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(Article begins on next page)

# A NOTE ON PRODUCT SETS OF RANDOM SETS

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ABSTRACT. Given two sets of positive integers  $A$  and  $B$ , let  $AB := \{ab : a \in A, b \in B\}$  be their *product set* and put  $A^k := A \cdots A$  ( $k$  times  $A$ ) for any positive integer  $k$ . Moreover, for every positive integer  $n$  and every  $\alpha \in [0, 1]$ , let  $\mathcal{B}(n, \alpha)$  denote the probabilistic model in which a random set  $A \subseteq \{1, \dots, n\}$  is constructed by choosing independently every element of  $\{1, \dots, n\}$  with probability  $\alpha$ . We prove that if  $A_1, \dots, A_s$  are random sets in  $\mathcal{B}(n_1, \alpha_1), \dots, \mathcal{B}(n_s, \alpha_s)$ , respectively,  $k_1, \dots, k_s$  are fixed positive integers,  $\alpha_i n_i \rightarrow +\infty$ , and  $1/\alpha_i$  does not grow too fast in terms of a product of  $\log n_j$ ; then  $|A_1^{k_1} \cdots A_s^{k_s}| \sim \frac{|A_1|^{k_1}}{k_1!} \cdots \frac{|A_s|^{k_s}}{k_s!}$  with probability  $1 - o(1)$ . This is a generalization of a result of Cilleruelo, Ramana, and Ramaré, who considered the case  $s = 1$  and  $k_1 = 2$ .

## 1. INTRODUCTION

Given two sets of positive integers  $A$  and  $B$ , let  $AB := \{ab : a \in A, b \in B\}$  be their *product set* and put  $A^k := A \cdots A$  ( $k$  times  $A$ ) for any positive integer  $k$ .

Problems involving the cardinalities of product sets have been considered by many researchers. For example, the study of  $M_n := |\{1, \dots, n\}^2|$  as  $n \rightarrow +\infty$  is known as the “multiplicative table problem” and was started by Erdős [2, 3]. The exact order of magnitude of  $M_n$  was determined by Ford [4] following earlier work of Tenenbaum [8]. Furthermore, Koukoulopoulos [7] provided uniform bounds for  $|\{1, \dots, n_1\} \cdots \{1, \dots, n_s\}|$  holding for a wide range of  $n_1, \dots, n_s$ . Cilleruelo, Ramana, and Ramaré [1] proved asymptotics or bounds for  $|(A \cap \{1, \dots, n\})^2|$  when  $A$  is the set of shifted prime numbers, the set of sums of two squares, or the set of shifted sums of two squares.

For every positive integer  $n$  and every  $\alpha \in [0, 1]$ , let  $\mathcal{B}(n, \alpha)$  denote the probabilistic model in which a random set  $A \subseteq \{1, \dots, n\}$  is constructed by choosing independently every element of  $\{1, \dots, n\}$  with probability  $\alpha$ . Cilleruelo, Ramana, and Ramaré [1] proved the following:

**Theorem 1.1.** *Let  $A$  be a random set in  $\mathcal{B}(n, \alpha)$ . If  $\alpha n \rightarrow +\infty$  and  $\alpha = o((\log n)^{-1/2})$ , then  $|A^2| \sim \frac{|A|^2}{2}$  with probability  $1 - o(1)$ .*

The contribution of this paper is the following generalization of Theorem 1.1.

**Theorem 1.2.** *Let  $A_1, \dots, A_s$  be random sets in  $\mathcal{B}(n_1, \alpha_1), \dots, \mathcal{B}(n_s, \alpha_s)$ , respectively; and let  $k_1, \dots, k_s$  be fixed positive integers. If  $\alpha_i n_i \rightarrow +\infty$  and*

$$\alpha_i = o\left(\left((\log n_1)^{k_1-1} \prod_{i=2}^s (\log n_i)^{k_i}\right)^{-(k_1+\dots+k_s-1)/2}\right),$$

for  $i = 1, \dots, s$ , then  $|A_1^{k_1} \cdots A_s^{k_s}| \sim \frac{|A_1|^{k_1}}{k_1!} \cdots \frac{|A_s|^{k_s}}{k_s!}$  with probability  $1 - o(1)$ .

## 2. NOTATION

We employ the Landau–Bachmann “Big Oh” and “little oh” notations  $O$  and  $o$ , as well as the associated Vinogradov symbol  $\ll$ , with their usual meanings. Any dependence of implied constants is explicitly stated or indicated with subscripts. For real random variables  $X$  and

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$Y$ , we say that “ $X = o(Y)$  with probability  $1 - o(1)$ ” if  $\mathbb{P}(|X| \geq \varepsilon|Y|) = o_\varepsilon(1)$  for every  $\varepsilon > 0$ , and that “ $X \sim Y$  with probability  $1 - o(1)$ ” if  $X = Y + o(Y)$  with probability  $1 - o(1)$ .

### 3. PRELIMINARIES

In this section we collect some preliminary results not directly related with product sets.

**Lemma 3.1.** *Let  $m$  be a positive integer. We have*

$$\sum_{a_1 \cdots a_m \leq x} \frac{1}{a_1 \cdots a_m} \ll_m (\log x)^m,$$

for all  $x \geq 2$ .

*Proof.* This is a standard application of Rankin’s method: For  $t := m/\log x$ , we have

$$\begin{aligned} \sum_{a_1 \cdots a_m \leq x} \frac{1}{a_1 \cdots a_m} &\leq x^t \sum_{a_1 \cdots a_m \leq x} \frac{1}{(a_1 \cdots a_m)^{1+t}} \leq x^t \left( \sum_{a=1}^{\infty} \frac{1}{a^{1+t}} \right)^m \\ &\leq x^t \left( 1 + \frac{1}{t} \right)^m \ll_m (\log x)^m, \end{aligned}$$

as claimed.  $\square$

The next lemma is an upper bound on the number of matrices of positive integers with bounded products of rows and columns.

**Lemma 3.2.** *Let  $m$  and  $n$  be positive integers. Then, for all  $x_1, \dots, x_n, y_1, \dots, y_m \geq 2$ , the number of  $m \times n$  matrices  $(c_{i,j})$  of positive integers satisfying  $\prod_{i=1}^m c_{i,h} \leq x_h$  and  $\prod_{j=1}^n c_{k,j} \leq y_k$ , for  $h = 1, \dots, n$  and  $k = 1, \dots, m$ , is at most*

$$(1) \quad O_{m,n} \left( \left( \prod_{i=1}^n x_i \prod_{j=1}^m y_j \right)^{1/2} \left( \prod_{i=1}^{n-1} \log x_i \right)^{m-1} \right)$$

*Proof.* We follow the same arguments of [5, p. 380], where the case  $m = n$  and  $x_1 = \dots = x_n = y_1 = \dots = y_m$  is proved.

The number of choices for  $c_{m,n}$  is at most

$$\min \left( \frac{x_n}{\prod_{i=1}^{m-1} c_{i,n}}, \frac{y_m}{\prod_{j=1}^{n-1} c_{m,j}} \right) \leq \left( \frac{x_n y_m}{\prod_{i=1}^{m-1} c_{i,n} \prod_{j=1}^{n-1} c_{m,j}} \right)^{1/2}.$$

We shall sum this latter quantity over all the choices of  $c_{i,n}$  and  $c_{m,j}$ , with  $i = 1, \dots, m-1$  and  $j = 1, \dots, n-1$ . Since  $c_{i,n} \leq y_i / \prod_{k=1}^{n-1} c_{i,k}$  and  $c_{m,j} \leq x_j / \prod_{h=1}^{m-1} c_{h,j}$ , we have

$$\sum_{c_{i,n}} \frac{1}{c_{i,n}^{1/2}} \ll \left( \frac{y_i}{\prod_{k=1}^{n-1} c_{i,k}} \right)^{1/2} \quad \text{and} \quad \sum_{c_{m,j}} \frac{1}{c_{m,j}^{1/2}} \ll \left( \frac{x_j}{\prod_{h=1}^{m-1} c_{h,j}} \right)^{1/2},$$

for  $i = 1, \dots, m-1$  and  $j = 1, \dots, n-1$ . Consequently,

$$\begin{aligned} \sum_{\substack{c_{1,n}, \dots, c_{m-1,n} \\ c_{m,1}, \dots, c_{m,n-1}}} \left( \frac{x_n y_m}{\prod_{i=1}^{m-1} c_{i,n} \prod_{j=1}^{n-1} c_{m,j}} \right)^{1/2} &\leq (x_n y_m)^{1/2} \prod_{i=1}^{m-1} \left( \sum_{c_{i,n}} \frac{1}{c_{i,n}^{1/2}} \right) \prod_{j=1}^{n-1} \left( \sum_{c_{m,j}} \frac{1}{c_{m,j}^{1/2}} \right) \\ &\ll_{m,n} \left( \prod_{j=1}^n x_j \prod_{i=1}^m y_i \right)^{1/2} \left( \prod_{h=1}^{m-1} \prod_{k=1}^{n-1} c_{h,k} \right)^{-1}. \end{aligned}$$

It remains only to sum over all the possibilities for  $c_{h,k}$ , with  $h = 1, \dots, m-1$  and  $k = 1, \dots, n-1$ . Thanks to Lemma 3.1, we have

$$\sum_{c_{h,k}} \left( \prod_{h=1}^{m-1} \prod_{k=1}^{n-1} c_{h,k} \right)^{-1} \leq \prod_{k=1}^{n-1} \sum_{c_{1,k} \cdots c_{m-1,k} \leq x_k} \frac{1}{c_{1,k} \cdots c_{m-1,k}} \ll_{m,n} \left( \prod_{k=1}^{n-1} \log x_k \right)^{m-1},$$

and the desired result follows.  $\square$

The next lemma is an upper bound for the number of solutions of a certain multiplicative equation with bounded factors.

**Lemma 3.3.** *Let  $m$  and  $n$  be positive integers. Then, for all  $x_1, \dots, x_n, y_1, \dots, y_m \geq 2$ , the number of solutions of the equation  $a_1 \cdots a_n = b_1 \cdots b_m$ , where  $a_1, \dots, a_n, b_1, \dots, b_m$  are positive integers satisfying  $a_i \leq x_i$  and  $b_j \leq y_j$ , for  $i = 1, \dots, n$  and  $j = 1, \dots, m$ , is at most (1).*

*Proof.* If  $a_1 \cdots a_n = b_1 \cdots b_m$  then there exists a  $m \times n$  matrix of positive integers  $(c_{i,j})$  such that  $a_h = \prod_{i=1}^m c_{i,h}$  and  $b_k = \prod_{j=1}^n c_{k,j}$ , for  $h = 1, \dots, n$  and  $k = 1, \dots, m$ . Indeed,  $a_1 \mid \prod_{i=1}^m b_i$  implies the existence of positive integers  $c_{1,1}, \dots, c_{m,1}$  such that  $a_1 = \prod_{i=1}^m c_{i,1}$  and  $c_{i,1} \mid b_i$ , for  $i = 1, \dots, m$ . Then  $a_2 \mid \prod_{i=1}^m b_i / c_{i,1}$ , which similarly implies the existence of positive integers  $c_{1,2}, \dots, c_{m,2}$  such that  $a_2 = \prod_{i=1}^m c_{i,2}$  and  $c_{i,1} c_{i,2} \mid b_i$ , for  $i = 1, \dots, m$ . Then  $a_3 \mid \prod_{i=1}^m b_i / (c_{i,1} c_{i,2})$ , and so on, until  $a_n = \prod_{i=1}^m b_i / (\prod_{j=1}^{n-1} c_{i,j})$ , when we set  $c_{i,n} := b_i / \prod_{j=1}^{n-1} c_{i,j}$  for  $i = 1, \dots, m$ . Applying Lemma 3.2 we get the desired result.  $\square$

#### 4. PROOF OF THEOREM 1.2

First, we need an asymptotic for the  $k$ th power of the size of a random set  $A$  in  $\mathcal{B}(n, \alpha)$ .

**Lemma 4.1.** *Let  $A$  be a random set in  $\mathcal{B}(n, \alpha)$ , and fix an integer  $k \geq 1$ . If  $\alpha n \rightarrow +\infty$ , then:*

- (i)  $\mathbb{E}(|A|^k) \sim (\alpha n)^k$ ; and
- (ii)  $|A|^k \sim (\alpha n)^k$  with probability  $1 - o_k(1)$ .

*Proof.* Clearly,  $|A|$  follows a binomial distribution with  $n$  trials and probability of success  $\alpha$ . Consequently, (i) is known (see, e.g., [6, Eq. (4.1)]). In turn, (i) implies that

$$\mathbb{V}(|A|^k) = \mathbb{E}(|A|^{2k}) - \mathbb{E}(|A|^k)^2 = o_k(\mathbb{E}(|A|^k)^2).$$

Hence, by Chebyshev's inequality, for every  $\varepsilon > 0$  we have

$$\mathbb{P}\left(|A|^k - \mathbb{E}(|A|^k) \geq \varepsilon \mathbb{E}(|A|^k)\right) \leq \frac{\mathbb{V}(|A|^k)}{(\varepsilon \mathbb{E}(|A|^k))^2} = o_{k,\varepsilon}(1),$$

so that  $|A|^k \sim \mathbb{E}(|A|^k) \sim (\alpha n)^k$  with probability  $1 - o_k(1)$ .  $\square$

The next lemma is an easy bound on the size of a product set.

**Lemma 4.2.** *Let  $A_1, \dots, A_s$  be finite sets of positive integers, and let  $k_1, \dots, k_s \geq 1$  be integers. Then*

$$\left| \prod_{i=1}^s A_i^{k_i} \right| \leq \prod_{i=1}^s \binom{|A_i| + k_i - 1}{k_i}.$$

*Proof.* The claim follows easily considering that  $\binom{|A|+k-1}{k}$  is the number of unordered  $k$ -tuples of elements from a set  $A$ .  $\square$

For the rest of this section, let  $A_1, \dots, A_s$  be random sets in  $\mathcal{B}(n_1, \alpha_1), \dots, \mathcal{B}(n_s, \alpha_s)$ , respectively; and let  $k_1, \dots, k_s$  be fixed positive integers. Also, assume  $\alpha_i n_i \rightarrow +\infty$  and

$$(2) \quad \alpha_i = o\left(\left(\left(\log n_1\right)^{k_1-1} \prod_{i=2}^s \left(\log n_i\right)^{k_i}\right)^{-(k_1+\dots+k_s-1)/2}\right),$$

for  $i = 1, \dots, s$ . For brevity, we will omit the dependence of implied constants from  $k_1, \dots, k_s$ .

**Lemma 4.3.** *We have  $\mathbb{E}(|A_1^{k_1} \cdots A_s^{k_s}|) \sim \frac{(\alpha_1 n_1)^{k_1}}{k_1!} \cdots \frac{(\alpha_s n_s)^{k_s}}{k_s!}$ .*

*Proof.* Hereafter, in operator subscripts, let  $\mathbf{a} := (\mathbf{a}_1, \dots, \mathbf{a}_s)$ , where each  $\mathbf{a}_i := \{a_{i,1}, \dots, a_{i,k_i}\}$  runs over the unordered  $k_i$ -tuples of elements of  $\{1, \dots, n_i\}$ . Also, put  $\|\mathbf{a}\| := \prod_{i=1}^s \prod_{j=1}^{k_i} a_{i,j}$ . With this notation, for each positive integer  $x$ , we have

$$\mathbb{P}(x \in A_1^{k_1} \cdots A_s^{k_s}) = \mathbb{P}\left(\bigvee_{\|\mathbf{a}\|=x} E_{\mathbf{a}}\right),$$

where

$$E_{\mathbf{a}} := \bigwedge_{i=1}^s (\mathbf{a}_i \subseteq A_i).$$

Consequently, by Bonferroni inequalities, we have

$$\begin{aligned} \mathbb{P}(x \in A_1^{k_1} \cdots A_s^{k_s}) &= \mathbb{P}\left(\bigvee_{\|\mathbf{a}\|=x}^* E_{\mathbf{a}}\right) + O\left(\sum_{\|\mathbf{a}\|=x}^{**} \mathbb{P}(E_{\mathbf{a}})\right) \\ &= \sum_{\|\mathbf{a}\|=x}^* \mathbb{P}(E_{\mathbf{a}}) + O\left(\sum_{\substack{\mathbf{a} \neq \mathbf{a}' \\ \|\mathbf{a}\|=\|\mathbf{a}'\|=x}}^* \mathbb{P}(E_{\mathbf{a}} \wedge E_{\mathbf{a}'})\right) + O\left(\sum_{\|\mathbf{a}\|=x}^{**} \mathbb{P}(E_{\mathbf{a}})\right), \end{aligned}$$

where the superscript  $*$  denotes the constraint  $|\mathbf{a}_i| = k_i$  for every  $i \in \{1, \dots, s\}$ , the superscript  $**$  denotes the complementary constrain  $|\mathbf{a}_i| < k_i$  for at least one  $i \in \{1, \dots, k\}$ , and  $\mathbf{a}' := (\mathbf{a}'_1, \dots, \mathbf{a}'_s)$  follows the same conventions of  $\mathbf{a}$ . Therefore,

$$\begin{aligned} (3) \quad \mathbb{E}(|A_1^{k_1} \cdots A_s^{k_s}|) &= \sum_{x \leq n_1^{k_1} \cdots n_s^{k_s}} \mathbb{P}(x \in A_1^{k_1} \cdots A_s^{k_s}) \\ &= \sum_{\mathbf{a}}^* \mathbb{P}(E_{\mathbf{a}}) + O\left(\sum_{\substack{\mathbf{a} \neq \mathbf{a}' \\ \|\mathbf{a}\|=\|\mathbf{a}'\|}}^* \mathbb{P}(E_{\mathbf{a}} \wedge E_{\mathbf{a}'})\right) + O\left(\sum_{\mathbf{a}}^{**} \mathbb{P}(E_{\mathbf{a}})\right). \end{aligned}$$

Since  $A_1, \dots, A_s$  are independent and each  $A_i$  belongs to  $\mathcal{B}(n_i, \alpha_i)$ , we have

$$\mathbb{P}(E_{\mathbf{a}}) = \bigwedge_{i=1}^s \mathbb{P}(\mathbf{a}_i \subseteq A_i) = \prod_{i=1}^s \alpha_i^{|\mathbf{a}_i|}.$$

Hence, for every positive integers  $m_1, \dots, m_s$ , with  $m_i \leq k_i$ , we have

$$\sum_{\mathbf{a}: |\mathbf{a}_i|=m_i} \mathbb{P}(E_{\mathbf{a}}) = \sum_{\mathbf{a}: |\mathbf{a}_i|=m_i} \prod_{i=1}^s \alpha_i^{m_i} = \prod_{i=1}^s \alpha_i^{m_i} \sum_{|\mathbf{a}_i|=m_i} 1 = \prod_{i=1}^s \alpha_i^{m_i} \binom{n_i}{m_i} \binom{k_i-1}{m_i-1},$$

where we used the fact that the number of unordered  $k$ -tuples of elements of  $\{1, \dots, n\}$  having cardinality equal to  $m$  is  $\binom{n}{m} \binom{k-1}{m-1}$ . Therefore,

$$(4) \quad \sum_{\mathbf{a}}^* \mathbb{P}(E_{\mathbf{a}}) \sim \prod_{i=1}^s \frac{(\alpha_i n_i)^{k_i}}{k_i!} \quad \text{and} \quad \sum_{\mathbf{a}}^{**} \mathbb{P}(E_{\mathbf{a}}) = o\left(\prod_{i=1}^s (\alpha_i n_i)^{k_i}\right),$$

as  $\alpha_i n_i \rightarrow +\infty$ , for  $i = 1, \dots, s$ . We have

$$(5) \quad \mathbb{P}(E_{\mathbf{a}} \wedge E_{\mathbf{a}'}) = \prod_{i=1}^s \mathbb{P}(\mathbf{a}_i \cup \mathbf{a}'_i \subseteq A_i) = \prod_{i=1}^s \alpha_i^{|\mathbf{a}_i \cup \mathbf{a}'_i|}.$$

Suppose that  $\mathbf{a}$  and  $\mathbf{a}'$ , with  $\mathbf{a} \neq \mathbf{a}'$  and  $\|\mathbf{a}\| = \|\mathbf{a}'\|$ , satisfy the condition of  $*$ , that is,  $|\mathbf{a}_i| = |\mathbf{a}'_i| = k_i$  for  $i = 1, \dots, s$ . We shall find an upper bound for (5). Clearly,  $|\mathbf{a}_i \cup \mathbf{a}'_i| \geq |\mathbf{a}_i| \geq k_i$  for  $i = 1, \dots, s$ . Moreover, since  $\mathbf{a} \neq \mathbf{a}'$ , there exists  $i_1 \in \{1, \dots, s\}$  such that  $\mathbf{a}_{i_1} \neq \mathbf{a}'_{i_1}$ .

Since  $|\mathbf{a}_{i_1}| = |\mathbf{a}'_{i_1}| = k_i$ , it follows that  $|\mathbf{a}_{i_1} \cup \mathbf{a}'_{i_1}| \geq k_{i_1} + 1$ . On the one hand, if there exists  $i_2 \in \{1, \dots, s\} \setminus \{i_1\}$  such that  $\mathbf{a}_{i_2} \neq \mathbf{a}'_{i_2}$ , then, similarly, we have  $|\mathbf{a}_{i_2} \cup \mathbf{a}'_{i_2}| \geq k_{i_2} + 1$ . Hence,

$$\mathbb{P}(E_{\mathbf{a}} \wedge E_{\mathbf{a}'}) \leq \alpha_{i_1} \alpha_{i_2} \prod_{i=1}^s \alpha_i^{k_i}.$$

On the other hand, if  $\mathbf{a}_i = \mathbf{a}'_i$  for every  $i \in \{1, \dots, s\} \setminus \{i_1\}$ , then from  $\|\mathbf{a}\| = \|\mathbf{a}'\|$  it follows that  $\prod_{j=1}^{k_{i_1}} a_{i_1,j} = \prod_{j=1}^{k_{i_1}} a'_{i_1,j}$ . In turn, this implies that  $|\mathbf{a}_{i_1} \cup \mathbf{a}'_{i_1}| \geq k_{i_1} + 2$ . Hence,

$$\mathbb{P}(E_{\mathbf{a}} \wedge E_{\mathbf{a}'}) \leq \alpha_{i_1}^2 \prod_{i=1}^s \alpha_i^{k_i}.$$

Therefore, using Lemma 3.3 and recalling (2), we obtain

$$\begin{aligned} (6) \quad \sum_{\substack{\mathbf{a} \neq \mathbf{a}' \\ \|\mathbf{a}\| = \|\mathbf{a}'\|}}^* \mathbb{P}(E_{\mathbf{a}} \wedge E_{\mathbf{a}'}) &\leq \left( \max_{1 \leq i, j \leq s} \alpha_i \alpha_j \right) \prod_{i=1}^s \alpha_i^{k_i} \sum_{\|\mathbf{a}\| = \|\mathbf{a}'\|} 1 \\ &\ll \left( \max_{1 \leq i, j \leq s} \alpha_i \alpha_j \right) \left( (\log n_1)^{k_1-1} \prod_{i=2}^s (\log n_i)^{k_i} \right)^{k_1 + \dots + k_s - 1} \prod_{i=1}^s (\alpha_i n_i)^{k_i} \\ &= o \left( \prod_{i=1}^s (\alpha_i n_i)^{k_i} \right). \end{aligned}$$

Finally, putting together (3), (4), and (6), we obtain the desired claim.  $\square$

*Proof of Theorem 1.2.* Define the random variable

$$X := \prod_{i=1}^s \binom{|A_i| + k_i - 1}{k_i} - \left| \prod_{i=1}^s A_i^{k_i} \right|.$$

Thanks to Lemma 4.2, we know that  $X$  is nonnegative. Moreover, from Lemma 4.1(i) and Lemma 4.3, it follows that

$$\mathbb{E}(X) = o \left( \prod_{i=1}^s (\alpha_i n_i)^{k_i} \right).$$

Hence, for every  $\varepsilon > 0$ , by Markov's inequality, we get

$$\mathbb{P} \left( X \geq \varepsilon \prod_{i=1}^s (\alpha_i n_i)^{k_i} \right) \leq \frac{\mathbb{E}(X)}{\varepsilon \prod_{i=1}^s (\alpha_i n_i)^{k_i}} = o_\varepsilon(1),$$

which in turn implies  $X = o(\prod_{i=1}^s (\alpha_i n_i)^{k_i})$  with probability  $1 - o(1)$ . Therefore, by Lemma 4.1(ii),

$$\left| \prod_{i=1}^s A_i^{k_i} \right| = \prod_{i=1}^s \binom{|A_i| + k_i - 1}{k_i} - X = \prod_{i=1}^s \frac{|A_i|^{k_i}}{k_i!} + o \left( \prod_{i=1}^s |A_i|^{k_i} \right),$$

with probability  $1 - o(1)$ , as claimed.  $\square$

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