V^b\hat{\boldsymbol{R}}^{cd}\epsilon_{abcd} - \overline{\Psi}^A\Gamma_a\Gamma_5\hat{\boldsymbol{\rho}}_A + {}^*\tilde{F}_{ab}V^b\hat{\boldsymbol{F}} - \frac{1}{12}\tilde{F}^{ef}\tilde{F}_{ef}V^bV^cV^d\epsilon_{abcd} = 0\,,\\
\delta\overline{\Psi}^A:&\quad 2\Gamma_aV^a\Gamma_5\hat{\boldsymbol{\rho}}_A - \epsilon_{AB}\Psi^{B*}\hat{\boldsymbol{F}} + \mathrm{i}\epsilon_{AB}\hat{\boldsymbol{F}}\Gamma_5\Psi^B = 0\,,\\
\delta\hat{A}:&\quad \mathrm{d}^*\hat{\boldsymbol{F}} - 2\mathrm{i}\epsilon^{AB}\overline{\Psi}_A\Gamma_5\hat{\boldsymbol{\rho}}_B = 0\,.
\end{aligned}\tag{7.71}$$

whereas by taking the variation of the full lagrangian, only the equations of motion for the spin connection and for the gravitino are modified and read

$$\delta\hat{\omega}^{ab}:\quad -\frac{1}{2}\hat{\mathcal{D}}\hat{\boldsymbol{R}}^{cd}\epsilon_{abcd} + \mathrm{i}\,\overline{\Psi}^A\Gamma_{ab}\Gamma_5\hat{\boldsymbol{\rho}}_A = 0\,, \tag{7.72}$$

$$\delta\overline{\Psi}^A:\quad \frac{\ell}{4}\Gamma^{ab}\Psi_A\hat{\boldsymbol{R}}^{cd}\epsilon_{abcd} - 2\mathrm{i}\ell\Gamma_5\hat{\mathcal{D}}\hat{\boldsymbol{\rho}}_A + \mathrm{i}\Gamma_5\hat{A}\epsilon_{AB}\hat{\boldsymbol{\rho}}^B + \Gamma_aV^a\Gamma_5\hat{\boldsymbol{\rho}}_A - \epsilon_{AB}\Psi^{B*}\hat{\boldsymbol{F}} = 0\,. \tag{7.73}$$

We again stress out that these two result exactly match up, upon use of the Bianchi identities.

Let us observe that the outcome of this analysis resembles the results of the pure GR case in [48]: in both cases the addition of counterterms allows to obtain a lagragian which is quadratic in the curvatures. In absence of fermions, the counterterms also prove to be useful for an asymptotical analysis of the theory and it is then natural to ask ourselves if the theory is finite and well behaves also in the supersymmetric case. This will be the main goal of the following Subsections and will heavily rely on the transformation rules of the fields: by using (5.8), the rheonomic parametrization and by finally projecting the fields on the bosonic hypersurface, one obtains

$$
\begin{aligned}
\delta V^a &= \hat{\mathcal{D}}p^a - j^{ab}V_b + \mathrm{i}\,\bar{\epsilon}_A\Gamma^a\Psi^A\,,\\
\delta\hat{\omega}^{ab} &= \hat{\mathcal{D}}j^{ab} + \frac{2}{\ell^2}\,p^{[a}V^{b]} + 2\,\tilde{R}^{ab}{}_{cd}p^cV^d + \overline{\Theta}^{ab}_{A|c}\Psi^Ap^c + \frac{1}{\ell}\,\bar{\epsilon}^A\Gamma^{ab}\Psi_A - \overline{\Theta}^{ab}_{A|c}\epsilon^AV^c\\
&\quad +\epsilon^{AB}\tilde{F}^{ab}\overline{\Psi}_A\epsilon_B + \mathrm{i}\,\epsilon^{AB}\,{}^*\tilde{F}^{ab}\overline{\Psi}_A\Gamma_5\epsilon_B\,,\\
\delta\Psi^A &= -\frac{1}{4}\,j^{ab}\Gamma_{ab}\Psi^A - \frac{\mathrm{i}}{2\ell}\,\Gamma_a\Psi^Ap^a + 2\,\tilde{\rho}^A_{ab}p^aV^b + \frac{\mathrm{i}}{2}\,\Gamma^a\Psi_Bp^b\tilde{F}_{ab}\epsilon^{AB} + \frac{1}{2}\,\Gamma_5\Gamma^a\Psi_B\,{}^*\tilde{F}_{ab}\,p^b\epsilon^{AB}\\
&\quad +\frac{\hat{\lambda}}{2\ell}\,\epsilon^{AB}\Psi_B + \hat{\mathcal{D}}\epsilon^A - \frac{1}{2\ell}\,\hat{A}\epsilon^{AB}\epsilon_B + \frac{\mathrm{i}}{2\ell}\,\Gamma_a\epsilon^AV^a - \frac{\mathrm{i}}{2}\,\epsilon^{AB}\tilde{F}_{ab}V^b\Gamma^a\epsilon_B\\
&\quad -\frac{1}{2}\,\epsilon^{AB}\,{}^*\tilde{F}_{ab}\,\Gamma_5\Gamma^a\epsilon_BV^b\,,\\
\delta\hat{A} &= \mathrm{d}\hat{\lambda} + 2\,\bar{\epsilon}^A\Psi^B\epsilon_{AB} + 2\,\tilde{F}_{ab}p^aV^b\,,
\end{aligned}
\tag{7.74}
$$

where $p^a$, $j^{ab}$, $\epsilon^A$, and $\hat{\lambda}$ correspond respectively to diffeomorphisms, local Lorentz transformations, supersymmetry and U(1) gauge transformations.
We now focus, as done for the AdS gravity case, on gauge fixings and asymptotical analysis of the fields.

## 7.4 Near-boundary analysis of Supergravity fields.

In the present Section, we are going to apply the holographic techniques outlined in Section 7.2 to the 4D Supergravity theory presented in in the previous one.
The transformation laws of the fields now depend on the local parameters $p^a$, $j^{ab}$, $\hat{\lambda}$ and $\epsilon_A$ and we will use this freedom to fix the radial components of the fields, which are unphysical, either being Lagrange multipliers or non-dynamic variables.
In particular, we want to choose a suitable generalisation of the gauge fixing conditions (7.17): the asymptotic behaviour of the vielbein remains the same as for gravity, because it is determined solely by the metric (7.11), whereas the behaviour of the spin connection can be evaluated from the vanishing of the supertorsion and receives also a contribution from fermions. The gravitini also act as a source for the electromagnetic field, determining the fall-off of the graviphoton connection.
We then first analyse the asymptotic behaviour of the gravitini: to this end, it is convenient to express them in terms of the chiral components with respect to the matrix $\Gamma^3$: $\Psi = \Psi_+ + \Psi_-$, where the eigenstates $\Psi_\pm$ of the matrix $\Gamma^3$ are defined by eq. (B.12).

We are interested in gravitini whose fall-off is as given in (6.9), namely $\Psi_{\mu\pm} = \mathcal{O}(z^{\mp 1/2})$ and $\Psi_{z\pm} = \mathcal{O}(z^{\pm 1/2})$: from a group theoretical point of view, the same result corresponds to the request of covariance with respect to the $OSp(2|4)$ group. In particular, the subgroup $SO(1,1) \subset OSp(2|4)$ that parametrizes radial rescalings in the bulk and dilations on the boundary defines a scaling ($\pm 1/2$). This requirement can be better written as

$$\Psi_{A\mu\pm} = \left(\frac{z}{\ell}\right)^{\mp\frac{1}{2}} \varphi_{A\mu\pm}(x,z)\,, \qquad \Psi_{Az\pm} = \left(\frac{z}{\ell}\right)^{\pm\frac{1}{2}} \varphi_{A\pm z}(x,z)\,, \tag{7.75}$$

where the Majorana fermions $\varphi_{A\mu\pm}$ and $\varphi_{Az\pm}$ are *regular* functions at the boundary and can be expanded as power series in $z$.
These relations are also consistent with the condition that singles out the spin $3/2$ components in the gravitini,

$$\Gamma^a \Psi_{A\hat{\mu}}\, V^{\hat{\mu}}_{\ a} = 0\,, \tag{7.76}$$

which in the FG frame (7.11) reads

$$\left(\Gamma^i \Psi_{A\mu}\right)_\pm V^\mu_{\ i} + \left(\Gamma^3 \Psi_{Az}\right)_\pm V^z_{\ 3} = 0\,.$$

In this thesis, we will be interested in asymptotic behaviours consistent with (7.76); however, let us notice that if we relax the conditions of FG metric, allowing instead for asymptotically non vanishing $V^3_\mu$, then more general asymptotics for the gravitini components $\Psi_{Az\pm}$ can in principle be considered.
Furthermore, since the supersymmetric transformation of the gravitini must be of the same order in $z$ as the gravitini themselves, one must have $\delta_\epsilon \Psi_{A\mu\pm} \sim \hat{\mathcal{D}}_\mu \epsilon_{A\pm} \sim \epsilon_{A\pm}$, namely

$$\epsilon_{A\pm} = \left(\frac{z}{\ell}\right)^{\mp\frac{1}{2}} H_{A\pm}(x,z)\,. \tag{7.77}$$

where $H_{A\pm}(x,z)$ is regular Majorana spinor.
With respect of the subgroup $SO(1,1)$, the bosonic fields $\hat{\omega}^{ij}$ and $\hat{A}$ have zero scaling, whereas $V^i$, $\omega^{i3}$ can be combined as

$$V^i_{\pm\hat{\mu}} = \frac{1}{2}\left(\ell\hat{\omega}^{i3}_{\hat{\mu}} \pm V^i_{\ \hat{\mu}}\right)\,, \tag{7.78}$$

having $\pm 1$ scaling. They asymptotically behave as

$$V^i_{\pm\mu} = \left(\frac{z}{\ell}\right)^{\mp 1} E^i_{\pm\mu}(x,z)\,, \tag{7.79}$$

where the regular functions $E^i_\pm$ have the following power expansion in $z$

$$\begin{aligned} E^i_{+\mu} &= E^i_{\ \mu} + \frac{z^2}{\ell^2}\,\frac{S^i_{\ \mu} - \tilde{S}^i_{\ \mu}}{2} + \frac{z^3}{\ell^3}\,\frac{\tau^i_{\ \mu} - 2\tilde{\tau}^i_{\ \mu}}{2} + \mathcal{O}(z^4)\,,\\ E^i_{-\mu} &= -\frac{\ell^2}{2}\,\mathcal{S}^i_{\ \mu} - \frac{z}{\ell}\,\frac{\tau^i_{\ \mu} + 2\tilde{\tau}^i_{\ \mu}}{2} + \mathcal{O}(z^2)\,. \end{aligned} \tag{7.80}$$

In order to relate the behaviour of the gravitini to the one of the graviphoton and the spin connection, we state for completeness the convention used in Appendix D and hereafter: unless stated differently, all regular functions on the boundary that appear here, $f = \{w^i,\ w^{ij},\ \varphi_{A\mu\pm},\ \varphi_{Az\pm},\ H_{A\pm}, \ldots\}$, are generically expanded in a power series

$$f(x,z) = \sum_{n=0}^{\infty} \left(\frac{z}{\ell}\right)^n f_{(n)}(x) = f_{(0)}(x) + \frac{z}{\ell} f_{(1)}(x) + \frac{z^2}{\ell^2} f_{(2)}(x) + \cdots . \tag{7.81}$$

Using these conventions, we turn to the behaviour of the $\hat{A}_z$ component of the graviphoton: the appropriate way of performing this analysis is by computing the equations of motion at various orders, as done in Appendix D.3. It turns out that there are only two possibilities, as shown in (D.44): either one allows for a $\varphi^A_{z-} \neq 0$ expressed in terms of a graviphoton, whose radial expansion starts at diverging order $\mathcal{O}(z^{-1})$ or one consistently sets $\varphi^A_{-z} = \hat{A}_{(-1)z} = 0$. Both choices lead to well-behaving equations of motion and transformation laws, but only the second one can be coupled with the condition (7.76). In the following we will mainly be interested in the choice

$$\Psi^A_{z-} = 0, \tag{7.82}$$

keeping in mind that, by relaxing (7.76), one could allow for different asymptotic behaviours of the fields: this could become relevant when dealing with graphene models and with the decomposition (6.18), which is actually incompatible with (7.76) itself.
A stronger condition $\Psi_{Az\pm} = 0$ was considered in [58] in the context of $\mathcal{N} = 1$ AdS$_4$ Supergravity. An advantage of having $\Psi_{Az+} \neq 0$ is to provide more freedom that could be used to simplify complicated fermionic expressions. We will see, though, that the presence of this particular field will not modify the asymptotic behaviour of the theory.

Finally, the asymptotic behaviour of the spin connection is computed in Appendix D.1, where it is found (see eqs. (D.7)) that $\hat{\omega}^{ab}_z \neq 0$, but it is still subleading on the boundary. The behaviour of the spin connection will then be expressed in terms of the following functions

$$\hat{\omega}^{i3}_z = w^i(x,z), \qquad \hat{\omega}^{ij}_z = \frac{z}{\ell} w^{ij}(x,z).$$

To sum up this discussion, the results of of Appendix D.1 and D.3 show that the gauge-fixing conditions have the form

$$\begin{aligned} V^3_{\ z} &= \frac{\ell}{z}, \qquad & \hat{\omega}^{i3}_z &= w^i(x,z), \qquad & \Psi_{\pm Az} &= \left(\frac{z}{\ell}\right)^{\pm\frac{1}{2}} \varphi_{\pm Az}(x,z), \\ V^i_{\ z} &= 0, \qquad & \hat{\omega}^{ij}_z &= \frac{z}{\ell} w^{ij}(x,z), \quad & \hat{A}_z &= \frac{\ell}{z} A_{(-1)z}(x) + \frac{z}{\ell} A_{(1)z}(x) + \mathcal{O}(z^3), \end{aligned} \tag{7.83}$$

where, for the sake of completeness, we distinguish the particular cases:

$$\begin{aligned} \Psi_{z\pm} \neq 0 \quad &\Rightarrow \quad \hat{A}_z = \mathcal{O}(1/z), \quad & w^i &= \mathcal{O}(1), \qquad & w^{ij} &= \mathcal{O}(1), \\ \Psi_{z-} = 0 \quad &\Rightarrow \quad \hat{A}_z = \mathcal{O}(z), \quad & w^i &= \mathcal{O}(z^2), \qquad & w^{ij} &= \mathcal{O}(1), \\ \Psi_{z\pm} = 0 \quad &\Rightarrow \quad \hat{A}_z = \mathcal{O}(z), \quad & w^i &= 0, \qquad & w^{ij} &= \mathcal{O}(1). \end{aligned} \tag{7.84}$$

When dealing with bispinors, we will opt for avoiding heavy notation and we will give different names to different orders in $z$, contrary to the convention chosen for quadrispinors (7.81): in particular

$$\begin{aligned}
\Psi^A_{+z} &= \sqrt{\frac{z}{\ell}}\,\varphi^A_{+z}(x,z) = \sqrt{\frac{z}{\ell}}\left[\begin{pmatrix}\psi^A_{+z}\\ 0\end{pmatrix} + \frac{z}{\ell}\begin{pmatrix}\zeta^A_{+z}\\ 0\end{pmatrix} + \mathcal{O}(z^2)\right], \\
\Psi^A_{-z} &= \sqrt{\frac{\ell}{z}}\,\varphi^A_{-z}(x,z) = \sqrt{\frac{\ell}{z}}\left[\begin{pmatrix}0\\ \psi^A_{-z}\end{pmatrix} + \frac{z}{\ell}\begin{pmatrix}0\\ \zeta^A_{-z}\end{pmatrix} + \mathcal{O}(z^2)\right].
\end{aligned} \tag{7.85}$$

It is important to emphasize that we assume that *the gauge-fixing functions do not transform under local transformations*. This is equivalent to the statement that their transformation law can always be reabsorbed in higher-order terms of the asymptotic transformations. However, the quantities $w^i(x)$ and $w^{ij}(x)$ are not gauge fixings, as they are obtained from the vanishing of the supertorsion condition. It is straightforward to check by varying the supertorsion that $\delta w^i$, $\delta w^{ij} \neq 0$ and that we can set $w^i = 0$ consistently (with $\delta w^i = 0$). However, if $w^i \neq 0$, then $\delta w^i \neq 0$ as well. The same is independently true for $w^{ij}$. Nonetheless, $\delta w^i$ and $\delta w^{ij}$ always appear at higher-order and they do not influence the near-boundary expressions.

The boundary fields are then expanded in the following way:

$$\begin{aligned}
V^i_{\ \mu} &= \frac{\ell}{z}\,E^i_{\ \mu} + \frac{z}{\ell}\,S^i_{\ \mu} + \frac{z^2}{\ell^2}\,\tau^i_{\ \mu} + \mathcal{O}(z^3)\,, \\
\hat{\omega}^{i3}_\mu &= \frac{1}{z}\,E^i_{\ \mu} - \frac{z}{\ell^2}\,\tilde{S}^i_{\ \mu} - \frac{2z^2}{\ell^3}\,\tilde{\tau}^i_{\ \mu} + \mathcal{O}(z^3)\,, \\
\hat{\omega}^{ij}_\mu &= \omega^{ij}_\mu(x,z) = \omega^{ij}_\mu + \frac{z}{\ell}\,\omega^{ij}_{(1)\mu} + \frac{z^2}{\ell^2}\,\omega^{ij}_{(2)\mu} + \mathcal{O}(z^3)\,, \\
\hat{A}_\mu &= A_\mu(x,z) = A_\mu + \frac{z}{\ell}\,A_{(1)\mu} + \frac{z^2}{\ell^2}\,A_{(2)\mu} + \mathcal{O}(z^3)\,, \\
\Psi^A_{\mu+} &= \sqrt{\frac{\ell}{z}}\,\varphi^A_{\mu+}(x,z) = \sqrt{\frac{\ell}{z}}\left[\begin{pmatrix}\psi^A_{\mu+}\\ 0\end{pmatrix} + \frac{z}{\ell}\begin{pmatrix}\zeta^A_{\mu+}\\ 0\end{pmatrix} + \frac{z^2}{\ell^2}\begin{pmatrix}\Pi^A_{\mu+}\\ 0\end{pmatrix} + \mathcal{O}(z^3)\right], \\
\Psi^A_{\mu-} &= \sqrt{\frac{z}{\ell}}\,\varphi^A_{\mu-}(x,z) = \sqrt{\frac{z}{\ell}}\left[\begin{pmatrix}0\\ \psi^A_{\mu-}\end{pmatrix} + \frac{z}{\ell}\begin{pmatrix}0\\ \zeta^A_{\mu-}\end{pmatrix} + \mathcal{O}(z^2)\right],
\end{aligned} \tag{7.86}$$

where all functions on the right hand side of the equations are finite at $z = 0$. Let us remark that we are allowing for linear terms in the expansion of spin connection and graviphoton, because in principle they could be switched on by fermions.
Moreover, since the spin connection cannot be completely determined by the vielbein, but also receives contributions by gravitini, the subleading terms in the expansion of $\hat{\omega}^{i3}_\mu$, $\tilde{S}^i_{\ \mu}$ and $\tilde{\tau}^i_{\ \mu}$ are different from the Riemannian counterparts $S^i_{\ \mu}$ and $\tau^i_{\ \mu}$. The boundary super-Schouten tensor is now defined as

$$\mathcal{S}^i_{\ \mu} = \frac{1}{\ell^2}\,(S^i_{\ \mu} + \tilde{S}^i_{\ \mu}) \tag{7.87}$$

and, as we will see, it is the gauge field associated to special conformal transformations. Similarly, we will see that the combination $-(\tau^i_{\ \mu} + 2\tilde{\tau}^i_{\ \mu})/\ell$ becomes the holographic stress tensor,

up to the fermionic terms. Let us observe that the super-Schouten tensor (7.87) is not just the bosonic Schouten tensor computed in presence of a torsion coming from spinors, but it is its supersymmetric generalisation, as we will see later.
Notice that there is now an obstruction to symmetrize $\mathcal{S}^i{}_\mu$ and the holographic stress tensor because the terms $\tilde{S}^i{}_\mu$ and $\tilde{\tau}^i{}_\mu$ are not a priori symmetric in presence of the gravitini.

Because of their importance in this discussion, let us concretely compute the antisymmetric parts of both quantities, $\mathcal{S}^i{}_\mu$ and $\tau^i{}_\mu + 2\tilde{\tau}^i{}_\mu$ and express them in terms of fermion bilinears. To explicitly perform these computations, one needs the following quantities, which arise from the asymptotic expansion

$$\begin{aligned} S_{\mu\nu} &= E_{i\mu} S^i{}_\nu \,, \qquad \tau_{\mu\nu} = E_{i\mu}\tau^i{}_\nu \,, \\ \tilde{S}_{\mu\nu} &= E_{i\mu}\tilde{S}^i{}_\nu \,, \qquad \tilde{\tau}_{\mu\nu} = E_{i\mu}\tilde{\tau}^i{}_\nu \,, \\ \mathcal{S}_{\mu\nu} &= E_{i\mu}\mathcal{S}^i{}_\nu \,. \end{aligned} \tag{7.88}$$

From (D.6), in the case $\varphi^A_{z-} = 0$, we have

$$\begin{aligned} \tilde{S}_{\mu\nu} &= S_{\nu\mu} - \ell\,\overline{\varphi}_{(0)A+[\mu}\varphi^A_{(0)-\nu]} + \mathrm{i}\ell\,\overline{\varphi}_{(0)A+(\nu}\Gamma_{\mu)}\varphi^A_{(0)+z} \,, \\ \tilde{\tau}_{\mu\nu} &= \frac{\tau_{\mu\nu} + 3\tau_{\nu\mu}}{4} + \frac{\ell}{2}\left(-\overline{\varphi}_{A+[\mu}\varphi^A_{-\nu]} + \mathrm{i}\overline{\varphi}^A_{+(\mu}\Gamma_{\nu)}\varphi_{A+z}\right)_{(1)} \,. \end{aligned} \tag{7.89}$$

from which we see that fermions do act as an obstruction to the antisimmetrisation of such coefficients

$$\begin{aligned} \tilde{S}_{[\mu\nu]} &= S_{[\nu\mu]} - \ell\,\overline{\varphi}_{(0)A+[\mu}\varphi^A_{(0)-\nu]} \,, \\ \tilde{\tau}_{[\mu\nu]} &= \frac{1}{2}\,\tau_{[\nu\mu]} - \frac{\ell}{2}\left(\overline{\varphi}_{(0)A+[\mu}\varphi^A_{(1)-\nu]} + \overline{\varphi}_{(1)A+[\mu}\varphi^A_{(0)-\nu]}\right) \,. \end{aligned} \tag{7.90}$$

We finally see from the definition of the super-Schouten tensor (7.87) that

$$\mathcal{S}_{\mu\nu} = \frac{2}{\ell^2}\,S_{(\mu\nu)} - \frac{1}{\ell}\,\overline{\varphi}_{(0)A+[\mu}\varphi^A_{(0)-\nu]} + \frac{\mathrm{i}}{\ell}\,\overline{\varphi}_{(0)A+(\nu}\Gamma_{\mu)}\varphi^A_{(0)+z} \,, \tag{7.91}$$

which implies, as expected, that

$$\begin{aligned} \mathcal{S}_{(\mu\nu)} &= \frac{2}{\ell^2}\,S_{(\mu\nu)} + \frac{\mathrm{i}}{\ell}\,\overline{\varphi}_{(0)A+(\mu}\Gamma_{\nu)}\varphi^A_{(0)+z} \,, \\ \mathcal{S}_{[\mu\nu]} &= -\frac{1}{\ell}\,\overline{\varphi}_{(0)A+[\mu}\varphi^A_{(0)-\nu]} \,. \end{aligned} \tag{7.92}$$

We observe that the antisymmetric part does not vanish for arbitrary fermions $\varphi^A_{\mu+}$: these components must remain arbitrary and unconstrained if we want an interesting boundary theory.
Similarly, at the next level in $z$, the combination $\tau_{\mu\nu} + 2\tilde{\tau}_{\mu\nu}$ behaves like

$$\tau_{\mu\nu} + 2\tilde{\tau}_{\mu\nu} \;=\; 3\tau_{(\mu\nu)} + \ell\,\Big(-\overline{\varphi}_{(0)A+[\mu}\varphi^A_{(1)-\nu]} - \overline{\varphi}_{(1)A+[\mu}\varphi^A_{(0)-\nu]}$$

$$+\,\mathrm{i}\overline{\varphi}^A_{(0)+(\mu}E^i_{\nu)}\Gamma_i\varphi_{(1)A+z}+\mathrm{i}\overline{\varphi}^A_{(0)+(\mu}E^i_{\nu)}\Gamma_i\varphi_{(1)A+z}\Big) \tag{7.93}$$

and it is not symmetric in general,

$$\tau_{[\mu\nu]}+2\tilde{\tau}_{[\mu\nu]}=-\ell\left(\overline{\varphi}_{(0)A+[\mu}\varphi^A_{(1)-\nu]}+\overline{\varphi}_{(1)A+[\mu}\varphi^A_{(0)-\nu]}\right). \tag{7.94}$$

We can now safely discuss gauge-fixing choices and study under which conditions they are preserved.

## 7.5 Gauge-fixing analysis

So far, we have chosen the radial components of the fields (7.83) in such a way that they generate the asymptotic expansion of the fields (7.86). To perform the analysis of these constraints, one needs the following observation.

**Rheonomic parametrizations.**

The transformation laws (7.74) explicitly depend on the contractions of the supercurvatures. The proper way to account for all contributions requires to know the near-boundary behaviour of the rheonomic parametrizations appearing there. The simplest way to proceed is to project the expressions (7.63) on spacetime and to identify their asymptotic behaviour with the one of the spacetime projections of the supercurvatures. One has to start from the $U(1)$ field strength, whose parametrization takes value on the 2-vielbein component only. One then proceeds to find $\tilde{\rho}^A_{ab}$ from the curvature of the gravitino, which can be further used to compute the coefficients $\Theta^{ab}_{A|c}$ and $\tilde{R}^{ab}_{cd}$ appearing in the last of (7.63).

By following this procedure, one can determine the asymptotic behaviour of all the supercovariant field-strengths, whose derivation is fully carried out in Appendix E. We sum up here the main results obtained there: For the parameters $\tilde{F}_{ab}$ and $\tilde{\rho}^A_{ab}$

$$\begin{aligned}
\tilde{F}_{ij}&=\mathcal{O}(z^3)\,, & \tilde{F}_{i3}&=-\frac{1}{2\ell}\left(\frac{z}{\ell}\right)^2A_{(1)\mu}E^\mu_i+\mathcal{O}(z^3)\,,\\
\tilde{\rho}^A_{ij+}&=\mathcal{O}(z^{5/2})\,, & \tilde{\rho}^A_{i3+}&=-\frac{1}{2\ell}\left(\frac{z}{\ell}\right)^{\frac{3}{2}}E^\mu_i\zeta^A_{\mu+}+\mathcal{O}(z^{5/2})\,,\\
\tilde{\rho}^A_{ij-}&=\mathcal{O}(z^{5/2})\,, & \tilde{\rho}^A_{i3-}&=\mathcal{O}(z^{5/2})\,,
\end{aligned} \tag{7.95}$$

which can be combined to express the radial power expansion of $\tilde{R}^{ab}_{cd}$, which reads

$$\begin{aligned}
\tilde{R}^{i3}{}_{jk}&=\frac{\mathrm{i}}{2\ell}\left(\frac{z}{\ell}\right)^2E^\mu_{[j}E^\nu_{k]}\overline{\psi}^A_{\mu+}\left(\gamma^i\zeta_{A\nu+}+\gamma^l\zeta_{A\rho+}E_{l\nu}E^{i\rho}\right)+\mathcal{O}(z^3)\,,\\
\tilde{R}^{ij}{}_{k3}&=-\frac{1}{2\ell}\left(\frac{z}{\ell}\right)^2E^\mu_k\left(\omega^{ij}_{(1)\mu}-\mathrm{i}\,\overline{\psi}^A_{\mu+}\gamma^{[i}E^{j]\nu}\zeta_{A\nu+}\right)+\mathcal{O}(z^3)\,,\\
\tilde{R}^{i3}{}_{j3}&=\mathcal{O}(z^3)\,,\qquad \tilde{R}^{ij}{}_{kl}=\mathcal{O}(z^3)\,.
\end{aligned} \tag{7.96}$$

It is worthwhile noticing that all expansions (7.95) and (7.96) are subleading in $z$ and, when they are slower than $\mathcal{O}(z^3)$, this is due to the linear terms in the connections, namely $A_{(1)\mu}$, $\omega^{ij}_{(1)\mu}$ and $\zeta^A_{\mu+}$.

**Residual symmetries.**

We can now look for the residual symmetries that leave the gauge fixing unaltered:

$$\delta V_z^a = 0\,, \qquad \delta \hat{A}_z = 0\,, \qquad \delta\Psi_{\pm Az} = 0\,, \qquad \delta\hat{\omega}_z^{i3} = \mathcal{O}(z^2), \qquad \delta\hat{\omega}_z^{ij} = \mathcal{O}(z). \tag{7.97}$$

As discussed previously, the gauge fixing conditions on the vielbein and gravitini actually determine the behaviour of the spin connection: in particular, a transformation of the radial components of the spin connection must not introduce new powers in the asymptotic expansion.

The corresponding parameters can be expanded as in eq. (7.81), where we keep the same notation for the leading orders of the bosonic parameters as in (7.52),

$$\begin{aligned}
p^i &= \frac{\ell}{z}\,\xi^i + \frac{z}{\ell}\,p^i_{(1)} + \frac{z^2}{\ell^2}\,p^i_{(2)} + \mathcal{O}(z^3),\\
p^3 &= -\ell\sigma + \frac{z}{\ell}\,p^3_{(1)} + \frac{z^2}{\ell^2}\,p^3_{(2)} + \frac{z^3}{\ell^3}\,p^3_{(3)} + \mathcal{O}(z^4),\\
j^{ij} &= \theta^{ij} + \frac{z}{\ell}\,j^{ij}_{(1)} + \frac{z^2}{\ell^2}\,j^{ij}_{(2)} + \frac{z^3}{\ell^3}\,j^{ij}_{(3)} + \mathcal{O}(z^4)\,,\\
j^{i3} &= \frac{1}{z}\,\xi^i + \frac{z}{\ell}\,j^{i3}_{(1)} + \frac{z^2}{\ell^2}\,j^{i3}_{(2)} + \mathcal{O}(z^3)\,,\\
\hat{\lambda} &= \lambda + \frac{z}{\ell}\,\lambda_{(1)} + \mathcal{O}(z^2)\,,\\
\epsilon^A_+ &= \sqrt{\frac{\ell}{z}}\,H_+(x,z) = \sqrt{\frac{\ell}{z}}\begin{pmatrix}\eta^A_+\\ 0\end{pmatrix} + \sqrt{\frac{z}{\ell}}\begin{pmatrix}\eta^A_{(1)+}\\ 0\end{pmatrix} + \mathcal{O}(z^{3/2})\,,\\
\epsilon^A_- &= \sqrt{\frac{z}{\ell}}H_-(x,z) = \sqrt{\frac{z}{\ell}}\begin{pmatrix}0\\ \eta^A_-\end{pmatrix} + \left(\frac{z}{\ell}\right)^{\frac{3}{2}}\begin{pmatrix}0\\ \eta^A_{(1)-}\end{pmatrix} + \mathcal{O}(z^{5/2})\,.
\end{aligned} \tag{7.98}$$

The subleading Lorentz parameters can be set to zero, as done in Subsection 7.2, by a proper symmetrisation of the vielbein coefficients in the expansion. We therefore take

$$j^{ij}_{(1)} = 0, \qquad j^{ij}_{(2)} = 0, \qquad j^{ij}_{(3)} = 0. \tag{7.99}$$

The higher-order coefficients can also be cancelled out, but they do not influence our results so we will not consider them here. The above conditions imply in (7.86) the following symmetry properties of the vielbein components,

$$\tau_{\mu\nu} = \tau_{\nu\mu}\,, \qquad S_{\mu\nu} = S_{\nu\mu}\,. \tag{7.100}$$

We start from the spin connection condition $\delta\hat{\omega}_z^{ij} = \mathcal{O}(z)$, which means that the finite terms must vanish. This computation yields

$$\partial_z j^{ij} - \frac{1}{\ell}\,\xi^\mu\omega^{ij}_{(1)\mu} + \frac{\mathrm{i}}{\ell}\,\xi^\mu\overline{\psi}^A_{\mu+}\gamma^{[i}E^{j]\nu}\zeta_{A\nu+} - \frac{\mathrm{i}}{\ell}\,\overline{\eta}^A_+\gamma^{[i}E^{j]\nu}\zeta_{A\nu+} + \mathcal{O}(z) = 0\,, \tag{7.101}$$

which amounts to solving the following algebraic equation

$$\xi^\mu \omega^{ij}_{(1)\mu} = \mathrm{i}\left(\xi^\mu \overline{\psi}^A_{\mu+} - \overline{\eta}^A_+\right)\gamma^{[i}E^{j]\nu}\zeta_{A\nu+}\,. \tag{7.102}$$

The leading order parameters $\xi^i$ and $\eta^A_+$ must remain arbitrary if we want a genuine and interesting boundary theory: this is instead an equation for the subleading fields $\omega^{ij}_{(1)\mu}$ and $\zeta_{A\mu+}$. Since we already know from Appendix D.1) that $\omega^{ij}_{(1)\mu} = 0$, we choose a particular solution

$$\zeta_{A\mu+} = 0\,, \tag{7.103}$$

which also appears in [58], in $\mathcal{N} = 1$ Supergravity. It is crucial that these fields remain zero after a generic local transformation, namely $\delta\omega^{ij}_{(1)\mu} = 0$ and $\delta\zeta_{A\mu+} = 0$. This consistency check will be performed later on.

Another constraint on the parameters arises from the fact that the FG coordinate frame (7.11) does not admit finite terms in the expansions of $V^i{}_\mu$ and $\hat{\omega}^{i3}_\mu$. Local invariance preserves this frame only if

$$0 = (\delta V^i_\mu)_{(0)} = -\frac{1}{\ell}\,E^i_\mu\, p^3_{(1)} \quad\Rightarrow\quad p^3_{(1)} = 0\,. \tag{7.104}$$

As a side note, this is consistent with the fact that $(\delta\hat{\omega}^{i3}_\mu)_{(0)} = -\frac{1}{\ell^2}\,E^i_\mu\, p^3_{(1)}$ must be zero as well.

On the other hand, the invariance of $\Psi^A_{\pm z}$ under transformations yields at the leading order

$$\begin{aligned}
0 \;&=\; \delta\Psi^A_{+z} \quad \overset{\text{order}\sqrt{\frac{\ell}{z}}}{\Longrightarrow} \quad 0 = \frac{1}{\ell}\left(\eta^A_{(1)+} - \xi^\mu\zeta^A_{\mu+}\right)\,, \\
0 \;&=\; \delta\Psi^A_{-z} \quad \overset{\text{order}\sqrt{\frac{z}{\ell}}}{\Longrightarrow} \quad 0 = \frac{1}{\ell}\left(\eta^A_{(1)-} - \xi^\mu\zeta^A_{\mu-}\right) + \frac{\mathrm{i}}{4\ell}\epsilon^{AB}A_{(1)\mu}\gamma^\mu\left(\eta_{B+} - \xi^\nu\psi_{B\nu+}\right)\,,
\end{aligned} \tag{7.105}$$

which can be solved using eq. (7.103) as

$$\eta^A_{(1)+} = 0\,, \qquad \eta^A_{(1)-} = \xi^\mu\zeta^A_{\mu-} - \frac{\mathrm{i}}{4}\,\epsilon^{AB}A_{(1)\mu}\gamma^\mu\left(\eta_{B+} - \xi^\nu\psi_{B\nu+}\right)\,. \tag{7.106}$$

From the conservation of the graviphoton gauge-fixing, we observe that

$$0 = \delta\hat{A}_z = \frac{1}{\ell}\,\lambda_{(1)} - \frac{1}{\ell}\,A_{(1)\mu}E^\mu_i\xi^i + \mathcal{O}(z) \quad\Rightarrow\quad \lambda_{(1)} = A_{(1)\mu}\xi^\mu\,. \tag{7.107}$$

We finally turn to the analysis of $\delta\hat{\omega}^{i3}_z = \mathcal{O}(z^2)$ and $\delta V^i_z = 0$: their finite order leads to the expression

$$0 = \delta V^i_{(0)z} = \ell\,\delta\hat{\omega}^{i3}_{(0)z} = j^{i3}_{(1)} + \frac{1}{\ell}\,p^i_{(1)} + w^{ij}_{(0)}\,\xi_j + \mathrm{i}\,\overline{\eta}_{+A}\gamma^i\,\psi^A_{+z}\,. \tag{7.108}$$

There are two unknown parameters, $p^i_{(1)}$ and $j^{i3}_{(1)}$ and only one equation: this leads to an arbitrary vector $K^i(x)$ in the solution, which will be associated to the special conformal transformations on $\partial\mathcal{M}$, as we will prove later. The solution for the first order parameters is

$$p^i_{(1)} = \ell m^i + \frac{\ell^2}{2}\,K^i \equiv b^i, \tag{7.109}$$

$$\ell j^{i3}_{(1)} = \ell m^i - \frac{\ell^2}{2}\, K^i \equiv -\tilde{b}^i\,,$$

where $m^i(x)$ is a function that depends on the gauge fixing

$$m^i(x) = -\frac{1}{2}\,\left(w^{ij}_{(0)}\xi_j + \mathrm{i}\,\overline{\eta}^A_+\gamma^i\psi_{Az+}\right)\,. \tag{7.110}$$

At linear order in $z$, by making use of the rheonomic parametrizations in Appendix E, one gets

$$\begin{aligned}
0 &= \delta V^i_{(1)z} = j^{i3}_{(2)} + \frac{2}{\ell}\, p^i_{(2)} + n^i\,,\\
0 &= \ell\,\delta\hat{\omega}^{i3}_{(1)z} = 2 j^{i3}_{(2)} + \frac{1}{\ell}\, p^i_{(2)} + s^i\,,
\end{aligned} \tag{7.111}$$

where we denoted

$$\begin{aligned}
n^i(x) &= w^{ij}_{(1)}\xi_j + \mathrm{i}\,\bar{\eta}_{+A}\gamma^i\,\zeta^A_{+z}\,,\\
s^i(x) &= -\frac{1}{\ell}\,\xi^\mu(\tau - 4\tilde{\tau})^i{}_\mu + \mathrm{i}\;\bar{\eta}_{+A}\gamma^i\,\zeta^A_{+z} - \xi^\mu E^{i\nu}\bar{\psi}_{+A\mu}\zeta^A_{-\nu} - \mathrm{i}\,\xi^\mu\bar{\psi}_{+A\mu}\gamma^i\zeta^A_{+z}\\
&\quad -\frac{\mathrm{i}}{4}\xi^\mu E^{i\nu}\epsilon_{AB}\overline{\psi}^A_{\mu+}\gamma^\rho\psi^B_{\nu+}A_{(1)\rho} + E^{i\mu}\overline{\eta}^A_+\left(\frac{\mathrm{i}}{4}\epsilon_{AB}\gamma^\rho\psi^B_{+\mu}A_{(1)\rho} + \zeta_{A-\mu}\right),
\end{aligned} \tag{7.112}$$

where the expression for the function $w^{ij}_{(1)}$ can be substituted from the vanishing supertorsion equation (D.13) in Appendix D.2,

$$w^{ij}_{(1)} = -\frac{2}{\ell}\,(\tau - \tilde{\tau})^{ij} - \mathrm{i}\,E^{\mu j}\overline{\psi}_{+A\mu}\gamma^i\,\zeta^A_{+z}\,. \tag{7.113}$$

The solution for the second order parameters $p^i_{(2)}$ and $j^{i3}_{(2)}$ is unique,

$$\begin{aligned}
p^i_{(2)} &= \frac{\ell}{3}\left(s^i - 2n^i\right)\,,\\
\ell j^{i3}_{(2)} &= \frac{\ell}{3}\left(n^i - 2s^i\right)\,.
\end{aligned} \tag{7.114}$$

In our computations, we will only need the following combination of the parameters,

$$\begin{aligned}
\ell j^{i3}_{(2)} - p^i_{(2)} &= \ell\left(n^i - s^i\right) = -\xi^\mu(\tau + 2\tilde{\tau})^i{}_\mu + \ell\xi^\mu E^{i\nu}\bar{\psi}_{+A\mu}\zeta^A_{-\nu}\\
&\quad + \frac{\mathrm{i}\ell}{4}\,\xi^\mu E^{i\nu}\epsilon_{AB}\overline{\psi}^A_{\mu+}\gamma^\rho\psi^B_{\nu+}A_{(1)\rho} - \ell E^{i\mu}\overline{\eta}^A_+\left(\frac{\mathrm{i}}{4}\,\epsilon_{AB}\gamma^\nu\psi^B_{+\mu}A_{(1)\nu} + \zeta_{A-\mu}\right).
\end{aligned} \tag{7.115}$$

After all the above considerations, the residual local parameters can be written as

$$\begin{aligned}
p^3 &= -\ell\sigma + \mathcal{O}(z^2)\,,\\
p^i &= \frac{\ell}{z}\,\xi^i + \frac{z}{\ell}\,b^i + \frac{z^2}{\ell^2}\,p^i_{(2)} + \mathcal{O}(z^3)\,,
\end{aligned}$$

$$
\begin{aligned}
j^{i3} &= \frac{1}{z}\,\xi^i - \frac{z}{\ell^2}\,\tilde{b}^i + \frac{z^2}{\ell^2}\,j^{i3}_{(2)} + \mathcal{O}(z^3)\,,\\
j^{ij} &= \theta^{ij} + \mathcal{O}(z^2)\,,\\
\hat{\lambda} &= \lambda + \frac{z}{\ell}\,A_{(1)\mu}\xi^\mu + \mathcal{O}(z^2)\,,\\
\epsilon^A_+ &= \sqrt{\frac{\ell}{z}}\begin{pmatrix}\eta^A_+\\ 0\end{pmatrix} + \mathcal{O}(z^{1/2})\,,\\
\epsilon^A_- &= \sqrt{\frac{z}{\ell}}\begin{pmatrix}0\\ \eta^A_-\end{pmatrix} + \mathcal{O}(z^{3/2})\,,
\end{aligned}
\tag{7.116}
$$

where the $p^i_{(2)}$ and $j^{i3}_{(2)}$ terms will play a role in cancellation of terms in the next step, but it will not influence the transformation law of the holographic fields. We also expect the conservation laws not to depend on $m^i$ because it is a gauge-fixing function. Without the gravitini, we have $b^i = \tilde{b}^i = \frac{\ell^2}{2}\,K^i$, $w^{ij} = 0$ and the result coincides with the pure AdS case (7.52).

We therefore see that the independent residual parameters in $\mathcal{N} = 2$ AdS$_4$ Supergravity are

$$
\sigma(x),\ \xi^i(x),\ \theta^{ij}(x)\,,\ \lambda(x)\,,\ \eta^A_\pm(x)
$$

and they are associated, respectively, to the dilatations, diffeomorphisms, Lorentz, Abelian and supersymmetric transformations in the holographically dual theory.
Let us notice that the parameters $b^i$ and $\tilde{b}^i$ are not on the same level as the others, because $b^i - \tilde{b}^i = 2\ell m^i$ is nonphysical and $b^i + \tilde{b}^i = \ell^2 K^i$ is not independent due to the condition

$$
0 = \delta V^3_\mu = -\ell\partial_\mu\sigma - \ell E^i_\mu K_i + \ell\xi_i\mathcal{S}^i{}_\mu + \overline{\eta}_{A+}\psi_{-A\mu} - \overline{\eta}_{A-}\psi_{+A\mu} + \mathcal{O}(z)\,, \tag{7.117}
$$

which allows to solve the finite term for $K^i$ as

$$
K^i = \frac{1}{\ell}\,E^{i\mu}\left(-\ell\partial_\mu\sigma + \ell\xi_j\,\mathcal{S}^j{}_\mu + \overline{\eta}_{A+}\psi^A_{-\mu} - \overline{\eta}_{A-}\psi^A_{+\mu}\right)\,. \tag{7.118}
$$

This confirms that $K^i$ is not an independent asymptotic parameter.
This last step completes the analysis of the gauge fixing conditions and we can now study the transformation laws of the holographic fields.

## 7.6 Transformation laws of the holographic fields

Determining the transformation laws of the boundary fields is a fundamental step in order to identify them as the sources of the boundary CFT and to obtain the Ward identities of the dual theory.
They analysis performed in the previous Subsection left us with the following power expansion of the fields

$$
V^i_\mu = \frac{\ell}{z}\,E^i_\mu + \frac{z}{\ell}\,S^i{}_\mu + \frac{z^2}{\ell^2}\,\tau^i{}_\mu + \mathcal{O}(z^3)\,,
$$

$$
\begin{aligned}
\hat{\omega}^{i3}_{\mu} &= \frac{1}{z}\,E^i_\mu - \frac{z}{\ell^2}\,\tilde{S}^i{}_\mu - \frac{2z^2}{\ell^3}\,\tilde{\tau}^i{}_\mu + \mathcal{O}(z^3)\;,\\
\hat{\omega}^{ij}_{\mu} &= \omega^{ij}_\mu + \frac{z^2}{\ell^2}\,\omega^{ij}_{(2)\mu} + \mathcal{O}(z^3)\,,\\
\hat{A}_\mu &= A_\mu + \frac{z}{\ell}\,A_{(1)\mu} + \frac{z^2}{\ell^2}\,A_{(2)\mu} + \mathcal{O}(z^3)\,,\\
\Psi^A_{\mu+} &= \sqrt{\frac{\ell}{z}}\left[\begin{pmatrix}\psi^A_{\mu+}\\ 0\end{pmatrix} + \frac{z^2}{\ell^2}\begin{pmatrix}\Pi^A_{\mu+}\\ 0\end{pmatrix} + \mathcal{O}(z^3)\right],\\
\Psi^A_{\mu-} &= \sqrt{\frac{z}{\ell}}\left[\begin{pmatrix}0\\ \psi^A_{\mu-}\end{pmatrix} + \frac{z}{\ell}\begin{pmatrix}0\\ \zeta^A_{\mu-}\end{pmatrix} + \mathcal{O}(z^2)\right].
\end{aligned}
\tag{7.119}
$$

From the bulk transformation law (7.74) and by writing the result in terms of boundary 1-forms in a basis $dx^\mu$ of $\partial M$, we find

$$
\begin{aligned}
\delta E^i &= \mathcal{D}\xi^i + \sigma E^i - \theta^{ij}E_j + \mathrm{i}\,\overline{\eta}^A_+\gamma^i\psi_{+A}\,,\\
\delta\omega^{ij} &= \mathcal{D}\theta^{ij} + 2\xi^{[i}\mathcal{S}^{j]} + 2\,K^{[i}E^{j]} + \frac{1}{\ell}\,\overline{\eta}^A_+\gamma^{ij}\psi_{-A} + \frac{1}{\ell}\,\overline{\eta}^A_-\gamma^{ij}\psi_{+A}\,,\\
\delta A &= \mathrm{d}\lambda + 2\epsilon_{AB}\,\overline{\eta}^A_+\psi^B_- + 2\epsilon_{AB}\,\overline{\eta}^A_-\psi^B_+\,,
\end{aligned}
\tag{7.120}
$$

for the bosonic fields and for the gravitino

$$
\begin{aligned}
\delta\psi_{+A} &= \mathcal{D}\eta_{A+} + \frac{\mathrm{i}}{\ell}\,E^i\gamma_i\eta_{A-} - \frac{\mathrm{i}}{\ell}\,\xi^i\gamma_i\psi_{-A} + \frac{1}{2}\,\sigma\psi_{+A}\\
&\quad -\frac{1}{4}\,\theta^{ij}\gamma_{ij}\varphi_{A+} + \frac{1}{2\ell}\,\lambda\epsilon_{AB}\,\psi^B_+ - \frac{1}{2\ell}\,A\,\epsilon_{AB}\eta^B_+\,.
\end{aligned}
\tag{7.121}
$$

Now, as a consequence of (7.69), the $OSp(2|4)$ supercurvatures vanish at the boundary. These conditions can be asymptotically expanded and read

$$
\begin{aligned}
\mathcal{R}^{ij} - 2\,E^{[i}\wedge\mathcal{S}^{j]} - \frac{1}{\ell}\,\overline{\psi}^A_+\wedge\gamma^{ij}\psi_{A-} &= 0\,,\\
\nabla\psi^A_+ + \frac{\mathrm{i}}{\ell}\,E^i\wedge\gamma_i\psi^A_- &= 0\,.
\end{aligned}
\tag{7.122}
$$

The first equation can be interpreted as saying that the supersymmetric generalisation of the Weyl tensor $W^{ij} = \mathcal{R}^{ij} - 2\,E^{[i}\wedge\mathcal{S}^{j]}$ vanishes on the boundary and can be solved in terms of the super-Schouten tensor as

$$
\mathcal{S}_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{4}\,g_{\mu\nu}\mathcal{R} - \frac{1}{\ell}\left(\overline{\psi}_{+A\,\rho}\gamma^\rho{}_\mu\psi_{-A\,\nu} - \overline{\psi}_{+A\,\nu}\gamma^\rho{}_\mu\psi_{-A\,\rho} - \frac{1}{2}\,g_{\mu\nu}\overline{\psi}_{+A\,\rho}\gamma^{\rho\lambda}\psi_{-A\,\lambda}\right). \tag{7.123}
$$

The above expression confirms the previous discussion regarding the difference between Schouten tensor and super-Schouten tensor.
The second equation can instead be solved in terms of $\psi^A_{\mu-}$ as

$$
\psi_{-A\,\mu} = -\frac{\ell}{2e_3}\,\epsilon^{\lambda\nu\rho}\gamma_\lambda\gamma_\mu\nabla_\nu\psi_{+A\,\rho}\,, \tag{7.124}
$$

by using the gamma matrix relation $\gamma_{\mu\nu} = \gamma_\mu\gamma_\nu - g_{\mu\nu}$.
These results show that $\mathcal{S}^i{}_\mu$ and $\psi_{-A\,\mu}$ are not independent fields since they can be expressed in terms of the supervielbein $(E^i{}_\mu, \psi_{+A\,\mu})$ and their curvatures.
Their transformation laws are therefore on a different level with respect to the other non-composite fields, as they also appear at a subleading order compared to the other fields:

$$\begin{aligned}
\delta\mathcal{S}^i &= \mathcal{D}K^i - \sigma\mathcal{S}^i - \theta^{ij}\mathcal{S}_j + \frac{2\mathrm{i}}{\ell^2}\,\overline{\eta}^A_-\Gamma^i\psi_{-A} + \mathcal{E}^i\,,\\
\delta\psi_{-A} &= \mathcal{D}\eta_{A-} + \frac{\mathrm{i}\ell}{2}\,\mathcal{S}^i\gamma_i\eta_{A+} - \frac{\mathrm{i}\ell}{2}\,K^i\gamma_i\psi_{+A} - \frac{1}{2}\,\sigma\psi_{-A}\\
&\quad -\frac{1}{4}\,\theta^{ij}\gamma_{ij}\varphi_{-A} + \frac{1}{2\ell}\,\lambda\epsilon_{AB}\psi^B_- - \frac{1}{2\ell}\,A\epsilon_{AB}\eta^B_- + \Sigma_A\,.
\end{aligned} \tag{7.125}$$

Eqs. (7.120)–(7.125), together with the transformation law of $B \equiv V^3_\mu \mathrm{d}x^\mu$ given by eq. (7.117), define the full set of $\mathcal{N}=2$ superconformal transformations on the boundary 1-forms $E^i$, $B$, $\mathcal{S}^i$, $\omega^{ij}$, $A$, $\psi_{\pm A}$. The factors $\ell$ ensure dimensional consistency of the equations with $[V^i{}_\mu] = L^0$, $[\mathcal{S}^i{}_\mu] = L^{-2}$, $[\psi^A_{\pm\mu}] = L^{-1/2}$, $[\xi^\mu] = [\xi^i] = L$ and $[\eta] = L^{1/2}$.
The contributions $\mathcal{E}^i = \mathcal{E}^i{}_\mu\,\mathrm{d}x^\mu$ and $\Sigma^A = \Sigma^A_\mu\,\mathrm{d}x^\mu$ come from the contraction of the curvatures and read

$$\begin{aligned}
\mathcal{E}^i{}_\mu &= \frac{2}{\ell}\,\tilde{R}^{i3}_{(3)jk}\xi^k E^j{}_\mu + \frac{1}{\ell}\,\overline{\Theta}^{i3}_{(5/2)-A|j}\left(\eta^A_+ E^j{}_\mu - \psi^A_{+\mu}\xi^j\right)\\
&= 2\xi^\nu\left(\mathcal{D}_{[\nu}\mathcal{S}^i{}_{\mu]} - \frac{\mathrm{i}}{\ell^2}\overline{\psi}^A_{\nu-}\gamma^i\psi_{\mu A-}\right) - \frac{1}{\ell}\overline{\Theta}^{i3}_{A|j-(5/2)}(\xi^\nu\psi^A_{\nu+} - \eta^A_+)E^j_\mu,\\
\Sigma^A_\mu &= 2E^\nu{}_{[i}E^\lambda{}_{j]}\left(\nabla_\nu\psi^A_{\lambda-} + \frac{\mathrm{i}\ell}{2}\,\mathcal{S}^k{}_\nu\gamma_k\psi^A_{\lambda+}\right)\xi^i E^j{}_\mu\,,
\end{aligned} \tag{7.126}$$

where we have used the expressions of the rheonomic parameters in Appendix E.
The spinor $\Sigma^A_\mu$ is the contraction of the Cottino, which is the leading term in the expansion of $\hat{\boldsymbol{\rho}}^A_-$:

$$\hat{\boldsymbol{\rho}}^A_{-\mu\nu} = \sqrt{\frac{z}{\ell}}\begin{pmatrix}0\\ \Omega^A_{\mu\nu}\end{pmatrix} + \mathcal{O}(z^{3/2})\,, \tag{7.127}$$

with $\Omega^A_{\mu\nu} = 2\nabla_{[\mu}\psi^A_{-\nu]} - \mathrm{i}\ell\,\gamma_i\psi^A_{+A[\mu}\mathcal{S}^i{}_{\nu]}$. The tensor $\mathcal{E}^i{}_\mu$ can be compared to the result described in the pure gravity case: the first term is the contraction of the super Cotton tensor

$$\mathcal{C}^i{}_{\mu\nu} = 2\mathcal{D}_{[\mu}\mathcal{S}^i{}_{\nu]} - \frac{2\mathrm{i}}{\ell^2}\,\overline{\psi}^A_{-[\mu}\gamma^i\psi_{-A|\nu]} \tag{7.128}$$

whereas the second term contains the contraction of the Cottino and of the curvature $\hat{\boldsymbol{\rho}}^A_{(1/2)\mu z+}$, as shown by

$$\Theta^{i3|j}_{(5/2)-A} = -2\mathrm{i}\gamma^{(i}\tilde{\rho}^{j)3}_{(5/2)A+} + \frac{1}{2}\,E^{i\mu}E^{j\nu}\Omega^A_{\mu\nu}\,. \tag{7.129}$$

In the end we get

$$\Sigma^A = i_\xi\Omega^A\,,$$

$$\mathcal{E}^i \;=\; i_\xi \mathcal{C}^i + \frac{1}{\ell}\left(\overline{\eta}^A_+ - \overline{\psi}_{+A\nu}\xi^\nu\right)\Theta^{i3}_{(5/2)-A|j}\,E^j\,. \tag{7.130}$$

We now observe that we could in principle fix the value of $\Pi^A_{\mu+}$ by solving $\hat{\boldsymbol{\rho}}^A_{(1/2)\mu z+} = 0$ and we could fix its form by suitably solving $\delta\Pi^A_{\mu+} = 0$ in terms of the subleading parameter $\eta^A_{(2)-}$. However, the result does not have observable consequences near the boundary, thus we will not proceed in this direction.

If we restrict the set of fields to $E^i_\mu$, $\psi_{+A\mu}$, $\omega^{ij}_\mu$, $A_\mu$ and the parameters to $\xi^i$, $\eta_{+A}$, $\theta^{ij}$ and $\lambda$, we see that $E^i_\mu$ transforms as a boundary vielbein, $\omega^{ij}_\mu$ as a boundary spin-connection and $\psi_{+A\mu}$ as a boundary gravitino, charged with respect to the SO(2) R-symmetry connection $A_\mu$. Correspondingly, the parameters $\xi^i$, $\eta_{+A}$, $\theta^{ij}$ and $\lambda$ are associated to boundary diffeomorphisms, supersymmetry, Lorentz and SO(2) gauge transformations, respectively.

On the other hand, the boundary function $\sigma$, with respect to which all the above fields have definite weight (1 for $E^i_\mu$, 1/2 for $\psi_{+A\mu}$, and 0 for $\omega^{ij}_\mu$ an $A_\mu$), is identified with the local parameter associated to Weyl dilatations because it produces rescaling of the vielbein, and therefore of the metric.

In the same fashion, the superconformal transformation is characterized by the local parameter $\eta_{-A}$, with the corresponding gauge field $\psi_{A-}$. The parameter $K^i$, although not independent within the gauge choice $V^3_\mu = 0$, corresponds to special conformal transformations, whose associated gauge connection is the super-Schouten tensor.

Let us conclude with a comment on the obtained transformation laws: the variations of the super-Schouten and of the conformino fields contain additional terms coming from the contraction of the curvatures. If we had chosen a gravity theory on global AdS these terms would be absent.
As we recall from the review of the geometric approach, their presence indicates that we are indeed studying a dynamical theory and that the theory is only locally invariant under the $OSp(2|4)$ group. This is perfectly analogous to what happens in GR without cosmological constant, which only locally is a Minkowski space having the Poincaré group as the symmetry group.
We will further discuss on this matter in the following Subsection 7.7.

**Consistency of the sub-leading gauge fixings.**

On top of the previous analysis of the asymptotic parameters, it remains to look for potential inconsistencies in having some linear terms vanishing, in particular $V^3_{(1)\mu} = \omega^{ij}_{(1)\mu} = \zeta^A_{\mu+} = 0$.
From (D.11) we see that we can solve $\delta\omega^{ij}_{(1)\mu}$ in terms of $\delta\zeta_{A\mu+}$

$$\delta\omega^{ij}_{(1)\mu} = \mathrm{i}\,E^{\nu i}E^{\lambda j}E_{k\mu}\,\delta\overline{\zeta}^A_{+[\nu}\gamma^k\psi^A_{\lambda]+} - 2\mathrm{i}\,E^{\nu[i}\delta\overline{\zeta}_{+A[\mu}\gamma^{j]}\psi^A_{\nu]+}\,.$$

It is then sufficient to look at $\delta V^3_{(1)\mu}$ and $\delta\zeta^A_{\mu+}$: the first relation reads

$$\delta V^3_{(1)\mu} = \left(-\frac{1}{\ell}p^i_{(2)} + j^{i3}_{(2)}\right)E_{i\mu} + \frac{1}{\ell}(\tau + 2\tilde{\tau})^i{}_\mu\xi_i + \overline{\eta}^A_+\zeta_{\mu-A} - \overline{\eta}^A_{(1)-}\psi_{\mu+A}$$

$$= \frac{2}{\ell}\xi^\nu(\tau + 2\tilde{\tau})_{[\nu\mu]} + 2\xi^\nu \overline{\psi}^A_{+[\nu}\zeta_{\mu]-A} = 0,$$

in virtue of (7.94). The remaining condition reads

$$\begin{aligned}
\delta\zeta^A_{\mu+} = &-\frac{\mathrm{i}}{\ell}\gamma_i\zeta^A_{-\mu}\xi^i - \frac{1}{2\ell}\psi^A_{\mu+}p^3_{(1)} + 2\tilde{\rho}^A_{(5/2)+ij}\xi^i E^j_\mu - \frac{1}{4\ell}\xi^\rho A_{(1)\rho}\epsilon^{AB}\psi_{B\mu+} \\
&+ \frac{\mathrm{i}}{4\ell}\gamma^i\psi_{B+\mu}\epsilon_{ijk}A_{(1)\rho}E^{k\rho}\xi^j + \frac{\lambda_{(1)}}{2\ell}\epsilon^{AB}\psi_{B+\mu} + \frac{\mathrm{i}}{\ell}\gamma_i\eta^A_{(1)-}E^i_\mu - \frac{1}{2\ell}A_{(1)\mu}\epsilon^{AB}\eta_{B+} \\
&+ \frac{1}{4\ell}A_{(1)\mu}\epsilon^A B\eta_{B+} - \frac{\mathrm{i}}{4\ell}\epsilon^{AB}\epsilon_{ijk}\gamma^i\eta_{B+}E^j_\mu A_{(1)\rho}E^{k\rho} = 0,
\end{aligned}$$

as all terms proportional to $A_{(1)\mu}$ and $\zeta_{\mu-}$ incredibly cancel out, by plugging in the expressions of $\tilde{\rho}^A_{ij}$, $\lambda_{(1)}$ and $\eta^A_{(1)-}$.

## 7.7 Superconformal currents in the holographic quantum theory

The asymptotic symmetries obtained in pure $\mathcal{N} = 2$ $AdS_4$ Supergravity are given by the three-dimensional superconformal transformations. According to the AdS/CFT correspondence, these are also asymptotic symmetries underlying the dual SCFT.
These transformations are expressed in terms of the local parameter $\theta^{ij}$ (associated to Lorentz transformations), $K^i$ (special conformal transformations), $\eta_{A+}$ (supersymmetric transformations), $\eta_{A-}$ (special superconformal transformations) and $\lambda$ (R-symmetry). The associated fields are the spin connection $\omega^{ij}_\mu$, the vielbein ${E^i}_\mu$, the dilatation gauge field $B_\mu$, the super-Schouten tensor ${\mathcal{S}^i}_\mu$, the gravitino $\psi^A_{+\mu}$, the conformino $\psi^A_{-\mu}$ and the graviphoton $A_\mu$.

The structure described above is summed up in the following table, where we also associate to each symmetry the corresponding quantum operator on the SCFT side.

| **Transformation** | **Local parameter** | **Source** | **Current** |
|---|---|---|---|
| Lorentz | $\theta^{ij}$ | $\omega^{ij}_\mu$ | ${J^\mu}_{ij} = 0$ |
| Translation | $\xi^i$ | $E^i_\mu$ | ${J^\mu}_i$ |
| Dilatation | $\sigma$ | $B_\mu = 0$ | ${J^\mu}_{(D)} = 0$ |
| Special conformal | $K^i$ | ${\mathcal{S}^i}_\mu$ | $J^\mu_{(K)i} = 0$ |
| Abelian R-symmetry | $\lambda$ | $A_\mu$ | $J^\mu$ |
| Supersymmetry | $\eta_{A+}$ | $\psi_{A+\mu}$ | ${J^\mu}_{A+}$ |
| Superconformal | $\eta_{A-}$ | $\psi_{A-\mu}$ | ${J^\mu}_{A-} = 0$ |

When all boundary fields (sources) are independent, the quantum operators are all non vanishing. However, when the sources are expressed in terms of other boundary fields, they become composite fields and the associated quantum operator vanish. In our case, the spin connection is completely determined by a constraint on the translation curvature (super-torsion), whereas the super-Schouten tensor and the conformino are expressed in terms the boundary conditions (7.123) and (7.124). At last, we fixed $B_\mu = {V^3}_\mu = 0$, which effectively eliminates the dilatation gauge field and the corresponding quantum operator.

As explained in the bosonic analysis, the fact that some currents are vanishing simply means that a part of the symmetry group is realised non-linearly. We now describe the properties of the three-dimensional superconformal group $\mathfrak{osp}(2|4)$ [59] and from this discussion we will be able to better understand our specific case.

The full $OSp(2|4)$ superalgebra describes the symmetries of the vacuum bulk theory and of its dual SCFT. From a gravity point of view, it is encoded in the definition of the vanishing curvatures $\hat{\mathbf{R}}^\Lambda = \{\hat{\mathbf{R}}^{ab}, \hat{\mathbf{R}}^a, \hat{\boldsymbol{\rho}}^A, \hat{\mathbf{F}}\}$,

$$\hat{\mathbf{R}}^\Lambda \equiv \mathrm{d}\boldsymbol{\mu}^\Lambda + \frac{1}{2} C_{\Sigma\Gamma}{}^\Lambda \, \boldsymbol{\mu}^\Sigma \wedge \boldsymbol{\mu}^\Gamma \,, \tag{7.131}$$

where $C_{\Sigma\Gamma}{}^\Lambda$ are the $\mathfrak{osp}(2|4)$ structure constants and $\boldsymbol{\mu}^\Lambda = \{\hat{\omega}^{ab}, V^a, \Psi_A, \hat{A}\}$ the Cartan 1-forms. These curvatures can be expressed as

$$\begin{aligned}
\hat{\mathbf{R}}^{ij} &= \hat{\mathcal{R}}^{ij} + \frac{4}{\ell^2} V_+^{[i} \wedge V_-^{j]} - \frac{1}{\ell} \overline{\Psi}_+^A \wedge \Gamma^{ij} \Psi_{A-} \,, \\
\hat{\mathbf{R}}_\pm^i &= \hat{\mathcal{D}} V_\pm^i \mp \frac{1}{\ell} V_\pm^i \wedge V^3 \mp \frac{\mathrm{i}}{2} \overline{\Psi}_\pm^A \wedge \Gamma^i \Psi_{A\pm} \,, \\
\hat{\mathbf{R}}^3 &= \mathrm{d}V^3 + \frac{2}{\ell} V_+^i \wedge V_{-i} + \overline{\Psi}_-^A \wedge \Psi_{A+} \,, \\
\hat{\mathbf{F}} &= \mathrm{d}\hat{A} - 2\epsilon_{AB} \overline{\Psi}_+^A \wedge \Psi_-^B \,, \\
\hat{\boldsymbol{\rho}}^A &= \hat{\mathcal{D}} \Psi_\pm^A \pm \frac{\mathrm{i}}{\ell} V_\pm^i \wedge \Gamma_i \Psi_\mp^A \pm \frac{1}{2\ell} V^3 \wedge \Psi_\pm^A - \frac{1}{2\ell} \epsilon_{AB} \hat{A} \wedge \Psi_\pm^B \,,
\end{aligned} \tag{7.132}$$

where $V^3$ is the 1-form associated with the Weyl transformations, $V_+^i$ the ones associated with the spacetime translations, $V_-^i$ with the conformal boosts, $\Psi_+^A$ with the supersymmetries, $\Psi_-^A$ with the superconformal transformations [60,61].
The very same algebra can be expressed in terms of the Cartan 1-forms describing the superconformal algebra in $d = 3$, which can be obtained as the leading order 1-form in the $z$-expansion of the above bulk quantities.
Let us summarize below the correspondence between the $D = 4$ gauge field and $d = 3$ superconformal field in the table below

| | | | |
|---|---|---|---|
| $\hat{\omega}^{ij}$ | $\to$ | $\omega^{ij}$ | Lorentz symmetry , |
| $V^3$ | $\to$ | $B$ | Weyl symmetry , |
| $V_+^i$ | $\to$ | $E^i$ | spacetime translations , |
| $V_-^i$ | $\to$ | $\mathcal{S}^i$ | conformal boosts , |
| $\Psi_+^A$ | $\to$ | $\psi_+^A$ | supersymmetry , |
| $\Psi_-^A$ | $\to$ | $\psi_-^A$ | superconformal symmetry , |
| $\hat{A}$ | $\to$ | $A$ | SO(2) R-symmetry . |

Indeed, if we define

$$B = \frac{1}{\ell} \left( V^3 - \ell \frac{\mathrm{d}z}{z} \right) = B_\mu(x) \, \mathrm{d}x^\mu \,, \tag{7.133}$$

which is non vanishing only for generalisations of the FG parametrization (7.11) and after rescaling the various fields by $z/\ell$ factors according to their $O(1,1)$ gradings one can rewrite the algebra in terms of SCFT quantities as

$$
\begin{aligned}
\mathbf{R}^{ij} &= \mathcal{R}^{ij} - 2\,E^{[i} \wedge \mathcal{S}^{j]} - \frac{1}{\ell}\,\overline{\psi}^A_+ \wedge \gamma^{ij}\psi_{A-}\,,\\
\mathbf{R}^i_+ &= \mathcal{D}E^i + B \wedge E^i - \frac{\mathrm{i}}{2}\,\overline{\psi}^A_+ \wedge \gamma^i\psi_{A+}\,,\\
\mathcal{C}^i \equiv -\frac{2}{\ell^2}\,\mathbf{R}^i_- &= \mathcal{D}\mathcal{S}^i - B \wedge \mathcal{S}^i - \frac{\mathrm{i}}{\ell^2}\,\overline{\psi}^A_- \wedge \gamma^i\psi_{A-}\,,\\
\mathbf{R} &= \mathrm{d}B - E^i \wedge \mathcal{S}_i + \frac{1}{\ell}\,\overline{\psi}^A_- \wedge \psi_{A+}\,,\\
\mathbf{F} &= \mathrm{d}A - 2\epsilon_{AB}\,\overline{\psi}^A_+ \wedge \psi^B_-\,,\\
\boldsymbol{\rho}^A_+ &= \mathcal{D}\psi^A_+ + \frac{1}{2}\,B \wedge \psi^A_+ + \frac{\mathrm{i}}{\ell}\,E^i \wedge \gamma_i\psi^A_- - \frac{1}{2\ell}\,\epsilon_{AB}A \wedge \psi^B_+\,,\\
\Omega^A \equiv \boldsymbol{\rho}^A_- &= \mathcal{D}\psi^A_- - \frac{1}{2}\,B \wedge \psi^A_- + \frac{\mathrm{i}\ell}{2}\,\mathcal{S}^i \wedge \gamma_i\psi^A_+ - \frac{1}{2\ell}\,\epsilon_{AB}A \wedge \psi^B_-\,,
\end{aligned}
\tag{7.134}
$$

where $\mathcal{D}$ is the usual Lorentz-covariant derivative. Notice that each $\mathcal{D}$ always appears in the combination $\mathcal{D} + \Delta B$ of the Weyl-covariant derivative, as naturally expected from a theory with local Weyl symmetry, where $\Delta$ is the scaling dimension of the corresponding field, $\Delta(E^i_\pm) = \pm 1$, $\Delta(\psi^A_\pm) = \pm\frac{1}{2}$, $\Delta(\mathcal{S}^i) = -1$ and $\Delta(\omega^{ij}) = \Delta(A) = \Delta(B) = 0$. This is a useful feature that can allow to reconstruct the $B$-term in the transformation laws (7.120)–(7.125), in the same way as it was done in the pure AdS gravity given by eqs. (7.60).
We notice that the curvatures in (7.134) are not all zero in our case: indeed we can recognise, in the $B = 0$ case, the non-vanishing super Cotton and the Cottino tensors, which appear in the variations of the boundary fields. From the above analysis, these additional terms can really be understood as contractions of curvatures and they signal that the dual quantum theory is a deformation of a theory truly invariant under the global $OSp(2|4)$ group.

We are now ready to explore the quantum symmetries in the theory dual to Supergravity.

**Superconformal currents**

In the AdS/CFT framework, the obtained boundary fields $\mathcal{J}^\Lambda(x) = \{E^i{}_\mu(x),\ \omega^{ij}_\mu(x),\ \psi_{+A\,\mu}(x),\ A_\mu(x)\}$ become sources for the corresponding operators in the dual quantum theory $J^\mu_\Lambda = \{J^\mu{}_i,\ J^\mu{}_{ij},\ J^\mu_{A+},\ J^\mu\}$. The latter are the energy-momentum tensor, spin current, supercurrent and $U(1)$-current, respectively and are identified with the expectation values of the Noether currents associated with the residual symmetries of the boundary action. However, we shall refrain from writing explicitly the expectation value symbol $\langle \ldots \rangle$.
According to the AdS/CFT correspondence, the bulk action in classical Supergravity approximation is identified with the effective action of the dual boundary theory as

$$
I_{\text{on}-\text{shell}}[E^i,\,\omega^{ij},\,\psi^A_+,\,A] = W[E^i,\,\omega^{ij},\,\psi^A_+,\,A] = -\mathrm{i}\,\ln(Z[E^i,\,\omega^{ij},\,\psi^A_+,\,A])\,. \tag{7.135}
$$

This identification allows to obtain the explicit expression of the currents and conservation laws that they have to satisfy. Let us start from the currents: in the derivation it is convenient to retain a four-dimensional notation. Indeed, by taking the variation of the above formula, we obtain

$$\delta W = \delta I_{\text{on-shell}} = \int\limits_{\partial\mathcal{M}} \left( -\frac{\ell^2}{4}\,\delta\hat{\omega}^{ab}\hat{\boldsymbol{R}}^{cd}\epsilon_{abcd} - 2\mathrm{i}\ell\delta\overline{\Psi}^A\Gamma_5\hat{\boldsymbol{\rho}}_A + \frac{1}{2}\,\delta\hat{A}\,{}^*\hat{\boldsymbol{F}} \right)\Bigg|_{z=\mathrm{d}z=0}^{\text{on-shell}} . \tag{7.136}$$

We can now asymptotically expand all fields in the above formula and write only the finite contributions, as all other contributions vanish when we set $z = \mathrm{d}z = 0$. The result has to be compared to the variation of the left hand side, which reads

$$\delta W = \int\limits_{\partial\mathcal{M}} \delta\mathcal{J}^\Lambda \wedge J_\Lambda = \int\limits_{\partial\mathcal{M}} \left( \delta E^i \wedge J_i + \frac{1}{2}\,\delta\omega^{ij}\wedge J_{ij} + \overline{J}^A_+ \wedge \delta\psi_{A+} + J\wedge\delta A \right) . \tag{7.137}$$

This procedure yields the explicit form of the currents

$$\begin{aligned}
J_i &= \frac{1}{2}\,\epsilon_{ijk}\left[\frac{2}{\ell}\,E^j\wedge(\tau^k + 2\tilde{\tau}^k) + \overline{\psi}^A_+\wedge\gamma^{jk}\zeta_{A-}\right],\\
J_{ij} &= 0\,,\\
J &= \frac{1}{2}\,\epsilon_{ijk}\,\tilde{F}^{i3}\,V^j\wedge V^k\Big|_{z=0}\,,\\
J^A_+ &= -2\,\mathrm{i}\,E^i\wedge\gamma_i\zeta^A_- + A_{(1)}\wedge\epsilon_{AB}\,\psi^B_+\,.
\end{aligned} \tag{7.138}$$

Here we find that the current associated to Lorentz transformation $J_{ij}$ vanishes, because the spin connection is a composite field, as we expect. The other composite fields (${\mathcal{S}^i}_\mu$ and $\psi_{A-\mu}$) have not been taken into account as sources.
The Hodge dual of the obtained expressions yields the Noether currents

$$\begin{aligned}
J^\mu_{\ i} &= -\frac{1}{\ell}\,\left((\tau^\mu_{\ i} + 2\tilde{\tau}^\mu_{\ i}) - E^\mu_{\ i}(\tau^k_{\ k} + 2\tilde{\tau}^k_{\ k})\right) + \frac{\mathrm{i}}{e_3}\,\epsilon^{\mu\nu\rho}\,\bar{\psi}^A_{+\nu}\gamma_i\zeta_{A-\rho}\,,\\
J^\mu_{A+} &= -\frac{2\mathrm{i}}{e_3}\,\epsilon^{\mu\nu\rho}\gamma_\nu\zeta_{A-\rho} + \frac{1}{e_3}\,\epsilon^{\mu\nu\rho}\,A_{(1)\nu}\epsilon_{AB}\,\psi^B_{+\rho}\,,\\
J^\mu &= -g^{\mu\nu}_{(0)}\,\tilde{F}_{\nu z} = \frac{1}{2\ell}\,g^{\mu\nu}_{(0)}\,A_{(1)\nu}\,,
\end{aligned} \tag{7.139}$$

where in the first equation the traces ${\tau^k}_k$, ${\tilde{\tau}^k}_k$ are defined using the vielbein tensor (e.g. ${\tau^k}_k \equiv {\tau^k}_\mu\,{E^\mu}_k$).
Let us conclude this analysis by recognising the holographic stress tensor $J_{\mu\nu} = J_{\mu i}{E^i}_\nu$: as a consistency check, we remember that in the bosonic case this quantity is traceless and this is confirmed by setting fermions to zero. In the supersymmetric case, the trace of $\tau_{\mu\nu} + 2\tilde{\tau}_{\mu\nu}$ is not necessarily zero and it has to be computed from the conservation law of the local Weyl symmetry. We now move on to the computation of the conservation laws that these operators must obey.

### Conservation laws

We now derive the explicit form of the conservation laws that the quantum operators have to satisfy. In the next paragraph we will prove that they are satisfied. The variation of the quantum action evaluated on the corresponding symmetry transformation of the fields must vanish: an integration by parts yields the following conservation laws

$$\begin{aligned}
\mathcal{D}J_i &= \mathcal{S}^j\, J_{ij} - \frac{\mathrm{i}}{\ell}\overline{J}^A_+\gamma_i\psi_{A-} + \mathcal{S}^k{}_i\, J_{kj}\, E^j - \frac{\mathrm{i}\ell}{2}\,\mathcal{S}^j{}_i\,\overline{J}^A_-\gamma_j\psi_{A+}\,, \\
\mathcal{D}J_{ij} &= 2\,E_{[i}\,J_{j]} - \frac{\mathrm{i}}{2}\,\overline{J}^A_+\gamma_{ij}\psi_{A+} - \frac{\mathrm{i}}{2}\,\overline{J}^A_-\gamma_{ij}\psi_{A-}\,, \\
0 &= \partial_\mu\left[E^{\mu i}\left(J_{ij}\,E^j - \frac{\mathrm{i}\ell}{2}\,\overline{J}^A_-\gamma_i\psi_{A+}\right)\right] + E^i\,J_i + \frac{1}{2}\,\overline{J}^A_+\,\psi_{A+} - \frac{1}{2}\,\overline{J}^A_-\,\psi_{A-}\,, \\
\mathrm{d}J &= \frac{1}{2\ell}\,\epsilon_{AB}\left(\overline{J}^A_+\,\psi_{B+} + \overline{J}^A_-\,\psi_{B-}\right)\,, \\
\nabla J_{A+} &= \frac{1}{2\ell}\,\gamma^{ij}\psi_{A-}\,J_{ij} + \mathrm{i}\,\gamma^i\,\psi_{A+}\,J_i - \frac{\mathrm{i}\ell}{2}\,\mathcal{S}^i\,\gamma_i\,J_{A-} + 2\,\epsilon_{AB}\,\psi_{B-}\,J + \frac{1}{\ell}\,\psi^i_{A-}\,J_{ij}\,E^j \\
&\quad - \frac{\mathrm{i}}{2}\,\psi^i_{A-}\,\overline{J}_{B-}\gamma_i\psi_{B+}\,, \qquad\qquad (7.140)\\
\nabla J_{A-} &= \frac{1}{2\ell}\,\gamma^{ij}\psi_{A+}\,J_{ij} + 2\,\epsilon_{AB}\,\psi_{B+}\,J - \frac{\mathrm{i}}{\ell}\,E^i\,\gamma_i\,J_{A+} - \frac{1}{\ell}\,\psi^i_{A+}\,J_{ij}\,E^j + \frac{\mathrm{i}}{2}\,\psi^i_{A+}\,\overline{J}_{B-}\gamma_i\psi_{B+}\,.
\end{aligned}$$

The above conservations laws reduce to the bosonic ones (7.59), as it can be checked in the absence of fermions and of the $U(1)$ gauge field. Moreover, we can notice that fermions are sources for the electromagnetic current $J$.
Let us now comment on the third equation in (7.140): since $J_{ij} = J^A_- = 0$, we have

$$E^i \wedge J_i = -\frac{1}{2}\,\overline{J}^A_+ \wedge \psi_{A+}\,. \qquad (7.141)$$

By plugging in the explicit expression of the currents, we find the trace of the bosonic part of the holographic stress tensor, namely

$$(2\tilde{\tau} + \tau)^l{}_l = -\mathrm{i}\,\ell\,\epsilon^{ijk}\bar{\psi}^A_{+j}\gamma_i\zeta_{A-\,k}\,, \qquad (7.142)$$

which can be proven to be consistent with the results achieved in [58]. We notice that the trace of the bosonic part is different from zero, but this is to be expected from the structure of the superalgebra. This does not mean that we have an anomaly, because an anomaly would imply having different contributions to $J^i \wedge E_i$ than the one given in eq. (7.141).

Similarly, both $J_{\mu\nu}$ and $\tau_{\mu\nu} + 2\tilde{\tau}_{\mu\nu}$ are not symmetric: the second conservation law gives the antisymmetric part as $E_{[i} \wedge J_{j]} = \frac{\mathrm{i}}{4}\overline{J}_+\gamma_{ij} \wedge \psi_+$. This indicates that, with our gauge fixing choice, $J_{\mu\nu}$ is not, as in pure gravity, the traceless Belinfante-Rosenfeld stress tensor. However, we know that, in principle, it is possible to use an ambiguity in definitions of Noether currents to construct so-called 'improved' stress tensor which would be symmetric and traceless.

#### The Ward identities

We now prove that the Ward identities are indeed satisfied, by requiring the effective action to be invariant under the superconformal transformations. It is important to notice that even if all the expressions are evaluated on shell in the bulk Supergravity, they will yield off shell identities for the QFT side, computed on the curved background.
This result will be achieved by noticing that we can identify the variation of the effettive action with the bulk/boundary gauge transformations written in terms of the gauge parameters $\hat{\Lambda}(x,z) = \{j^{ab}, p^a, \epsilon^A_\pm, \hat{\lambda}\}$ and $\Lambda(x) = \{\theta^{ij}, \xi^i, \sigma, \eta^A_\pm, \lambda\}$, namely

$$\delta W \equiv \delta_\Lambda W = \delta_{\hat{\Lambda}} W\Big|^{\text{on-shell}}_{z=\mathrm{d}z=0} . \tag{7.143}$$

The above expression vanishes provided that the coefficient of the independent symmetry parameters vanish as well. This method will thus allow to obtain the four dimensional expression of the Ward identities and clearly makes use of an already renormalised quantum effective action.
Let us start by integrating (7.136) by parts

$$\begin{aligned}\delta W = \int\limits_{\partial\mathcal{M}} \Bigg[ & \frac{\ell^2}{4} j^{ab} \mathcal{D}\hat{\boldsymbol{R}}^{cd}\epsilon_{abcd} - \frac{\ell^2}{4}\left(\frac{2}{\ell^2}p^a V^b + \frac{1}{\ell}\overline{\epsilon}^A\Gamma^{ab}\Psi_A\right)\hat{\boldsymbol{R}}^{cd}\epsilon_{abcd} + 2\mathrm{i}\ell\overline{\epsilon}^A\Gamma_5\mathcal{D}\hat{\boldsymbol{\rho}}_A \\ & - 2\mathrm{i}\ell\left(\frac{1}{4}j^{ab}\overline{\Psi}^A\Gamma_{ab} + \frac{\mathrm{i}}{2\ell}p^a\overline{\Psi}^A\Gamma_a + \frac{1}{2\ell}\lambda\epsilon^{AB}\overline{\Psi}_B - \frac{1}{2\ell}\hat{A}\epsilon^{AB}\overline{\epsilon}_B - \frac{\mathrm{i}}{2\ell}\overline{\epsilon}^A\Gamma_a V^a\right)\Gamma_5\hat{\boldsymbol{\rho}}_A \\ & - \frac{1}{2}\lambda\ \mathrm{d}\,{}^*\hat{\boldsymbol{F}} + \overline{\epsilon}^A\Psi^B\epsilon_{AB}\,{}^*\hat{\boldsymbol{F}}\Bigg]\Bigg|^{\text{on-shell}}_{z=\mathrm{d}z=0} . \end{aligned} \tag{7.144}$$

By making use of the Bianchi identities (7.62), we obtain

$$\begin{aligned}\delta W = \int\limits_{\partial\mathcal{M}} \Bigg[ & \frac{\ell}{4} j^{ab}\overline{\Psi}^A\Gamma^{cd}\hat{\boldsymbol{\rho}}_A\epsilon_{abcd} - \frac{\ell^2}{4}\left(\frac{2}{\ell^2}p^a V^b + \frac{1}{\ell}\overline{\epsilon}^A\Gamma^{ab}\Psi_A\right)\hat{\boldsymbol{R}}^{cd}\epsilon_{abcd} \\ & + 2\mathrm{i}\ell\left(\frac{1}{2\ell}\hat{A}\epsilon^{AB}\overline{\epsilon}_A\Gamma_5\hat{\boldsymbol{\rho}}_B - \frac{\mathrm{i}}{2\ell}\overline{\epsilon}^A\Gamma_5\Gamma_a\hat{\boldsymbol{\rho}}_A V^a + \frac{1}{4}\hat{\boldsymbol{R}}^{ab}\overline{\epsilon}^A\Gamma_5\Gamma_{ab}\Psi_A - \frac{1}{2\ell}\epsilon^{AB}\hat{\boldsymbol{F}}\overline{\epsilon}_A\Gamma_5\Psi_B\right) \\ & - 2\mathrm{i}\ell\left(\frac{1}{4}j^{ab}\overline{\Psi}^A\Gamma_{ab} + \frac{\mathrm{i}}{2\ell}p^a\overline{\Psi}^A\Gamma_a + \frac{1}{2\ell}\lambda\epsilon^{AB}\overline{\Psi}_B - \frac{1}{2\ell}\hat{A}\epsilon^{AB}\overline{\epsilon}_B - \frac{\mathrm{i}}{2\ell}\overline{\epsilon}^A\Gamma_a V^a\right)\Gamma_5\hat{\boldsymbol{\rho}}_A \\ & - \frac{1}{2}\lambda\ \mathrm{d}^*\hat{\boldsymbol{F}} + \overline{\epsilon}^A\Psi^B\epsilon_{AB}\,{}^*\hat{\boldsymbol{F}}\Bigg]\Bigg|^{\text{on-shell}}_{z=\mathrm{d}z=0} . \end{aligned} \tag{7.145}$$

Let us start from the Lorentz transformations: the coefficient in front of $j^{ab}$ is given by

$$\frac{\ell}{4}\,j^{ab}\overline{\Psi}^A\Gamma^{cd}\hat{\boldsymbol{\rho}}_A\,\epsilon_{abcd} - \frac{\mathrm{i}\ell}{2}\,j^{ab}\overline{\Psi}^A\Gamma_{ab}\Gamma_5\hat{\boldsymbol{\rho}}_A = 0 \tag{7.146}$$

and identically vanishes, due to the identity (B.7) of four-dimensional gamma matrices.

As for the terms containing $p^a$, one finds, up to terms which vanish in the $z \to 0$ limit,

$$-\frac{1}{2}\, p^a V^b \hat{\boldsymbol{R}}^{cd}\epsilon_{abcd} + p^a \overline{\Psi}^A \Gamma_a \Gamma_5 \hat{\boldsymbol{\rho}}_A\,. \tag{7.147}$$

The above expression disappears at the boundary by effect of the equations of motion (see (7.71)),

$$-\frac{1}{2}\, p^a V^b \hat{\boldsymbol{R}}^{cd}\epsilon_{abcd} + p^a \overline{\Psi}^A \Gamma_a \Gamma_5 \hat{\boldsymbol{\rho}}_A = \frac{1}{2}\, p^a \epsilon_{abcd} V^b \left( \hat{F}^{cd} \hat{\boldsymbol{F}} - \frac{1}{6} \hat{F}_{ef} \hat{F}^{ef} V^c V^d \right), \tag{7.148}$$

since the two terms on the right hand side are zero at $z = 0$.

The terms involving the parameter $\epsilon_A$ are given by

$$\begin{aligned}
&\mathrm{i}\hat{A}\epsilon^{AB}\overline{\epsilon}_A \Gamma_5 \hat{\boldsymbol{\rho}}_B + \overline{\epsilon}^A \Gamma_5 \Gamma_a \hat{\boldsymbol{\rho}}_A V^a + \frac{\mathrm{i}\ell}{2} \hat{\boldsymbol{R}}^{ab} \overline{\epsilon}^A \Gamma_5 \Gamma_{ab} \hat{\boldsymbol{\rho}}_A - \mathrm{i}\epsilon^{AB} \hat{\boldsymbol{F}} \overline{\epsilon}_A \Gamma_5 \Psi_B \\
&- \frac{\ell}{4} \overline{\epsilon}^A \Gamma^{ab} \Psi_A \hat{\boldsymbol{R}}^{cd} \epsilon_{abcd} + \mathrm{i}\hat{A}\epsilon^{AB} \overline{\epsilon}_B \Gamma_5 \hat{\boldsymbol{\rho}}_A - \overline{\epsilon}^A \Gamma_a \Gamma_5 \hat{\boldsymbol{\rho}}_A V^a + \overline{\epsilon}^A \Psi^B \epsilon_{AB}\, {}^*\hat{\boldsymbol{F}} \\
&= \overline{\epsilon}^A (-2\Gamma_a V^a \Gamma_5 \hat{\boldsymbol{\rho}}_A + \epsilon_{AB} \Psi^B\, {}^*\hat{\boldsymbol{F}} - \mathrm{i}\epsilon_{AB} \hat{\boldsymbol{F}} \Gamma_5 \Psi^B)
\end{aligned} \tag{7.149}$$

and all vanish as a consequence of the equations of motion of the gravitino (7.71).

Finally, the terms depending on $\hat{\lambda}$ and find

$$\hat{\lambda} \left( -\frac{1}{2}\, \mathrm{d}^* \hat{\boldsymbol{F}} - \mathrm{i}\epsilon^{AB} \overline{\Psi}_B \Gamma_5 \hat{\boldsymbol{\rho}}_A \right), \tag{7.150}$$

which vanishes by virtue of the gauge field equations of motion in (7.71).

This concludes the proof of $\delta W = 0$, which, as said, corresponds to proving that the equations (7.140), which were derived from $\delta W = 0$ in the three-dimensional notation, are indeed satisfied. This can be seen as a consequence of the absence of any anomaly, in particular conformal anomaly, in $d = 3$.
Note that, in the above derivation, we have neglected the curvature-contraction terms occurring in the general expression of the symmetry variations of the fields which one can check to give vanishing contributions at the boundary.
We now conclude this Section with some comments on the obtained results and on the possible future developments.

## 7.8 Discussion

In this Section we developed in detail the holographic framework for a $\mathcal{N} = 2$ pure $\mathrm{AdS}_4$ Supergravity in first order formalism, including all the contributions in the fermionic fields. This analysis generalises the results of [58, 62, 63] and includes a general discussion of the gauge-fixing conditions on the bulk fields, which then lead to the asymptotic symmetries at the boundary. The corresponding currents of the boundary theory are constructed and shown to satisfy the associated Ward identities, once the field equations of the bulk theory are imposed.

We then conclude that the boundary terms originally introduced in [46] to restore the Supersymmetry invariance of the action also manage to regularise the theory from an asymptotic point of view. In this sense, the vanishing of the curvatures at the boundary (7.69), which is a direct consequence of the way the theory has been regularised, proved to be the key to obtain a finite resulting dual theory. They indeed cancel all diverging terms appearing in (7.136) and allow to derive the super-Schouten tensor, genearlising the bosonic expression and the conformino.
These results then prove that this procedure actually generalised the topological regularisation to the supersymmetric case.

By working in first order formalism, we were able to keep the full superconformal structure of the vacuum theory manifest in principle, even if only a part of it is realized as a symmetry of the theory on $\partial M$, as the rest appears as a non-linear realization on $\partial M$.
It would be now interesting to choose specific solutions on the gravity side: for example we could consider a special choice for which $\psi_{A-\mu} \propto \psi_{A+\mu}$. We expect the dual theory to describe fluctuations around a maximally symmetric three dimensional background, AdS$_3$, dS$_3$ or Mink$_3$ depending on the explicit form of Schouten tensor and conformino. Such theory would be obtained as a suitable projection on the $OSp(2|4)$ asymptotic symmetry group. This research line would be interesting for including the mentioned Unconventional Supersymmetry in the gauge/gravity duality framework.
For this specific reason, we also refrained from imposing $\gamma^\mu \psi_{\pm\mu} = 0$ in the dual theory, because this would mean setting the spin-1/2 field $\chi$ to zero. We shall pursue this objective in a future investigation, where it may be necessary to generalise the FG choice, which at the same time could allow for non vanishing spin-1/2 fields and which would put special conformal transformations on the same level as other transformations.
Furthermore, as anticipated, the results presented in this Section are a first step towards an extension to $\mathcal{N} > 2$ bulk Supergravity, which would be relevant for extending the work [34] to a holographic context.

# A Differential form conventions and Levi Civita symbol

In this Appendix we state some useful formulas used throughout the text involving differential forms. Given a smooth $m$-dimensional manifold $M$, a $p$-form is locally expressed in terms of differentials $\mathrm{d}x^\mu$ in the following way

$$\omega^{(p)} = \frac{1}{p!}\,\omega_{\mu_1\ldots\mu_p}\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_p}, \tag{A.1}$$

where the Einstein summation convention is understood. If the starting manifold is endowed with a metric structure, the Hodge dual is a map ${}^*: \Omega^p(M)\to\Omega^{m-p}(M)$ can be defined as

$${}^*\omega = \frac{\sqrt{g}}{p!(m-p)!}\epsilon_{\mu_1\ldots\mu_m}\omega^{\mu_1\ldots\mu_p}\mathrm{d}x^{\mu_{p+1}}\wedge\ldots\wedge\mathrm{d}x^{\mu_m}.$$

The Levi-Civita Symbol appearing in the above definition and in Part II of this thesis is defined as

$$\epsilon_{0123} = -\epsilon^{0123} = 1 \tag{A.2}$$

and it is related to the $d=3$ symbol in the following way

$$\epsilon_{ijk} := \epsilon_{ijk3}, \qquad \epsilon^{ijk} := -\epsilon^{ijk3} \implies \epsilon_{012} = \epsilon^{012} = 1. \tag{A.3}$$

# B Gamma matrices and spinors conventions

In this Appendix, we clarify the convention used for spinors and gamma matrices, appearing in the text. The $D=5$ gamma matrices and charge conjugation matrix appearing in (6.1) are defined as

$$\tilde{\Gamma}_a := i\Gamma_a\Gamma_5, \qquad \tilde{\Gamma}_4 := \Gamma_5, \qquad C_5 := \tilde{\Gamma}_0\tilde{\Gamma}_4 = \Gamma_0, \tag{B.1}$$

in terms of the $D=4$ gamma matrices, which are given here in terms of the Pauli matrices as

$$\begin{aligned} &\Gamma^i = \sigma_1\otimes\gamma^i\,, \quad \gamma^0 = \sigma_2\,, \quad \gamma^1 = i\sigma_1\,, \quad \gamma^2 = i\sigma_3\,, \\ &\Gamma^3 = i\sigma_3\otimes\mathbf{1}\,, \quad \Gamma_5 = i\,\Gamma^0\Gamma^1\Gamma^2\Gamma^3 = -\sigma_2\otimes\mathbf{1} = \begin{pmatrix} \mathbb{0} & \mathrm{i}\mathbb{1}_2 \\ -\mathrm{i}\mathbb{1}_2 & \mathbb{0}\end{pmatrix}. \end{aligned} \tag{B.2}$$

Here $\eta^{ab} = \mathrm{diag}(+,-,-,-)$ and $a$, $b,\ldots = 0,1,2,3$ and the Clifford algebra is given by

$$\{\Gamma^a,\Gamma^b\} = 2\eta^{ab}\mathbf{1}_{4\times4}. \tag{B.3}$$

The matrices defined in (B.1) satisfy the following relations

$$\tilde{\Gamma}_{ab} = \Gamma_{ab}, \qquad \tilde{\Gamma}_{a4} = i\Gamma_a, \qquad C_5 = C_5^{-1} = -C_5^t = -C_5^*, \qquad C_5^{-1}\tilde{\Gamma}_{\mathcal{A}}C_5 = (\tilde{\Gamma}_{\mathcal{A}})^T, \tag{B.4}$$

where $\mathcal{A} = (a, 4)$. In particular, the five dimensional charge conjugation matrix $C_5$ acts on four dimensional gamma matrices in the usual way

$$C_5^{-1}\Gamma_a C_5 = -(\Gamma_a)^T. \tag{B.5}$$

In these cases we will just indicate the charge conjugation matrix as $C$. These last relation allows to derive a general property of the antisymmetric product of $k$ gamma matrices

$$(C\Gamma^{a_1 \ldots a_k})^T = -(-1)^{\frac{k(k+1)}{2}} C\Gamma^{a_1 \ldots a_k}, \tag{B.6}$$

which is particularly useful, when ones exchanges the order of spinors in bilinears. Since in four dimensions one cannot have more than 4 antisymmetrised different gamma matrices, one has the following important relation

$$\frac{1}{2}\,\epsilon_{abcd}\,\Gamma^{cd} = \mathrm{i}\Gamma_{ab}\Gamma_5\,, \tag{B.7}$$

where the Levi-Civita symbol has been defined in the previous Appendix.

Let us now focus on the spinor conventions: the Dirac conjugate of a four dimensional spinor is given by

$$\bar{\Psi} := \Psi^\dagger\Gamma_0 = -i\Psi^\dagger\tilde{\Gamma}_0\tilde{\Gamma}_4, \tag{B.8}$$

whereas Majorana spinors satisfy the reality condition

$$\Psi = \Psi^* = -C_5\overline{\Psi}^t. \tag{B.9}$$

Gravitini are Majorana spinors, which are also 1-forms, which have to be expanded along both bosonic and fermionic direction of superspace, after the Principal bundle breakdown has been performed.
Let us state here properties of bilinears involving gravitini: they descend from (B.6) and read

$$\begin{aligned}
\overline{\Psi}_A\Phi_B &= -\overline{\Phi}_B\Psi_A,\\
\overline{\Psi}_A\Gamma^a\Phi_B &= \overline{\Phi}_B\Gamma^a\Psi_A\\
\overline{\Psi}_A\Gamma^{ab}\Phi_B &= \overline{\Phi}_B\Gamma^{ab}\Psi_A\\
\overline{\Psi}_A\Gamma^a\Gamma_5\Phi_B &= -\overline{\Phi}_B\Gamma^a\Gamma_5\Psi_A\\
\overline{\Psi}_A\Gamma_5\Phi_B &= -\overline{\Phi}_B\Gamma_5\Psi_A,
\end{aligned} \tag{B.10}$$

where $A, B$ are R-symmetry indices.
For the purposes of this thesis, gravitini are decomposed with respect to the $\Gamma^3$ matrix, instead of the usual $\Gamma_5$: this is due to the fact that we always consider asymptotic limits along the radial direction identified by the $a = 3$ component. We define the projectors

$$\mathbb{P}_\pm = \frac{\mathbb{1} \mp \mathrm{i}\Gamma^3}{2} \quad \Rightarrow \quad \mathbb{P}_\pm\Psi_\pm = \Psi_\pm\,, \quad \overline{\Psi}_\pm = \overline{\Psi}_\pm\mathbb{P}_\mp\,, \tag{B.11}$$

provided that $\Gamma^3$ has eigenvalues

$$-\,\mathrm{i}\Gamma^3\Psi_\pm = \pm\Psi_\pm\,. \tag{B.12}$$

For the sake of completeness, we state here the $D = 3$ relation corresponding to (B.7)

$$\gamma^{ij} = i\epsilon^{ijk}\gamma_k, \tag{B.13}$$

where $i, j = 0, 1, 2$.

# C Radial foliation and Gaussian coordinates

Let us consider a pseudo Riemannian spacetime manifold $M$ and introduce a radial foliation $(\Sigma_r)_{r\in\mathbb{R}}$ in such a way that each hypersurface is described by some coordinate system $x^\mu$ and that $x^{\hat{\mu}} = (r, x^\mu)$ is a well-behaved local coordinate system on $M$ [64]. This choice introduces a natural basis on $T_pM$

$$\partial_r, \qquad \partial_\mu,$$

whose dual basis on $T_p^*M$ is $\mathrm{d}x^{\hat{\mu}}$. In particular, we have $\mathrm{d}r(\partial_r) = 1$. We now write the following vector identity

$$\boldsymbol{\partial}_r = N\mathbf{n} + \boldsymbol{\beta}, \tag{C.1}$$

where $\mathbf{n}$ is a normal spacelike vector whose norm is $g(\mathbf{n}, \mathbf{n}) = -\mathbf{1}$, $N$ is a positive real function called lapse and $\boldsymbol{\beta} = \beta^\mu\partial_\mu$ is a vector tangent to $\Sigma_r$ called shift vector. The identity (C.1) states that the normal vector $N\mathbf{n}$ and the vector induced by the radial coordinate $r$ in general differ by an horizontal vector $\boldsymbol{\beta}$. They only coincide if the coordinates of the foliation satisfy $x^\mu = const$.

We can now compute the components of the metric:

$$\begin{aligned} g_{rr} &= g(\boldsymbol{\partial}_r, \boldsymbol{\partial}_r) = -N^2 + \beta^\mu\beta_\mu, \\ g_{r\mu} &= g(\boldsymbol{\partial}_r, \boldsymbol{\partial}_\mu) = \beta_\mu, \\ g_{\mu\nu} &= g(\boldsymbol{\partial}_\mu, \boldsymbol{\partial}_\nu) \equiv \gamma_{\mu\nu}. \end{aligned} \tag{C.2}$$

The obtained metric is then

$$g_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g_{rr} & g_{rr} \\ g_{rr} & g_{\mu\nu} \end{pmatrix} = \begin{pmatrix} -N^2 + \beta^\mu\beta_\mu & \beta_\mu \\ \beta_\mu & \gamma_{\mu\nu} \end{pmatrix}, \tag{C.3}$$

which can be equivalently rewritten as

$$g = -N^2\,\mathrm{d}r^2 + \gamma_{\mu\nu}(\mathrm{d}x^\mu + \beta^\mu\mathrm{d}r)(\mathrm{d}x^\nu + \beta^\nu\mathrm{d}r). \tag{C.4}$$

The lapse function and the shift vector are a redundancy of the theory, as the choice of the foliation is arbitrary. The metric written above can be further simplified by choosing the gauge fixing $\boldsymbol{\beta} = 0$ and $N = 1$. This means that $\boldsymbol{\partial}_r$ truly coincides with the unit normal vector $\mathbf{n}$. The obtained coordinates are called Gaussian normal (radial) coordinates and the spacetime metric tensor can be simplified to

$$g = -\mathrm{d}r^2 + \gamma_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu. \tag{C.5}$$

# D Asymptotic expansions

## D.1 Spin connection

In pure AdS$_4$ gravity, a spin connection $\mathring{\omega}^{ab}_{\hat{\mu}}(x,z)$ satisfies the torsion constraint $\hat{T}^a_{\hat{\mu}\hat{\nu}} = \mathring{\mathcal{D}}_{\hat{\mu}} V^a_{\hat{\nu}} - \mathring{\mathcal{D}}_{\hat{\nu}} V^a_{\hat{\mu}} = 0$. In the supersymmetric case, the vielbein satisfies instead the super-torsion constraint $\hat{\mathbf{R}}^a_{\hat{\mu}\hat{\nu}} = \hat{\mathcal{D}}_{\hat{\mu}} V^a_{\hat{\nu}} - \hat{\mathcal{D}}_{\hat{\nu}} V^a_{\hat{\mu}} - i\overline{\Psi}_{A[\hat{\mu}}\Gamma^a\Psi_{A\hat{\nu}]} = 0$, where the contribution of the gravitini can be taken into account as a contorsion term in the spin connection

$$\hat{\omega}^{ab} = \mathring{\omega}^{ab} + C^{ab}\,, \quad C^{ab} = C^{ab}{}_{\hat{\mu}}\, dx^{\hat{\mu}}\,. \tag{D.1}$$

We now explicitly evaluate this contribution: from the decomposition

$$\hat{\mathcal{D}}_{\hat{\mu}} V^a_{\hat{\nu}} = \mathring{\mathcal{D}}_{\hat{\mu}} V^a_{\hat{\nu}} + C^a{}_{\hat{\nu}\hat{\mu}}\,, \tag{D.2}$$

we find the following expression

$$\hat{\mathbf{R}}^a_{\hat{\mu}\hat{\nu}} = 0 \quad \Leftrightarrow \quad C_{\hat{\lambda}[\hat{\mu}\hat{\nu}]} = -\frac{\mathrm{i}}{2}\,\overline{\Psi}^A_{\hat{\mu}}\Gamma_{\hat{\lambda}}\Psi_{A\hat{\nu}}\,.$$

The solution is obtained by taking different permutations of the expression above

$$C_{\hat{\lambda}\hat{\mu}\hat{\nu}} = \frac{\mathrm{i}}{2}\,\overline{\Psi}^A_{\hat{\lambda}}\Gamma_{\hat{\mu}}\Psi_{A\hat{\nu}} - \frac{\mathrm{i}}{2}\,\overline{\Psi}^A_{\hat{\mu}}\Gamma_{\hat{\lambda}}\Psi_{A\hat{\nu}} + \frac{\mathrm{i}}{2}\,\overline{\Psi}^A_{\hat{\lambda}}\Gamma_{\hat{\nu}}\Psi_{A\hat{\mu}}\,, \tag{D.3}$$

which can be restated in the following way

$$C^{ab}{}_{\hat{\mu}} = \frac{\mathrm{i}}{2}\,V^{a\hat{\nu}}\overline{\Psi}^A_{\hat{\nu}}\Gamma^b\Psi_{A\hat{\mu}} - \frac{\mathrm{i}}{2}\,V^{b\hat{\nu}}\overline{\Psi}^A_{\hat{\nu}}\Gamma^a\Psi_{A\hat{\mu}} + \frac{\mathrm{i}}{2}\,V^{a\hat{\nu}}V^{b\hat{\lambda}}V_{c\hat{\mu}}\,\overline{\Psi}^A_{\hat{\nu}}\Gamma^c\Psi_{A\hat{\lambda}}\,. \tag{D.4}$$

Note that, since $\overline{\Psi}^A_{\hat{\nu}}\Gamma^c\Psi_{A\hat{\lambda}} = -\overline{\Psi}^A_{\hat{\lambda}}\Gamma^c\Psi_{A\hat{\nu}}$, the tensor $C^{ab}{}_{\hat{\mu}}$ is explicitly antisymmetric in $ab$.

The various components of the contorsion asymptotically depend on fermions in the following way

$$\begin{aligned}
C^{i3}{}_z &= \hat{E}^{i\mu}\left(\overline{\varphi}^A_{+\mu}\varphi_{A-z} - \frac{z^2}{\ell^2}\,\overline{\varphi}^A_{-\mu}\varphi_{A+z}\right) + \frac{\mathrm{i}}{2}\left(\overline{\varphi}^A_{-z}\Gamma^i\varphi_{A-z} + \frac{z^2}{\ell^2}\,\overline{\varphi}^A_{+z}\Gamma^i\varphi_{A+z}\right)\,,\\
C^{ij}{}_z &= \frac{\mathrm{i}z}{\ell}\,\hat{E}^{\mu[i}\left(\overline{\varphi}^A_{+\mu}\Gamma^{j]}\varphi_{A+z} + \overline{\varphi}^A_{-\mu}\Gamma^{j]}\varphi_{A-z}\right) + \frac{z}{2\ell}\,\hat{E}^{i\mu}\,\hat{E}^{j\nu}\,\left(\overline{\varphi}^A_{-\mu}\varphi_{A+\nu} - \overline{\varphi}^A_{+\mu}\varphi_{A-\nu}\right)\,,\\
C^{i3}{}_\mu &= \frac{z}{2\ell}\,\hat{E}^{i\nu}\left(\overline{\varphi}^A_{+\nu}\varphi_{A-\mu} - \overline{\varphi}^A_{-\nu}\varphi_{A+\mu}\right) + \frac{\mathrm{i}z}{2\ell}\left(\overline{\varphi}^A_{+z}\Gamma^i\varphi_{A+\mu} + \overline{\varphi}^A_{-z}\Gamma^i\varphi_{A-\mu}\right)\\
&\quad - \frac{\mathrm{i}z}{2\ell}\,\hat{E}^{i\nu}\hat{E}_{j\mu}\left(\overline{\varphi}^A_{+\nu}\Gamma^j\varphi_{A+z} + \overline{\varphi}^A_{-\nu}\Gamma^j\varphi_{A-z}\right)\,,\\
C^{ij}{}_\mu &= \mathrm{i}\hat{E}^{\nu[i}\left(\overline{\varphi}^A_{+\nu}\Gamma^{j]}\varphi_{A+\mu} + \frac{z^2}{\ell^2}\,\overline{\varphi}^A_{-\nu}\Gamma^{j]}\varphi_{A-\mu}\right) + \frac{\mathrm{i}}{2}\,\hat{E}^{i\nu}\hat{E}^{j\lambda}\hat{E}_{k\mu}\left(\overline{\varphi}^A_{+\nu}\Gamma^k\varphi_{A+\lambda} + \frac{z^2}{\ell^2}\,\overline{\varphi}^A_{-\nu}\Gamma^k\varphi_{A-\lambda}\right)
\end{aligned} \tag{D.5}$$

and from eq. (7.27) for the full spin-connection we obtain

$$\hat{\omega}^{i3}_z \;=\; \left(\overline{\varphi}^{Ai}_{+} + \frac{\mathrm{i}}{2}\,\overline{\varphi}^A_{-z}\Gamma^i\right)\varphi_{A-z} + \frac{z^2}{\ell^2}\left(-\overline{\varphi}^{Ai}_{-} + \frac{\mathrm{i}}{2}\overline{\varphi}^A_{+z}\Gamma^i\right)\varphi_{A+z}\,,$$

$$
\begin{aligned}
\hat{\omega}^{ij}_z &= \frac{z}{\ell}\left(\mathrm{i}\overline{\varphi}^{A[i}_{+}\Gamma^{j]}\varphi_{A+z} + \mathrm{i}\overline{\varphi}^{A[i}_{-}\Gamma^{j]}\varphi_{A-z} + \overline{\varphi}^{A[i}_{-}\varphi^{j]}_{A+}\right),\\
\hat{\omega}^{i3}_\mu &= \frac{1}{z}\,\hat{E}^i_\mu - \frac{1}{2}\,k_{\mu\nu}\hat{E}^{i\nu} + \frac{z}{2\ell}\Big(\overline{\varphi}^{Ai}_{+}\varphi_{A-\mu} - \overline{\varphi}^{Ai}_{-}\varphi_{A+\mu} + \mathrm{i}\overline{\varphi}^{A}_{+z}\Gamma^i\varphi_{A+\mu}\\
&\quad - \mathrm{i}\overline{\varphi}^{Ai}_{+}\Gamma_\mu\varphi_{A+z} + \mathrm{i}\overline{\varphi}^{A}_{-z}\Gamma^i\varphi_{A-\mu} - \mathrm{i}\overline{\varphi}^{Ai}_{-}\Gamma_\mu\varphi_{A-z}\Big), \qquad\qquad \text{(D.6)}\\
\hat{\omega}^{ij}_\mu &= \mathring{\omega}^{ij}_\mu + \mathrm{i}\overline{\varphi}^{A[i}_{+}\Gamma^{j]}\varphi_{A+\mu} + \frac{\mathrm{i}}{2}\,\overline{\varphi}^{Ai}_{+}\Gamma_\mu\varphi^j_{A+} + \frac{z^2}{\ell^2}\left(\mathrm{i}\overline{\varphi}^{A[i}_{-}\Gamma^{j]}\varphi_{A-\mu} + \frac{\mathrm{i}}{2}\,\overline{\varphi}^{Ai}_{-}\Gamma_\mu\varphi^j_{A-}\right).
\end{aligned}
$$

We see that the $\mathcal{O}(1/z)$ term of $\hat{\omega}^{i3}_\mu$ is not modified by the fermions and this is consistent with the asymptotically AdS behaviour of the extrinsic curvature.
The most general gauge fixing with $\Psi_{\pm z} \neq 0$, is

$$
\begin{aligned}
\hat{\omega}^{i3}_z &= w^i(x,z)\,,\\
\hat{\omega}^{ij}_z &= \frac{z}{\ell}\,w^{ij}(x,z)\,, \qquad\qquad \text{(D.7)}
\end{aligned}
$$

where $w^i, w^{ij} = \mathcal{O}(1)$ and the boundary fields are

$$
\begin{aligned}
\hat{\omega}^{i3}_\mu &= \frac{1}{z}\,E^i_\mu - \frac{z}{\ell^2}\,\tilde{S}^i_\mu - \frac{2z^2}{\ell^3}\,\tilde{\tau}^i_\mu + \mathcal{O}(z^3)\,,\\
\hat{\omega}^{ij}_\mu &= \omega^{ij}_\mu + \frac{z}{\ell}\omega^{ij}_{\mu(1)} + \frac{z^2}{\ell^2}\,\omega^{ij}_{(2)\mu} + \frac{z^3}{\ell^3}\,\omega^{ij}_{(3)\mu} + \mathcal{O}(z^4)\,, \qquad\qquad \text{(D.8)}
\end{aligned}
$$

where now $S^i_\mu \neq \tilde{S}^i_\mu$, $\tau^i_\mu \neq \tilde{\tau}^i_\mu$ and $\omega^{ij}_\mu \neq \mathring{\omega}^{ij}_\mu$.

As particular cases, let us notice that when $\Psi^A_{-z} = 0$ and $\Psi^A_{+z} \neq 0$, the behaviour (D.6) yields $w^i = \mathcal{O}(z^2)$ and all other components remain the same. Furthermore, if we set to zero both components $\Psi^A_{\pm z} = 0$, we have exactly $w^i = 0$.

This behaviour of $w^i$ and $w^{ij}$ that we just described is summed up in the table (7.84).

## D.2 The supercurvatures

In this Subsection we evaluate, without imposing any gauge fixing conditions on the radial components of the gravitini $\Psi^A_{z\pm}$, the first contributions in the asymptotic expansion of the super field strengths, decomposing them with respect to a world-volume basis on the four-dimensional spacetime. Let us generically denote by $\hat{R}^\Lambda = \{\hat{\mathbf{R}}^{ab}, \hat{\mathbf{R}}^a, \hat{\boldsymbol{\rho}}^A, \hat{\mathbf{F}}\}$ any 2-form field strengths given by eq. (7.61),

$$
\hat{R}^\Lambda = \frac{1}{2}\,\hat{R}^\Lambda_{\hat{\mu}\hat{\nu}}\mathrm{d}x^{\hat{\mu}} \wedge \mathrm{d}x^{\hat{\nu}} = \frac{1}{2}\,\hat{R}^\Lambda_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu + \hat{R}^\Lambda_{\mu z}\mathrm{d}x^\mu \wedge \mathrm{d}z\,. \qquad\qquad \text{(D.9)}
$$

We use the following notation for the supercurvature expansion

$$
\hat{R}^\Lambda_{\hat{\mu}\hat{\nu}} = \sum_{n=n_{\min}}^{\infty}\left(\frac{z}{\ell}\right)^n \hat{R}^\Lambda_{(n)\hat{\mu}\hat{\nu}}\,. \qquad\qquad \text{(D.10)}
$$

where $n_{\min}$ denotes the minimal power of $\frac{z}{\ell}$ in the expansion, that is the order of the most divergent term.

From the supertorsion constraint $\hat{\mathbf{R}}^a_{\hat{\mu}\hat{\nu}} = 2\,\hat{\mathcal{D}}_{[\hat{\mu}} V^a_{\hat{\nu}]} - \mathrm{i}\overline{\Psi}^A_{\hat{\mu}} \Gamma^a \Psi^A_{\hat{\nu}} = 0$, with $n_{\min} = -1$, we find the following expansion coefficients in terms of the boundary quantities

$$
\begin{aligned}
\hat{\mathbf{R}}^i_{(-1)\mu\nu} &= 2\,\mathcal{D}_{[\mu} E^i{}_{\nu]} - \mathrm{i}\,\overline{\psi}^A_{+[\mu}\gamma^i\psi^A_{\nu]+} = 0\,, \\
\hat{\mathbf{R}}^i_{(0)\mu\nu} &= 2\,\omega^{ij}_{(1)[\mu} E_{j|\nu]} - 2\,\mathrm{i}\,\overline{\zeta}^A_{+[\mu}\gamma^i\psi^A_{\nu]+} = 0\,, \\
\hat{\mathbf{R}}^i_{(1)\mu\nu} &= 2\,\mathcal{D}_{[\mu} S^i{}_{\nu]} + 2\,\omega^{ij}_{(2)[\mu} E_{j|\nu]} \\
&\quad -\mathrm{i}\left(\overline{\zeta}^A_{+[\mu}\gamma^i\zeta^A_{\nu]+} + 2\,\overline{\Pi}^A_{+[\mu}\gamma^i\psi^A_{\nu]+} + \overline{\psi}^A_{-[\mu}\gamma^i\psi^A_{\nu]-}\right) = 0\,, \\
\hat{\mathbf{R}}^i_{(2)\mu\nu} &= 2\,\mathcal{D}_{[\mu}\tau^i{}_{\nu]} + 2\,\omega^{ij}_{(1)[\mu} S_{j|\nu]} + 2\,\omega^{ij}_{(3)[\mu} E_{j|\nu]} \\
&\quad -2\,\mathrm{i}\left(\overline{\zeta}^A_{+[\mu}\gamma^i\Pi^A_{\nu]+} + \overline{\Delta}^A_{+[\mu}\gamma^i\psi^A_{\nu]+} + \overline{\zeta}^A_{-[\mu}\gamma^i\psi^A_{\nu]-}\right) = 0\,,
\end{aligned}
\tag{D.11}
$$

where we identified $\Delta^A_{+\mu} = \psi^A_{(3)+\mu}$. Note that the last equation gives the expression for $\omega^{ij}_{(3)\mu}$ in the supersymmetric case. The next supertorsion components to be expanded are $\hat{\mathbf{R}}^i_{\mu z}$, which behave as

$$
\begin{aligned}
\hat{\mathbf{R}}^i_{(0)\mu z} &= \frac{1}{2\ell}\left(\tilde{S}^i{}_\mu - S^i{}_\mu\right) - \frac{1}{2}\,w^{ij}_{(0)} E_{j\mu} \\
&\quad -\frac{\mathrm{i}}{2}\left(\overline{\psi}_{A+\mu}\gamma^i\psi_{A+z} + \overline{\psi}_{A-\mu}\gamma^i\psi_{A-z}\right) = 0\,,
\end{aligned}
\tag{D.12}
$$

$$
\begin{aligned}
\hat{\mathbf{R}}^i_{(1)\mu z} &= \frac{1}{\ell}\left(\tilde{\tau}^i{}_\mu - \tau^i{}_\mu\right) - \frac{1}{2}\,w^{ij}_{(1)} E_{j\mu} - \frac{\mathrm{i}}{2}\left(\overline{\psi}_{A+\mu}\gamma^i\zeta_{A+z}\right. \\
&\quad \left. + \overline{\psi}_{A-\mu}\gamma^i\zeta_{A-z} + \overline{\zeta}_{A+\mu}\gamma^i\psi_{A+z} + \overline{\zeta}_{A-\mu}\gamma^i\psi_{A-z}\right) = 0\,.
\end{aligned}
\tag{D.13}
$$

The $\hat{\mathbf{R}}^3$ components of the supertorsion start at $n_{\min} = 0$ and in the vicinity of the boundary read

$$
\begin{aligned}
\hat{\mathbf{R}}^3_{(0)\mu\nu} &= -\frac{2}{\ell}\left(S_{[\mu\nu]} - \tilde{S}_{[\nu\mu]}\right) - 2\,\mathrm{i}\,\overline{\psi}_{A+[\mu}\psi_{A-\nu]} = 0\,, \\
\hat{\mathbf{R}}^3_{(1)\mu\nu} &= -\frac{2}{\ell}\left(\tau_{[\mu\nu]} - 2\,\tilde{\tau}_{[\nu\mu]}\right) - 2\,\mathrm{i}\left(\overline{\psi}_{A+[\mu}\zeta_{A-\nu]} + \overline{\zeta}_{A+[\mu}\psi_{A-\nu]}\right) = 0\,,
\end{aligned}
\tag{D.14}
$$

while the projected to $\mathrm{d}z \wedge \mathrm{d}x^\mu$ are

$$
\begin{aligned}
\hat{\mathbf{R}}^3_{(-1)\mu z} &= \frac{1}{2}\,w^i_{(0)} E_{i\mu} - \frac{\mathrm{i}}{2}\,\overline{\psi}_{A+\mu}\psi_{A-z} = 0\,, \\
\hat{\mathbf{R}}^3_{(0)\mu z} &= \frac{1}{2}\,w^i_{(1)} E_{i\mu} - \frac{\mathrm{i}}{2}\left(\overline{\psi}_{A+\mu}\zeta_{A-z} + \overline{\zeta}_{A+\mu}\psi_{A-z}\right) = 0\,, \\
\hat{\mathbf{R}}^3_{(1)\mu z} &= \frac{1}{2}\,w^i_{(0)} S_{i\mu} + \frac{1}{2}\,w^i_{(2)} E_{i\mu} - \frac{\mathrm{i}}{2}\,\overline{\psi}_{A-\mu}\psi_{A+z} \\
&\quad -\frac{\mathrm{i}}{2}\left(\overline{\psi}_{A+\mu}\Pi_{A-z} + \overline{\zeta}_{A+\mu}\zeta_{A-z} + \overline{\Pi}_{A+\mu}\psi_{A-z}\right) = 0\,,
\end{aligned}
\tag{D.15}
$$

where $\Pi^A_{-z} = \psi^A_{(2)-z}$. The last equation gives the expression for $w^i_{(2)}$.

Focusing now on the AdS supersurvature $\hat{\mathbf{R}}^{ij} = \mathcal{R}^{ij} + \frac{4}{\ell^2} V_+^{[i} V_-^{j]} - \frac{1}{\ell}\left(\overline{\Psi}_+^A \Gamma^{ij} \Psi_-^A + \overline{\Psi}_-^A \Gamma^{ij} \Psi_+^A\right)$, which starts at $n_{\min} = 0$, we get

$$
\begin{aligned}
\hat{\mathbf{R}}^{ij}_{(0)|\mu\nu} &= 2\,\mathcal{R}^{ij}_{\mu\nu} - 4\,E^{[i}{}_{[\mu}\mathcal{S}^{j]}{}_{\nu]} - \frac{2}{\ell}\,\overline{\psi}^A_{-\mu}\gamma^{ij}\psi^A_{+\nu} = 0\,, \\
\hat{\mathbf{R}}^{ij}_{(1)\mu\nu} &= 2\,\mathcal{D}_{[\mu}\omega^{ij}_{(1)|\nu]} - \frac{4}{\ell^2}\,E^{[i}{}_{[\mu}(\tau^{j]}{}_{\nu]} + 2\tilde{\tau}^{j]}{}_{\nu]}) \\
&\quad -\frac{2}{\ell}\left(\overline{\psi}^A_{-[\mu}\gamma^{ij}\zeta^A_{+\nu]} + \overline{\psi}^A_{+[\mu}\gamma^{ij}\zeta^A_{-\nu]}\right)\,, \\
\hat{\mathbf{R}}^{ij}_{(-1)\mu z} &= E^{[i}{}_{\mu}w^{j]}_{(0)} - \frac{1}{2\ell}\,\overline{\psi}^A_{+\mu}\gamma^{ij}\psi^A_{-z}\,, \\
\hat{\mathbf{R}}^{ij}_{(0)\mu z} &= -\frac{1}{2\ell}\left(-2\ell\,E^{[i}{}_{\mu}w^{j]}_{(1)} + \omega^{ij}_{(1)\mu} + \overline{\psi}^A_{+\mu}\gamma^{ij}\zeta^A_{-z}\right)\,, \\
\hat{\mathbf{R}}^{ij}_{(1)\mu z} &= \frac{1}{2}\,\mathcal{D}_\mu w^{ij}_{(0)} - \frac{1}{\ell}\,\omega^{ij}_{(2)\mu} - \tilde{S}^{[i}{}_{\mu}w^{j]}_{(0)} \\
&\quad -\frac{1}{2\ell}\left(\overline{\psi}^A_{-\mu}\gamma^{ij}\psi^A_{+z} + \overline{\psi}^A_{+\mu}\gamma^{ij}\Pi^A_{-z} + \overline{\Pi}^A_{+\mu}\gamma^{ij}\psi^A_{-z}\right)\,.
\end{aligned}
\tag{D.16}
$$

Next, from $\hat{\mathbf{R}}^{i3} = \hat{\mathcal{D}}\omega^{i3} - \frac{1}{\ell^2}\,V^i V^3 - \frac{\mathrm{i}}{2\ell}\left(\overline{\Psi}_+^A\Gamma^i\Psi_+^A - \overline{\Psi}_-^A\Gamma^i\Psi_-^A\right)$, we find $n_{\min} = 1$ and

$$
\begin{aligned}
\hat{\mathbf{R}}^{i3}_{(1)\mu\nu} &= -\frac{2}{\ell}\,\mathcal{D}_{[\mu}\tilde{S}^i{}_{\nu]} + \frac{2}{\ell}\,\omega^{ij}_{(2)[\mu}E_{j|\nu]} + \frac{\mathrm{i}}{\ell}\left(\overline{\psi}^A_{-[\mu}\gamma^i\psi^A_{-\nu]} - \overline{\zeta}^A_{+[\mu}\gamma^i\zeta^A_{+\nu]}\right. \\
&\quad \left. -2\,\overline{\Pi}^A_{+[\mu}\gamma^i\psi^A_{+\nu]}\right)\,, \\
\hat{\mathbf{R}}^{i3}_{(0)\mu z} &= \frac{1}{2}\,\mathcal{D}_\mu w^i_{(0)} + \frac{\mathrm{i}}{\ell}\,\overline{\psi}^A_{-\mu}\gamma^i\psi^A_{-z}\,, \\
\hat{\mathbf{R}}^{i3}_{(1)\mu z} &= \frac{1}{2}\,\mathcal{D}_\mu w^i_{(1)} + \omega^{(1)|ij}_{\mu}w_{(0)|j} + \frac{1}{2\,\ell^2}\,(2\tilde{\tau}^i{}_\mu + \tau^i{}_\mu) + \frac{1}{2\ell}\,w^{ij}_{(0)}S_{j|\mu} \\
&\quad +\frac{\mathrm{i}}{\ell}\left(\overline{\psi}^A_{-\mu}\gamma^i\zeta^A_{-z} + \overline{\zeta}^A_{-\mu}\gamma^i\psi^A_{-z}\right)\,,
\end{aligned}
\tag{D.17}
$$

where we have also exploited the vanishing supertorsion equations (D.12) and (D.13).

As regards to the graviphoton super field strength $\hat{\mathbf{F}} = \mathrm{d}\hat{A} - 2\,\epsilon_{AB}\overline{\Psi}_{+A}\Psi_{-B}$, we obtain $n_{\min} = 0$ and

$$
\begin{aligned}
\hat{\mathbf{F}}_{(0)\mu\nu} &= 2\,\partial_{[\mu}A_{\nu]} - 4\,\epsilon_{AB}\overline{\psi}^A_{+[\mu}\psi^B_{-\nu]} = 0\,, \\
\hat{\mathbf{F}}_{(1)\mu\nu} &= 2\,\partial_{[\mu}A_{(1)\nu]} - 4\left(\overline{\psi}^A_{+[\mu}\zeta^B_{-\nu]} + \overline{\zeta}^A_{+[\mu}\psi^B_{-\nu]}\right)\epsilon_{AB}\,, \\
\hat{\mathbf{F}}_{(-1)\mu z} &= \frac{1}{2}\,\partial_\mu A_{(-1)z} - \overline{\psi}^A_{+\mu}\psi^B_{-z}\epsilon_{AB}\,, \\
\hat{\mathbf{F}}_{(0)\mu z} &= \frac{1}{2}\,\partial_\mu A_{(0)z} - \frac{1}{2\ell}\,A_{(1)\mu} - \overline{\psi}^A_{+\mu}\zeta^B_{-z}\epsilon_{AB}\,, \\
\hat{\mathbf{F}}_{(1)\mu z} &= \frac{1}{2}\,\partial_\mu A_{(1)z} - \frac{1}{\ell}\,A_{(2)\mu} - \left(\overline{\psi}^A_{-\mu}\psi^B_{+z} + \overline{\psi}^A_{+A\mu}\Pi^B_{-z}\right)\epsilon_{AB}\,.
\end{aligned}
\tag{D.18}
$$

At last, we analyse the gravitini supercurvatures: the positive chirality reads $\hat{\boldsymbol{\rho}}_{+A} = \mathrm{d}\Psi_{+A} + \frac{1}{4}\,\hat{\omega}^{ij}\Gamma_{ij}\Psi_{+A} - \frac{1}{2\ell}\,\epsilon_{AB}\hat{A}\Psi_{+B} + \frac{\mathrm{i}}{\ell}\,V_+^i\Gamma_i\Psi_{-A} - \frac{1}{2\ell}\,\Psi_{+A}V^3$ which starts at $n_{\min} = -1/2$

and leads to

$$
\begin{aligned}
\hat{\boldsymbol{\rho}}_{(-1/2)+A\mu\nu} &= 2\,\nabla_{[\mu}\psi^A_{+\nu]} + \frac{2\mathrm{i}}{\ell}\,\gamma_{[\mu}\psi^A_{-\nu]} = 0\,, \qquad\qquad (\mathrm{D.19})\\
\hat{\boldsymbol{\rho}}_{(1/2)+A\mu\nu} &= 2\,\nabla_{[\mu]}\zeta_{+A\nu]} + \frac{2}{\ell}\,\gamma_{[\mu}\zeta_{-A\nu]} + \frac{1}{2}\,\gamma_{ij}\omega^{ij}_{(1)[\mu}\psi_{+A\nu]} - \frac{1}{\ell}\,A_{(1)[\mu}\psi_{+B\nu]}\epsilon_{AB}\,,\\
\hat{\boldsymbol{\rho}}_{(-3/2)+A\mu z} &= \frac{\mathrm{i}}{2\ell}\left(\gamma_\mu\psi_{-Az} - \frac{\mathrm{i}}{2}A_{(-1)z}\psi_{+B\mu}\epsilon_{AB}\right),\\
\hat{\boldsymbol{\rho}}_{(-1/2)+A\mu z} &= \frac{\mathrm{i}}{2\ell}\gamma_\mu\zeta_{-Az} + \frac{1}{4\ell}\,\left(A_{(0)z}\psi_{+B\mu} + A_{(-1)z}\zeta_{+B\mu}\right)\epsilon_{AB} - \frac{1}{2\ell}\,\zeta_{+A\mu}\,,\\
\hat{\boldsymbol{\rho}}_{(1/2)+A\mu z} &= \frac{1}{2}\,\nabla_\mu\psi_{+Az} - \frac{1}{8}\,w^{ij}\gamma_{ij}\psi_{+A\mu} + \frac{\mathrm{i}}{4\ell}\,\left(S^i{}_\mu - \tilde{S}^i{}_\mu\right)\gamma_i\psi_{-Az} - \frac{1}{\ell}\,\Pi_{+A\mu}\\
&\quad -\frac{\mathrm{i}}{4}\,w^i\gamma_i\psi_{-A\mu} + \frac{\mathrm{i}2\ell}{2\ell}\,\gamma_\mu\Pi_{-Az} + \frac{1}{4\ell}\,\left(A_{(1)z}\psi_{+B\mu} + A_{(-1)z}\Pi_{+B\mu} + A_{(0)z}\zeta_{+B\mu}\right)\epsilon_{AB}\,.
\end{aligned}
$$

Finally, the negative chirality gravitino curvature $\hat{\boldsymbol{\rho}}_{-A} = \mathrm{d}\Psi_{-A} + \frac{1}{4}\,\hat{\omega}^{ij}\Gamma_{ij}\Psi_{-A} - \frac{1}{2\ell}\,\epsilon_{AB}\hat{A}\Psi_{-B} - \frac{\mathrm{i}}{\ell}V^i_-\Gamma_i\Psi_{+A} + \frac{1}{2\ell}\,\Psi_{-A}V^3$ has $n_{\min} = +1/2$ and reads

$$
\begin{aligned}
\hat{\boldsymbol{\rho}}_{(1/2)-A\mu\nu} &= 2\,\nabla_{[\mu}\psi_{-A\nu]} - \mathrm{i}\,\ell\,\gamma_i\psi_{+A[\mu}\mathcal{S}^i{}_{\nu]}\,,\\
\hat{\boldsymbol{\rho}}_{(3/2)-A\mu\nu} &= \nabla_{[\mu}\zeta_{-A\nu]} - \frac{\mathrm{i}}{2}\,\ell\,\gamma_i\zeta_{+A[\mu}\mathcal{S}^i{}_{\nu]} + \frac{1}{4}\,\omega^{ij}_{(1)|[\mu}\gamma_{ij}\psi_{-A\nu]}\\
&\quad -\frac{1}{2\ell}\,A_{(1)[\mu}\psi_{-B\nu]}\epsilon_{AB} + \frac{\mathrm{i}}{2\ell}\,\left(\tau^i{}_{[\mu} + 2\,\tilde{\tau}^i{}_{[\mu}\right)\gamma_i\psi_{+A\nu]}\,,\\
\hat{\boldsymbol{\rho}}_{(-1/2)-A\mu z} &= \frac{1}{2}\,\nabla_\mu\psi_{-Az} + \frac{1}{4\ell}\,A_{(-1)z}\,\epsilon_{AB}\psi_{-B\mu} + \frac{\mathrm{i}}{4}\gamma_i w^i_{(0)}\psi_{+A\mu}\,,\\
\hat{\boldsymbol{\rho}}_{(1/2)-A\mu z} &= \frac{1}{2}\,\nabla_\mu\zeta_{-Az} + \frac{1}{4\ell}\,A_{(0)z}\,\epsilon_{AB}\psi_{-B\mu} + \frac{\mathrm{i}}{4}\gamma_i w^i_{(1)}\psi_{+A\mu}\\
&\quad -\frac{1}{2\ell}\,\zeta_{-A\mu} + \frac{1}{4\ell}\,A_{(-1)z}\zeta_{-B\mu}\epsilon_{AB}\,. \qquad\qquad (\mathrm{D.20})
\end{aligned}
$$

We observe that the $\hat{\mathbf{R}}^\Lambda_{(n_{\min})\mu\nu}$ components of $\hat{\mathbf{R}}^\Lambda = \{\hat{\mathbf{R}}^{ab}, \hat{\mathbf{R}}^a, \hat{\boldsymbol{\rho}}^A, \hat{\mathbf{F}}\}$ define the curvatures $\{\mathbf{R}^{ij}, \mathbf{R}^i, \boldsymbol{\rho}^A, \mathbf{F}, \mathcal{C}^i, \Omega^A\}$ of the $\mathcal{N}=2$ superconformal group OSp(2|4) discussed in Subsection 7.7 and given by eqs. (7.134). These expressions would all vanish in a vacuum theory having OSp(2|4) isometries, but in our case the negative grading curvatures $\hat{\mathbf{R}}^{i3}_{\mu\nu}$ and $\hat{\boldsymbol{\rho}}_{-A\mu\nu}$ do not vanish and lead to contraction terms in the variation of the boundary fields, as we see in the main text.

### D.3 Equations of motion of the graviphoton

Here we analyse the relation between gauge fixing and asymptotic behaviour of the fields by using radial field equations. In Appendix D.1, a similar problem was discussed for the spin connection using the vanishing of the supertorsion.

Radial evolution of the graviphoton is given by the respective field equation in (7.71) that, in components has the form

$$
\hat{\mathcal{D}}_{\hat{\nu}}\hat{\mathbf{F}}^{\hat{\nu}\hat{\mu}} = \frac{\mathrm{i}}{e}\,\epsilon^{\hat{\mu}\hat{\nu}\hat{\lambda}\hat{\rho}}\,\overline{\Psi}^A_{\hat{\nu}}\Gamma^5\hat{\boldsymbol{\rho}}^B_{\hat{\lambda}\hat{\rho}}\,\epsilon_{AB}\,. \qquad (\mathrm{D.21})
$$

Using the conventions (A.3) and (7.21), the component $\hat{\mu} = \mu$ acquires the form

$$\hat{\mathcal{D}}_\nu \hat{\mathbf{F}}^{\nu\mu} + \hat{\mathcal{D}}_z \hat{\mathbf{F}}^{z\mu} = -\frac{\mathrm{i}}{e}\,\epsilon^{\mu\nu\lambda}\left(2\overline{\Psi}{}^A_\nu \Gamma_5 \hat{\boldsymbol{\rho}}^B_{\lambda z} + \overline{\Psi}{}^A_z \Gamma_5 \hat{\boldsymbol{\rho}}^B_{\nu\lambda}\right)\epsilon_{AB}\,. \tag{D.22}$$

For convenience, we factorise the relevant field strength components as

$$\begin{aligned} \hat{\mathbf{F}}^{z\mu} &= -\left(\tfrac{z}{\ell}\right)^4 g^{\mu\nu}\hat{\mathbf{F}}_{z\nu}\,, & \hat{\boldsymbol{\rho}}^A_{\mu z\pm} &= \left(\tfrac{z}{\ell}\right)^{\pm\frac{1}{2}}\Xi^A_{\mu\pm}\,, \\ \hat{\mathbf{F}}^{\mu\nu} &= \left(\frac{z}{\ell}\right)^4 F^{\mu\nu}\,, & \hat{\boldsymbol{\rho}}^A_{\mu\nu\pm} &= \left(\tfrac{z}{\ell}\right)^{\mp\frac{1}{2}}\Xi^A_{\mu\nu\pm}\,, \end{aligned} \tag{D.23}$$

where $\hat{\mathbf{F}}_{\mu\nu} = F_{\mu\nu}$ and the tensors $\hat{\mathbf{F}}_{z\mu}$, $F_{\mu\nu}$, $\Xi^A_{\mu\pm}$ and $\Xi^A_{\mu\nu\pm}$ have to be expanded in power series in $z$. The metric $g_{\mu\nu}(x,z)$ and its inverse $g^{\mu\nu}$ rise and lower the spacetime indices on $\partial\mathcal{M}$. Recalling the FG metric (7.11) and the tensor $k_{\mu\nu} = \partial_z g_{\mu\nu}$ introduced by eq. (7.28), as well as using the Christoffel symbols

$$\begin{aligned} \hat{\Gamma}^\mu_{\nu z} &= -\frac{1}{z}\,\delta^\mu_\nu + \frac{1}{2}\,k^\mu_\nu\,, & \hat{\Gamma}^\mu_{zz} &= 0 = \hat{\Gamma}^z_{z\mu}\,, \\ \hat{\Gamma}^z_{\mu\nu} &= -\frac{1}{z}\,g_{\mu\nu} + \frac{1}{2}\,k_{\mu\nu}\,, & \hat{\Gamma}^z_{zz} &= -\frac{1}{z}\,, \end{aligned} \tag{D.24}$$

the radial graviphoton equation becomes

$$\begin{aligned} &\mathcal{D}_\nu F^{\nu\mu} - \left(k^{\mu\nu} - \frac{k}{2}\,g^{\mu\nu}\right)\hat{\mathbf{F}}_{\nu z} + g^{\mu\nu}\partial_z \hat{\mathbf{F}}_{\nu z} \\ = \;& -\frac{\mathrm{i}}{\hat{e}_3}\,\epsilon^{\mu\nu\lambda}\left(2\,\overline{\varphi}{}^A_{+\nu}\Gamma_5 \Xi^B_{\lambda+} + 2\,\overline{\varphi}{}^A_{-\nu}\Gamma_5\Xi^B_{\lambda-} + \overline{\varphi}{}^A_{+z}\Gamma_5\Xi^B_{\nu\lambda+} + \overline{\varphi}{}^A_{-z}\Gamma_5\Xi^B_{\nu\lambda-}\right)\epsilon_{AB}\,. \end{aligned} \tag{D.25}$$

Now we calculate $\hat{\mathbf{F}}_{\mu z}$, $\Xi^A_{\mu\pm}$ and $\Xi^A_{\mu\nu\pm}$ defined in (D.23). Evaluation of the components

$$\begin{aligned} \hat{\mathbf{F}}^{\hat{\mu}\hat{\nu}} &= \hat{g}^{\hat{\mu}\hat{\alpha}}\hat{g}^{\hat{\nu}\hat{\beta}}\left(\partial_{\hat{\alpha}}\hat{A}_{\hat{\beta}} - \partial_{\hat{\beta}}\hat{A}_{\hat{\alpha}} - 2\epsilon_{AB}\,\overline{\Psi}{}^A_{\hat{\alpha}}\Psi^B_{\hat{\beta}}\right), \\ \hat{\boldsymbol{\rho}}^A_{\hat{\mu}\hat{\nu}} &= 2\hat{\mathcal{D}}_{[\hat{\mu}}\Psi^A_{\hat{\nu}]} - \frac{1}{\ell}\,\epsilon_{AB}\hat{A}_{[\hat{\mu}}\Psi^B_{\hat{\nu}]} - \frac{\mathrm{i}}{\ell}\,\Gamma_a\Psi^A_{[\hat{\mu}}V^a_{\hat{\nu}]} \end{aligned} \tag{D.26}$$

leads to

$$\begin{aligned} \hat{\mathbf{F}}_{\mu z} &= \partial_\mu \hat{A}_z - \partial_z A_\mu - \frac{2\ell}{z}\,\epsilon_{AB}\,\overline{\varphi}{}^A_{+\mu}\varphi^B_{z-} - \frac{2z}{\ell}\,\epsilon_{AB}\,\overline{\varphi}{}^A_{-\mu}\varphi^B_{z+}\,, \\ F^{\mu\nu} &= g^{\mu\alpha}g^{\nu\beta}\left(\mathcal{F}_{\alpha\beta} - 4\epsilon_{AB}\,\overline{\varphi}{}^A_{+\alpha}\varphi^B_{-\beta}\right) = 0 \end{aligned} \tag{D.27}$$

and, by means of the rescalings (7.79), we get

$$\begin{aligned} \Xi^A_{\mu\pm} &= \mathcal{D}_\mu\varphi^A_{\pm z} - \frac{1}{4}\left(\frac{z}{\ell}\right)^{1\mp1} w^{ij}\Gamma_{ij}\varphi^A_{\pm\mu} - \left(\frac{z}{\ell}\right)^{\mp1}\partial_z\varphi^A_{\pm\mu} - \frac{1}{2\ell}\,\epsilon_{AB}\,A_\mu\varphi^B_{\pm z} \\ &\quad \mp\frac{\mathrm{i}}{2}\,w^i\Gamma_i\varphi^A_{\mp\mu} + \frac{1}{2\ell}\left(\frac{z}{\ell}\right)^{\mp1}\epsilon_{AB}\,\hat{A}_z\varphi^B_{\pm\mu} \pm \frac{\mathrm{i}}{\ell}\left(\frac{z}{\ell}\right)^{\mp2} E^i_{\pm\mu}\Gamma_i\varphi^A_{\mp z}\,, \\ \Xi^A_{\mu\nu\pm} &= 2\mathcal{D}_{[\mu}\varphi^A_{\nu]\pm} \pm \frac{2\mathrm{i}}{\ell}\,E^i_{\pm[\mu}\Gamma_i\varphi^A_{\nu]\mp} - \frac{1}{\ell}\,\epsilon_{AB}\,A_{[\mu}\varphi^B_{\nu]\pm}\,. \end{aligned} \tag{D.28}$$

We also assume that the gauge-fixing functions are

$$\begin{aligned}
\hat{A}_z &= \frac{\ell}{z}\,A_{(-1)z} + A_{(0)z} + \frac{z}{\ell}\,A_{(1)z} + \mathcal{O}(z^3)\,, \\
\hat{A}_\mu &= \frac{\ell}{z}\,A_{(-1)\mu} + A_\mu + \frac{z}{\ell}\,A_{(1)\mu} + \frac{z^2}{\ell^2}\,A_{(2)\mu} + \mathcal{O}(z^3)\,, \\
\varphi^A_{+\mu} &= \varphi^A_{(0)+\mu} + \frac{z}{\ell}\,\varphi^A_{(1)+\mu} + \mathcal{O}(z^2)\,,
\end{aligned} \tag{D.29}$$

in general allowing the linear terms (in contrast to eq. (7.86) valid in pure gravity), where $E_\pm$ expand as (7.80), and we find

$$\begin{aligned}
\hat{\mathbf{F}}_{\mu z} &= \frac{\ell}{z^2}\,A_{(-1)\mu} + \frac{\ell}{z}\left(\partial_\mu A_{(-1)z} - 2\epsilon_{AB}\,\overline{\varphi}^A_{(0)+\mu}\varphi^B_{(0)z-}\right) + \mathcal{O}(1)\,, \\
\Xi^A_{\mu+} &= \frac{\ell}{2z^2}\left(A_{(-1)z}\,\epsilon^{AB}\,\varphi_{B+\mu} + 2\mathrm{i}E^i_\mu\Gamma_i\varphi^A_{(0)-z}\right) + \frac{1}{z}\left(\frac{1}{2}\,\epsilon_{AB}\,A_{(0)z}\varphi^B_{(0)+\mu} - \varphi^A_{(1)+\mu}\right) + \mathcal{O}(1)\,, \\
& F^{\mu\nu},\ \Xi^A_{\mu-},\ \Xi^A_{\mu\nu\pm} = \mathcal{O}(1)\,.
\end{aligned} \tag{D.30}$$

Remembering that $k_{\mu\nu} = \mathcal{O}(z)$, the graviphoton equation (D.25) then yields

$$\begin{aligned}
\frac{\ell}{z^3} \quad &: \quad A_{(-1)\mu} = 0\,, \\
\frac{\ell}{z^2} \quad &: \quad \partial_\mu A_{(-1)z} = \left(2\,\overline{\varphi}^A_{(0)+\mu} - \frac{1}{e_3}\,g_{(0)\mu\sigma}\,\epsilon^{\sigma\nu\lambda}E^i_\lambda\overline{\varphi}^A_{(0)+\nu}\Gamma_5\Gamma_i\right)\epsilon_{AB}\varphi^B_{(0)-z}\,, \\
\frac{1}{z} \quad &: \quad 0 = \epsilon^{\mu\nu\lambda}\,\overline{\varphi}^A_{(0)+\nu}\Gamma_5\left(\frac{1}{2}\,A_{(0)z}\varphi^A_{(0)+\mu} + \varphi^B_{(1)+\mu}\epsilon_{AB}\right)
\end{aligned} \tag{D.31}$$

and all other terms are finite. We used the fact that the term $\overline{\varphi}^A_{+\nu}\Gamma_5\varphi_{A+\lambda}$ is symmetric in $(\nu\lambda)$ so it vanishes when contracted with $\epsilon^{\sigma\nu\lambda}$. From the last equation in (D.31), when $\varphi^A_{(0)+\mu} \neq 0$ (and otherwise), we can choose a particular solution $A_{(0)z} = 0$, $\varphi^A_{(1)+\mu} \equiv \binom{\zeta^A_{\mu+}}{0} = 0$, which is in agreement with eq. (7.103) obtained in the main text. This choice was also taken in [58] in the context of $\mathcal{N} = 1$ Supergravity. Then (D.29) implies

$$\begin{aligned}
\hat{A}_z &= \frac{\ell}{z}\,A_{(-1)z} + \frac{z}{\ell}\,A_{(1)z} + \mathcal{O}(z^3)\,, \\
\hat{A}_\mu &= A_\mu + \frac{z}{\ell}\,A_{(1)\mu} + \frac{z^2}{\ell^2}\,A_{(2)\mu} + \mathcal{O}(z^3)\,, \\
\varphi^A_{+\mu} &= \varphi^A_{(0)+\mu} + \mathcal{O}(z^2)\,.
\end{aligned} \tag{D.32}$$

We also conclude that the gauge-fixing functions $A_{(-1)z}$ and $\varphi^B_{(0)-z}$ are correlated, which is consistent with the table (7.84). In addition, the boundary graviphoton does not acquire divergent terms of the form $1/z$ even when $\varphi^A_{(0)z-} \neq 0$. We have not considered the logarithmic terms here.

The graviphoton curvature behaves in the following way on the boundary,

$$\hat{\mathbf{F}}_{\mu z} = \frac{\ell}{z}\left(\partial_\mu A_{(-1)z} - 2\epsilon_{AB}\,\overline{\varphi}^A_{(0)+\mu}\varphi^B_{(0)-z}\right) - \frac{1}{\ell}\,A_{(1)\mu} + \mathcal{O}(z)\,,$$

$$\hat{\mathbf{F}}_{\mu\nu} \;=\; \mathcal{F}_{\mu\nu} - 4\epsilon_{AB}\,\overline{\varphi}^A_{+[\mu}\varphi^B_{-\nu]} = 0\,. \tag{D.33}$$

This shows that it is possible to have the components $\hat{\mathbf{F}}_{\mu z} \neq 0$ on the boundary $z=0, \mathrm{d}z=0$, with a suitable gauge choice which changes the asymptotics.

## D.4 Equations of motion of the gravitini

The equation of motion that describes the dynamics of gravitini (7.71) in components has the form

$$0 = \epsilon^{\hat{\mu}\hat{\nu}\hat{\lambda}\hat{\tau}}\left(V^a_{\hat{\mu}}\Gamma_a\Gamma_5\hat{\boldsymbol{\rho}}_{A\hat{\nu}\hat{\lambda}} + \frac{\mathrm{i}}{2}\,\epsilon_{AB}\hat{\mathbf{F}}_{\hat{\mu}\hat{\nu}}\Gamma_5\Psi^B_{\hat{\lambda}}\right) + e\,\epsilon_{AB}\Psi^B_{\hat{\lambda}}\,\hat{\mathbf{F}}^{\hat{\lambda}\hat{\tau}}\,, \tag{D.34}$$

where the expression of the Hodge dual was used. The radial expansion of the gravitini is given by the components $\hat{\tau}=\mu$ which, with the conventions in Appendix A and (7.21), leads to

$$\begin{aligned} 0 \;=\;& \epsilon^{\mu\nu\lambda}\left(-V^3_z\Gamma^3\Gamma_5\hat{\boldsymbol{\rho}}_{A\nu\lambda} - 2V^i_\nu\Gamma_i\Gamma_5\hat{\boldsymbol{\rho}}_{Az\lambda} + \frac{\mathrm{i}}{2}\,\epsilon_{AB}\hat{\mathbf{F}}_{\nu\lambda}\Gamma_5\Psi^B_z + \mathrm{i}\,\epsilon_{AB}\hat{\mathbf{F}}_{z\nu}\Gamma_5\Psi^B_\lambda\right) \\ & + e\,\epsilon_{AB}\left(\Psi^B_z\,\hat{\mathbf{F}}^{z\mu} + \Psi^B_\nu\,\hat{\mathbf{F}}^{\nu\mu}\right)\,. \end{aligned} \tag{D.35}$$

Projecting the last expression along the two chiralities through the projector $\mathbb{P}_\pm$ defined in (B.11), we find

$$\begin{aligned} 0 \;=\;& \epsilon^{\mu\nu\lambda}\left(\mp\mathrm{i}V^3_z\Gamma_5\hat{\boldsymbol{\rho}}_{\mp A\nu\lambda} - 2V^i_\nu\Gamma_i\Gamma_5\hat{\boldsymbol{\rho}}_{\pm Az\lambda} + \frac{\mathrm{i}}{2}\,\epsilon_{AB}\hat{\mathbf{F}}_{\nu\lambda}\Gamma_5\Psi^B_{\mp z} + \mathrm{i}\,\epsilon_{AB}\hat{\mathbf{F}}_{z\nu}\Gamma_5\Psi^B_{\mp\lambda}\right) \\ & + e\,\epsilon_{AB}\left(\Psi^B_{\pm z}\,\hat{\mathbf{F}}^{z\mu} + \Psi^B_{\pm\nu}\,\hat{\mathbf{F}}^{\nu\mu}\right)\,. \end{aligned} \tag{D.36}$$

Now we can use eqs. (D.23), (7.21), (7.83) and (7.119), to obtain the equation expressed in terms of the auxiliary quantities with known asymptotic behaviour,

$$\begin{aligned} 0 \;=\;& \left(\frac{z}{\ell}\right)^{\pm\frac{1}{2}-1}\epsilon^{\mu\nu\lambda}\left(\mp\mathrm{i}\,\Gamma_5\Xi^A_{\nu\lambda\mp} + 2\hat{E}^i_\nu\Gamma_i\Gamma_5\Xi^A_{\lambda\pm}\right) \\ & + \left(\frac{z}{\ell}\right)^{\pm\frac{1}{2}}\epsilon_{AB}\left(-\mathrm{i}\,\epsilon^{\mu\nu\lambda}\Gamma_5\varphi^B_{\mp\lambda} + e_3\,g^{\mu\nu}\varphi^B_{\pm z}\right)\hat{\mathbf{F}}_{\nu z} \\ & + \left(\frac{z}{\ell}\right)^{\mp\frac{1}{2}}\epsilon_{AB}\left(\frac{\mathrm{i}}{2}\,\epsilon^{\mu\nu\lambda}F_{\nu\lambda}\Gamma_5\varphi^B_{\mp z} + e_3\,F^{\nu\mu}\varphi^B_{\pm\nu}\right)\,. \end{aligned} \tag{D.37}$$

All tensors appearing above are finite, except $\hat{\mathbf{F}}_{\mu z}$ and $\Xi^A_{\mu+}$. With this at hand, we identify the leading orders of the $z$-component of the gravitini equations of motion (looking at the two projections separately). By requiring the most divergent terms to vanish (that are $(\ell/z)^{5/2}$ and $(\ell/z)^{3/2}$ in the two chiralities), we get

$$\begin{aligned} 0 \;=\;& \epsilon^{ijk}\left(A_{(-1)z}\,\epsilon_{AB}\,\Gamma_i\varphi^B_{(0)+\mu}E^\mu_j + 2\mathrm{i}\Gamma_{ij}\varphi_{A(0)-z}\right)\,, \\ 0 \;=\;& \epsilon^{\mu\nu\lambda}\left(\mathrm{i}\,\Xi^A_{(0)\nu\lambda+} - 2E^i_\nu\Gamma_i\Xi^A_{(0)\lambda-}\right) \end{aligned} \tag{D.38}$$

$$+\epsilon_{AB}\left(-\mathrm{i}\,\epsilon^{\mu\nu\lambda}\varphi^B_{(0)+\lambda}+e_{3(0)}\,g^{\mu\nu}_{(0)}\Gamma_5\varphi^B_{(0)-z}\right)\left(\partial_\nu A_{(-1)z}-2\epsilon_{AC}\,\overline{\varphi}^A_{(0)+\nu}\varphi^C_{(0)-z}\right)\,,$$

where we multiplied the equations by $\Gamma_5$. Since $\partial_\nu A_{(-1)z}$ is correlated with $\varphi^A_{(0)-z}$ through the condition (D.31), it can be used in the second equation.

It turns out that we can solve the gauge fixing functions from the first equation in (D.38), in terms of the dynamic fields. Contracting it by $\epsilon_{ki'j'}$, it acquires an equivalent form

$$0=-A_{(-1)z}\,\epsilon_{AB}\,E^\mu_{[i}\Gamma_{j]}\varphi^B_{(0)+\mu}+2\mathrm{i}\Gamma_{ij}\varphi_{A(0)-z}\,. \tag{D.39}$$

We can contract the above equation by $\Gamma^{ij}$ and use the contractions of the gamma matrices, which in this case become $\Gamma_i\Gamma^i=3$, $\Gamma^{ij}\Gamma_j=2\Gamma^i$ and $\Gamma^{ij}\Gamma_{ij}=-6$. As a result, we obtain a solution which relates the gauge fixing $\varphi^A_{(0)-z}$ with the gauge fixing $A_{(-1)z}$,

$$\varphi^A_{(0)-z}=\frac{\mathrm{i}}{6}\,A_{(-1)z}\epsilon^{AB}\,\Gamma^i\varphi_{B(0)+\mu}E^\mu_{\ i}\,. \tag{D.40}$$

Then second equation in (D.31) becomes a linear differential equation in $A_{(-1)z}$. One possible solution is $A_{(-1)z}=0$ that, from eq. (D.40), yields $\varphi^A_{(0)-z}=0$. On the other hand, when $A_{(-1)z}\neq 0$, we can solve $\varphi^A_{(0)+\mu}$ from the first equation in (D.38) as

$$A_{(-1)z}\varphi^A_{(0)+\mu}=2\mathrm{i}\,E^i_\mu\Gamma_i\varphi^B_{(0)-z}\epsilon_{AB} \tag{D.41}$$

and the differential equation becomes

$$A_{(-1)z}\partial_\mu A_{(-1)z}=2\mathrm{i}\,E_{k\mu}\overline{\varphi}^A_{(0)-z}\left(2\Gamma^k+\epsilon^{ijk}\Gamma_5\Gamma_{ij}\right)\varphi^A_{(0)-z}=0\,, \tag{D.42}$$

where the last zero is due to antisymmetry of the fermionic bilinears, namely $\overline{\varphi}^A_{(0)-z}\Gamma^k\varphi^A_{(0)-z}\equiv 0$ and $\overline{\varphi}^A_{(0)-z}\Gamma_5\Gamma_{ij}\varphi^A_{(0)-z}\equiv 0$ so that each term in the sum vanishes independently. The only solution of the above equation is $A_{(-1)z}=const.$

Let us observe that taking $A_{(-1)z}=0$ and plugging $\varphi^A_{(1)\mu}=0$ into the last equation in (D.31), we are left with $A_{(0)z}=0$, meaning that, in this case, $A_{(0)z}=0$, $\varphi^A_{(1)+\mu}=0$ is actually the only solution to the aforesaid equation.
On the other hand, if we take $A_{(-1)z}\neq 0$ and use (D.41) and $\varphi^A_{(1)+\mu}=0$ into the last equation of (D.31), we obtain

$$A_{(0)z}E^\lambda_{\ k}\,\overline{\varphi}^A_{(0)-z}\Gamma^k\varphi^A_{(0)-z}=0\,, \tag{D.43}$$

which is identically satisfied since $\overline{\varphi}^A_{(0)-z}\Gamma^k\varphi^A_{(0)-z}=0$. In particular, this means that, in this case, the last equation in (D.31) is solved by (D.41) and $\varphi^A_{(1)+\mu}=0$, without forcing $A_{(0)z}$ to vanish.

Summing up the results, the following gauge fixings for $A_z$ and $\varphi^A_{-z}$ are allowed:

$$\begin{array}{lll} A_{(-1)z}=0\,, & A_{(0)z}=0\,, & \varphi^A_{(0)-z}=0\,,\\ A_{(-1)z}=const\,, & A_{(0)z}\neq 0\,, & \varphi^A_{(0)-z}=\frac{\mathrm{i}}{6}\,A_{(-1)z}\,\epsilon^{AB}\Gamma^\mu\varphi_{B(0)+\mu}\,, \end{array} \tag{D.44}$$

where the first line can be seen as a special case of the general solution given in the second line. If one imposes the condition $\Gamma^{\hat{\mu}}\Psi_{\hat{\mu}} = 0$ as in [58], then eq. (D.40) implies $\psi_{-z} = 0$ and therefore $A_{(-1)z} = 0$ as the only solution.

In this text, we mostly focus on the case $\varphi^A_{(0)-z} = 0$ and we actually take the whole gauge-fixing function $\Psi^A_{-z}$ to vanish.
At the end, let us recall that, in our approach, the gauge-fixing functions are invariant under the gauge transformations ($\delta\hat{A}_z = 0$). Thus, the above solutions are consistent because, since $A_{(-1)z}$ is constant, it also implies $\delta A_{(-1)z} = 0$ for the asymptotic transformations.

## E The rheonomic parametrizations

In this Section we presentWe finally study the asymptotic expansion of the rheonomic parametrizations $\tilde{R}^{ab}{}_{cd}$, $\tilde{\rho}^A_{ab}$ and $\tilde{F}_{ab}$. The procedure is the one described in the main text the applied gauge fixing corresponds to $A_{(-1)z} = 0$ and $\Psi^A_{z-} = 0$.

We start from the graviphoton field strength

$$\hat{\mathbf{F}} = \mathrm{d}\hat{A} - \overline{\Psi}_A\Psi_B\epsilon^{AB} = \hat{F}_{ab}V^aV^b. \tag{E.1}$$

By expanding both sides of this equation onto the basis $\mathrm{d}x^{\hat{\mu}} \wedge \mathrm{d}x^{\hat{\nu}}$, one can derive the explicit expression of the rheonomic parametrizations

$$\begin{aligned}\tilde{F}_{ij} &= \left(\frac{z}{\ell}\right)^3 E^\mu_{[i}E^\nu_{j]}\left(\partial_\mu A_{(1)\nu} - 2\epsilon_{AB}\overline{\psi}^A_{\mu+}\zeta^B_{\nu-} - 2\epsilon_{AB}\overline{\zeta}^A_{\mu+}\psi^B_{\nu-}\right) + \mathcal{O}(z^4),\\ 2\tilde{F}_{i3} &= -\frac{1}{\ell}\left(\frac{z}{\ell}\right)^2 A_{(1)\mu}E^\mu_i + \left(\frac{z}{\ell}\right)^3\left(\partial_\mu A_{(1)z} - \frac{2}{\ell}A_{(2)\mu} + 2\epsilon_{AB}\overline{\psi}^A_{z+}\psi^B_{\mu-}\right)E^\mu_i + \mathcal{O}(z^4),\end{aligned} \tag{E.2}$$

where we have used that $\hat{\mathbf{F}}_{\mu\nu} = \mathcal{O}(z)$.

We now focus on the curvature of the gravitini. The analysis is the same as for the gauge field strength: we take the expression of the curvature and its rheonomic expansion

$$\begin{aligned}\hat{\boldsymbol{\rho}}^A &= \mathrm{d}\Psi^A + \frac{1}{4}\Gamma_{ab}\hat{\omega}^{ab}\Psi^A - \frac{1}{2\ell}\hat{A}\epsilon^{AB}\Psi_B - \frac{\mathrm{i}}{2\ell}\Gamma_a\Psi^AV^a\\ &= \tilde{\rho}^A_{ab}V^aV^b - \frac{\mathrm{i}}{2}\Gamma^a\Psi_BV^b\tilde{F}_{ab}\epsilon^{AB} - \frac{1}{4}\Gamma_5\Gamma^a\Psi_BV^b\tilde{F}^{cd}\epsilon^{AB}\epsilon_{abcd}\end{aligned} \tag{E.3}$$

and expand this relation onto the basis $\mathrm{d}x^{\hat{\mu}} \wedge \mathrm{d}x^{\hat{\nu}}$ to obtain

$$\begin{aligned}\tilde{\rho}^A_{ij+} &= \left(\frac{z}{\ell}\right)^{\frac{5}{2}} E^\mu_{[i}E^\nu_{j]}\Big(\nabla_\mu\zeta^A_{\nu+} + \frac{\mathrm{i}}{\ell}E^k_\mu\gamma_k\zeta^A_{\nu-} + \frac{1}{4}\omega^{kl}_{(1)\mu}\gamma_{kl}\psi^A_{\nu+} - \frac{1}{4\ell}A_{(1)\mu}\psi_{\nu+B}\epsilon^{AB}\\ &\quad + \frac{\mathrm{i}}{4\ell}\epsilon_{lmn}\gamma^l\psi_{B\mu+}E^m_\nu E^{\rho n}A_{(1)\rho}\epsilon^{AB}\Big) + \mathcal{O}(z^{7/2}),\\ 2\tilde{\rho}^A_{i3+} &= -\frac{1}{\ell}\left(\frac{z}{\ell}\right)^{\frac{3}{2}} E^\mu_i\zeta^A_{\mu+} + \left(\frac{z}{\ell}\right)^{\frac{5}{2}} E^\mu_i\Big(\nabla_\mu\psi^A_{z+} - \frac{1}{4}w^{jk}_{(0)}\gamma_{jk}\psi^A_{\mu+} + \frac{1}{2\ell}\epsilon^{AB}A_{(1)z}\psi_{B\mu+}\end{aligned}$$

$$- \frac{2}{\ell}\Pi^A_{\mu+}\Bigg) + \mathcal{O}(z^{7/2}),$$

$$\tilde{\rho}^A_{ij-} = \left(\frac{z}{\ell}\right)^{\frac{5}{2}} E^\mu_{[i}E^\nu_{j]}\left(\nabla_\mu\psi^A_{\nu-} + \frac{\mathrm{i}\ell}{2}\mathcal{S}^k{}_\mu\gamma_k\psi^A_{\nu+}\right) + \mathcal{O}(z^{7/2})$$

$$2\tilde{\rho}^A_{i3-} = -\left(\frac{z}{\ell}\right)^{\frac{5}{2}} E^\mu_i\left(\frac{1}{\ell}\zeta^A_{\mu-} + \frac{\mathrm{i}}{4\ell}\epsilon^{AB}\gamma^j\psi_{B\mu+}A_{(1)\nu}E^\nu_j\right) + \mathcal{O}(z^{7/2}), \tag{E.4}$$

where we used that $\hat{\boldsymbol{\rho}}^A_{\mu\nu} = \mathcal{O}(z^{1/2})$.

This result allows to compute the spinor-tensor

$$\Theta^{ab|c}_A = -2\mathrm{i}\Gamma^{[a}\tilde{\rho}^{b]c}_A + \mathrm{i}\Gamma^c\tilde{\rho}^{ab}_A\,, \tag{E.5}$$

as an intermediate step necessary to find the remaining parametrizations. In particular we obtain

$$\Theta^{ij|k}_{A+} = \mathrm{i}\left(\frac{z}{\ell}\right)^{\frac{5}{2}}\left(-\gamma^iE^{[j\mu}E^{k]\nu} + \gamma^jE^{[i\mu}E^{k]\nu} + \gamma^kE^{[i\mu}E^{j]\nu}\right)\left(\nabla_\mu\psi_{A\nu-} + \frac{\mathrm{i}\ell}{2}\mathcal{S}^l{}_\mu\gamma_l\psi_{A\nu+}\right)$$
$$+ \mathcal{O}(z^{7/2}),$$

$$\Theta^{ij|3}_{A+} = -\mathrm{i}\left(\frac{z}{\ell}\right)^{\frac{5}{2}}\gamma^{[i}E^{j]\mu}\left(\frac{1}{\ell}\zeta_{A\mu-} + \frac{\mathrm{i}}{4\ell}\epsilon_{AB}\gamma^k\psi^B_{\mu+}A_{(1)\rho}E^\rho_k\right) - \left(\frac{z}{\ell}\right)^{\frac{5}{2}}E^{[i\mu}E^{j]\nu}\Bigg(\nabla_\mu\zeta_{A\nu+}$$
$$+ \frac{\mathrm{i}}{\ell}E^k_\mu\gamma_k\zeta_{A\nu-} + \frac{1}{4}\omega^{kl}_{(1)\mu}\gamma_{kl}\psi_{A\nu+} - \frac{1}{4\ell}A_{(1)\mu}\psi_{\nu+B}\epsilon^{AB} + \frac{\mathrm{i}}{4\ell}\epsilon_{klm}\gamma^k\psi^B_{\mu+}E^l_\nu E^{\rho m}A_{(1)\rho}\epsilon_{AB}\Bigg)$$
$$+ \mathcal{O}(z^{7/2}),$$

$$\Theta^{i3|j}_{A+} = \mathrm{i}\left(\frac{z}{\ell}\right)^{\frac{5}{2}}\gamma^{(i}E^{j)\mu}\left(\frac{1}{\ell}\zeta_{A\mu-} + \frac{\mathrm{i}}{4\ell}\epsilon_{AB}\gamma^k\psi^B_{\mu+}A_{(1)\nu}E^\nu_k\right) - \left(\frac{z}{\ell}\right)^{\frac{5}{2}}E^{[i\mu}E^{j]\nu}\Bigg(\nabla_\mu\zeta_{A\nu+}$$
$$+ \frac{\mathrm{i}}{\ell}E^k_\mu\gamma_k\zeta_{A\nu-} + \frac{1}{4}\omega^{kl}_{(1)\mu}\gamma_{kl}\psi_{A\nu+} - \frac{1}{4\ell}A_{(1)\mu}\psi_{\nu+B}\epsilon^{AB} + \frac{\mathrm{i}}{4\ell}\epsilon_{klm}\gamma^k\psi^B_{\mu+}E^l_\nu E^{\rho m}A_{(1)\rho}\epsilon_{AB}\Bigg)$$
$$+ \mathcal{O}(z^{7/2}),$$

$$\Theta^{i3|3}_{A+} = -\frac{1}{\ell}\left(\frac{z}{\ell}\right)^{\frac{3}{2}}\zeta_{A\mu+}E^{\mu i} + \left(\frac{z}{\ell}\right)^{\frac{5}{2}}E^{i\mu}\Bigg(\nabla_\mu\psi_{Az+} - \frac{1}{4}w^{jk}_{(0)}\gamma_{jk}\psi_{A\mu+} + \frac{1}{2\ell}\epsilon_{AB}A_{(1)z}\psi^B_{\mu+}$$
$$- \frac{2}{\ell}\Pi_{A\mu+}\Bigg) + O(z^{7/2}),$$

$$\Theta^{ij|k}_{A-} = \mathrm{i}\left(\frac{z}{\ell}\right)^{\frac{5}{2}}\left(-\gamma^iE^{[j\mu}E^{k]\nu} + \gamma^jE^{[i\mu}E^{k]\nu} + \gamma^kE^{[i\mu}E^{j]\nu}\right)\Bigg(\nabla_\mu\zeta_{A\nu+} + \frac{\mathrm{i}}{\ell}E^l_\mu\gamma_l\zeta_{A\nu-}$$
$$+ \frac{1}{4}\omega^{lm}_{(1)\mu}\gamma_{lm}\psi_{A\nu+} - \frac{1}{4\ell}A_{(1)\mu}\psi_{\nu+B}\epsilon^{AB} + \frac{\mathrm{i}}{4\ell}\epsilon_{lmn}\gamma^l\psi^B_{\mu+}E^m_\nu E^{\rho n}A_{(1)\rho}\epsilon_{AB}\Bigg) + \mathcal{O}(z^{7/2}),$$

$$\Theta^{ij|3}_{A-} = -\frac{\mathrm{i}}{\ell}\left(\frac{z}{\ell}\right)^{\frac{3}{2}}\gamma^{[i}E^{j]\mu}\zeta_{A\mu+} + \mathrm{i}\left(\frac{z}{\ell}\right)^{\frac{5}{2}}\gamma^{[i}E^{j]\mu}\Bigg(\nabla_\mu\psi_{Az+} - \frac{1}{4}w^{kl}_{(0)}\gamma_{kl}\psi_{A\mu+}$$
$$+ \frac{1}{2\ell}\epsilon_{AB}A_{(1)z}\psi^B_{\mu+} - \frac{2}{\ell}\Pi_{A\mu+}\Bigg) + \left(\frac{z}{\ell}\right)^{\frac{5}{2}}E^{[i\mu}E^{j]\nu}\left(\nabla_\mu\psi_{A\nu-} + \frac{\mathrm{i}\ell}{2}\mathcal{S}^k{}_\mu\gamma_k\psi_{A\nu+}\right) + O(z^{7/2}),$$

$$\Theta^{i3|j}_{A-} = \frac{\mathrm{i}}{\ell}\left(\frac{z}{\ell}\right)^{\frac{3}{2}}\gamma^{(i}E^{j)\mu}\zeta_{A\mu+} - \mathrm{i}\left(\frac{z}{\ell}\right)^{\frac{5}{2}}\gamma^{(i}E^{j)\mu}\Bigg(\nabla_\mu\psi_{Az+} - \frac{1}{4}w^{kl}_{(0)}\gamma_{kl}\psi_{A\mu+} + \frac{1}{2\ell}\epsilon_{AB}A_{(1)z}\psi^B_{\mu+}$$

$$
-\frac{2}{\ell}\Pi_{A\mu+}\bigg)+\left(\frac{z}{\ell}\right)^{\frac{5}{2}}E^{[i\mu}E^{j]\nu}\left(\nabla_\mu\psi_{A\nu-}+\frac{\mathrm{i}\ell}{2}\mathcal{S}^k{}_\mu\gamma_k\psi_{A\nu+}\right)+O(z^{7/2}),
$$

$$
\Theta^{i3|3}_{A-}=\left(\frac{z}{\ell}\right)^{\frac{5}{2}}E^{i\mu}\left(\frac{1}{\ell}\zeta_{A\mu-}+\frac{\mathrm{i}}{4\ell}\epsilon_{AB}\gamma^j\psi^B_{\mu+}A_{(1)\rho}E^\rho_j\right)+O(z^{7/2}).
$$

We are now ready to compute the rheonomic parametrization of the curvature $\hat{\boldsymbol{R}}^{ab}$. Since

$$
\begin{aligned}
\hat{\boldsymbol{R}}^{ab}&=\mathrm{d}\hat{\omega}^{ab}+\hat{\omega}^{ac}\hat{\omega}_c{}^b-\frac{1}{\ell^2}V^aV^b-\frac{1}{2\ell}\overline{\Psi}^A\Gamma^{ab}\Psi_A\\
&=\tilde{R}^{ab}{}_{cd}V^cV^d-\overline{\Theta}^{ab}_{A|c}\Psi_AV^c-\frac{1}{2}\overline{\Psi}_A\Psi_B\epsilon_{AB}\tilde{F}^{ab}-\frac{\mathrm{i}}{4}\epsilon^{abcd}\overline{\Psi}_A\Gamma_5\Psi_B\epsilon_{AB}\tilde{F}_{cd},
\end{aligned}\tag{E.6}
$$

applying the usual procedure yields

$$
\begin{aligned}
\tilde{R}^{i3}{}_{jk}&=\frac{\mathrm{i}}{2\ell}\left(\frac{z}{\ell}\right)^2E^\mu_{[j}E^\nu_{k]}\overline{\psi}^A_{\mu+}\gamma^i\zeta_{A\nu+}+\frac{\mathrm{i}}{2\ell}\left(\frac{z}{\ell}\right)^2E^\mu_{[j}E^\nu_{k]}\overline{\psi}^A_{\mu+}\gamma^l\zeta_{A\rho+}E_{l\nu}E^{i\rho}\\
&+\frac{1}{\ell}\left(\frac{z}{\ell}\right)^3E^\mu_{[j}E^\nu_{k]}\bigg\{-\mathcal{D}_\mu\tilde{S}^i_\nu+\omega^i_{(2)l\mu}E^l_\nu-\mathrm{i}\overline{\Pi}^A_{\mu+}\gamma^i\psi_{A\nu+}-\frac{\mathrm{i}}{2}\overline{\zeta}^A_{\mu+}\gamma^i\zeta_{A\nu+}\\
&+\frac{\mathrm{i}}{2}\overline{\psi}^A_{\mu-}\gamma^i\psi_{A-\nu}+\overline{\psi}^A_{\mu+}E_{l\nu}\bigg[-\mathrm{i}\gamma^{(i}E^{l)\rho}\bigg(\nabla_\rho\psi_{Az+}-\frac{1}{4}w^{mn}_{(0)}\gamma_{mn}\psi_{A\rho+}\\
&+\frac{1}{2\ell}\epsilon_{AB}A_{(1)z}\psi^B_{\rho+}-\frac{2}{\ell}\Pi_{A\rho+}\bigg)+E^{[i\rho}E^{l]\sigma}\left(\nabla_\rho\psi_{A\sigma-}+\frac{\mathrm{i}\ell}{2}\mathcal{S}^m{}_\rho\gamma_m\psi_{A\sigma+}\right)\bigg]\bigg\}+\mathcal{O}(z^4),\\
2\tilde{R}^{i3}{}_{j3}&=\left(\frac{z}{\ell}\right)^3E^\mu_j\bigg\{-\frac{1}{\ell}w^i_{(1)k}E^k{}_\mu+\frac{1}{\ell^2}\left(4\tilde{\tau}^i_\mu-\tau^i_\mu\right)-\frac{\mathrm{i}}{\ell}\overline{\zeta}^A_{\mu+}\gamma^i\psi_{Az+}-\frac{\mathrm{i}}{\ell}\overline{\psi}^A_{\mu+}\gamma^i\zeta_{Az+}\\
&+\frac{1}{\ell}\overline{\psi}^A_{\mu-}\zeta_{A\nu+}E^{\nu i}-\overline{\psi}^A_{\mu+}E^{i\nu}\left(\frac{1}{\ell}\zeta_{A\nu-}+\frac{\mathrm{i}}{4\ell}\epsilon_{AB}\gamma^l\psi^B_{\nu+}A_{(1)\rho}E^\rho_l\right)\bigg\}+\mathcal{O}(z^4),
\end{aligned}\tag{E.7}
$$

$$
\begin{aligned}
\tilde{R}^{ij}{}_{kl}&=\left(\frac{z}{\ell}\right)^3E^\mu_{[k}E^\nu_{l]}\bigg\{\partial_\mu\omega^{ij}_{(1)\nu}+\omega^i{}_{(1)m\mu}\omega^{mj}{}_\nu+\omega^i{}_{m\mu}\omega^{mj}_{(1)\nu}-\frac{2}{\ell^2}(\tau^{[i}_\mu+2\tilde{\tau}^{[i}_\mu)E^{j]}_\nu\\
&-\frac{1}{\ell}\left(\overline{\psi}^A_{\mu+}\gamma^{ij}\zeta_{A\nu-}+\overline{\zeta}^A_{\mu+}\gamma^{ij}\psi_{A\nu-}\right)+\mathrm{i}E_{m\nu}\overline{\psi}^A_{\mu+}\bigg(-\gamma^iE^{[j\rho}E^{m]\sigma}+\gamma^jE^{[i\rho}E^{m]\sigma}\\
&+\gamma^mE^{[i\rho}E^{j]\sigma}\bigg)\bigg(\nabla_\rho\zeta_{A\sigma+}+\frac{\mathrm{i}}{\ell}E^n_\rho\gamma_n\zeta_{A\sigma-}+\frac{1}{4}\omega^{np}_{(1)\rho}\gamma_{np}\psi_{A\sigma+}\\
&-\frac{1}{4\ell}A_{(1)\rho}\psi^B_{\sigma+}\epsilon_{AB}+\frac{\mathrm{i}}{4\ell}\epsilon_{npq}\gamma^n\psi^B_{\rho+}E^p_\sigma E^{\lambda q}A_{(1)\lambda}\epsilon_{AB}\bigg)\bigg\}+\mathcal{O}(z^4),\\
2\tilde{R}^{ij}{}_{k3}&=-\left(\frac{z}{\ell}\right)^2E^\mu_k\left(\frac{1}{\ell}\omega^{ij}_{(1)\mu}-\frac{\mathrm{i}}{\ell}\overline{\psi}^A_{\mu+}\gamma^{[i}E^{j]\nu}\zeta_{A\nu+}\right)\\
&+\left(\frac{z}{\ell}\right)^3E^\mu_k\bigg\{\partial_\mu w^{ij}-\frac{2}{\ell}\omega^{ij}_{(2)\mu}+\omega^i{}_{l\mu}w^{lj}_{(0)}-w^i{}_l\omega^{lj}{}_\mu+\frac{1}{\ell}\left(E^i_\mu w^j_{(0)}-w^i_{(0)}E^j_\mu\right)\\
&+\frac{1}{\ell}\overline{\psi}^A_{z+}\gamma^{ij}\psi_{A\mu-}-\overline{\psi}^A_{\mu+}\bigg[\mathrm{i}\gamma^{[i}E^{j]\nu}\bigg(\nabla_\nu\psi_{Az+}-\frac{1}{4}w^{lm}_{(0)}\gamma_{lm}\psi_{A\nu+}+\frac{1}{2\ell}\epsilon_{AB}A_{(1)z}\psi^B_{\nu+}\\
&-\frac{2}{\ell}\Pi_{A\nu+}\bigg)+E^{[i\nu}E^{j]\rho}\left(\nabla_\nu\psi_{A\rho-}+\frac{\mathrm{i}\ell}{2}\mathcal{S}^l{}_\nu\gamma_l\psi_{A\rho+}\right)\bigg]\bigg\}+\mathcal{O}(z^4).
\end{aligned}
$$

To obtain the above formulas, we used $\hat{\boldsymbol{R}}^{ab}_{\mu\nu} = \mathcal{O}(z)$ and that the supertorsion is zero (see, in particular, (D.12)).

## References


[1] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. **2** (1998), 253-291 doi:10.4310/ATMP.1998.v2.n2.a2 [arXiv:hep-th/9802150 [hep-th]].

[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from noncritical string theory,” Phys. Lett. B **428** (1998), 105-114 doi:10.1016/S0370-2693(98)00377-3 [arXiv:hep-th/9802109 [hep-th]]

[3] A. Sen, “Universality of the tachyon potential,” JHEP **12** (1999), 027 doi:10.1088/1126-6708/1999/12/027 [arXiv:hep-th/9911116 [hep-th]].

[4] E. Witten, “Noncommutative Geometry and String Field Theory,” Nucl. Phys. B **268** (1986), 253-294 doi:10.1016/0550-3213(86)90155-0.

[5] K. Ohmori, “A Review on tachyon condensation in open string field theories,” [arXiv:hep-th/0102085 [hep-th]].

[6] M. Schnabl, “Analytic solution for tachyon condensation in open string field theory,” Adv. Theor. Math. Phys. **10** (2006) no.4, 433-501 doi:10.4310/ATMP.2006.v10.n4.a1 [arXiv:hep-th/0511286 [hep-th]].

[7] T. Erler and C. Maccaferri, “String Field Theory Solution for Any Open String Background,” JHEP **10** (2014), 029 doi:10.1007/JHEP10(2014)029 [arXiv:1406.3021 [hep-th]].

[8] I. Ellwood and M. Schnabl, “Proof of vanishing cohomology at the tachyon vacuum,” JHEP **02** (2007), 096 doi:10.1088/1126-6708/2007/02/096 [arXiv:hep-th/0606142 [hep-th]].

[9] N. Berkovits, “SuperPoincare invariant superstring field theory,” Nucl. Phys. B **450** (1995), 90-102 doi:10.1016/0550-3213(95)00259-U [arXiv:hep-th/9503099 [hep-th]].

[10] T. Erler, “Analytic solution for tachyon condensation in Berkovits‘ open superstring field theory,” JHEP **11** (2013), 007 doi:10.1007/JHEP11(2013)007 [arXiv:1308.4400 [hep-th]].

[11] T. Erler, C. Maccaferri and R. Noris, “Taming boundary condition changing operator anomalies with the tachyon vacuum,” JHEP **06** (2019), 027 doi:10.1007/JHEP06(2019)027 [arXiv:1901.08038 [hep-th]].

[12] T. Erler and C. Maccaferri, “String field theory solution for any open string background. Part II,” JHEP **01** (2020), 021 doi:10.1007/JHEP01(2020)021 [arXiv:1909.11675 [hep-th]].

[13] R. Blumenhagen, D. Lüst and S. Theisen, “Basic concepts of string theory,” doi:10.1007/978-3-642-29497-6.

[14] D. Tong, “String Theory,” [arXiv:0908.0333 [hep-th]].

[15] J. Polchinski, “String theory. Vol. 1: An introduction to the bosonic string,” doi:10.1017/CBO9780511816079.

[16] J. Polchinski, “String theory. Vol. 2: Superstring theory and beyond,” doi:10.1017/CBO9780511618123.

[17] D. Friedan, E. J. Martinec and S. H. Shenker, “Conformal Invariance, Supersymmetry and String Theory,” Nucl. Phys. B **271** (1986), 93-165 doi:10.1016/0550-3213(86)90356-1.

[18] C. R. Preitschopf, C. B. Thorn and S. A. Yost, “SUPERSTRING FIELD THEORY,” Nucl. Phys. B **337** (1990), 363-433 doi:10.1016/0550-3213(90)90276-J.

[19] I. Y. Arefeva, P. B. Medvedev and A. P. Zubarev, “Background Formalism for Superstring Field Theory,” Phys. Lett. B **240** (1990), 356-362 doi:10.1016/0370-2693(90)91112-O.

[20] T. Erler and M. Schnabl, “A Simple Analytic Solution for Tachyon Condensation,” JHEP **10** (2009), 066 doi:10.1088/1126-6708/2009/10/066 [arXiv:0906.0979 [hep-th]].

[21] Y. Okawa, “Analytic methods in open string field theory,” Prog. Theor. Phys. **128** (2012), 1001-1060 doi:10.1143/PTP.128.1001

[22] Y. Okawa, “Comments on Schnabl’s analytic solution for tachyon condensation in Witten’s open string field theory,” JHEP **04** (2006), 055 doi:10.1088/1126-6708/2006/04/055 [arXiv:hep-th/0603159 [hep-th]].

[23] T. Erler, “Tachyon Vacuum in Cubic Superstring Field Theory,” JHEP **01** (2008), 013 doi:10.1088/1126-6708/2008/01/013 [arXiv:0707.4591 [hep-th]].

[24] I. Y. Aref’eva, A. S. Koshelev, D. M. Belov and P. B. Medvedev, “Tachyon condensation in cubic superstring field theory,” Nucl. Phys. B **638** (2002), 3-20 doi:10.1016/S0550-3213(02)00472-8 [arXiv:hep-th/0011117 [hep-th]].

[25] T. Erler, “The Identity String Field and the Sliver Frame Level Expansion,” JHEP **11** (2012), 150 doi:10.1007/JHEP11(2012)150 [arXiv:1208.6287 [hep-th]].

[26] T. Erler, “Exotic Universal Solutions in Cubic Superstring Field Theory,” JHEP **04** (2011), 107 doi:10.1007/JHEP04(2011)107 [arXiv:1009.1865 [hep-th]].

[27] Y. Ne’eman and T. Regge, “Gravity and Supergravity as Gauge Theories on a Group Manifold,” Phys. Lett. B **74** (1978), 54-56 doi:10.1016/0370-2693(78)90058-8

[28] L. Castellani, R. D’Auria and P. Fre, “Supergravity and superstrings: A Geometric perspective. Vol. 1: Mathematical foundations,”

[29] L. Castellani, R. D’Auria and P. Fre, “Supergravity and superstrings: A Geometric perspective. Vol. 2: Supergravity,”

[30] L. Castellani, “Supergravity in the Group-Geometric Framework: A Primer,” Fortsch. Phys. **66** (2018) no.4, 1800014 doi:10.1002/prop.201800014 [arXiv:1802.03407 [hep-th]].

[31] Y. Choquet-Bruhat, C. Dewitt-Morette "Analysis, manifolds, and physics,"

[32] M. Nakahara, "Geometry, topology and physics,"

[33] S. Kobayashi, K. Nomizu, "Fundations of Differential Geometry, Volume I",

[34] L. Andrianopoli, B. L. Cerchiai, R. D'Auria, A. Gallerati, R. Noris, M. Trigiante and J. Zanelli, "$\mathcal{N}$-extended $D = 4$ supergravity, unconventional SUSY and graphene," JHEP **01** (2020), 084 doi:10.1007/JHEP01(2020)084 [arXiv:1910.03508 [hep-th]].

[35] H. Kleinert, "Gauge fields in condensed matter. Vol. 2: Stresses and defects. Differential geometry, crystal melting."

[36] P. D. Alvarez, M. Valenzuela and J. Zanelli, "Supersymmetry of a different kind," JHEP **04** (2012), 058 doi:10.1007/JHEP04(2012)058 [arXiv:1109.3944 [hep-th]].

[37] A. Achucarro and P. K. Townsend, "A Chern-Simons Action for Three-Dimensional anti-De Sitter Supergravity Theories," Phys. Lett. B **180** (1986), 89 doi:10.1016/0370-2693(86)90140-1.

[38] G. W. Gibbons and S. W. Hawking, "Action Integrals and Partition Functions in Quantum Gravity," Phys. Rev. D **15** (1977), 2752-2756 doi:10.1103/PhysRevD.15.2752.

[39] J. W. York, Jr., "Role of conformal three geometry in the dynamics of gravitation," Phys. Rev. Lett. **28** (1972), 1082-1085 doi:10.1103/PhysRevLett.28.1082.

[40] L. Andrianopoli, B. L. Cerchiai, R. D'Auria and M. Trigiante, "Unconventional supersymmetry at the boundary of $\mathrm{AdS}_4$ supergravity," JHEP **04** (2018), 007 doi:10.1007/JHEP04(2018)007 [arXiv:1801.08081 [hep-th]].

[41] G. W. Semenoff, "Condensed Matter Simulation of a Three-dimensional Anomaly," Phys. Rev. Lett. **53** (1984), 2449 doi:10.1103/PhysRevLett.53.2449

[42] F. D. M. Haldane, "Model for a Quantum Hall Effect without Landau Levels: Condensed-Matter Realization of the 'Parity Anomaly'," Phys. Rev. Lett. **61** (1988), 2015-2018 doi:10.1103/PhysRevLett.61.2015.

[43] R. Noris and L. Fatibene, "Spin frame transformations and Dirac equations," [arXiv:1910.04634 [math.DG]].

[44] H. Blaine Lawson, Jr., Marie-Louise Michelsohn "Spin Geometry".

[45] L. Andrianopoli, B. L. Cerchiai, R. Matrecano, O. Miskovic, R. Noris, R. Olea, L. Ravera and M. Trigiante, "$\mathcal{N} = 2$ $\mathrm{AdS}_4$ supergravity, holography and Ward identities," [arXiv:2010.02119 [hep-th]].

[46] L. Andrianopoli and R. D'Auria, "N=1 and N=2 pure supergravities on a manifold with boundary," JHEP **08** (2014), 012 doi:10.1007/JHEP08(2014)012 [arXiv:1405.2010 [hep-th]].

[47] S. W. MacDowell and F. Mansouri, "Unified Geometric Theory of Gravity and Supergravity," Phys. Rev. Lett. **38** (1977) 739 [Erratum-ibid. **38** (1977) 1376].

[48] O. Miskovic and R. Olea, "Topological regularization and self-duality in four-dimensional anti-de Sitter gravity," Phys. Rev. D **79** (2009), 124020 doi:10.1103/PhysRevD.79.124020 [arXiv:0902.2082 [hep-th]].

[49] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," Int. J. Theor. Phys. **38** (1999), 1113-1133 doi:10.1023/A:1026654312961 [arXiv:hep-th/9711200 [hep-th]].

[50] K. Skenderis, "Lecture notes on holographic renormalization," Class. Quant. Grav. **19**, 5849 (2002) [hep-th/0209067].

[51] V. Balasubramanian and P. Kraus, "A Stress tensor for Anti-de Sitter gravity," Commun. Math. Phys. **208** (1999), 413-428 doi:10.1007/s002200050764 [arXiv:hep-th/9902121 [hep-th]].

[52] S. de Haro, S. N. Solodukhin and K. Skenderis, "Holographic reconstruction of spacetime and renormalization in the AdS / CFT correspondence," Commun. Math. Phys. **217** (2001), 595-622 doi:10.1007/s002200100381 [arXiv:hep-th/0002230 [hep-th]].

[53] C. Imbimbo, A. Schwimmer, S. Theisen and S. Yankielowicz, "Diffeomorphisms and holographic anomalies," Class. Quant. Grav. **17** (2000), 1129-1138 doi:10.1088/0264-9381/17/5/322 [arXiv:hep-th/9910267 [hep-th]].

[54] R. Penrose and W. Rindler, "SPINORS AND SPACE-TIME. VOL. 2: SPINOR AND TWISTOR METHODS IN SPACE-TIME GEOMETRY," doi:10.1017/CBO9780511524486.

[55] J. D. Brown and M. Henneaux, "Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity," Commun. Math. Phys. **104** (1986), 207-226 doi:10.1007/BF01211590.

[56] R. L. Arnowitt, S. Deser and C. W. Misner, "The Dynamics of general relativity," Gen. Rel. Grav. **40** (2008), 1997-2027 doi:10.1007/s10714-008-0661-1 [arXiv:gr-qc/0405109 [gr-qc]].

[57] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, P. Fre and T. Magri, "N=2 supergravity and N=2 superYang-Mills theory on general scalar manifolds: Symplectic covariance, gaugings and the momentum map," J. Geom. Phys. **23** (1997), 111-189 doi:10.1016/S0393-0440(97)00002-8 [arXiv:hep-th/9605032 [hep-th]].

[58] A. J. Amsel and G. Compere, "Supergravity at the boundary of AdS supergravity," Phys. Rev. D **79** (2009), 085006 doi:10.1103/PhysRevD.79.085006 [arXiv:0901.3609 [hep-th]].

[59] E. Fradkin and A. A. Tseytlin, "Conformal Supergravity," Phys. Rept. **119** (1985), 233-362

[60] L. Castellani, A. Ceresole, R. D'Auria, S. Ferrara, P. Fre and M. Trigiante, "G / H M-branes and AdS(p+2) geometries," Nucl. Phys. B **527** (1998), 142-170 [hep-th/9803039].

[61] G. Dall'Agata, D. Fabbri, C. Fraser, P. Fre, P. Termonia and M. Trigiante, "The Osp(8|4) singleton action from the supermembrane," Nucl. Phys. B **542** (1999) 157 [hep-th/9807115].

[62] I. Papadimitriou, "Supercurrent anomalies in 4d SCFTs," JHEP **07** (2017), 038 [arXiv:1703.04299 [hep-th]].

[63] I. Papadimitriou, "Supersymmetry anomalies in $\mathcal{N} = 1$ conformal supergravity," JHEP **04** (2019), 040 [arXiv:1902.06717 [hep-th]].

[64] E. Gourgoulhon, "3+1 formalism and bases of numerical relativity," [arXiv:gr-qc/0703035 [gr-qc]].