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# $G$ -DEFORMATIONS OF MAPS INTO PROJECTIVE SPACE

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ABSTRACT.  $G$ -deformability of maps into projective space is characterised by the existence of certain Lie algebra valued 1-forms. This characterisation gives a unified way to obtain well known results regarding deformability in different geometries.

## 1. INTRODUCTION

It is well known that isothermic surfaces are the only surfaces in conformal geometry that admit non-trivial second order deformations [13] and that  $R$ - and  $R_0$ -surfaces are the only surfaces in projective geometry that admit non-trivial second order deformations [11, 17]. In [27] it is shown that  $\Omega$ - and  $\Omega_0$ -surfaces are the only surfaces in Lie sphere geometry that admit non-trivial second order deformations. Motivated by these results we investigate  $G$ -deformations of smooth maps into  $G$ -invariant submanifolds of projective space  $\mathbb{P}(V)$ , where  $G$  is a group acting linearly on  $V$ . This method quickly recovers the aforementioned results regarding deformability in the context of gauge theory.

The examples studied in this paper are all examples of  $R$ -spaces [33]. The author believes that the main theorem of this paper can be used to study deformations in general  $R$ -spaces and intends to do so in subsequent work.

It should be noted that Cartan's method of moving frames was utilised in [19, 22] to outline methods for considering deformations of submanifolds of general homogeneous spaces. A different approach is used in this paper that is more suited to recovering gauge-theoretic characterisations of certain classes of surfaces.

We start by recalling the definition of  $k$ -th order deformations of maps into homogeneous spaces [19, 22]. Let  $N$  be a manifold on which a Lie group  $G$ , with Lie algebra  $\mathfrak{g}$ , acts smoothly and let  $F : \Sigma \rightarrow N$  be a smooth map from a manifold  $\Sigma$  into  $N$ .

**Definition 1.1.** Let  $k \in \mathbb{N} \cup \{0\}$ . We say that  $\hat{F} : \Sigma \rightarrow N$  is a  $k^{\text{th}}$ -order  $G$ -deformation of  $F$  if there exists a smooth map  $g : \Sigma \rightarrow G$  such that for all  $p \in \Sigma$

$$g^{-1}(p)\hat{F} \quad \text{and} \quad F$$

agree to order  $k$  at  $p$ . The map  $g$  is called a  $k$ -th order  $G$ -deformation of  $F$ .

If  $F$  and  $\hat{F}$  are congruent, i.e.,  $\hat{F} = AF$  for some  $A \in G$ , we say that the deformation is trivial. A map  $F : \Sigma \rightarrow N$  is said to be  $G$ -deformable of order  $k$  if it admits a non-trivial  $k$ -th order  $G$ -deformation, otherwise  $F$  is said to be  $G$ -rigid to  $k$ -th order.

*Remark 1.2.* Note that the notion of “agreeing to order  $k$ ” means that the projections into any chart agree to order  $k$ .

*Remark 1.3.*  $k$ -th order contact at a point is transitive, i.e., if  $\phi_1$  and  $\phi_2$  agree to  $k$ -th order at a point  $p$  and  $\phi_2$  and  $\phi_3$  agree to  $k$ -th order at  $p$ , then  $\phi_1$  and  $\phi_3$  agree to  $k$ -th order at  $p$ .

Clearly, if  $\hat{F}$  is a  $k$ -th order  $G$ -deform of  $F$  then we may write  $\hat{F} = gF$  for the given  $k$ -th order  $G$ -deformation  $g : \Sigma \rightarrow G$ . In this way we may recover  $\hat{F}$  from the deformation  $g$ . Furthermore, for any  $A \in G$ , it is clear that  $Ag$  is a  $k$ -th order deformation of  $F$  if and only if  $g$  is a  $k$ -th order deformation of  $F$ . This leads us to the following definition:

**Definition 1.4.**  $\eta \in \Omega^1(\mathfrak{g})$  is a  $k$ -th order infinitesimal deformation of  $F$  if  $\eta$  satisfies the Maurer-Cartan equation and  $g$  is a  $k$ -th order  $G$ -deformation of  $F$  for any  $g : \Sigma \rightarrow G$  satisfying  $g^{-1}dg = \eta$ .

The following lemma concerns the uniqueness of the map  $g : \Sigma \rightarrow G$  defining a  $G$ -deform:

**Lemma 1.5.** Let  $\hat{F} : \Sigma \rightarrow S$  be a  $k$ -th order  $G$ -deform of  $F$  of each other via  $g : \Sigma \rightarrow G$  and. Then  $\hat{F}$  is a  $k$ -th order  $G$ -deform of  $F$  via  $\tilde{g} : \Sigma \rightarrow G$  as well if and only if  $F$  is a  $k$ -th order deform of itself via  $h := g^{-1}\tilde{g}$ .

*Proof.* Since  $\hat{F}$  is a  $k$ -th order  $G$ -deform of  $F$  via  $g$ , we have that for each  $p \in \Sigma$ ,  $g^{-1}(p)\hat{F}$  agrees to  $k$ -th order with  $F$  at  $p$ . Let  $\tilde{g} : \Sigma \rightarrow G$  and define  $h := g^{-1}\tilde{g}$ . Then since  $h^{-1}(p)$  is constant, one has that  $h^{-1}(p)g^{-1}(p)\hat{F}$  agrees to order  $k$  with  $h^{-1}(p)F$  at  $p$ . It follows by Remark 1.3 that  $h^{-1}(p)F$  agrees to order  $k$  with  $F$  at  $p$  if and only if  $\tilde{g}^{-1}(p)\hat{F} = h^{-1}(p)g^{-1}(p)\hat{F}$  agrees to order  $k$  with  $F$  at  $p$ .  $\square$

We will only be interested in deformations that are non-trivial. We thus have the following result:

**Lemma 1.6.** Suppose that  $\hat{F} : \Sigma \rightarrow S$  is a  $k$ -th order  $G$ -deform of  $F$  via  $g : \Sigma \rightarrow G$ . Then this is a trivial deformation if and only if  $g = Ah$  where  $A \in G$  and  $h : \Sigma \rightarrow G$  such that  $F$  is a  $k$ -th order  $G$ -deform of itself via  $h : \Sigma \rightarrow G$ .

*Proof.* This follows by Lemma 1.5 and noting that if  $\hat{F} = AF$  for some  $A \in G$  then  $\hat{F}$  is a  $k$ -th order  $G$  deform of  $F$  via  $A$ .  $\square$

## 2. DEFORMATIONS IN PROJECTIVE SPACE

Suppose that  $V$  is a vector space with projectivisation  $\mathbb{P}(V)$  and suppose that  $G$  is a Lie group acting linearly on  $V$ .

**Proposition 2.1.**  $\phi, \hat{\phi} : \Sigma \rightarrow \mathbb{P}(V)$  agree to order  $k$  at  $p \in \Sigma$  if and only if for any  $v_0 \in V^*$ , the sections  $\sigma, \hat{\sigma}$  of  $\phi$  and  $\hat{\phi}$ , respectively, such that

$$v_0(\sigma) = v_0(\hat{\sigma}) = 1$$

agree to order  $k$  at  $p$  on the open set where they are defined.

*Proof.*  $\phi$  and  $\hat{\phi}$  agree to order  $k$  at  $p$  if and only if in any chart of  $\mathbb{P}(V)$  they agree to order  $k$  at  $p$ . Let  $U := \mathbb{P}(V) \setminus \mathbb{P}(\ker v_0)$ . Then  $U$  is an open subset of  $\mathbb{P}(V)$  and

$$\psi : U \rightarrow V, \quad [u] \mapsto u,$$

where  $u \in [u]$  satisfies  $v_0(u) = 1$ , defines a chart  $(U, \psi)$  on  $\mathbb{P}(V)$ . Thus,  $\phi$  and  $\hat{\phi}$  agreeing to order  $k$  at  $p$  in this chart is equivalent to  $\sigma := \psi(\phi)$  and  $\hat{\sigma} := \psi(\hat{\phi})$

agreeing to order  $k$  at  $p$ . The result follows as the collection of charts defined by all  $v_0 \in V^*$  is an atlas for  $\mathbb{P}(V)$ .  $\square$

Let  $S$  be a  $G$ -invariant submanifold of  $\mathbb{P}(V)$ .  $k$ -th order contact of two maps in  $S$  is equivalent to  $k$ -th order contact as maps into  $\mathbb{P}(V)$ . Therefore we may use Proposition 2.1 to study contact in  $S$ . Let  $F : \Sigma \rightarrow S$  be a smooth map from a manifold  $\Sigma$  into  $S$ .

To simplify our exposition in this section, we shall use the following notation: let  $j, k \in \mathbb{Z}$  and define  $S_{j,k} := \{j, \dots, k\}$  if  $j \leq k$  and  $S_{j,k} := \emptyset$  if  $k < j$ . Let  $W$  be a vector bundle over  $\Sigma$ , suppose that  $X_j, \dots, X_k \in \Gamma T\Sigma$  and let  $\sigma \in \Gamma W$ . Then for  $J \subset S_{j,k}$  with  $J = \{j_1 < \dots < j_l\}$  we let

$$d_{X_J}\sigma := d_{X_{j_1}}(d_{X_{j_2}} \dots (d_{X_{j_l}}\sigma)),$$

and

$$d_{X_\emptyset}\sigma := \sigma.$$

We will repeatedly use the Leibniz rule, i.e., if  $\sigma, \xi \in \Gamma W$  and  $J \subset S_{j,k}$ , then

$$d_{X_J}(\sigma \otimes \xi) = \sum_{K \subset J} (d_{X_K}\sigma) \otimes (d_{X_{J \setminus K}}\xi).$$

The following lemma allows us to characterise deformability of a map  $g : \Sigma \rightarrow G$  in terms of its Maurer-Cartan form:

**Lemma 2.2.** *Let  $k \in \mathbb{N}$  and suppose that  $g$  is a  $(k-1)$ -th order deformation of  $F$ . Then  $F$  and  $g^{-1}(p)gF$  agree to order  $k$  at  $p \in \Sigma$  if and only if for any  $v_0 \in V^*$  and  $Y, X_1, \dots, X_{k-1} \in \Gamma T\Sigma$ ,*

$$\theta(Y)d_{X_{S_{1,k-1}}}\sigma = \sum_{K \subset S_{1,k-1}} v_0(\theta(Y)d_{X_K}\sigma)d_{X_{S_{1,k-1} \setminus K}}\sigma,$$

at  $p$ , where  $\theta = g^{-1}dg$  and  $\sigma \in \Gamma F$  such that  $v_0(\sigma) = 1$ .

*Proof.* We shall use strong induction on  $k$ . Consider the case  $k = 1$ :  $F$  and  $g^{-1}(p)gF$  agree to order 1 at  $p$  if and only if for any  $v_0 \in V^*$ ,  $v_0(g^{-1}(p)g\sigma)$  and  $g^{-1}(p)g\sigma$  agree to order 1 at  $p$  where  $\sigma \in \Gamma F$  such that  $v_0(\sigma) = 1$ . This holds if and only if for any  $Y \in T_p\Sigma$ ,

$$g^{-1}(p)d_Y(g\sigma) = d_Y(v_0(g^{-1}(p)g\sigma)\sigma).$$

Now using the Leibniz rule and that  $\theta_p(Y) = g^{-1}(p)d_Yg$ , this holds if and only if

$$\theta_p(Y)\sigma + d_Y\sigma = v_0(\theta_p(Y)\sigma)\sigma + d_Y\sigma.$$

Noting that  $d_\emptyset\sigma = \sigma$ , we see that the proposition holds when  $k = 1$ .

Let  $n \in \mathbb{N}$  and assume that the proposition holds for all  $k < n$  and assume that  $F$  and  $\hat{F}$  are  $(n-1)$ -th order deformations of each other. Let  $Y, X_1, \dots, X_{n-1} \in \Gamma T\Sigma$ . Then for any  $K \subset \{1, \dots, n-1\}$  with  $|K| < n-1$  we have, by our inductive hypothesis,

$$(1) \quad \theta(Y)d_{X_K}\sigma = \sum_{L \subset K} v_0(\theta(Y)d_{X_L}\sigma)d_{X_{K \setminus L}}\sigma.$$

Since  $F$  and  $\hat{F}$  are  $(n-1)$ -th order deformations of each other we have that for any  $v_0 \in V^*$  and  $X_1, \dots, X_{n-1} \in \Gamma T\Sigma$ ,

$$g^{-1}d_{X_{S_{1,n-1}}}g\sigma - \sum_{K \subset S_{1,n-1}} v_0(g^{-1}d_{X_K}g\sigma)d_{X_{S_{1,n-1} \setminus K}}\sigma = 0,$$

where  $\sigma \in \Gamma f$  such that  $v_0(\sigma) = 1$ . Differentiating at  $p$  with respect to  $X_0 \in \Gamma T\Sigma$  we get, using the Leibniz rule and that  $d_Y g^{-1} = -\theta(Y)g^{-1}$ ,

$$\begin{aligned} 0 &= -\theta_p(X_0)g^{-1}(p)d_{X_{S_{1,n-1}}}g\sigma + g^{-1}(p)d_{X_0}d_{X_{S_{1,n-1}}}g\sigma \\ &+ \sum_{K \subset S_{1,n-1}} [v_0(\theta_p(X_0)g^{-1}(p)d_{X_K}g\sigma)d_{X_{S_{1,n-1} \setminus K}}\sigma \\ &- v_0(g^{-1}(p)d_{X_0 X_K}g\sigma)d_{X_{S_{1,n-1} \setminus K}}\sigma - v_0(g^{-1}(p)d_{X_K}g\sigma)d_{X_0 X_{S_{1,n-1} \setminus K}}\sigma] \\ &= -\theta_p(X_0)g^{-1}(p)d_{X_{S_{1,n-1}}}g\sigma + d_{X_{S_0, n-1}}(g^{-1}(p)g\sigma) \\ &+ \sum_{K \subset S_{1,n-1}} v_0(\theta_p(X_0)g^{-1}(p)d_{X_K}g\sigma)d_{X_{S_{1,n-1} \setminus K}}\sigma - d_{X_{S_0, n-1}}(v_0(g^{-1}(p)g\sigma)\sigma). \end{aligned}$$

Thus,  $v_0(g^{-1}(p)g\sigma)\sigma$  and  $g^{-1}(p)g\sigma$  agree to order  $n$  at  $p$  if and only if

$$(2) \quad \theta_p(X_0)g^{-1}(p)d_{X_{S_{1,n-1}}}g\sigma = \sum_{K \subset S_{1,n-1}} v_0(\theta_p(X_0)g^{-1}(p)d_{X_K}g\sigma)d_{X_{S_{1,n-1} \setminus K}}\sigma.$$

Now,  $v_0(g^{-1}(p)g\sigma)\sigma$  and  $g^{-1}(p)g\sigma$  agree up to order  $n-1$  at  $p$ , thus for any  $K \subset S_{1,n-1}$ ,

$$g^{-1}(p)d_{X_K}g\sigma = d_{X_K}(v_0(g^{-1}(p)g\sigma)\sigma) = \sum_{L \subset K} v_0(g^{-1}(p)d_{X_L}g\sigma)d_{X_{K \setminus L}}\sigma.$$

Thus, (2) becomes

$$\begin{aligned} 0 &= -\theta_p(X_0) \sum_{K \subset S_{1,n-1}} v_0(g^{-1}(p)d_{X_K}g\sigma)d_{X_{S_{1,n-1} \setminus K}}\sigma \\ &+ \sum_{K \subset S_{1,n-1}} \sum_{L \subset K} v_0(\theta_p(X_0)v_0(g^{-1}(p)d_{X_L}g\sigma)d_{X_{K \setminus L}}\sigma)d_{X_{S_{1,n-1} \setminus K}}\sigma \\ &= - \sum_{K \subset S_{1,n-1}} v_0(g^{-1}(p)d_{X_K}g\sigma)\theta_p(X_0)d_{X_{S_{1,n-1} \setminus K}}\sigma \\ &+ \sum_{K \subset S_{1,n-1}} \sum_{L \subset K} v_0(g^{-1}(p)d_{X_L}g\sigma)v_0(\theta_p(X_0)d_{X_{K \setminus L}}\sigma)d_{X_{S_{1,n-1} \setminus K}}\sigma. \end{aligned}$$

After relabelling we have that

$$\begin{aligned} 0 &= \sum_{K \subset S_{1,n-1}} v_0(g^{-1}(p)d_{X_K}g\sigma)(-\theta_p(X_0)d_{X_{S_{1,n-1} \setminus K}}\sigma \\ &+ \sum_{L \subset (S_{1,n-1} \setminus K)} v_0(\theta_p(X_0)d_{X_L}\sigma)d_{X_{(S_{1,n-1} \setminus K) \setminus L}}\sigma). \end{aligned}$$

Using the inductive hypothesis (1) we then have

$$0 = -\theta_p(X_0)d_{X_{S_{1,n-1}}}\sigma + \sum_{K \subset S_{1,n-1}} v_0(\theta_p(X_0)d_{X_K}\sigma)d_{X_{S_{1,n-1} \setminus K}}\sigma.$$

Hence, the result holds for the case  $k = n$ . Therefore, by induction the result is proved.  $\square$

Applying Lemma 2.2 recursively, one obtains the following theorem:

**Theorem 2.3.**  $\eta \in \Omega^1(\mathfrak{g})$  is a  $k$ -th order infinitesimal deformation of  $F$  if and only if  $\eta$  satisfies the Maurer Cartan equation and for all  $r \in \{0, \dots, k-1\}$ ,  $v_0 \in V^*$  and  $Y, X_1, \dots, X_r \in \Gamma T\Sigma$ ,

$$\eta(Y)d_{X_{S_{1,r}}} \sigma = \sum_{K \subset S_{1,r}} v_0(\eta(Y)d_{X_K} \sigma)d_{X_{S_{1,r} \setminus K}} \sigma,$$

where  $\sigma \in \Gamma F$  such that  $v_0(\sigma) = 1$ .

We now wish to find an invariant characterisation of deformability in terms of the Maurer-Cartan form, i.e., a characterisation that does not require charts. Essentially this achieved by taking the characterisation of Theorem 2.3 and successively applying the Leibniz rule. Let  $r \in \{0, \dots, k-1\}$ ,  $Y, X_1, \dots, X_r \in \Gamma T\Sigma$  and  $v_0 \in V^*$ . For  $I, J \subset \{1, \dots, r\}$ , contemplate the following equation:

$$(3) \quad (d_{X_I} \eta(Y))d_{X_J} \sigma = \sum_{K \subset J} v_0((d_{X_I} \eta(Y))d_{X_K} \sigma)d_{X_{J \setminus K}} \sigma,$$

where  $\sigma \in \Gamma F$  such that  $v_0(\sigma) = 1$ .

**Lemma 2.4.** Suppose that for all  $I, J \subset \{1, \dots, r\}$  with  $|I| + |J| < r$ , (3) holds. Then (3) holds for all  $I, J \subset \{1, \dots, r\}$  with  $|I| = i \in \{0, \dots, r\}$  and  $|I| + |J| = r$  if and only if (3) holds for all  $I, J \subset \{1, \dots, r\}$  with  $|I| = i+1$  and  $|I| + |J| = r$ .

*Proof.* Suppose that (3) holds for all  $I, J \subset \{1, \dots, r\}$  with  $|I| = i \in \{0, \dots, r\}$  and  $|I| + |J| = r$ . Let  $I, J \subset \{1, \dots, r\}$  with  $|I| = i+1$  and  $|I| + |J| = r$ . Without loss of generality, assume that  $\min I < \min J$ . Let  $a$  denote the smallest element of  $I$  and  $\tilde{I} := I \setminus \{a\}$ . Then by our assumption

$$(d_{X_{\tilde{I}}} \eta(Y))d_{X_J} \sigma = \sum_{K \subset J} v_0((d_{X_{\tilde{I}}} \eta(Y))d_{X_K} \sigma)d_{X_{J \setminus K}} \sigma.$$

Differentiating this with respect to  $X_a$  at  $p$  and using the Leibniz rule we have that

$$\begin{aligned} & (d_{X_I} \eta(Y))d_{X_J} \sigma + (d_{X_I} \eta(Y))d_{X_{\{a\} \cup J}} \sigma \\ &= \sum_{K \subset J} (v_0((d_{X_I} \eta(Y))d_{X_K} \sigma)d_{X_{J \setminus K}} \sigma + v_0((d_{X_{\tilde{I}}} \eta(Y))d_{X_{\{a\} \cup K}} \sigma)d_{X_{J \setminus K}} \sigma \\ &+ \sum_{K \subset J} v_0((d_{X_{\tilde{I}}} \eta(Y))d_{X_K} \sigma)d_{X_{\{a\} \cup J \setminus K}} \sigma) \\ &= \sum_{K \subset J} v_0((d_{X_I} \eta(Y))d_{X_K} \sigma)d_{X_{J \setminus K}} \sigma + \sum_{L \subset \{a\} \cup J} v_0((d_{X_{\tilde{I}}} \eta(Y))d_{X_L} \sigma)d_{X_{\{a\} \cup J \setminus L}} \sigma. \end{aligned}$$

By our supposition,

$$(d_{X_{\tilde{I}}} \eta(Y))d_{X_{\{a\} \cup J}} \sigma = \sum_{L \subset \{a\} \cup J} v_0((d_{X_{\tilde{I}}} \eta(Y))d_{X_L} \sigma)d_{X_{\{a\} \cup J \setminus L}} \sigma.$$

Thus,

$$(d_{X_I} \eta(Y))d_{X_J} \sigma = \sum_{K \subset J} (v_0((d_{X_I} \eta(Y))d_{X_K} \sigma)d_{X_{J \setminus K}} \sigma).$$

A similar argument can be used to prove the converse.  $\square$

**Corollary 2.5.** Suppose that for all  $I, J \subset \{1, \dots, r\}$  with  $|I| + |J| < r$ , (3) holds. Then if (3) holds for all  $I, J \subset \{1, \dots, r\}$  with  $|I| = i \in \{0, \dots, r\}$  and  $|I| + |J| = r$ , then (3) holds for all  $I, J \subset \{1, \dots, r\}$  with  $|I| + |J| = r$ .

We are now in a position to state the following invariant version of Theorem 2.3:

**Theorem 2.6.**  $\eta \in \Omega^1(\mathfrak{g})$  is a  $k$ -th order infinitesimal deformation of  $F$  if and only if  $\eta$  satisfies the Maurer-Cartan equation and

$$(4) \quad \eta(Y)F \leq F, \quad (d_{X_1}\eta(Y))F \leq F, \quad \dots, \quad (d_{X_1 \dots X_{k-1}}\eta(Y))F \leq F,$$

for all  $Y, X_1, \dots, X_{k-1}, \in \Gamma T\Sigma$ .

*Proof.* Firstly, notice that (4) is equivalent to (3) with  $|I| = r \in \{0, \dots, k-1\}$  and  $|J| = 0$ , for any choice of  $v_0 \in V^*$ .

Suppose that  $\eta$  is a  $k$ -th order infinitesimal deformation of  $F$  and let  $g : \Sigma \rightarrow G$  such that  $g^{-1}dg = \eta$ . Then by Theorem 2.3, for any  $r \in \{0, \dots, k-1\}$ ,  $Y, X_1, \dots, X_r \in \Gamma T\Sigma$  and  $v_0 \in V^*$ , we have that (3) holds for all  $I, J \subset \{1, \dots, r\}$  with  $|I| = 0$  and  $|J| = r$ . By Corollary 2.5 it then follows that (3) holds for all  $I, J \subset \{1, \dots, r\}$  with  $|I| = r$  and  $|J| = 0$ .

Conversely, suppose that  $\eta$  satisfies the Maurer-Cartan equation and, for any  $r \in \{0, \dots, k-1\}$ ,  $Y, X_1, \dots, X_r \in \Gamma T\Sigma$  and  $v_0 \in V^*$ , (3) holds for all  $I, J \subset \{1, \dots, r\}$  with  $|I| = r$  and  $|J| = 0$ . Then by Corollary 2.5, (3) holds for all  $I, J \subset \{1, \dots, r\}$  with  $|I| = 0$  and  $|J| = r$ . By Theorem 2.3 it then follows that  $\eta$  is a  $k$ -th order infinitesimal deformation of  $F$ .  $\square$

### 3. PROJECTIVE 3-SPACE

Cartan [11] investigated projective deformability and rigidity of surfaces in projective 3-space. Modern references on this topic include [1, 17, 20, 23]. It was shown in [17] that the class of second order deformable surfaces in projective 3-space can be split naturally into two subclasses:  $R$ - and  $R_0$ -surfaces. A modern account of this can be found in [15] and a gauge theoretic approach for these surfaces was developed in [14]. In this section we will use the results from Section 2 to study these notions.

So let us consider projective 3-space  $\mathbb{P}(\mathbb{R}^4)$  with transformation group  $SL(4)$ . Suppose that  $\Sigma$  is a 2-dimensional manifold and let  $F : \Sigma \rightarrow \mathbb{P}(\mathbb{R}^4)$  be a smooth map. We can view  $F$  as a rank 1 subbundle of the trivial bundle  $\underline{\mathbb{R}}^4 := \Sigma \times \mathbb{R}^4$ . Let  $F^{(1)}$  denote derived bundle of  $F$ , i.e., the set of sections of  $F$  and derivatives of sections of  $F$ . Assuming that  $F$  is an immersion is equivalent to assuming that  $F^{(1)}$  is a rank 3 subbundle of the trivial bundle. Let  $T_1, T_2$  denote the (possibly complex conjugate) asymptotic directions of  $F$ , i.e., for any  $X \in \Gamma T_1, Y \in \Gamma T_2$  and  $\sigma \in \Gamma F$ ,

$$d_X d_X \sigma, d_Y d_Y \sigma \in \Gamma F^{(1)}.$$

We will make the further assumption that the derived bundle  $F^{(2)}$  of  $F^{(1)}$  satisfies  $F^{(2)} = \underline{\mathbb{R}}^4$ . In other words, for  $X \in \Gamma T_1, Y \in \Gamma T_2$  and  $\sigma \in \Gamma F$ ,  $d_X d_Y \sigma$  never belongs to  $F^{(1)}$ .

**3.1. Second order deformations.** We will now investigate when  $F$  admits non-trivial second order  $SL(4)$ -deformations. By Theorem 2.3,  $\eta \in \Omega^1(\mathfrak{sl}(4))$  is a second order infinitesimal deformation of  $F$  if and only if  $\eta$  satisfies the Maurer-Cartan equation and for all  $v_0 \in (\mathbb{R}^4)^*$  and  $X, Y \in \Gamma T\Sigma$

$$(5) \quad \eta\sigma = v_0(\eta\sigma)\sigma$$

and

$$(6) \quad \eta(X)d_Y\sigma = v_0(\eta(X)\sigma)d_Y\sigma + v_0(\eta(X)d_Y\sigma)\sigma,$$

where  $\sigma \in \Gamma F$  such that  $v_0(\sigma) = 1$ .

Suppose that  $\eta$  is such a second order infinitesimal deformation. Let  $X \in \Gamma T_1$  and  $Y \in \Gamma T_2$ . By equation (6) we have that

$$\eta(X)d_X\sigma = v_0(\eta(X)\sigma)d_X\sigma + v_0(\eta(X)d_X\sigma)\sigma.$$

Differentiating this in the  $Y$  direction gives

$$\begin{aligned} (d_Y\eta(X))d_X\sigma + \eta(X)d_{YX}\sigma &= d_Y(v_0(\eta(X)\sigma))d_X\sigma + v_0(\eta(X)\sigma)d_{YX}\sigma \\ &\quad + d_Y(v_0(\eta(X)d_X\sigma))\sigma + v_0(\eta(X)d_X\sigma)d_Y\sigma. \end{aligned}$$

Since  $\eta$  satisfies the Maurer-Cartan equation, one deduces that the left hand side of this equation is

$$\eta(X)d_{YX}\sigma \text{ mod } F^{(1)}.$$

Whereas the right hand side is

$$v_0(\eta(X)\sigma)d_{YX}\sigma \text{ mod } F^{(1)}.$$

Similarly, one can show that

$$\eta(Y)d_{YX}\sigma = v_0(\eta(Y)\sigma)d_{YX}\sigma \text{ mod } F^{(1)}.$$

Using that  $\{\sigma, d_X\sigma, d_Y\sigma, d_{YX}\sigma\}$  forms a basis for  $\mathbb{P}(\mathbb{R}^4)$  and that  $\eta$  takes values in  $\mathfrak{sl}(4)$  and is thus trace free, we must have that  $v_0(\eta\sigma) = 0$ . Therefore,

$$\eta F = 0 \quad \text{and} \quad \eta F^{(1)} \leq \Omega^1(F).$$

Conversely if  $\eta$  satisfies

$$\eta F = 0 \quad \text{and} \quad \eta F^{(1)} \leq \Omega^1(F)$$

then clearly (5) and (6) hold and thus  $\eta$  is a second order infinitesimal deformation of  $F$ .

One can show (see [31, Lemma 3.21]) that an  $\eta \in \Omega^1(\mathfrak{sl}(4))$  of the above form satisfies the Maurer-Cartan equation if and only if  $\eta$  is closed. Thus, we have arrived at the following proposition:

**Proposition 3.1.**  *$\eta \in \Omega^1(\mathfrak{sl}(4))$  is a second order infinitesimal deformation of  $F$  if and only if  $\eta$  is closed and satisfies  $\eta F = 0$  and  $\eta F^{(1)} \leq \Omega^1(F)$ .*

We will now investigate the uniqueness and triviality of second order deformations. According to Lemma 1.5 and Lemma 1.6, this is determined by second order deformations,  $h : \Sigma \rightarrow G$ , between  $F$  and itself. By Proposition 3.1, such a  $h$  satisfies

$$(7) \quad hF = F, \quad \theta_h F = 0 \quad \text{and} \quad \theta_h F^{(1)} \leq \Omega^1(F),$$

where  $\theta_h := h^{-1}dh$ . Now  $hF = F$  implies that for any  $\sigma \in \Gamma F$ ,

$$h\sigma = \lambda\sigma$$

for a smooth function  $\lambda$ . Thus, for any  $X \in \Gamma T\Sigma$

$$(d_X h)\sigma + h d_X \sigma = \lambda d_X \sigma + (d_X \lambda)\sigma.$$

Using that  $\theta_h F = 0$

$$h d_X \sigma = \lambda d_X \sigma + (d_X \lambda)\sigma.$$

Differentiating this condition with respect to  $Y \in \Gamma T\Sigma$  we have that

$$h d_{YX} \sigma = \lambda d_{YX} \sigma + (d_Y \lambda) d_X \sigma + (d_X \lambda) d_Y \sigma + (d_{YX} \lambda) \sigma - (d_Y h) d_X \sigma.$$

Then, since  $h$  takes values in  $\mathrm{SL}(4)$  and  $\theta_h F^{(1)} \leq \Omega^1(F)$ , we must have that  $\lambda = \pm 1$ . Furthermore,

$$h|_{F^{(1)}} = \pm id|_{F^{(1)}} \quad \text{and} \quad h|_{\mathbb{R}^4/F} = \pm id|_{\mathbb{R}^4/F}.$$

Thus, we may write

$$h = \pm(id + \xi),$$

where  $\xi$  satisfies  $\xi|_{F^{(1)}} = 0$  and  $im\xi \leq F$ . Clearly  $\xi$  is trace-free, so  $\xi \in \Gamma\mathfrak{sl}(4)$ . Hence,  $h = \pm \exp(\xi)$ . Conversely, given an  $h$  of such a form, one can easily check that (7) is satisfied. Thus we obtain the following lemmata:

**Lemma 3.2.** *Second order deformations between two maps  $F, \hat{F} : \Sigma \rightarrow \mathbb{P}(\mathbb{R}^4)$  are determined up to right multiplication by  $\pm \exp(\xi)$ , for any  $\xi \in \Gamma\mathfrak{sl}(4)$  satisfying  $\xi|_{F^{(1)}} = 0$  and  $im\xi \leq F$ .*

**Lemma 3.3.**  *$\eta$  is a trivial second order infinitesimal deformation of  $F$  if and only if  $\eta = d\xi$ , where  $\xi \in \Gamma\mathfrak{sl}(4)$  satisfying  $\xi|_{F^{(1)}} = 0$  and  $im\xi \leq F$ .*

We have therefore proved the main theorem of this subsection:

**Theorem 3.4.**  *$F : \Sigma \rightarrow \mathbb{P}(\mathbb{R}^4)$  is deformable of order two if and only if there exists  $\eta \in \Omega^1(\mathfrak{sl}(4))$ , such that  $\eta$  is closed,*

$$\eta F = 0, \quad \eta F^{(1)} \leq \Omega^1(F)$$

and  $\eta \neq d\xi$  for any  $\xi \in \Gamma\mathfrak{sl}(4)$  satisfying  $\xi|_{F^{(1)}} = 0$  and  $im\xi \leq F$ .

In Section 6 we shall see that the deformability of a map into  $\mathbb{P}(\mathbb{R}^4)$  coincides with deformability of its contact lift. In that setting the triviality of deformations can be identified by the vanishing of a certain two-tensor.

By using the gauge theoretic definition of  $R$ -/ $R_0$ -surfaces given in [14], one recovers the following classical result:

**Corollary 3.5** ([11, 17]).  *$R$ -surface and  $R_0$ -surfaces are the only second order deformable surfaces of projective geometry.*

**3.2. Third order deformations.** We shall now show that rigidity occurs at third order in projective 3-space. Suppose that  $\eta$  is a third order infinitesimal deformation of  $F$ . Then by Theorem 3.4,  $\eta$  is closed and satisfies

$$\eta F = 0 \quad \text{and} \quad \eta F^{(1)} \leq \Omega^1(F).$$

Furthermore, by Theorem 2.3, for any  $v_0 \in (\mathbb{R}^4)^*$  and  $X, Y, Z \in \Gamma T\Sigma$ ,

$$\begin{aligned} \eta(X)d_Y Z \sigma &= v_0(\eta(X)d_Y Z \sigma)\sigma + v_0(\eta(X)d_Y \sigma)d_Z \sigma \\ &\quad + v_0(\eta(X)d_Z \sigma)d_Y \sigma + v_0(\eta(X)\sigma)d_Y Z \sigma, \end{aligned}$$

where  $\sigma \in \Gamma f$  such that  $v_0(\sigma) = 1$ . Now suppose that  $Y$  is an asymptotic direction of  $F$  and  $Z = Y$ . Then  $d_Y Z \sigma \in \Gamma F^{(1)}$  and thus  $\eta(X)d_Y Z \sigma \in \Gamma F$ . Hence,  $v_0(\eta(X)d_Y \sigma) = 0$ . Therefore,  $\eta F^{(1)} = 0$ . We will now use that  $\eta$  is closed to show that  $\eta = 0$ : suppose that  $X, Y, Z \in \Gamma T\Sigma$ . Then, as  $\eta$  is closed, we have that for any  $\sigma \in \Gamma F$

$$d\eta(X, Y)d_Z \sigma = 0.$$

Since  $\eta|_{F^{(1)}} = 0$ , this is equivalent to

$$\eta(X)d_Y Z \sigma - \eta(Y)d_X Z \sigma = 0.$$

Assume now that  $X$  and  $Y$  are distinct asymptotic directions of  $F$ . Then setting  $Z = Y$  implies that  $\eta(Y)d_{XY}\sigma = 0$ , since  $d_{YY}\sigma \in \Gamma F^{(1)}$ . Similarly, setting  $Z = X$  implies that  $\eta(X)d_{YX}\sigma = 0$ , which in turn implies that  $\eta(X)d_{XY}\sigma = 0$ . Therefore as  $\{\sigma, d_X\sigma, d_Y\sigma, d_{XY}\sigma\}$  is a basis for  $\mathbb{R}^4$ ,  $\eta = 0$ . Thus we have proved the following classically known theorem:

**Theorem 3.6.** *Surfaces in projective 3-space are rigid to third order.*

#### 4. HYPERSURFACES IN THE CONFORMAL $n$ -SPHERE

In this section we will apply the results of Section 2 to examine deformations of hypersurfaces in conformal geometry. For a modern treatment of conformal geometry see for example [2, 3, 6, 7, 21, 25, 24].

Let  $n \in \mathbb{N}$ . Then we may view  $\mathbb{S}^n$  as the projective light cone  $\mathbb{P}(\mathcal{L})$  of  $\mathbb{R}^{n+1,1}$ , which is acted upon transitively by the orthogonal group  $O(n+1, 1)$ . Suppose that  $F : \Sigma \rightarrow \mathbb{P}(\mathcal{L})$  is an immersion, where  $\Sigma$  is an  $(n-1)$ -dimensional manifold. We will view  $F$  as a null line subbundle of  $\mathbb{R}^{n+1,1}$ . Note that as  $F$  is an immersion, the derived bundle  $F^{(1)}$  of  $F$  is a codimension 1 subbundle of  $F^\perp$ . Let  $V$  be a sphere congruence enveloped by  $F$ , i.e.,  $V$  is a bundle of  $(n, 1)$ -planes such that  $F^{(1)} \leq V$ . Then let  $\tilde{F}$  be a null-line subbundle of  $V$  complementary to  $F$ , i.e.,  $F \oplus \tilde{F}$  is a  $(1, 1)$ -subbundle of  $V$ . Let  $U := (F \oplus \tilde{F})^\perp \cap V$ . Then  $F^{(1)} = F \oplus U$  and  $F^\perp = F \oplus U \oplus V^\perp$ . We now have a splitting

$$\mathbb{R}^{n+1,1} = F \oplus \tilde{F} \oplus U \oplus V^\perp,$$

and thus a splitting of  $\wedge^2 \mathbb{R}^{n+1,1}$ :

$$\wedge^2 \mathbb{R}^{n+1,1} = F \wedge U \oplus F \wedge V^\perp \oplus U \wedge U \oplus U \wedge V^\perp \oplus F \wedge \tilde{F} \oplus \tilde{F} \wedge U \oplus \tilde{F} \wedge V^\perp.$$

**4.1. Second order deformations.** By Theorem 2.3,  $\eta \in \Omega^1(\mathfrak{o}(n+1, 1))$  is a second order infinitesimal deformation of  $F$  if and only if  $\eta$  satisfies the Maurer-Cartan equation, and for all  $v_0 \in (\mathbb{R}^{n+1,1})^*$  and  $X, Y \in \Gamma T\Sigma$

$$(8) \quad \eta\sigma = v_0(\eta\sigma)\sigma \quad \text{and} \quad \eta(X)d_Y\sigma = v_0(\eta(X)\sigma)d_Y\sigma + v_0(\eta(X)d_Y\sigma)\sigma,$$

where  $\sigma \in \Gamma F$  such that  $v_0(\sigma) = 1$ . From the skew-symmetry of  $\eta$  it follows that  $v_0(\eta\sigma) = 0$ . Thus, (8) holds if and only if

$$\eta F = 0 \quad \text{and} \quad \eta F^{(1)} \leq \Omega^1(F),$$

or equivalently

$$\eta F = 0 \quad \text{and} \quad \eta U \leq \Omega^1(F).$$

This clearly holds if and only if

$$\eta \in \Omega^1(F \wedge U \oplus F \wedge V^\perp) = \Omega^1(F \wedge F^\perp).$$

Now  $F \wedge F^\perp$  is a bundle of abelian subalgebras of  $\mathfrak{o}(n+1, 1)$ . Therefore,  $[\eta \wedge \eta] = 0$  and the condition that  $\eta$  satisfies the Maurer-Cartan equation reduces to  $\eta$  being closed.

We shall now investigate the uniqueness and triviality of second order deformations. According to Lemma 1.5 and Lemma 1.6, this is determined by second order deformations,  $h : \Sigma \rightarrow G$ , between  $F$  and itself, i.e.,  $h$  satisfies  $hF = F$  and  $\theta_h := h^{-1}dh \in \Omega^1(F \wedge F^\perp)$ . Thus, for any section  $\sigma \in \Gamma F$ ,  $h\sigma = \lambda\sigma$ , for some smooth function  $\lambda$ . Differentiating this along  $X \in \Gamma T\Sigma$  gives

$$(d_X h)\sigma + h d_X \sigma = (d_X \lambda)\sigma + \lambda d_X \sigma.$$

But since  $\theta_h F = 0$ , we have that

$$hd_X\sigma = (d_X\lambda)\sigma + \lambda d_X\sigma.$$

The orthogonality of  $h$  then gives that  $\lambda = \pm 1$ . Furthermore  $h|_{F^{(1)}} = \pm id|_{F^{(1)}}$  and so for any  $\nu \in \Gamma F^{(1)}$ ,  $h\nu = \pm\nu$ . Differentiating this condition along  $Y \in \Gamma T\Sigma$  gives that

$$(d_Y h)\nu + hd_Y\nu = \pm d_Y\nu.$$

Then since  $\theta_h F^\perp \leq F$ , we have that  $h|_{F^{(2)}} \equiv \pm id|_{F^{(2)}} \text{ mod } F$ . Now,  $F^{(2)} := (F^{(1)})^{(1)} = \underline{\mathbb{R}}^{n+1,1}$ , so we may write

$$h = \pm id + \xi,$$

where  $\xi|_{F^{(1)}} = 0$  and  $im\xi \leq F$ . From the orthogonality of  $h$  one may deduce that  $\xi$  is skew-symmetric. Combined with  $\xi|_{F^{(1)}} = 0$  and  $im\xi \leq F$ , this can only hold if  $\xi = 0$ . We therefore have the following lemmata:

**Lemma 4.1.** *Suppose that  $g_1$  and  $g_2$  are second order deformations between  $F$  and  $\hat{F}$ . Then  $g_1 = \pm g_2$ .*

**Lemma 4.2.**  *$\eta$  is a trivial second order infinitesimal deformation of  $F$  if and only if  $\eta = 0$ .*

We have thus arrived at the main theorem of this subsection:

**Theorem 4.3.**  *$F : \Sigma \rightarrow \mathbb{P}(\mathcal{L})$  is deformable of order two if and only if there exists a closed non-zero one-form  $\eta$  taking values in  $F \wedge F^\perp$ .*

In [5] it is shown that an  $\eta$  satisfying the conditions of Theorem 4.3 does not exist for  $n > 3$ . In the case of  $n = 3$ , using the gauge-theoretic definition of isothermic surfaces (see for example [7, 10]), one recovers the classically known result:

**Corollary 4.4** ([13]). *Isothermic surfaces are the only second order deformable surfaces in the conformal 3-sphere.*

*Remark 4.5.* In [12, 28], the deformability of submanifolds in the conformal  $n$ -sphere with codimension greater than one was considered. In this case it is shown that, although isothermic surfaces are deformable to second order, a generic second order deformable surface is not isothermic.

In [32] it was proved that more can be said about where  $\eta$  takes values:

**Proposition 4.6.** *If  $\eta \in \Omega^1(F \wedge F^\perp)$  is closed then  $\eta \in \Omega^1(F \wedge F^{(1)})$ .*

**4.2. Third order deformations.** We will now show that rigidity occurs at third order in the conformal 3-sphere. Suppose that  $\eta$  is a third order infinitesimal deformation of  $F$ . Then by Proposition 4.6,  $\eta \in \Omega^1(F \wedge F^{(1)})$ . Furthermore, by Theorem 2.6, for all  $X, Y, Z \in \Gamma T\Sigma$ ,

$$(d_Y \eta(Z))\sigma = \xi\sigma \quad \text{and} \quad (d_X d_Y \eta(Z))\sigma \in \Gamma F,$$

for some smooth function  $\xi$ . Using the Leibniz rule, one then deduces that

$$(d_Y \eta(Z))d_X\sigma = \xi d_X\sigma \text{ mod } F,$$

where  $\sigma \in \Gamma F$  such that  $v_0(\sigma) = 1$ . The skew-symmetry of  $(d_Y \eta(Z))$  implies that  $\xi = 0$ . Hence,  $(d_Y \eta(Z))\sigma = 0$ . By the Leibniz rule this implies that  $\eta(Z)d_Y\sigma = 0$  and thus  $\eta F^{(1)} = 0$ . Therefore,  $\eta = 0$  and it follows that:

**Theorem 4.7.** *A surface in the conformal 3-sphere is rigid to third order.*

## 5. LEGENDRE MAPS

In this section we study the deformability of contact elements in Lie sphere geometry and projective geometry. This problem has been studied in [4, 15, 16, 18, 27].

Let  $s, t \in \mathbb{N}$  such that  $(s, t) = (3, 3)$  or  $(s, t) = (4, 2)$ . Consider  $\mathbb{R}^{s,t}$  and let  $\mathcal{L}^5$  denote the 5-dimensional lightcone of this space. Let  $\mathcal{Z}$  denote the Grassmannian of null two dimensional subspaces of  $\mathbb{R}^{s,t}$ .  $\mathcal{Z}$  is acted upon transitively by  $G = O(s, t)$ . We say that a smooth map  $f : \Sigma \rightarrow \mathcal{Z}$  is a *Legendre map* if  $f^{(1)} \leq f^\perp$  and at every  $p \in \Sigma$ , if  $X \in T_p \Sigma$  such that  $d_X \sigma \in f(p)$  for all sections  $\sigma \in \Gamma f$ , then  $X = 0$ . We may view a Legendre map as rank 2 null subbundle on the trivial bundle  $\underline{\mathbb{R}}^{s,t} := \Sigma \times \mathbb{R}^{s,t}$ .

It was shown in [8] that a Legendre map naturally equips  $T\Sigma$  with a conformal structure. In the case that  $(s, t) = (4, 2)$  this conformal structure at each point either vanishes or has signature  $(1, 1)$ , however in the case of  $(s, t) = (3, 3)$ , any signature is possible. From this point onwards we shall make the assumption that the signature of this conformal structure is  $(1, 1)$  at each point. In this case we may denote by  $T_1$  and  $T_2$  the null subbundles of this conformal structure. Our Legendre map then admits two special rank 1 subbundles  $s_1$  and  $s_2$ , called the *curvature sphere congruences of  $f$* , such that

$$d_X \sigma_1, d_Y \sigma_2 \in \Gamma f,$$

for all  $\sigma_1 \in \Gamma s_1$ ,  $\sigma_2 \in \Gamma s_2$ ,  $X \in \Gamma T_1$  and  $Y \in \Gamma T_2$ . We may then form a splitting of the trivial bundle  $\underline{\mathbb{R}}^{s,t}$  as  $\underline{\mathbb{R}}^{s,t} = S_1 \oplus_\perp S_2$ , where

$$(9) \quad S_1 := \langle \sigma_1, d_Y \sigma_1, d_Y d_Y \sigma_1 \rangle \quad \text{and} \quad S_2 := \langle \sigma_2, d_X \sigma_2, d_X d_X \sigma_2 \rangle.$$

This is called the *Lie cyclide splitting*. For  $i \in \{1, 2\}$ , let  $f_i$  denote the set of sections of  $f$  and derivatives of  $f$  along  $T_i$ . One then has that  $f_i$  is a rank 3 subbundle of  $f^\perp$  and furthermore

$$f^\perp / f = f_1 / f \oplus_\perp f_2 / f,$$

with each  $f_i / f$  inheriting a non-degenerate metric from that of  $\mathbb{R}^{s,t}$ .

We identify  $f$  with the map  $F : \Sigma \rightarrow Z$ , defined by  $F = \wedge^2 f$ , where  $Z$  is the subset of  $\mathbb{P}(\wedge^2 \mathbb{R}^{s,t})$  defined by

$$Z := \{[v \wedge w] : v, w \in \mathcal{L} \text{ and } (v, w) = 0\}.$$

$Z$  is acted upon smoothly and transitively by  $O(s, t)$  via

$$A[v \wedge w] = [Av \wedge Aw].$$

Let  $\tilde{f} : \Sigma \rightarrow \mathcal{Z}$  be complementary to  $f$ , i.e.,  $f \oplus \tilde{f}$  is a rank 4 bundle with signature  $(2, 2)$ . Let  $U = (f \oplus \tilde{f})^\perp$ . Then we have a splitting of  $\underline{\mathbb{R}}^{s,t}$ :

$$\underline{\mathbb{R}}^{s,t} = (f \oplus \tilde{f})^\perp \oplus_\perp U.$$

This induces a splitting of  $\wedge^2 \underline{\mathbb{R}}^{s,t}$ :

$$\wedge^2 \underline{\mathbb{R}}^{s,t} = \wedge^2 f \oplus f \wedge U \oplus f \wedge \tilde{f} \oplus \wedge^2 U \oplus \tilde{f} \wedge U \oplus \wedge^2 \tilde{f}.$$

**5.1. Second order deformations.** By Theorem 2.6,  $\eta \in \Omega^1(\underline{\mathfrak{o}}(s,t))$  is a second order infinitesimal deformation if and only if  $\eta$  satisfies the Maurer-Cartan equation and

$$(10) \quad \eta F \leq \Omega^1(F) \quad \text{and} \quad (d_X \eta(Y))F \leq F,$$

for all  $X, Y \in \Gamma T\Sigma$ . Now  $\eta F \leq \Omega^1(F)$  if and only if for linearly independent  $\sigma, \xi \in \Gamma f$ ,

$$(\eta\sigma) \wedge \xi + \sigma \wedge (\eta\xi) = \eta(\sigma \wedge \xi) \in \Omega^1(F).$$

Since  $\sigma$  and  $\xi$  are linearly independent this is equivalent to

$$\eta f \leq \Omega^1(f).$$

Similarly, one can show that  $(d_X \eta(Y))F \leq F$  is equivalent to  $(d_X \eta(Y))f \leq f$ . By the Leibniz rule, this holds if and only if for any section  $\sigma \in \Gamma f$ ,

$$(11) \quad d_X(\eta(Y)\sigma) - \eta(Y)d_X\sigma \in \Gamma f.$$

Now, if we assume that  $X$  is a curvature direction, i.e.,  $X \in \Gamma T_i$  for some  $i \in \{1, 2\}$ , then  $\eta f \leq \Omega^1(f)$  implies that  $d_X(\eta(Y)\sigma) \in \Gamma f_i$ . Furthermore,  $\eta(Y)d_X\sigma$  is orthogonal to  $d_X\sigma$ . Therefore, as the metric on  $\mathbb{R}^{s,t}$  restricts to a non-degenerate metric on  $f_i/f$ , we can deduce that

$$d_X(\eta(Y)\sigma), \eta(Y)d_X\sigma \in \Gamma f.$$

Now,  $d_X(\eta(Y)\sigma) \in \Gamma f$  if and only if  $\eta(Y)\sigma \in \Gamma s_i$ . Since this holds for all  $i \in \{1, 2\}$ , one has that  $\eta f \equiv 0$ . Also,  $\eta(X)d_Y\sigma \in \Gamma f$  implies that  $\eta f^{(1)} \leq \Omega^1(f)$ . Thus,  $\eta U \leq \Omega^1(f)$ . Finally,

$$\eta f \equiv 0 \quad \text{and} \quad \eta U \leq \Omega^1(f)$$

if and only if

$$\eta \in \Omega^1(\wedge^2 f \oplus f \wedge U) = \Omega^1(f \wedge f^\perp).$$

One can easily check that the converse is true, i.e., given  $\eta \in \Omega^1(f \wedge f^\perp)$  satisfying the Maurer-Cartan equation, (10) holds.

The following proposition was proved in [30] in the case that  $(s, t) = (4, 2)$ . Using analogous arguments one can show that it holds in the case that  $(s, t) = (3, 3)$  as well.

**Proposition 5.1.** *Suppose that  $\eta \in \Omega^1(f \wedge f^\perp)$ . Then  $\eta$  satisfies the Maurer-Cartan equation if and only if it is closed. Furthermore,  $\eta(T_i) \leq f \wedge f_i$  and  $[\eta \wedge \eta] = 0$ .*

Thus, we have arrived at the following proposition:

**Proposition 5.2.**  *$\eta \in \Omega^1(\underline{\mathfrak{o}}(s,t))$  is a second order infinitesimal deformation of  $f$  if and only if  $\eta$  is closed and takes values in  $f \wedge f^\perp$ .*

We now wish to determine the uniqueness and triviality of such deformations. Following Lemma 1.5 and Lemma 1.6, we investigate second order deformations  $h : \Sigma \rightarrow O(s, t)$  between  $F$  and itself. By Proposition 5.2, such a  $h$  is characterised by

$$(12) \quad hF = F \quad \text{and} \quad \theta_h := h^{-1}dh \in \Omega^1(f \wedge f^\perp).$$

Furthermore,  $hF = F$  if and only if  $hf = f$ . Let  $\sigma_i \in \Gamma s_i$  be a lift of one of the curvature spheres of  $f$ . Then, since  $hf = f$  we have that

$$h\sigma_i = \nu,$$

for some  $\nu \in \Gamma f$ . Differentiating this condition with respect to the curvature direction  $X \in \Gamma T_i$  yields

$$(d_X h)\sigma_i + h d_X \sigma_i = d_X \nu.$$

Since  $\theta_h \in \Omega^1(f \wedge f^\perp)$ , we have that  $(d_X h)\sigma_i = 0$  and thus

$$h d_X \sigma_i = d_X \nu.$$

Since  $d_X \sigma_i \in \Gamma f$  and  $h f = f$ , we must have that  $d_X \nu \in \Gamma f$ . Thus,  $\nu \in \Gamma s_i$ . Therefore, for some smooth function  $\lambda$  we have that  $h \sigma_i = \lambda \sigma_i$ . Differentiating this condition gives for all  $Z \in \Gamma T \Sigma$ ,

$$(13) \quad (d_Z h)\sigma_i + h d_Z \sigma_i = (d_Z \lambda)\sigma_i + \lambda d_Z \sigma_i.$$

Then the orthogonality of  $h$  and that  $\theta_h f \equiv 0$  implies that  $\lambda = \pm 1$ . Therefore,  $h|_{s_i} = \pm id|_{s_i}$ . We then have two cases to consider either  $h|_f = \pm id|_f$  or  $h|_{s_1} = \pm id|_{s_1}$  and  $h|_{s_2} = \mp id|_{s_2}$ .

**Lemma 5.3.** *Suppose that  $h|_{s_1} = \pm id|_{s_1}$  and  $h|_{s_2} = \mp id|_{s_2}$ . Then  $S_1$  and  $S_2$  are constant.*

*Proof.* Let  $\sigma_1 \in \Gamma s_1$  and  $\sigma_2 \in \Gamma s_2$  and let  $X \in \Gamma T_1$  and  $Y \in \Gamma T_2$ . Then

$$d_X \sigma_1 = \alpha_1 \sigma_1 + \beta_1 \sigma_2 \quad \text{and} \quad d_Y \sigma_2 = \alpha_2 \sigma_1 + \beta_2 \sigma_2,$$

for smooth functions  $\alpha_1, \alpha_2, \beta_1, \beta_2$ . Now

$$\pm(\alpha_1 \sigma_1 + \beta_1 \sigma_2) = \pm d_X \sigma_1 = d_X (h \sigma_1) = (d_X h)\sigma_1 + h d_X \sigma_1 = \pm \alpha_1 \sigma_1 \mp \beta_1 \sigma_2,$$

since  $\theta_h f \equiv 0$ . Thus  $\beta_1 = 0$ . Similarly, one can show that  $\alpha_2 = 0$ . Then, since  $X \in \Gamma T_1$  and  $Y \in \Gamma T_2$  are arbitrary. Thus,  $d_X \sigma_1 \in \Gamma s_1$  and  $d_Y \sigma_2 \in \Gamma s_2$  and one deduces from (9) that  $S_1$  and  $S_2$  are constant.  $\square$

$S_1$  and  $S_2$  can only be constant if  $f$  is a Dupin cyclide. In that case we may define  $\rho \in \mathcal{O}(s, t)$  such that  $\rho$  restricts to the identity on  $S_1$  and minus the identity on  $S_2$ . One then has that  $\tilde{h} := \rho h$  is a second order deformation between  $F$  and itself satisfying  $\tilde{h}|_f = \pm id|_f$ .

So let us now assume that  $h|_f = \pm id|_f$ . Then by (13),  $h|_{f^{(1)}} = \pm id|_{f^{(1)}}$ . By differentiating this condition again one finds that  $h|_{f^{(2)}/f} = \pm id|_{f^{(2)}/f}$ . Therefore we may write

$$h = \pm(id + \xi),$$

where  $\xi$  satisfies  $\xi(\mathbb{R}^{s,t}) \leq f$  and  $\xi f^\perp \equiv 0$ . Since  $\xi(\mathbb{R}^{s,t}) \leq f$ , we have that  $(\xi v, \xi w) = 0$  for all  $v, w \in \Gamma \mathbb{R}^{s,t}$ . The orthogonality of  $h$  then implies that  $\xi$  is skew-symmetric. Combining this with the fact that  $\xi(\mathbb{R}^{s,t}) \leq f$  and  $\xi f^\perp \equiv 0$  gives that  $\xi \in \Gamma(\wedge^2 f)$ . Hence,  $h = \pm \exp(\xi)$ .

Conversely, it is straightforward to check that if  $h = \pm \exp(\xi)$ , for some  $\xi \in \Gamma(\wedge^2 f)$ , then  $h$  satisfies (12). We have thus arrived at the following lemmata:

**Lemma 5.4.** *Suppose that  $f$  and  $\hat{f}$  are second order deformations of each other via  $g_1$  and  $g_2$ . Then in the case that  $f$  is not a Dupin cyclide we have that  $g_2 = \pm g_1 \exp(\xi)$  for some  $\xi \in \Gamma(\wedge^2 f)$ . In the case that  $f$  is a Dupin cyclide, either  $g_2 = \pm g_1 \exp(\xi)$  or  $g_2 = \pm \rho g_1 \exp(\xi)$ .*

**Lemma 5.5.**  *$\eta$  is a trivial second order infinitesimal deformation of  $f$  if and only if  $\eta = d\xi$  for some  $\xi \in \Gamma(\wedge^2 f)$ .*

As shown in [30], since  $\sigma \mapsto \eta(X)d_Y\sigma$  defines an endomorphism  $f \rightarrow f$ , there is a quadratic differential

$$q(X, Y) = \text{tr}(\sigma \mapsto \eta(X)d_Y\sigma)$$

associated to closed one-forms taking values in  $f \wedge f^\perp$ . It turns out that we may use  $q$  to determine the triviality of  $\eta$ :

**Lemma 5.6.**  $q = 0$  if and only if  $\eta = d\xi$  for some  $\xi \in \Gamma(\wedge^2 f)$ .

*Proof.* We may write an arbitrary closed one-form  $\eta \in \Omega^1(f \wedge f^\perp)$  as

$$\eta = \alpha \sigma_1 \wedge d\sigma_1 + \beta \sigma_2 \wedge d\sigma_1 + \gamma \sigma_1 \wedge d\sigma_2 + \delta \sigma_2 \wedge d\sigma_2 \text{ mod } \Omega^1(\wedge^2 f)$$

for  $\sigma_1 \in \Gamma s_1$ ,  $\sigma_2 \in \Gamma s_2$  and some smooth functions  $\alpha, \beta, \gamma, \delta$ . The quadratic differential of  $\eta$  is then

$$q = -\alpha(d\sigma_1, d\sigma_1) - \delta(d\sigma_2, d\sigma_2).$$

Since  $(d\sigma_1, d\sigma_1) \in \Gamma(T_2^*)^2$  and  $(d\sigma_2, d\sigma_2) \in \Gamma(T_1^*)^2$ , one has that  $q = 0$  if and only if  $\alpha = \delta = 0$ . One can clearly see that if  $\eta = d\xi$ , for some  $\xi := \lambda\sigma_1 \wedge \sigma_2$ , then  $\alpha = \delta = 0$ . On the other hand, if  $\alpha = \delta = 0$ , then the closure of  $\eta$  implies that  $\beta = -\gamma$  and moreover  $\eta = d(\beta\sigma_2 \wedge \sigma_1)$ . Hence  $\eta = d\xi$  for  $\xi := \beta\sigma_2 \wedge \sigma_1$ .  $\square$

We thus obtain the main theorem of this section:

**Theorem 5.7.**  $f : \Sigma \rightarrow \mathcal{Z}$  is deformable to second order if and only if there exists a closed one-form  $\eta$  taking values in  $f \wedge f^\perp$  such that  $q \neq 0$ .

Using the gauge theoretic definition of  $\Omega$ - and  $\Omega_0$ -surfaces of [30], one recovers the following result:

**Corollary 5.8** ([27]).  $\Omega$ - and  $\Omega_0$ -surfaces are the only second order deformable surfaces of Lie sphere geometry.

*Remark 5.9.* In [9, 27] it was shown how second order deformable maps in Lie sphere geometry yield deformable maps in conformal and Laguerre geometry. For more information about deformability in Laguerre geometry, see [26, 29].

**5.2. Third order deformations.** In this subsection we shall show that rigidity occurs at third order for Legendre maps. Suppose that  $\eta$  is a third order infinitesimal deformation of  $F$ . Then by Theorem 5.7,  $\eta \in \Omega^1(f \wedge f^\perp)$  and  $\eta$  is closed. Now by Theorem 2.6, for  $X, Y, Z \in \Gamma T\Sigma$ ,

$$(d_X d_Y \eta(Z))F \leq F.$$

or, equivalently,

$$(14) \quad (d_X d_Y \eta(Z))f \leq f.$$

Let  $\sigma \in \Gamma f$  and assume that  $X$  is a curvature direction of  $f$ , i.e.,  $X \in \Gamma T_i$  for  $i \in \{1, 2\}$ . Then by the Leibniz rule, equation (14) implies that

$$(15) \quad d_X((d_Y \eta(Z))\sigma) - (d_Y \eta(Z))d_X \sigma \in \Gamma f.$$

Now since  $(d_Y \eta(Z))\sigma \in \Gamma f$ , we have that  $d_X((d_Y \eta(Z))\sigma) \in \Gamma f_i$ . Furthermore, as  $d_Y \eta(Z)$  is skew-symmetric,  $(d_Y \eta(Z))d_X \sigma$  is orthogonal to  $d_X \sigma$ . Thus, equation (15) holds if and only if

$$d_X((d_Y \eta(Z))\sigma) \in \Gamma f \quad \text{and} \quad (d_Y \eta(Z))d_X \sigma \in \Gamma f.$$

Now  $d_X((d_Y\eta(Z))\sigma) \in \Gamma f$  implies that

$$(d_Y\eta(Z))\sigma \in \Gamma s_i.$$

Since  $i$  was arbitrary, we then have that  $(d_Y\eta(Z))\sigma = 0$ . By the Leibniz rule this implies that

$$d_Y(\eta(Z)\sigma) - \eta(Z)d_Y\sigma = 0,$$

and since  $\eta(Z)f = 0$ , we have that

$$\eta(Z)d_Y\sigma = 0.$$

Hence,  $\eta f^\perp \equiv 0$  and thus  $\eta \in \Omega^1(\wedge^2 f)$ . One can then check that  $\eta$  being closed implies that  $\eta \equiv 0$ . We have thus arrived at the following result:

**Theorem 5.10.** *Legendre maps are rigid to third order.*

## 6. PROJECTIVE APPLICABILITY REVISITED

It is well known that surfaces in projective space  $F : \Sigma \rightarrow \mathbb{P}(\mathbb{R}^4)$  can be represented by their contact lifts in  $\mathbb{R}^{3,3}$ :

$$f = F \wedge F^{(1)}.$$

The derived bundle of this contact lift is

$$f^{(1)} = F^{(1)} \wedge F^{(1)} + F \wedge \underline{\mathbb{R}}^4.$$

Recall also that there is an isomorphism  $\phi : \mathfrak{sl}(4) \rightarrow \mathfrak{o}(3,3)$ , defined by

$$\phi(A)(v \wedge w) = Av \wedge w + v \wedge Aw.$$

Since  $\phi$  is constant,  $\phi$  intertwines the trivial connections on  $\underline{\mathfrak{sl}(4)}$  and  $\underline{\mathfrak{o}(3,3)}$ . Let  $\Theta \leq \underline{\mathfrak{sl}(4)}$  denote the subbundle of  $\underline{\mathfrak{sl}(4)}$  such that  $A \in \Gamma\Theta$  if and only if

$$AF = 0 \quad \text{and} \quad AF^{(1)} \leq F.$$

Then  $\phi$  yields an isomorphism between  $\Theta$  and  $f \wedge f^\perp$ . Since  $\phi$  is constant one has that closed 1-forms taking values in  $\Theta$  are in one-to-one correspondence with closed one forms taking values in  $f \wedge f^\perp$ . Furthermore, if we let  $\Psi$  denote the subbundle of  $\Theta$  defined by  $A \in \Gamma\Psi$  if and only if

$$AF^{(1)} = 0 \quad \text{and} \quad im A \leq F,$$

then  $\phi$  yields an isomorphism between  $\Psi$  and  $\wedge^2 f$ . Thus, one deduces that the triviality of second order infinitesimal deformations is preserved by  $\phi$ . We have thus recovered the classical result of Fubini [18]:

**Theorem 6.1.** *A surface in projective 3-space is deformable of order two if and only if its contact lift is deformable of order two.*

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