THE GEOMETRY OF CONFORMAL TIMELIKE GEODESICS IN THE EINSTEIN UNIVERSE

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Abstract. This paper studies the geometry of the critical points of the simplest conformally invariant variational problem for timelike curves in the n-dimensional Einstein universe. Such critical curves are referred to as conformal timelike geodesics. The functional defining the variational problem is the Lorentz analogue of the conformal arclength functional in Möbius geometry. We compute the Euler–Lagrange equations and show that the trajectory of a conformal timelike geodesic is constrained into some totally umbilical Einstein universe of dimension 2, 3, or 4. The case of dimension 2 leads to orbits of 1-parameter groups of Lorentz Möbius transformations, while that of dimension 3 has been dealt with in [8]. In this paper, we discuss the case of conformal timelike geodesics in the 4-dimensional Einstein universe whose trajectories are not contained in any lower dimensional totally umbilical Einstein universe. It is shown that such curves can be explicitly integrated by quadratures and explicit expressions in terms of elliptic functions and integrals are provided.

1. Introduction

Let $\mathcal{E}^{1,n-1}$ denote the conformal compactification of Minkowski n-space ($n > 1$), realized as the set of oriented null lines through the origin in pseudo-Euclidean space $\mathbb{R}^{2,n}$. Topologically, $\mathcal{E}^{1,n-1}$ is the sphere product $S^1 \times S^{n-1}$. The restriction to $\mathcal{E}^{1,n-1}$ of the pseudo-Euclidean structure of $\mathbb{R}^{2,n}$ makes $\mathcal{E}^{1,n-1}$ into an oriented, time-oriented, Lorentz manifold with product metric $-dt^2 + g_{S^{n-1}}$, where dt^2 is the standard metric on S^1 and $g_{S^{n-1}}$ that on S^{n-1}. The Lorentz manifold $\mathcal{E}^{1,n-1}$ is known in the literature as the compact *Einstein universe*.[1] The role of $\mathcal{E}^{1,n-1}$ in conformal Lorentz geometry is similar to that of the conformal n-sphere S^n (the Möbius n-space) in conformal Riemannian geometry [2, 12, 16]. In particular, all Robertson–Walker spacetimes, including the Lorentz spaceforms, can be conformally realized as open domains of $\mathcal{E}^{1,n-1}$ [9, 19]. Conformal Lorentz geometry, in both its intrinsic and extrinsic aspects, has played an important role since the work

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[1] Actually, $\mathcal{E}^{1,n-1}$ is the double covering of the space that in [2, 12] is called Einstein universe. The advantage of working with the compact model $\mathcal{E}^{1,n-1}$, instead that with its universal covering $\mathbb{R} \times S^{n-1}$ (the Einstein static universe), is that its group of (restricted) conformal transformations is a Lie group of matrices (cf. [10] for more details), which greatly simplifies computation.

of H. Weyl in general relativity [42] and of W. Blaschke and G. Thomsen in the classical geometries of Laguerre, Möbius and Lie [4, 6, 20, 21]. In the 1980s, the subject has been considered in the twistor approach to gravity by Penrose and Rindler [34] and more recently in the study of cyclic cosmological models [32, 33, 39, 40] and in the regularization of the Kepler problem [16]. For other applications of conformal Lorentz geometry, we refer to [1].

Let us begin by recalling some facts about the conformal geometry of curves in S^n which are comparatively relevant to our study. It is well known that along a generic curve in the conformal sphere S^n there exists a canonical conformally invariant arc element whose integral defines a conformally invariant functional, called the arclength functional. This construction is classical and goes back to the work of a number authors of the first decades of the 19th century, including Fialkov, Haantjes, Liebmann, Takasu, Vessiot, *et al.* [11, 18, 25, 37, 41]. In [28], E. Musso characterized the critical curves of the arclength functional for the case of S^3 and provided explicit expressions for the critical curves in terms of elliptic functions. Extending this work to S^n, in [26], M. Magliaro, L. Mari and M. Rigoli proved that every critical curve in S^n lies in a totally umbilical 4-sphere S^4 and obtained explicit expressions for the critical curves in S^4. Interestingly enough, it follows from the results in [26] that any closed critical curve in S^n lies in some $S^3 \subset S^n$, up to Möbius transformation. The question of existence and properties of closed critical curves for the arclength functional in S^3 has recently been addressed by E. Musso and L. Nicolodi in [29, 30].

The purpose of this paper is to study the timelike curves in $\mathcal{E}^{1,n-1}$ which are critical for the *conformal strain functional*, the Lorentz analogue of the arclength functional for curves in S^n. These curves are referred to as *conformal timelike geodesics*, or *conformal worldlines*. The conformal strain functional defines the simplest conformally invariant variational problem on the space of generic timelike curves (i.e., timelike curves without conformal vertices) and generalizes the functional for timelike curves in $\mathcal{E}^{1,2}$ considered in [8]. Proceeding by the method of moving frames in analogy with [26, 28], we compute the variational equations satisfied by the conformal worldlines. We then prove that the conformal worldlines in $\mathcal{E}^{1,n-1}$ are constrained into some totally umbilical Einstein universe of dimension 2, 3, or 4. This is the Lorentz analogue of the result for the critical curves of the arclength functional in S^n mentioned above [26]. Now, the case of conformal worldlines in a 3-dimensional Einstein universe has been partially investigated in [8]. Moreover, it can be shown that the trajectories of the conformal worldlines in a 2-dimensional Einstein universe are orbits of 1-parameter groups of Lorentz Möbius transformations. The focus of this paper is on the conformal worldlines of $\mathcal{E}^{1,3}$ whose trajectories are not contained in any lower dimensional totally umbilical Einstein universe, the so-called *linearly full conformal worldlines*. We will prove that they can be integrated by quadratures and will compute explicit expressions in terms of elliptic functions and elliptic integrals.

More specifically, the results are organized as follows. In Section 2, we collect some basic facts about conformal Lorentz geometry [2, 16, 31] and develop for timelike curves in $\mathcal{E}^{1,n-1}$ the classical approach to the conformal geometry of curves in Möbius space [11, 18, 27, 35, 36, 37, 38]. We define the conformal strain for a timelike curve and the associated notion of conformal vertex. We then introduce the infinitesimal conformal strain and the conformal strain functional for generic timelike curves.

In Section 3, we use the moving frame method to compute the Euler–Lagrange equations of the conformal strain functional. This is the content of Theorem A. Next, we show that the trajectory of a conformal worldline is constrained in a totally umbilical Einstein universe of dimension 2, 3 or 4. This is proved in Theorem B.

In Section 4, we study the linearly full conformal worldlines of $\mathcal{E}^{1,3}$. For a linearly full worldline $\gamma \subset \mathcal{E}^{1,3}$, we construct a canonical lift to the (restricted) conformal group, called the *canonical conformal frame field* of γ, and define the three conformal curvatures of γ (cf. Proposition 3). We then use the variational equations to show that the curvatures are either constant, in which case the trajectory of γ is an orbit of a 1-parameter group of (restricted) conformal transformations (cf. Remark 7), or can be expressed in terms of Jabobi's elliptic functions. This is done in Proposition 5. As a consequence of Proposition 5, it is shown that any congruent class of conformal worldlines can be represented by a model worldline, the so-called *standard configuration* (cf. Definition 12). This implies that the conformal equivalence classes of linearly full conformal worldlines can be parametrized by three real parameters (cf. Proposition 6). We then define the *momentum operator* of a worldline γ. This is an element of the Lie algebra of the (restricted) conformal group, which is intrinsically defined by the worldline. The existence of the momentum is a consequence of the conformal invariance of the strain functional and of the Nöther conservation theorem (cf. Remark 8). A conformal worldline is called regular, exceptional, or singular depending on whether the momentum is a regular, exceptional, or singular element of the Lie algebra. Proposition 7 shows that a worldline can only be either *regular*, or *exceptional*. Next, for both types of worldlines, we define the integrating factor of an eigenvalue of the momentum and construct the principal vectors of the eigenvalues (cf. Proposition 8). The rather technical and lengthy calculations for the explicit determination of the integrating factors are given in the final Appendix.

In Section 5, we use the integrating factors and the principal vectors to integrate by quadratures the trajectories of the linearly full conformal worldlines of $\mathcal{E}^{1,3}$ with nonconstant curvatures. This is the content of Theorems C and D. At the end of the section, the theoretical aspects underlying the integration by quadratures are briefly discussed. We also explain why, contrary to what happens in $\mathcal{E}^{1,2}$, where there are countably many closed worldlines with nonconstant curvatures (cf. [8]), a linearly full conformal worldline in $\mathcal{E}^{1,3}$ with nonconstant curvatures cannot be closed. Finally, we show that the trajectory of a conformal worldline is invariant under the action of an infinite cyclic group of conformal transformations.

2. Conformal geometry of a timelike curve

2.1. **The Einstein universe and its restricted conformal group.** Let $\mathcal{E}^{1,n-1}$ denote the n-dimensional submanifold of $\mathbb{R}^{n+2}$, $n > 1$, defined by the equations $x_0^2 + x_1^2 = 1$ and $x_2^2 + \cdots + x_{n+1}^2 = 1$. As a manifold, $\mathcal{E}^{1,n-1}$ is the Cartesian product $S^1 \times S^{n-1}$. The restriction to $\mathcal{E}^{1,n-1}$ of the quadratic form

$$g = -dx_0^2 - dx_1^2 + dx_2^2 + \cdots + dx_{n+1}^2$$

induces on $\mathcal{E}^{1,n-1}$ a Lorentz pseudo-metric g_e. The normal bundle of $\mathcal{E}^{1,n-1}$ is spanned by the restrictions of the vector fields $\mathbf{n}_1 = x_0\partial_{x_0} + x_1\partial_{x_1}$ and $\mathbf{n}_2 =$

$x_2\partial_{x_2} + \cdots + x_{n+1}\partial_{x_{n+1}}$. Thus, contracting $dx_0 \wedge \cdots \wedge dx_{n+1}$ with $\mathbf{n}_1$ and $\mathbf{n}_2$, yields a volume form on $\mathcal{E}^{1,n-1}$ which in turn defines an orientation. The vector field $-x_1\partial_{x_0} + x_0\partial_{x_1}$ is tangent to $\mathcal{E}^{1,n-1}$ and induces a unit timelike vector field on $\mathcal{E}^{1,n-1}$. We time-orient $\mathcal{E}^{1,n-1}$ by requiring that such a vector field is future-directed.

Definition 1. The Lorentz manifold $(\mathcal{E}^{1,n-1}, g_e)$, with the above orientation and time-orientation, is called the *n-dimensional Einstein universe*.

In order to describe the conformal geometry of $\mathcal{E}^{1,n-1}$, it is convenient to consider in $\mathbb{R}^{n+2}$ the coordinates

$$y_0 = \frac{1}{\sqrt{2}}(x_0 + x_{n+1}), \quad y_1 = x_1, \dots, y_n = x_n, \quad y_{n+1} = \frac{1}{\sqrt{2}}(-x_0 + x_{n+1}).$$

The corresponding basis of $\mathbb{R}^{n+2}$ is denoted by $(E_0, \dots, E_{n+1})$ and is called the *standard* (*light-cone*) *basis* of $\mathbb{R}^{n+2}$. With respect to $(y_0, \dots, y_{n+1})$, the scalar product associated to g can be written as

$$\langle Y, Y' \rangle = -(y_0 y'_{n+1} + y_{n+1} y'_0) - y_1 y'_1 + \sum_{j=2}^{n} y_j y'_j. \tag{2.1}$$

In addition, $dV = dy_0 \wedge \cdots \wedge y_{n+1}$ defines a positive volume form. From now on, we will use light-cone coordinates and we will think of the elements $V \in \mathbb{R}^{n+2}$ as their coordinate column vectors with respect to $(E_0, \dots, E_{n+1})$. For each $V \in \mathbb{R}^{n+2}$, $V \neq 0$, $[V]$ will denote the oriented line spanned by V. By the mapping $\mathcal{E}^{1,n-1} \ni V \mapsto [V]$, $\mathcal{E}^{1,n-1}$ can be identified with the manifold of isotropic oriented lines through the origin of $\mathbb{R}^{n+2}$ (null rays). Using this identification, the identity connected component $\mathrm{A}^{\uparrow}_{+}(2,n)$ of the pseudo-orthogonal group of (2.1) acts transitively and effectively on the left of $\mathcal{E}^{1,n-1}$ by $\mathbf{X}[V] = [\mathbf{X}\, V]$. This action preserves the oriented, time-oriented conformal Lorentz structure of $\mathcal{E}^{1,n-1}$. It is a classical result that, if $n > 2$, every restricted conformal transformation of $\mathcal{E}^{1,n-1}$ is induced by a unique element of $\mathrm{A}^{\uparrow}_{+}(2,n)$ [7, 12]. For this reason, we call $\mathrm{A}^{\uparrow}_{+}(2,n)$ the *restricted conformal group of the n-dimensional Einstein universe*.

Remark 1. To distinguish the connected component of the identity we proceed as in [16]. Let $\mathcal{C} \subset \bigwedge^2(\mathbb{R}^{n+2})$ be the cone of all isotropic bi-vectors, i.e., the non-zero decomposable elements $V \wedge W$ of $\bigwedge^2(\mathbb{R}^{n+2})$ such that $\langle V, V \rangle = \langle W, W \rangle = \langle V, W \rangle = 0$. The function

$$\mathfrak{V} : \mathcal{C} \ni V \wedge W \mapsto dV(V, W, E_1, \dots, E_n, E_{n+1} - E_0) \in \mathbb{R}$$

never vanishes and the half cones

$$\mathcal{C}_+ = \{V \wedge W \in \mathcal{C} : \mathfrak{V}(V \wedge W) > 0\}, \quad \mathcal{C}_- = \{V \wedge W \in \ : \mathfrak{V}(V \wedge W) < 0\}$$

are the two connected components of $\mathcal{C}$. Therefore, $\mathrm{A}^{\uparrow}_{+}(2,n)$ is the group of all pseudo-orthogonal matrices $\mathbf{B}$ of the scalar product (2.1), such that $\det \mathbf{B} = 1$ and $\mathbf{B}\,\mathcal{C}_+ = \mathcal{C}_+$.

Let

$$\mathtt{m} = (\mathtt{m}_{ji}), \quad \mathtt{m}_{ji} := \langle \mathrm{E}_j, \mathrm{E}_i \rangle, \quad i, j = 0, \dots, n+1. \tag{2.2}$$

Then the column vectors $B_0, \dots, B_{n+1}$ of a matrix $\mathbf{B} \in \mathrm{A}^{\uparrow}_{+}(2,n)$ form a *light-cone basis* of $\mathbb{R}^{n+2}$, i.e., a positive-oriented basis such that

$$\langle B_i, B_j \rangle = \mathtt{m}_{ji}, \quad i,j = 0,\dots,n+1, \quad B_0 \wedge (B_1 + B_2) \in \mathcal{C}_+.$$

This allows us to identify $\mathrm{A}^{\uparrow}_{+}(2,n)$ with the manifold of all light-cone bases of $\mathbb{R}^{n+2}$.

Let $\mathbb{R}^{2,n}$ denote $\mathbb{R}^{n+2}$ equipped with the scalar product (2.1), the volume form dV, and the positive half cone $\mathcal{C}_+$. Differentiation of the $\mathbb{R}^{2,n}$-valued maps

$$\mathcal{B}_j : \mathrm{A}^{\uparrow}_{+}(2,n) \ni B \mapsto B_j \in \mathbb{R}^{2,n}, \quad j = 0,\dots,n+1,$$

yields

$$d\mathcal{B}_j = \sum_{i=0}^{n+1} \mu^i_j \mathcal{B}_i,$$

where μ^i_j are left-invariant 1-forms. The conditions $\langle B_j, B_i \rangle = \mathtt{m}_{ji}$ imply that $\mu = (\mu^i_j)$ takes values in the Lie algebra of $\mathrm{A}^{\uparrow}_{+}(2,n)$, namely

$$\mathfrak{a}(2,n) = \left\{ X \in \mathfrak{gl}(n+2,\mathbb{R}) \mid {}^tX\mathtt{m} + \mathtt{m}X = 0 \right\}.$$

As a consequence, we can write

$$\mu = \begin{pmatrix} \mu^0_0 & -\mu^1_{n+1} & {}^t\mu_{n+1} & 0 \\ \mu^1_0 & 0 & {}^t\mu_1 & \mu^1_{n+1} \\ \mu_0 & \mu_1 & \tilde{\mu} & \mu_{n+1} \\ 0 & -\mu^1_0 & {}^t\mu_0 & -\mu^0_0 \end{pmatrix},$$

where

$$\mu_0 = {}^t(\mu^2_0, \dots, \mu^n_0), \quad \mu_1 = {}^t(\mu^2_1, \dots, \mu^n_1), \quad \mu_{n+1} = {}^t(\mu^2_{n+1}, \dots, \mu^n_{n+1})$$

and ${}^t\tilde{\mu} + \tilde{\mu} = 0$. The left-invariant 1-forms μ^0_0, μ^j_0, μ^j_{n+1}, $j = 1,\dots,n$, and μ^i_j, $1 \le i < j = 1,\dots,n$, are linearly independent and span the dual of the Lie algebra $\mathfrak{a}(2,n)$. They satisfy the *Maurer–Cartan equations*

$$d\mu^i_j = -\sum_{k=0}^{n+1} \mu^i_k \wedge \mu^k_j, \quad i,j = 0,\dots,n+1. \tag{2.3}$$

Remark 2. Let $\mathbb{M}^{1,n-1}$ be *Minkowski n-space*, i.e., the affine space $\mathbb{R}^n$ with the Lorentzian scalar product

$$(\mathbf{p},\mathbf{q}) = -p_1q_1 + p_2q_2 + \cdots + p_nq_n.$$

The map

$$\mathbf{j}(\mathbf{p}) = \left[{}^t\!\left(1, p_1, \dots, p_n, \frac{(\mathbf{p},\mathbf{p})}{2}\right) \right] \in \mathcal{E}^{1,n-1} \tag{2.4}$$

is a conformal embedding whose image is said the *Minkowski-chamber* of $\mathcal{E}^{1,n-1}$. Let $\mathrm{P}^{\uparrow}_{+}(1,n-1) = \mathbb{M}^{1,n-1} \rtimes \mathrm{SO}^{\uparrow}_{+}(1,n-1)$ be the restricted Poincaré group of $\mathbb{M}^{1,n-1}$. For each $(\mathbf{p},\mathrm{L}) \in \mathrm{P}^{\uparrow}_{+}(1,n-1)$, let ${}^*\mathbf{p} := (-p_1, p_2, \dots, p_n)$. The matrix

$$\mathbf{B}(\mathbf{p},\mathrm{L}) = \begin{pmatrix} 1 & 0 & 0 \\ \mathbf{p} & \mathrm{L} & 0 \\ \frac{{}^*\mathbf{p}\mathbf{p}}{2} & {}^*\mathbf{p}\mathrm{L} & 1 \end{pmatrix}$$

belongs to $\mathrm{A}^{\uparrow}_{+}(2,n)$ and

$$\mathbf{J} : \mathrm{P}^{\uparrow}_{+}(1,n-1) \ni (\mathbf{p},\mathrm{L}) \mapsto \mathbf{B}(\mathbf{p},\mathrm{L})) \in \mathrm{A}^{\uparrow}_{+}(2,n) \tag{2.5}$$

is a faithful representation. Similarly, one can build conformal embeddings of de Sitter and Anti-de Sitter n-spaces into $\mathcal{E}^{1,n-1}$. Also Robertson–Walker n-spacetimes can be conformally realized as open submanifolds of the Einstein universe [19].

Definition 2. A $(2+h)$-dimensional subspace $\mathbb{W} \subset \mathbb{R}^{2,n}$, $h = 1, \dots, n-1$, is called *Lorentzian* if the restriction of $\langle\, ,\rangle$ to $\mathbb{W}$ is nondegenerate and of signature $(2,h)$. The set $\mathcal{E}(\mathbb{W})$ of all null rays belonging to $\mathbb{W}$ is a h-dimensional totally umbilical Lorentzian submanifold of $\mathcal{E}^{1,n-1}$. We call $\mathcal{E}(\mathbb{W})$ a *h-dimensional Lorentz cycle* of $\mathcal{E}^{1,n-1}$. The cycle $\mathcal{E}(\mathbb{W})$ endowed with the induced conformal structure is a conformal Lorentz manifold equivalent to a h-dimensional Einstein universe.

2.2. **Timelike curves.** Let $\gamma : I \subset \mathbb{R} \to \mathcal{E}^{1,n-1}$ be a parametrized timelike curve, $n \geq 3$. A *null lift* of γ is a map $\Gamma : I \to \mathbb{R}^{2,n}$, such that $\gamma(t) = [\Gamma(t)]$, for every $t \in I$. Since γ is a timelike immersion, a null lift Γ of γ satisfies

$$\Gamma \wedge \Gamma'_{|t} \neq 0, \quad \langle \Gamma_{|t}, \Gamma_{|t}\rangle = \langle \Gamma_{|t}, \Gamma'_{|t}\rangle = 0, \quad \langle \Gamma'_{|t}, \Gamma'_{|t}\rangle < 0,$$

for every $t \in I$. For each $k = 1, \dots, n+1$, the kth *osculating space* of γ at $\gamma(t)$ is the linear subspace $\mathcal{T}^k(\gamma)|_t \subset \mathbb{R}^{2,n}$ spanned by the vectors $\Gamma_{|t}$, $\Gamma'_{|t}$, $\dots$, $\Gamma^k{}_{|t}$. Clearly, this definition is independent of the choice of the null lift. The totality of osculating spaces defines the kth osculating sheaf

$$\mathcal{T}^k(\gamma) = \left\{(t, V) \in I \times \mathbb{R}^{2,n} \;\mid\; V \in \mathcal{T}^k(\gamma)|_t\right\}.$$

We have the following.

Lemma 1. *For each $t \in I$, $\mathcal{T}^2(\gamma)|_t$ is a 3-dimensional Lorentzian subspace of $\mathbb{R}^{2,n}$.*

Proof. Choose Γ so that $\langle \Gamma', \Gamma'\rangle = -1$. Differentiating $\langle \Gamma, \Gamma'\rangle = 0$ and $\langle \Gamma', \Gamma'\rangle = -1$, we get $\langle \Gamma, \Gamma''\rangle = 1$ and $\langle \Gamma', \Gamma''\rangle = 0$. This implies that $\Gamma_{|t}, \Gamma'_{|t}$ and $\Gamma''_{|t}$ are linearly independent, for each $t \in I$. Next, let

$$A_1 = \frac{1}{2}\Big(1 + \langle \Gamma'', \Gamma''\rangle\Big)\Gamma - \Gamma'', \quad A_2 = \Gamma' \quad A_3 = \frac{1}{2}\Big(1 - \langle \Gamma'', \Gamma''\rangle\Big)\Gamma + \Gamma''.$$

Then, $(A_{1|t}, A_{2|t}, A_{3|t})$ is a basis of $\mathcal{T}^2(\gamma)|_t$, such that

$$\begin{aligned} &\langle A_{i|t}, A_{j|t}\rangle = 0, \quad i \neq j, \\ &\langle A_{1|t}, A_{1|t}\rangle = \langle A_{2|t}, A_{2|t}\rangle = -\langle A_{3|t}, A_{3|t}\rangle = -1, \end{aligned}$$

as required. □

The above lemma implies that $\mathcal{T}^k(\gamma)|_t$ is a Lorentzian subspace of $\mathbb{R}^{2,n}$, for each $k \geq 2$, and for each $t \in I$. The Lorentzian cycle $\mathcal{E}(\mathcal{T}^k(\gamma)|_t)$ is denoted by $\mathcal{E}^k(\gamma)|_t$. We call $\mathcal{E}^k(\gamma)|_t$ the kth *osculating cycle* of γ at $\gamma(t)$. The orthogonal complement $\mathcal{N}^k(\gamma)|_t$ of $\mathcal{T}^k(\gamma)|_t$ is a spacelike subspace such that $\mathbb{R}^{2,n} = \mathcal{T}^k(\gamma)|_t \oplus \mathcal{N}^k(\gamma)|_t$. We call $\mathcal{N}^k(\gamma)|_t$ the kth *normal space* at $\gamma(t)$. The totality of kth normal spaces defines the kth *normal sheaf*

$$\mathcal{N}^k(\gamma) = \left\{(t, V) \in I \times \mathbb{R}^{2,n} \;\mid\; V \in \mathcal{N}^k(\gamma)|_t\right\}.$$

Let $\mathrm{pr}_{(k)}|_t$ denote the orthogonal projection of $\mathbb{R}^{n+2}$ onto $\mathcal{N}^k(\gamma)|_t$.

Definition 3. The *conformal strain* of a timelike curve γ is the quartic differential

$$\mathcal{Q}_\gamma = \frac{\langle \mathrm{pr}_{(2)}(\Gamma'''), \mathrm{pr}_{(2)}(\Gamma''')\rangle}{|\langle \Gamma', \Gamma'\rangle|} dt^4. \tag{2.6}$$

The conformal strain is independent of the choice of the null lift Γ. In addition, if γ and $\widetilde{\gamma}$ are two *equivalent timelike curves*,[2] then $\mathcal{Q}_{\widetilde{\gamma}} = h^*(\mathcal{Q}_\gamma)$.

Definition 4. If $\mathcal{Q}(\gamma)|_t = 0$, the point $\gamma(t)$ is said to be a *conformal vertex*. A timelike curve without conformal vertices is said to be *generic*. If $\mathcal{Q}_\gamma = 0$, the curve is said to be *totally degenerate*.

Remark 3. If γ is totally degenerate, then $\mathcal{E}^1(\gamma)|_t$ is constant and the trajectory of γ is contained in a 1-dimensional conformal cycle.

Following [8], it can be shown that $\mathcal{E}^1(\gamma)|_t$ has second order analytic contact with γ at $\gamma(t)$. Moreover, $\gamma(t)$ is a conformal vertex if and only if the order of contact is strictly greater than 2. This underlines the fact that the conformal strain measures the infinitesimal distortion of the curve from its osculating cycle.

Definition 5. If γ is generic, there exists a unique null lift Γ, referred to as the *canonical lift*, such that $\mathcal{Q}_\gamma = \langle \Gamma', \Gamma' \rangle^2 dt^4$. The smooth positive function $\upsilon_\gamma = |\langle \Gamma', \Gamma' \rangle|^{1/2}$ is called the *conformal strain density* and the exterior differential 1-form $\sigma_\gamma = \upsilon_\gamma dt$ is called the *infinitesimal conformal strain* (or *conformal arc element*) of γ. By construction, σ_γ is invariant under the action of the restricted conformal group and orientation preserving changes of parameter. In particular, any generic timelike curve can be parametrized is such a way that $\upsilon_\gamma = 1$. In this case, we say that the curve is *parametrized by conformal parameter*, which will be usually denoted by u. For a smooth map $f : I \to \mathbb{R}^h$, by $\dot{f} = \upsilon_\gamma^{-1} f'$ we define the *derivative of f with respect to the conformal arc element.*

Remark 4. The 1-form σ_γ is the Lorentzian analogue of the conformal arc element of a curve in the 3-dimensional round sphere [23, 26, 28, 30] and generalizes the analogue notion for a generic timelike curve in the $(1+2)$-Einstein universe [8].

Let γ be a generic timelike curve and let Γ be its canonical lift. Then, for every t, $\dim(\mathcal{T}^3(\gamma)|_t) = 4$, and hence $\mathcal{T}^3(\gamma)$ is a vector bundle. Furthermore, the cross sections

$$M_0 = \Gamma, \quad M_1 = \frac{1}{|\langle \Gamma', \Gamma' \rangle|^{1/2}} \Gamma', \quad M_2 = \frac{\mathrm{pr}_{(2)}(\Gamma''')}{\langle \mathrm{pr}_{(2)}(\Gamma'''), \mathrm{pr}_{(2)}(\Gamma''') \rangle^{1/2}}, \tag{2.7}$$

and

$$M_{n+1} = -\frac{1}{\langle \Gamma, \Gamma'' \rangle} \Gamma'' + \frac{\langle \Gamma', \Gamma'' \rangle}{\langle \Gamma, \Gamma'' \rangle} \Gamma' + \frac{1}{2} \Big(\frac{\langle \Gamma'', \Gamma'' \rangle}{\langle \Gamma, \Gamma'' \rangle^2} - \frac{\langle \Gamma'', \Gamma' \rangle^2}{\langle \Gamma, \Gamma'' \rangle^2} \Big) \Gamma \tag{2.8}$$

give rise to a *canonical trivialization* (M_0, M_1, M_2, M_{n+1}) of $\mathcal{T}^3(\gamma)$, such that

$$\begin{aligned} &\langle M_0, M_0 \rangle = \langle M_{n+1}, M_{n+1} \rangle = \langle M_0, M_1 \rangle = \langle M_0, M_2 \rangle = 0, \\ &\langle M_1, M_{n+1} \rangle = \langle M_2, M_{n+1} \rangle = \langle M_1, M_2 \rangle = 0, \\ &\langle M_0, M_{n+1} \rangle = \langle M_1, M_1 \rangle = -\langle M_2, M_2 \rangle = -1. \end{aligned} \tag{2.9}$$

In particular, $M_{0|_t} \wedge (M_{1|_t} + M_{2|_t})$ is an isotropic bi-vector, for every $t \in I$. If we revert the orientation along the curve, this bi-vector changes sign.

This implies that any generic timelike curve has an *intrinsic orientation*, defined by $M_{0|_t} \wedge (M_{1|_t} + M_{2|_t}) \in \mathcal{C}_+$, for every $t \in I$.

[2] I.e., $\widetilde{\gamma} = B \cdot (\gamma \circ h)$, where $B \in \mathrm{A}_+^\uparrow(1, n+1)$ and h is a change of parameter.

Assumption 2. From now on, we implicitly assume that all timelike curves are generic and equipped with their intrinsic orientations. For any such a curve γ, we denote by Γ its canonical null lift.

Definition 6. The scalar product (2.1) induces a metric structure on the vector bundle $\mathcal{N}^3(\gamma)$. The *covariant derivative* of a cross section $V : I \to \mathcal{N}^3(\gamma)$, with respect to the conformal arc element, is defined by

$$D(V) = \mathrm{pr}_{(3)}(\dot{V}) : I \to \mathcal{N}^3(\gamma). \tag{2.10}$$

The normal bundles $\mathcal{N}^2(\gamma)$ and $\mathcal{N}^3(\gamma)$ possess two *canonical cross sections*, denoted by W_γ and S_γ, respectively. The cross section S_γ is defined by

$$S_\gamma = \mathrm{pr}_{(3)}(\dot{M}_2). \tag{2.11}$$

Next, let

$$h_1 = \langle \dot{M}_1, M_{n+1} \rangle, \quad h_2 = \sqrt{\langle S_\gamma, S_\gamma \rangle}. \tag{2.12}$$

We define W_γ by

$$W_\gamma := (\dot{h}_1 - 3h_2\dot{h}_2)M_2 - (h_2^2 - 2h_1)S_\gamma + D^2(S_\gamma). \tag{2.13}$$

Note that

$$\begin{aligned}
\mathcal{T}^4(\gamma)|_t &= \mathrm{span}(M_0|_t, M_1|_t, M_2|_t, S_\gamma|_t, M_{n+1}|_t),\\
\mathcal{T}^5(\gamma)|_t &= \mathrm{span}(M_0|_t, M_1|_t, M_2|_t, S_\gamma|_t, D(S_\gamma)|_t, M_{n+1}|_t),
\end{aligned}$$

for every $t \in I$.

2.3. **First and second order frame fields.** A *first order frame field* along γ is a smooth map

$$A = (A_0, \dots, A_{n+1}) : I \to \mathrm{A}_+^\uparrow(2, n),$$

such that A_0 is a null lift and $A_0' \in \mathrm{span}(A_0, A_1)$. First order frame fields do exist along any timelike curve. If A is a first order frame field, then any other is given by

$$\widetilde{A} = A\, X(r, x, y, R),$$

where

$$r,\, x : I \to \mathbb{R},\ x > 0, \quad y : I \to \mathbb{R}^{n-1}, \quad R : I \to \mathrm{SO}(n-1)$$

are smooth functions and

$$X(r, x, y, R) = \begin{pmatrix} r & -x & {}^t yR & \frac{{}^t yy - x^2}{2} \\ 0 & 1 & 0 & x/r \\ 0 & 0 & R & y/r \\ 0 & 0 & 0 & r^{-1} \end{pmatrix}. \tag{2.14}$$

Definition 7. A first order frame field is said to be of *second order* if

$$A_0 = M_0, \quad A_1 = M_1, \quad A_2 = M_2, \quad A_{n+1} = M_{n+1}.$$

Second order frame fields do exist along any generic, timelike curve with its intrinsic orientation. Note that $(A_3, \dots, A_n)$ is a trivialization of the third-order normal bundle of γ.

If A is a second order frame field, then

$$A' = A \begin{pmatrix} 0 & -h_1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & h_1 \\ 0 & 0 & 0 & -{}^t s & 1 \\ 0 & 0 & s & \phi & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix} \upsilon_\gamma, \tag{2.15}$$

where $\phi = (\phi^j_i)_{i,j=3,\dots,n} : I \to \mathfrak{o}(n-2)$ is a smooth map, h_1 is as in (2.12), and $s = {}^t(s^3, \dots, s^n)$ is defined by

$$S_\gamma = \sum_{j=3}^n s^j A_j.$$

If $V = \sum_{j=3}^n V^j A_j$ is a cross-section of $\mathcal{N}^3(\gamma)$, then

$$DV = \sum_{j=3}^n \Big(\dot{V}^j + \sum_{i=3}^n \phi^j_i V^i \Big) A_j.$$

This implies that

$$DS_\gamma = \sum_{j=3}^n s^j_{(1)} A_j, \quad D^2(S_\gamma) = \sum_{j=3}^n s^j_{(2)} A_j, \tag{2.16}$$

and

$$W_\gamma = \big(\dot{h}_1 - 3\, {}^t\! s\, \dot{s}\big) A_2 + \sum_{j=3}^n \big(s^j_{(2)} + (2h_1 - {}^t\! s\, s) s^j\big) A_j, \tag{2.17}$$

where $s_{(1)}$, $s_{(2)} : I \to \mathbb{R}^{n-3}$ are defined by

$$s^j_{(1)} := \dot{s}^j + \sum_{i=3}^n \phi^j_i s^i, \quad s^j_{(2)} := \dot{s}^j_{(1)} + \sum_{i=3}^n \phi^j_i s^i_{(1)}.$$

2.4. **The strain functional and the conformal worldlines.** For a generic timelike curve γ, we have seen that there is a canonical arc element σ_γ on I. If $K \subset I$ is a closed interval in I, we can define the *total strain functional*

$$\mathcal{S}_K(\gamma) = \int_K \sigma_\gamma$$

on the space of smooth, generic, timelike immersions of I into $\mathcal{E}^{1,n-1}$.

Definition 8. We say that a generic timelike curve γ is a *conformal timelike geodesic*, or a *conformal worldline*, if for any closed interval $K \subset I$ and for any smooth variation

$$\mathbf{g} : I \times (-\epsilon, \epsilon) \ni (t, \tau) \mapsto \gamma_\tau(t) \in \mathcal{E}^{1,n-1},$$

with $\gamma_0 = \gamma$ and $\mathrm{supp}(\mathbf{g}) \subset K$,[3]

$$\frac{d}{d\tau} \left(\mathcal{S}_K(\gamma_\tau)\right)|_{\tau=0} = 0.$$

[3] $\mathrm{supp}(\mathbf{g})$ denotes the support of the variation, i.e., the closure of the set of all $t \in I$ such that $\gamma_\tau(t) \neq \gamma(t)$, for some $\tau \in (-\epsilon, \epsilon)$.

3. The variational equations

The purpose of this section is to prove the following.

Theorem A. *A generic timelike curve γ equipped with its intrinsic orientation is a conformal timelike geodesic if and only if W_γ (cf. (2.13)) vanishes identically.*

Proof. First, we prove that a generic timelike curve with $W_\gamma = 0$ is a conformal worldline. Without loss of generality, we may assume that γ is parametrized by conformal parameter. By (2.13), it follows that $W_\gamma = 0$ if and only if

$$\dot{h}_1 = 3h_2\dot{h}_2, \quad D^2(S_\gamma) = (h_2^2 - 2h_1)S_\gamma. \tag{3.1}$$

Let $A : I \to \mathrm{A}_+^\uparrow(2,n)$ be a fixed second order frame field and

$$\mathbf{g} : I \times (-\epsilon,\epsilon) \ni (u,\tau) \mapsto \gamma_\tau(u) \in \mathcal{E}^{1,n-1}$$

be a compactly supported variation of γ, such that $\mathrm{supp}(\mathbf{g}) \subseteq K$. By possibly shrinking the interval $(-\epsilon,\epsilon)$, we may assume that, for every $\tau \in (-\epsilon,\epsilon)$, γ_τ is generic and equipped with its intrinsic orientation. Then, there is a differentiable map

$$\mathcal{A} : I \times (-\epsilon,\epsilon) \ni (u,\tau) \mapsto \mathcal{A}_\tau(u) \in \mathrm{A}_+^\uparrow(2,n),$$

such that $\mathcal{A}_0 = A$ and $\mathcal{A}_\tau$ is a second order frame field along γ_τ, for every $\tau \in (-\epsilon,\epsilon)$. This implies that there exist smooth maps $Q, \Lambda : I \times (-\epsilon,\epsilon) \to \mathfrak{a}(2,n)$ such that

$$\mathcal{A}^{-1}d\mathcal{A} = Q du + \Lambda d\tau,$$

where

$$Q = \begin{pmatrix} 0 & -m & v & 0 & 0 \\ v & 0 & 0 & 0 & m \\ 0 & 0 & 0 & -{}^t p & v \\ 0 & 0 & p & \psi & 0 \\ 0 & -v & 0 & 0 & 0 \end{pmatrix},$$

and

$$\Lambda = \begin{pmatrix} \lambda_0^0 & -\lambda_{n+1}^1 & \lambda_{n+1}^2 & {}^t\lambda_{n+1} & 0 \\ \lambda_0^1 & 0 & \lambda_1^2 & {}^t\lambda_1 & \lambda_{n+1}^1 \\ \lambda_0^2 & \lambda_1^2 & 0 & -{}^t\lambda_2 & \lambda_{n+1}^2 \\ \lambda_0 & \lambda_1 & \lambda_2 & \lambda & \lambda_{n+1} \\ 0 & -\lambda_0^1 & \lambda_0^2 & {}^t\lambda_0 & -\lambda_0^0 \end{pmatrix},$$

for smooth maps

$$\begin{aligned} &v,\, m,\, \lambda_0^0,\, \lambda_0^1,\, \lambda_0^2,\, \lambda_{n+1}^1,\, \lambda_{n+1}^2 : I \times (-\epsilon,\epsilon) \to \mathbb{R}, \\ &p,\, \lambda_0,\, \lambda_1,\, \lambda_2,\, \lambda_{n+1} : I \times (-\epsilon,\epsilon) \to \mathbb{R}^{n-2}, \\ &\psi,\, \lambda : I \times (-\epsilon,\epsilon) \to \mathfrak{o}(n-2). \end{aligned}$$

Let the restrictions to $I \cong I \times \{0\}$ of the above maps be denoted by the same symbols with an over bar. Note that $\mathrm{supp}(\overline{\Lambda}) \subseteq K$. The cross section

$$V_{\mathbf{g}} = \sum_{a=2}^{n} \overline{\lambda_0^a} A_a : I \to \mathcal{N}^2(\gamma)$$

is independent of the choice of A and $\mathcal{A}$. We call $V_{\mathbf{g}}$ the *infinitesimal variation* of $\mathbf{g}$. From $\dot{A} = A\,\overline{Q}$ and the identity (2.15), we obtain $\overline{\upsilon} = 1$, $\overline{m} = h_1$, $\overline{p} = s$ and $\overline{\psi} = \phi$. From the Maurer–Cartan equations, it follows that

$$\partial_u \Lambda - \partial_\tau Q = \Lambda\, Q - Q\, \Lambda =: [\Lambda, Q]. \tag{3.2}$$

Using (3.2), we compute[4]

$$\begin{array}{rcl}
(\partial_\tau \upsilon)\,|_I & = & \overline{\lambda^0_0} + \dot{\overline{\lambda^1_0}}, \\
\overline{\lambda^0_0} & = & \dfrac{1}{2}\left(\dot{\overline{\lambda^0_2}} - \dot{\overline{\lambda^1_0}} - \overline{\lambda_{n+1}} \cdot s - h_1 \overline{\lambda^2_1}\right), \\
\dot{\overline{\lambda_0}} & = & \overline{\lambda_1} - \overline{\lambda^2_0}\, s - \phi\, \overline{\lambda_0}, \\
\dot{\overline{\lambda_1}} & = & -h_1 \overline{\lambda_0} - \overline{\lambda_{n+1}} - \overline{\lambda^2_1}\, s - \phi\, \overline{\lambda_1}, \\
\dot{\overline{\lambda_{n+1}}} & = & h_1 \overline{\lambda_1} + \overline{\lambda_2} - \overline{\lambda^0_2}\, s - \phi\, \overline{\lambda_{n+1}}, \\
\dot{\overline{\lambda^2_0}} & = & \overline{\lambda^2_1} + s \cdot \overline{\lambda_0}.
\end{array}$$

Integrating by parts, taking into account that $\operatorname{supp}\big(\overline{\lambda^i_j}\big) \subset K$, $i, j = 0, \dots, n+1$, and that ϕ is skew-symmetric, it follows from the above identities that

$$\begin{aligned}
&\frac{d}{d\tau}\left(\mathcal{S}_K(\gamma_\tau)\right)|_{\tau=0} = \int_K (\partial_\tau \upsilon|_I)\, du = \int_K \big(\overline{\lambda^0_0} + \dot{\overline{\lambda^1_0}}\big)\, du = \int_K \overline{\lambda^0_0}\, du \\
&= -\frac{1}{2}\int_K \Big(s \cdot (-\dot{\overline{\lambda_1}} - h_1 \overline{\lambda_0} - \overline{\lambda^2_1} s - \phi\, \overline{\lambda_1}) + h_1 \big(\dot{\overline{\lambda^2_0}} - s \cdot \overline{\lambda_0}\big)\Big) du \\
&= \frac{1}{2}\int_K \Big(- \dot{s} \cdot \overline{\lambda_1} + h_1 s \cdot \overline{\lambda_0} + (s \cdot s)\overline{\lambda^2_1} + s \cdot \phi\, \overline{\lambda_1} + \dot{h}_1 \overline{\lambda^2_0} + h_1 s \cdot \overline{\lambda_0}\Big) du \\
&= -\frac{1}{2}\int_K \dot{s} \cdot (\dot{\overline{\lambda_0}} + \overline{\lambda^2_0} s + \phi\, \overline{\lambda_0})\, du + \frac{1}{2}\int_K s \cdot \phi\, (\dot{\overline{\lambda_0}} + \overline{\lambda^2_0} s + \phi\, \overline{\lambda_0})\, du \\
&\quad + \frac{1}{2}\int_K \Big(h_1 s \cdot \overline{\lambda_0} + ({}^t s\, s)(\dot{\overline{\lambda^2_0}} - s \cdot \overline{\lambda_0}) + \dot{h}_1 \overline{\lambda^2_0} + h_1 s \cdot \overline{\lambda_0}\Big) du \\
&= \frac{1}{2}\int_K \Big(\overline{\lambda^2_0}\,(\dot{h}_1 - 3\, {}^t s\, \dot{s}) + \overline{\lambda_0} \cdot \big(\dot{s}_{(1)} + \phi\, s_{(1)} - ({}^t s\, s - 2h_1)s\big)\Big) du \\
&= \frac{1}{2}\int_K \Big(\overline{\lambda^2_0}\,(\dot{h}_1 - 3\, {}^t s\, \dot{s}) + \overline{\lambda_0} \cdot \big(s_{(2)} + (2h_1 - {}^t s\, s)s\big)\Big)\, du \\
&= \frac{1}{2}\int_K \langle V_{\mathbf{g}}, W_\gamma \rangle du.
\end{aligned}$$

This implies the result.

Next, we show that, for each $u_0 \in I$, there exists an open interval $J \subset I$ containing u_0, such that, for every smooth function $\rho : I \to \mathbb{R}$ with compact support $K \subset J$ and every $j = 2, \dots, n$, there exists a compactly supported variation $\mathbf{g}$, such that $V_{\mathbf{g}} = \rho A_j$. This clearly implies that a conformal worldline satisfies $W_\gamma = 0$.

Using the conformal invariance of the functional, without loss of generality, we may suppose that $\gamma(u)$ belongs to the Minkowski chamber, for every u lying in an open interval $J \subset I$ containing u_0. Then, $\gamma|_J = \mathbf{j} \circ \alpha$, where $\alpha : J \to \mathbb{M}^{1,n-1}$ is a timelike curve of Minkowski space. Let $\mathbf{t} : J \to \mathbb{M}^{1,n-1}$ be the future-directed

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[4] For $v, w \in \mathbb{R}^{n-2}$, thought of as column vectors, $v \cdot w = {}^t v w$ denotes the usual dot product.

timelike unit tangent vector along α and let

$$N(\alpha) = \{(u, \mathbf{v}) \in J \times \mathbb{M}^{1,n-1} \mid {}^*\mathbf{v}\,\mathbf{t}_{|u} = 0\}$$

be the normal bundle of α, equipped with the metric covariant derivative $\nabla(\mathbf{v}) := \mathrm{pr}(\dot{\mathbf{v}})$.[5] Let $(\mathbf{b}_2, \dots, \mathbf{b}_n)$ be a flat orthogonal trivialization of $N(\alpha)$, such that $(\mathbf{t}, \mathbf{b}_2, \dots, \mathbf{b}_n)$ is positive-oriented. Then $(\alpha, \mathbf{t}, \mathbf{b}_2, \dots, \mathbf{b}_n) : J \to P_+^\uparrow(1, n-1)$ is a lift of α to the restricted Poincaré group,[6] and

$$F = \mathbf{J} \circ (\alpha, \mathbf{t}, \mathbf{b}_2, \dots, \mathbf{b}_n) : J \to \mathrm{A}_+^\uparrow(2, n)$$

is a first order frame field along $\gamma|_J$. Let $A : J \to \mathrm{A}_+^\uparrow(2, n)$ be a second order frame field along γ. Then $A = F\,X(r, x, y, R)$, where

$$x,\, r : J \to \mathbb{R}, r > 0, \quad y : J \to \mathbb{R}^{n-1}, \quad R = (R^i_j)_{i,j=2,\dots,n} : J \to \mathrm{SO}(n-1)$$

are smooth maps and $X(r, x, y, R)$ is as in (2.14). Now, let

$$m = {}^t(m^2, \dots, m^n) = \frac{\rho}{r}\,{}^t(R^2_j, \dots, R^n_j).$$

It is now an easy matter to check that the variation

$$\begin{aligned} &\gamma_\tau(u) = \mathbf{j} \circ (\alpha|_u + \tau \sum_{j=2}^{n} m^j(u)\mathbf{b}_j|_u), \quad (u, \tau) \in J \times (-\epsilon, \epsilon), \\ &\gamma_\tau(u) = \gamma(u), \quad (u, \tau) \in (I \setminus J) \times (-\epsilon, \epsilon), \end{aligned}$$

satisfies the required properties. This concludes the proof of the Theorem A. □

The next theorem shows that the conformal timelike geodesics (conformal worldlines) lie in some Einstein universe of dimension 2, 3, or 4.

Theorem B. *The trajectory of a conformal timelike geodesic is contained in an m-dimensional Lorentzian cycle, where m can be* 2, 3, *or* 4.

Proof. Let γ be a conformal worldline parametrized by conformal parameter. Let $A = (A_0, \dots, A_{n+1})$ be a second order frame field along γ. Then

$$\mathcal{T}^5(\gamma)|_u = \mathrm{span}(A_0|_u, A_1|_u, A_2|_u, S_\gamma|_u, DS_\gamma|_u, A_{n+1}|_u),$$

for every $u \in I$. From (2.15), we have

$$\begin{aligned} &\dot{A}_0 = A_1, \quad \dot{A}_1 = -h_1 A_0 - A_{n+1}, \\ &\dot{A}_2 = A_0 + S_\gamma, \quad \dot{A}_{n+1} = h_1 A_1 + A_2, \end{aligned} \tag{3.3}$$

and

$$\dot{A}_j = -s^j A_2 + \sum_{k=3}^{n} \phi^k_j A_k, \quad j = 3, \dots, n. \tag{3.4}$$

The first equation in (3.1) yields

$$(h_2)^2 = \frac{2}{3}h_1 + c_1, \quad c_1 \in \mathbb{R}. \tag{3.5}$$

The remaining equations in (3.1) can be written as

$$\dot{s} = -\phi\, s + s_{(1)}, \quad \dot{s}_{(1)} = -(\frac{4}{3}h_1 - c_1)s - \phi\, s_{(1)} \tag{3.6}$$

[5]Here, $\mathrm{pr}|_u$ denotes the orthogonal projection of $\mathbb{M}^{1,n-1}$ onto $N(\alpha)|_u$.

[6]$\mathbf{J}$ is the faithful representation of $P_+^\uparrow(1, n)$ into the conformal group.

or, equivalently, in the form

$$(3.7)\qquad \begin{cases} \frac{d}{du}(S_\gamma) = DS_\gamma - (\frac{2}{3}h_1 + c_1)A_2, \\ \frac{d}{du}(DS|_\gamma) = -\Big(\frac{4}{3}h_1 - c_1\Big)S_\gamma - \frac{1}{3}\dot{h}_1 A_2. \end{cases}$$

From (3.6), it follows that $(s, s_{(1)})$ is a solution of the linear system

$$\frac{d}{du}(s \wedge s_{(1)}) = -(\phi\, s) \wedge s_{(1)} - s \wedge (\phi\, s_{(1)}).$$

Hence, two possibilities may occur:

- Case I : $s|_u \wedge s_{(1)}|_u \neq 0$, for every $u \in I$;
- Case II : $s \wedge s_{(1)}$ vanishes identically.

Case I. The vectors fields S_γ and DS_γ are everywhere linearly independent. Then, the osculating spaces $\mathcal{T}^5(\gamma)|_u$ are 6-dimensional. Using (3.3) and (3.7), we obtain

$$\frac{d}{du}(A_0 \wedge A_1 \wedge A_2 \wedge S_\gamma \wedge DS_\gamma \wedge A) = 0.$$

Hence, $\mathcal{T}^5(\gamma)|_u$ coincides with a fixed 6-dimensional Lorentzian subspace $\mathbb{W}_\gamma \subseteq \mathbb{R}^{2,n}$, for each $u \in I$. This implies that the trajectory of γ is contained in the 4-dimensional Lorentzian cycle $\mathcal{E}(\mathbb{W}_\gamma)$.

Case II. Since $(s, s_{(1)})$ is a solution of the linear system (3.6), there are two possibilities:

- Case II.1 : $s = s_{(1)} = 0$;
- Case II.2 : ${}^t s\, s + {}^t s_{(1)}\, s_{(1)} > 0$.

Case II.1. If $s = \dot{s} = 0$, then (3.3) implies

$$\frac{d}{du}(A_0 \wedge A_1 \wedge A_2 \wedge A_{n+1}) = 0.$$

Therefore, $\mathcal{T}^3(\gamma)|_u$ coincides with a fixed 4-dimensional Lorentzian subspace $\mathbb{W}_\gamma \subseteq \mathbb{R}^{2,n}$, for every $u \in I$. Hence, the trajectory of γ is contained in the 2-dimensional Lorentzian cycle $\mathcal{E}(\mathbb{W}_\gamma)$.

Case II.2 If ${}^t s\, s + {}^t s_{(1)}\, s_{(1)} > 0$ and $s \wedge s_{(1)} = 0$, then $\mathrm{span}(S_\gamma|_u, DS_\gamma|_u)$ is a 1-dimensional spacelike subspace, for every $u \in I$. Thus

$$\mathcal{P} = \big\{(u, V) \in I \times \mathbb{R}^{2,n} \ \mid \ V \in \mathrm{span}(S_\gamma|_u, DS_\gamma|_u)\big\}$$

is a rank 1 vector bundle. Let P be a unit length cross section of $\mathcal{P}$. The identity $s \wedge s_{(1)} = 0$ implies that S_γ and DS_γ are both proportional to P. Let $\widetilde{I} = I \setminus I_*$, where I_* is the discrete set $\{u \in I \mid S_\gamma|_u = 0\}$. On $\widetilde{I}$, the 5th and 4th order osculating bundles are spanned by $A_0, A_1, A_2, P, A_{n+1}$. This implies the existence of a smooth function $f : \widetilde{I} \to \mathbb{R}$, such that

$$\dot{P}|_{\widetilde{I}} = f P|_{\widetilde{I}}, \quad \mathrm{mod}(A_0, A_1, A_2, A_{n+1}),$$

Taking into account the previous identity, from (3.3) we get

$$(3.8)\qquad \frac{d}{du}(A_0 \wedge A_1 \wedge A_2 \wedge P \wedge A_{n+1})|_{\widetilde{I}} = f(A_0 \wedge A_1 \wedge A_2 \wedge P \wedge A_{n+1})|_{\widetilde{I}}.$$

Let $\mathrm{Gr}_5(\mathbb{R}^{n+2})$ be the Grassmannian of 5-dimensional subspaces of $\mathbb{R}^{n+2}$. From (3.8), it follows that the map

$$I \ni u \mapsto \mathrm{span}(A_0|_u \wedge A_1|_u \wedge A_2|_u \wedge P|_u \wedge A_{n+1}|_u) \in \mathrm{Gr}_5(\mathbb{R}^{n+2})$$

is constant on $\widetilde{I}$, and hence on I by continuity. Then, the 4th order osculating spaces coincide with a fixed 5-dimensional Lorentzian subspace $\mathbb{W}_\gamma \subseteq \mathbb{R}^{2,n}$. Hence, the trajectory of γ is contained in the 3-dimensional Lorentzian cycle $\mathcal{E}(\mathbb{W}_\gamma)$. □

By Theorem B, if γ is a conformal worldline, then three possibilities may occur:

- γ is trapped in a 4-dimensional Einstein universe and there are no 3-dimensional Lorentzian cycles containing the trajectory of γ. In this case, we say that γ is a *linearly full conformal worldline*.
- the trajectory is trapped in a 3-dimensional Lorentzian cycle but does not lie in any 2-dimensional Lorentzian cycle.
- the trajectory is trapped in a 2-dimensional Lorentzian cycle.

4. Linearly full conformal worldlines

In this and in the next section, we will focus on the linearly full conformal worldlines of a 4-dimensional Einstein universe $\mathcal{E}^{1,3}$. We suppose that the curves are parametrized by conformal parameter and oriented by the intrinsic orientation.

4.1. **Canonical frames and conformal curvatures.**

Proposition 3. *Let $\gamma : I \to \mathcal{E}^{1,3}$ be a conformal worldline. Then, there exists a unique second order conformal frame $B : I \to A^{\uparrow}_{+}(2,3)$, such that*

$$\dot{B} = B\,\mathcal{K}, \tag{4.1}$$

with

$$\mathcal{K} = \begin{pmatrix} 0 & -k_1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & k_1 \\ 0 & 0 & 0 & -k_2 & 0 & 1 \\ 0 & 0 & k_2 & 0 & -k_3 & 0 \\ 0 & 0 & 0 & k_3 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.2}$$

where k_1, k_2, $k_3 : I \to \mathbb{R}$ are smooth functions with $k_2 > 0$ and $k_3 \neq 0$. In addition, if k_1, k_2 and k_3 are not constant, then there exist c_1, c_2, $c_3 \in \mathbb{R}$, $c_3 \neq 0$, such that

$$k_1 = \frac{3}{2}k_2^2 + c_1, \quad k_3 = c_3 k_2^{-2}, \quad \dot{k}_2^2 + k_2^4 + c_3^2 k_2^{-2} + 2c_1 k_2^2 + c_2 = 0. \tag{4.3}$$

Proof. Let (M_0, M_1, M_2, M_5) be the canonical trivialization of $\mathcal{T}^3(\gamma)$ and $S_\gamma : I \to \mathcal{N}^3(\gamma)$ be the cross section defined in (2.11). Let

$$B_3 = S_\gamma / \sqrt{\langle S_\gamma, S_\gamma \rangle}$$

and consider the unique unit cross section $B_4 : I \to \mathcal{N}^4(\gamma)$, such that

$$\det(M_0, M_1, M_2, M_3, B_4, M_5) > 0.$$

If we let

$$B_0 = M_0, \quad B_1 = M_1, \quad B_2 = M_2, \quad B_5 = M_5,$$

the map $B = (B_0, \ldots, B_5)$ is a second order frame field along γ. In view of (2.15), we have

$$\dot{B}_0 = B_1, \quad \dot{B}_1 = -k_1 B_0 - B_5, \quad \dot{B}_5 = k_1 B_1 + B_2, \tag{4.4}$$

where $k_1 = h_1$.[7] Since D is a metric covariant derivative, DB_3 is a multiple of B_4, and hence $DB_3 = k_3B_4$, for some nonzero function k_3. From (2.15), we have

$$\dot{B}_2 = B_0 + k_2B_3, \tag{4.5}$$

where $k_2 = \sqrt{\langle S_\gamma, S_\gamma\rangle} > 0$. Taking into account (3.7), we then obtain

$$\dot{B}_3 = -k_2B_2 + k_3B_4. \tag{4.6}$$

Moreover, since B takes values in $\mathrm{A}^{\uparrow}_{+}(2,4)$, using (4.4)–(4.6), we obtain

$$\dot{B}_4 = -k_3B_3. \tag{4.7}$$

Combining (4.4)–(4.7), it follows that B satisfies (4.1), as required.

Next, suppose that the functions k_1, k_2 and k_3 are not constant. The first and second covariant derivatives of S_γ can be written as

$$DS_\gamma = \dot{k}_2B_3 + k_2k_3B_4, \quad D^2S_\gamma = (\ddot{k}_2 - k_2k_3^2)B_3 + (2\dot{k}_2k_3 + k_3\dot{k}_2)B_4.$$

Thus, from the variational equation $W_\gamma = 0$, it follows that

$$\ddot{k}_2 = k_2(k_3^2 + k_2^2 - 2k_1), \quad k_2\dot{k}_3 + 2k_3\dot{k}_2 = 0, \quad \dot{k}_1 = 3k_2\dot{k}_2. \tag{4.8}$$

The second and third equations in (4.8) imply the existence of two constants c_1 and $c_3 \neq 0$, such that

$$k_1 = \frac{3}{2}k_2^2 + c_1, \quad k_3 = c_3k_2^{-2}. \tag{4.9}$$

Substituting (4.9) into (4.8), we find

$$\ddot{k}_2 = -2k_2^3 + c_3^2k^{-3} - 4k_2 - 2c_1k_2,$$

which implies the existence of a constant c_2, such that

$$\dot{k}_2^2 + k_2^4 + c_3^2k_2^{-2} + 2c_1k_2^2 + c_2 = 0.$$

□

Definition 9. We call B, introduced in the proof of Proposition 3, the *canonical conformal frame*. The functions k_1, k_2, k_3 are called the *conformal curvatures*, while the constants c_1, c_2, c_3 are referred to as the *characters* of the worldline.

Remark 5. Conversely, if k_1, k_2, $k_3 : I \to \mathbb{R}$ are smooth functions satisfying (4.3), by solving the linear system (4.1) with initial condition $\mathrm{B}(u_0) \in \mathrm{A}^{\uparrow}_{+}(2,4)$, we get a smooth map $\mathrm{B} : I \to \mathrm{A}^{\uparrow}_{+}(2,4)$. The curve $\gamma = [B_0]$ is a linearly full conformal worldline with conformal curvatures k_1, k_2, k_3 and canonical conformal frame B. Any other worldline $\widetilde{\gamma}$ with the same curvatures is congruent to γ, with respect to the restricted conformal group, i.e., there exists $\mathbf{X} \in \mathrm{A}^{\uparrow}_{+}(2,4)$, such that $\widetilde{\gamma} = \mathbf{X}\cdot\gamma$.

Remark 6. The sign ambiguity of the third curvature can be removed by the following argument. If k_3 is the third conformal curvature of γ and if $\mathbf{X}$ is an orientation–preserving and time-reversing conformal transformation, then $\mathbf{X}\cdot\gamma(-u)$ is a conformal worldline with conformal curvatures $k_1(-u)$, $k_2(-u)$ and $-k_3(-u)$. Therefore, up to a time-reversing conformal transformation, k_3 can be considered positive.

Definition 10. The sign of the third curvature is called the *conformal helicity* of the worldline. By the previous remark, it is not restrictive to consider conformal worldlines with positive helicity. From now on, positive helicity is assumed.

[7] We refer to (2.12) for the definition of h_1.

Remark 7. If k_1, k_2 and k_3 are constant, the variational equations imply $k_1 = (k_2^2 + k_3^2)/2$. In this case $\mathcal{K}$ (cf. (4.2)) is a fixed element of the Lie algebra $\mathfrak{a}(2,4)$ and γ is congruent to the orbit through $[{}^t(1,0,0,0,0)]$ of the 1-parameter group of conformal transformations $\mathbb{R} \ni u \mapsto \mathrm{Exp}(u\mathcal{K}) \in A_+^{\uparrow}(2,4)$. Thus, the determination of the conformal worldlines with constant curvatures is reduced to the computation of the exponentials of the matrices $u\mathcal{K}$. From a computational point of view, this requires a detailed analysis of possible orbit types of the infinitesimal generator $\mathcal{K}$.

4.2. **Conformal curvatures in terms of Jacobi's elliptic functions.** In the following, we suppose that the conformal curvatures are not constant.

Lemma 4. *If c_1, c_2, c_3 are the characters of γ, then the 3-order polynomial $Q_1(t) = t^3+2c_1t^2+c_2t+c_3^2$ has three distinct real roots e_1, e_2, e_3, such that $e_1 < 0 < e_2 < e_3$.*

Proof. Let e_1, e_2, e_3 be the roots of Q_1. Then,

$$\dot{k}_2^2 + k_2^4 + c_3^2 k_2^{-2} + 2c_1 k_2^2 + c_2 = 0$$

implies

$$(k_2\dot{k}_2)^2 = -(k_2^2 - e_1)(k_2^2 - e_2)(k_2^2 - e_3) \tag{4.10}$$

and

$$c_1 = -\frac{1}{2}(e_1 + e_2 + e_3), \quad c_2 = e_1e_2 + e_1e_3 + e_2e_3, \quad c_3^2 = -e_1e_2e_3. \tag{4.11}$$

If two roots are the complex conjugate one of the other, say e_2 and e_3, the third equation of (4.11) implies $e_1 < 0$. Hence the right hand side of (4.10) is strictly negative. This contradicts the fact that k_2 is non constant. If Q_1 has a double root, say $e_2 = e_3$, then the third equation of (4.11) implies $e_1 < 0$. So, as in the previous case, the right hand side of (4.10) is strictly negative, which is a contradiction. Thus the roots must be real and distinct. We choose the ordering $e_1 < e_2 < e_3$. By the third equation of (4.11), two possibilities may occur: either $e_1 < e_2 < e_3 < 0$, or $e_1 < 0 < e_2 < e_3$. In the first case, the right hand side of (4.10) is negative, which is a contradiction. Thus $e_1 < 0 < e_2 < e_3$, as claimed. □

Definition 11. We say that e_1, e_2, e_3 are the *phase parameters* of the worldline. According to (4.11), the characters can be computed from the phase parameters.

Next, we let

$$\ell_1 = e_2\ell_3, \quad \ell_2 = e_1\ell_4, \quad \ell_3 = e_3 - e_1, \quad \ell_4 = e_3 - e_2, \quad m = \ell_4/\ell_3, \tag{4.12}$$

and denote by $K(m)$ and $\mathrm{sn}(-,m)$ the complete integral of the first kind and the Jacobi's sn-function with parameter m.[8]

Proposition 5. *Let γ be a conformal worldline with phase parameters $e_1 < 0 < e_2 < e_3$. Then*

$$k_2(u) = \sqrt{\frac{\ell_1 - \ell_2\mathrm{sn}^2(\sqrt{\ell_3}u + u_0, m)}{\ell_3 - \ell_4\mathrm{sn}^2(\sqrt{\ell_3}u + u_0, m)}} \tag{4.13}$$

and

$$k_1 = \frac{3}{2}k_2^2 - \frac{1}{2}(e_1 + e_2 + e_3), \quad k_3 = \sqrt{-e_1e_2e_3}\, k_2^{-2}, \tag{4.14}$$

[8] The parameter m is the square of the modulus k of the elliptic function. In the literature, the notation $\mathrm{sn}(-,k)$ is also used to denote the sn-function with modulus k.

where u_0 is a constant.

Proof. Let $f(u) = k_2(u/\sqrt{\ell_3})$. Then, by (4.10), we have

$$\dot{f}^2 = -\frac{4}{e_3 - e_1}(f - e_1)(f - e_2)(f - e_3).$$

If $e_1 < 0 < e_2 < e_3$, the general solution of the equation above (cf. [3], page77) is

$$f(u) = \frac{\ell_1 - \ell_2 \mathrm{sn}^2(u + u_0, m)}{\ell_3 - \ell_4 \mathrm{sn}^2(u + u_0, m)},$$

where u_0 is a constant. This implies (4.13). We conclude the proof by observing that (4.14) is an immediate consequence of (4.9) and (4.11). □

The above proposition has the following consequences: (1) k_2 is a strictly positive, even periodic function, with period

$$\omega = 2K(m)/\sqrt{\ell_3}; \tag{4.15}$$

(2) the parametrizations by conformal parameter of a worldline are defined on the whole real line. In addition, by possibly shifting the independent variable, the constant u_0 in (4.13) can be put equal to zero. This means that a conformal worldline admits a parametrization by conformal parameter, such that $k_2(0) = \sqrt{\ell_1/\ell_3}$. This is referred to as the *canonical parametrization.*

Definition 12. We say that a conformal worldline γ is a *standard configuration* if: (1) γ is parametrized by the canonical parameter; and (2) the canonical frame B of γ satisfies the initial condition $B_{|_0} = I_6$.[9] Clearly, any conformal worldline is conformally equivalent to a unique standard configuration.

The above discussion can be summarized in the following.

Proposition 6. *The standard configurations are in 1-1 correspondence with the points of the domain*

$$\mathcal{F} = \{\mathbf{e} = (e_1, e_2, e_3) \in \mathbb{R}^3 : e_1 < 0 < e_2 < e_3\} \subset \mathbb{R}^3.$$

In other words, for each $\mathbf{e} = (e_1, e_2, e_3) \in \mathcal{F}$, there exists a unique conformal worldline γ in its standard configuration with phase parameters (e_1, e_3, e_3).

4.3. **The momentum operator.** Let γ be a conformal worldline in standard configuration with curvatures k_1, k_2, k_3. Let $\mathcal{H} : Y \to \mathfrak{a}(2,4)$ be the map given by

$$\mathcal{H} = \begin{pmatrix} 0 & -1 & -(k_2^2 - k_1) & \dot{k}_2 & k_2 k_3 & 0 \\ 0 & 0 & 0 & -k_2 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & -(k_2^2 - k_1) \\ 0 & -k_2 & 0 & 0 & 0 & \dot{k}_2 \\ 0 & 0 & 0 & 0 & 0 & k_2 k_3 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}. \tag{4.16}$$

Then, (4.13) and (4.14) imply

$$\dot{\mathcal{H}} = [\mathcal{H}, \mathcal{K}]. \tag{4.17}$$

This equation together with $B' = B\,\mathcal{K}$ yields

$$B_{|_u}\, \mathcal{H}_{|_u}\, B^{-1}{}_{|_u} = \mathcal{H}_{|_0}, \quad \forall\, u \in \mathbb{R}. \tag{4.18}$$

[9] Here, I_6 denotes the 6×6 identity matrix.

Definition 13. We let $\mathfrak{m} := \mathcal{H}_{|0}$ and call $\mathfrak{m}$ the *momentum operator* of γ.

Remark 8. We give a brief explanation of the conceptual origin of the momentum operator. The first step is the construction of the *momentum space* and of the *Euler–Lagrange exterior differential system* [14, 15]. In our specific situation, the momentum space is the 19-dimensional manifold $Z = A_+^{\uparrow}(2,4) \times \mathfrak{K}$, where

$$\mathfrak{K} = \left\{ \mathfrak{k} = (\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3, \dot{\mathfrak{k}}_2) \in \mathbb{R}^4 \mid \mathfrak{k}_2 > 0, \mathfrak{k}_3 > 0 \right\} \subset \mathbb{R}^4.$$

The restricted conformal group acts freely on the left of Z by

$$L_Y(X, \mathfrak{k}) = (YX, \mathfrak{k}), \quad \forall\, X, Y \in A_+^{\uparrow}(2,4), \forall\, \mathfrak{k} \in \mathfrak{K}.$$

The Euler–Lagrange differential system is the $A_+^{\uparrow}(2,4)$-invariant Pfaffian differential ideal $\mathcal{I} \subset \Omega^*(Z)$ generated by the 1-forms

$$\mu_0^2, \quad \mu_0^3, \quad \mu_0^4, \quad \mu_1^2, \quad \mu_1^3, \quad \mu_1^4, \quad \mu_5^3, \quad \mu_5^4, \quad \mu_2^4, \quad \mu_0^0$$
$$\mu_5^2 - \mu_0^1, \quad \mu_5^1 - \mathfrak{k}_1\mu_0^1, \quad \mu_2^3 - \mathfrak{k}_2\mu_0^1, \quad \mu_3^4 - \mathfrak{k}_3\mu_0^1$$

and

$$d\dot{\mathfrak{k}}_2 - \mathfrak{k}_2(\mathfrak{k}_3^2 + \mathfrak{k}_2^2 - 2\mathfrak{k}_1)\mu^1, \quad d\mathfrak{k}_2 - \dot{\mathfrak{k}}_2\mu_0^1, \quad d\mathfrak{k}_1 - 3\mathfrak{k}_2\dot{k}_2\mu_0^1, \quad \mathfrak{k}_2 d\mathfrak{k}_3 + 2\mathfrak{k}_2\dot{\mathfrak{k}}_2\mu_0^1.$$

The independence condition of the system is the invariant 1-form μ_0^1. The integral curves of $(\mathcal{I}, \mu_0^1)$ can be constructed as follows. Let γ be a conformal worldline[10] with curvatures k_1, k_2, k_3 and canonical frame B. The lift $\mathfrak{b} = (B, k_1, k_2, k_3, \dot{k}_2) : \mathbb{R} \to Z$ of γ to Z is said to be the *extended frame field* along γ. Proposition 3 tells us that the integral curves of $(\mathcal{I}, \mu_0^1)$ are the extended frame fields of the worldlines. Using the Maurer–Cartan equations (2.3), we see that

$$\zeta = \frac{1}{2}(\mu_0^1 + \mu_5^2 + (\mathfrak{k}_2^2 - \mathfrak{k}_1)\mu_0^2 - \dot{\mathfrak{k}}_2\mu_0^3 - \mathfrak{k}_2\mathfrak{k}_3\mu_0^4 + \mathfrak{k}_2\mu_1^3) \tag{4.19}$$

is an invariant contact 1-form, such that the integral curves of its characteristic vector field X_ζ are the extended frames of the worldlines.[11] We can think of ζ as a map into the dual space $\mathfrak{a}(2,4)^*$ of the conformal Lie-algebra. If $\natural : \mathfrak{a}^*(4,2) \to \mathfrak{a}(2,4)$ denotes the pairing defined by the Killing form, then $\zeta^\natural = \natural \circ \zeta : Z \to \mathfrak{a}(2,4)$ is an equivariant map which in our context plays the role of the Legendre transformation. If $\mathfrak{b}$ is the extended frame field of γ, then $\mathcal{H}$ coincides with $\zeta^\natural \circ \mathfrak{b}$. The momentum map of the action of $A_+^{\uparrow}(2,3)$ on (Z, ζ) is given by

$$\widehat{\mathfrak{m}} : (X, \mathfrak{k}) \in Z \to ad_X^*(\zeta|_{(B,\mathfrak{k})}) \in \mathfrak{a}^*(2,4).$$

By *Nöther's conservation theorem* for a Hamiltonian action on a contact manifold, $\widehat{\mathfrak{m}}$ is constant along the characteristic curves. Thus, if $\mathfrak{b}$ is the extended frame of a worldline γ, then $\widehat{\mathfrak{m}}_\gamma = \widehat{\mathfrak{m}} \circ \mathfrak{b}$ is constant and $\widehat{\mathfrak{m}}_\gamma^\natural$ is the momentum operator of γ. This explains the geometrical origin of the momentum operator of a worldline.

Hereafter we will adopt the following notations:

- $P_\mathfrak{m}$ is the characteristic polynomial of $\mathfrak{m}$;
- $\mathcal{S}_\mathfrak{m}$ is the spectrum of $\mathfrak{m}$, where $\mathfrak{m}$ is viewed as an endomorphism of $\mathbb{C}^6$;
- for each $\lambda \in \mathcal{S}_\mathfrak{m}$, $n_1(\lambda)$ is the multiplicity of λ as a root of $P_\mathfrak{m}$ and $n_2(\lambda)$ is the complex dimension of the $\mathfrak{m}$-eigenspace $\mathbb{V}_\lambda$ of λ.

[10] Recall that γ is linearly full, with positive helicity and parametrized by the conformal parameter

[11] The characteristic vector field X_ζ is defined by $\zeta(X_\zeta) = 1$, $\iota_{X_\zeta} d\zeta = 0$.

Definition 14. A conformal worldline is said to be *regular, exceptional*, or *singular* depending on whether $\mathfrak{m}$ is a regular, exceptional, or singular element of the Lie algebra $\mathfrak{a}(2,4)$. In other words: (1) γ is regular if $\mathcal{S}_{\mathfrak{m}}$ consists of six elements; (2) γ is exceptional if $\mathcal{S}_{\mathfrak{m}}$ has less that six elements and $n_2(\lambda)=1$, for every $\lambda\in\mathcal{S}_{\mathfrak{m}}$; (3) γ is singular if $n_2(\lambda)>1$, for some λ.

From (4.18) it follows that $P_{\mathfrak{m}}$ coincides with the characteristic polynomial of $\mathcal{H}_{|_u}$, for every $u\in\mathbb{R}$. By (4.3), we get

$$P_{\mathfrak{m}}(t)=t^6+2c_1t^4+(c_2+1)t^2+c_3^2=Q_1(t^2)+t^2. \tag{4.20}$$

Let $Q_2(t)$ be the third-order polynomial $Q_1(t)+t^2$. Since the roots of $Q_1(t)$ are the phase parameters $e_1<0<e_2<e_3$, we infer that three possibilities may occur:

- Q_2 has three distinct real roots ρ_1, ρ_2, ρ_3, such that $e_1<\rho_1<0<e_2<\rho_2<\rho_3<e_3$;
- Q_2 has one negative real root ρ_1 with $e_1<\rho_1$ and two complex conjugate roots $\rho_2=\mu+i\nu$, $\nu>0$ and $\rho_3=\mu-i\nu$;
- Q_2 has one simple real root ρ_1 such that $e_1<\rho_1<0$ and a double real root $\rho_2=\rho_3$, $e_2<\rho<e_3$.

Let[12]

$$\begin{aligned}&\lambda_0=i\sqrt{|\rho_1|},\quad \lambda_1=-i\sqrt{|\rho_1|},\quad \lambda_2=\sqrt{\rho_2},\\&\lambda_3=-\sqrt{\rho_2},\quad \lambda_4=\sqrt{\rho_3},\quad \lambda_5=-\sqrt{\rho_3}.\end{aligned} \tag{4.21}$$

In the first two cases the eigenvalues of $P_{\mathfrak{m}}$ are simple and

$$\mathcal{S}_{\mathfrak{m}}=\{\lambda_0,\dots,\lambda_5\}. \tag{4.22}$$

While, in the third case we have

$$\mathcal{S}_{\mathfrak{m}}=\{\lambda_0,\dots,\lambda_3\} \tag{4.23}$$

and the two real eigenvalues λ_2 and λ_3 are the double roots of $P_{\mathfrak{m}}$.

We can prove the following.

Proposition 7. *A conformal worldline is either regular or exceptional.*

Proof. For every $\lambda\in\mathcal{S}_{\mathfrak{m}}$, let $L_\lambda=(L^0_\lambda,\dots,L^5_\lambda):\mathbb{R}\to\mathbb{C}^6$ be defined by

$$\begin{cases}L^0_\lambda=\lambda(\lambda^2-k_2^2)(\lambda^2+k_1-k_2^2),\\ L^1_\lambda=\lambda(\lambda-k_2\dot{k}_2),\\ L^2_\lambda=-\lambda^2(\lambda^2-k_2^2),\\ L^3_\lambda=\lambda(\lambda\dot{k}_2-k_2),\\ L^4_\lambda=k_2k_3(\lambda^2-k_2^2),\\ L^5_\lambda=\lambda(\lambda^2-k_2^2).\end{cases} \tag{4.24}$$

The map L_λ is real-analytic and periodic, with period ω. Let D_λ be its zero set. If $\lambda\notin\mathbb{R}$, then $D_\lambda=\emptyset$. If $\lambda\in\mathbb{R}$, instead, we have

$$D_\lambda=\begin{cases}D^+_\lambda=\{n\omega+p_\lambda,n\in\mathbb{Z}\},\quad \lambda>0,\\ D^-_\lambda=\{n\omega-p_\lambda,n\in\mathbb{Z}\},\quad \lambda<0,\end{cases} \tag{4.25}$$

[12]If $z\in\mathbb{C}\setminus\mathbb{R}^+$, then $\sqrt{z}$ is the determination of the square root with positive imaginary part.

where $p_\lambda \in (0,\omega)$ is given by

$$p_\lambda = \mathrm{sn}^{-1}\left(\frac{1}{\alpha}, m\right), \quad \alpha = \sqrt{\frac{\ell_2 - \lambda^2\ell_4}{\ell_1 - \lambda^2\ell_3}}. \tag{4.26}$$

The λ-eigenspace $\mathcal{V}_\lambda|_u$ of $\mathcal{H}|_u$ is spanned by $L_\lambda|_u$, for every $u \notin D_\lambda$. Since D_λ is a discrete set, this implies that $\dim(\mathcal{V}_\lambda|_u) = 1$, for every $u \in \mathbb{R}$. On the other hand, $\mathfrak{m}$ and $\mathcal{H}|_u$ belongs to the same adjoint orbit, for every u. Thus, also the eigenspaces of the momentum are 1-dimensional. This proves the result. $\square$

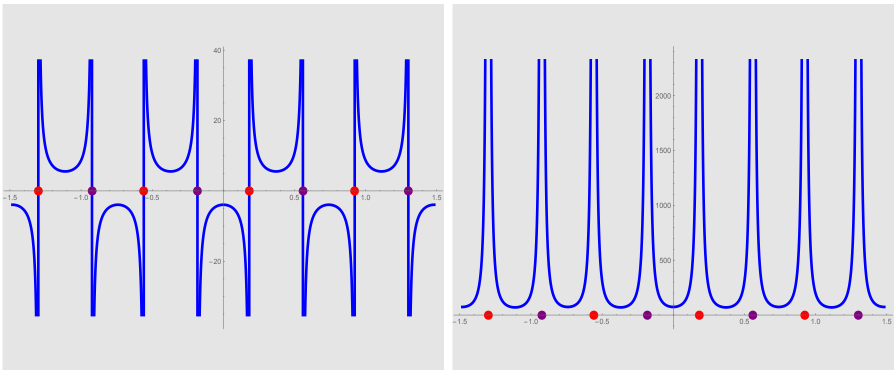

FIGURE 1. The graphs of the functions s_λ (on the left) and r_λ (on the right), $\lambda \in \mathbb{R}$.

4.4. **Integrating factors and principal vectors.** Let γ be the standard configuration of a linearly full conformal worldline with nonconstant curvatures. Denote by $e_1 < 0 < e_2 < e_3$ its phase parameters and by $B : \mathbb{R} \to A_+^\uparrow(2,4)$ its canonical frame field. For each $\lambda \in \mathcal{S}_\mathfrak{m}$, consider the functions

$$r_\lambda = \frac{k_2\dot{k}_2 + \lambda}{k_2^2 - \lambda^2}, \quad s_\lambda = \frac{\lambda^2 + k_2^2 - 2\lambda k_2\dot{k}_2}{(\lambda^2 - k_2^2)^2}. \tag{4.27}$$

If $\lambda \notin \mathbb{R}$, the functions r_λ and s_λ are periodic, complex-valued and real-analytic; if $\lambda \in \mathbb{R}$, r_λ and s_λ are periodic, real-valued and real-analytic on the complement of the discrete set $\widetilde{D}_\lambda = D_\lambda^+ \cup D_\lambda^-$; their absolute values tend to infinity when u approaches one of the points of $\widetilde{D}_\lambda$ (see Figure 1). For notational consistency, we put $\widetilde{D}_\lambda = \emptyset$, when $\lambda \notin \mathbb{R}$.

Definition 15. A primitive $\delta_\lambda : \mathbb{R} \setminus \widetilde{D}_\lambda \to \mathbb{C}$ of r_λ is said to be an *integrating factor of the first kind* for the eigenvalue λ if

- $\delta_\lambda|_0 = 0$;
- $e^{-\delta_\lambda} L_\lambda : \mathbb{R} \setminus \widetilde{D}_\lambda \to \mathbb{C}^6$ extends to a real-analytic map $\mathbb{R} \to \mathbb{C}^6$.

Remark 9. The integrating factors of the first kind are quasi-periodic functions, with quasi-period 2ω. If $\lambda \notin \mathbb{R}$, the function δ_λ is a regular, complex-valued function. If $\lambda \in \mathbb{R}$, the integrating factor δ_λ is real-analytic on the complement of the discrete set D_λ and its imaginary part is locally constant. The function e^{δ_λ} is

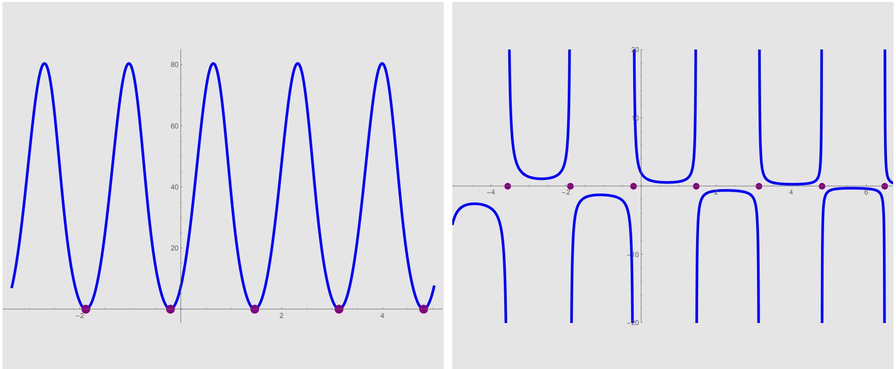

FIGURE 2. The graphs of the functions $\|L_\lambda\|^2$ (on the left) and $e^{-\delta_\lambda}$ (on the right), $\lambda \in \mathbb{R}$.

real-valued, with singularities at the points of D_λ (see Figure 2). The singularities of $e^{-\delta_\lambda}$ compensate the zeroes of the functions L^j_λ so that the products $e^{-\delta_\lambda}L^j_\lambda$ are regular, real-analytic maps (see Figure 3). The evaluation of the integrating factors in terms of elliptic integrals and the Jacobi theta functions is analyzed in the appendix. The explicit expression of the integrating factor of the first kind for a non-real eigenvalue is given in (6.4) while the integrating factor of a real eigenvalue can be found in (6.7).

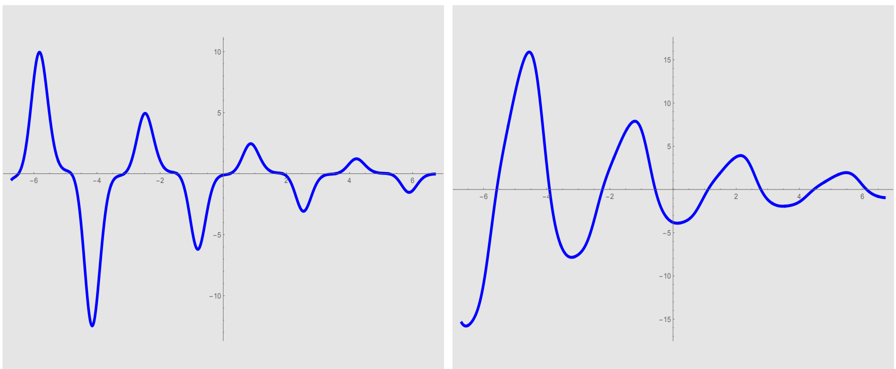

FIGURE 3. The graphs of the functions $e^{-\delta_\lambda}L^0_\lambda$ (on the left) and $e^{-\delta_\lambda}L^3_\lambda$ (on the right), $\lambda \in \mathbb{R}$.

Let λ be a multiple root of $P_{\mathfrak{m}}$. We set

$$\widehat{D}_\lambda = \{-\mathrm{sign}(\lambda)p + n\omega : n \in \mathbb{Z}\}$$

and we define $T_\lambda = (T^0_\lambda, \dots, T^5_\lambda) : \mathbb{R} \setminus \widehat{D}_\lambda \to \mathbb{R}^6$ by

$$\begin{cases} T^0_\lambda = \frac{1}{2}(\lambda^2 - k_2^2)(6\lambda^2 + 2c_1 + k_2^2), \\ T^1_\lambda = \frac{k_2(k_2^2\dot{k}_2 + \lambda^2\dot{k}_2 - 2\lambda k_2)}{\lambda^2 - k_2^2}, \\ T^2_\lambda = -2\lambda(\lambda^2 - k_2^2), \\ T^3_\lambda = \frac{k_2(\lambda^2 + k_2^2 - 2\lambda k_2 \dot{k}_2)}{\lambda^2 - k_2^2}, \\ T^4_\lambda = 0, \\ T^5_\lambda = (\lambda^2 - k_2^2). \end{cases} \tag{4.28}$$

Remark 10. The map T_λ is periodic with period ω, is real-analytic on the complement of $\widehat{D}_\lambda$ and tends to $\pm\infty$ when u tends to a point of $\widehat{D}_\lambda$. It vanishes at the point of D_λ (see Figure 4)

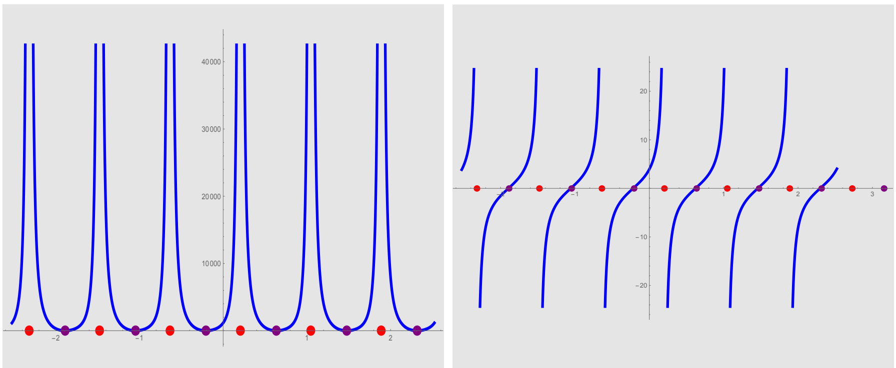

FIGURE 4. The graphs of the functions $\|T_\lambda\|^2$ (on the left) and η_λ (on the right) when $\lambda \in \mathbb{R}$ is a multiple root of $P_{\mathfrak{m}}$.

Definition 16. A primitive $\eta_\lambda : \mathbb{R} \setminus \widetilde{D}_\lambda \to \mathbb{R}$ of s_λ is said an *integrating factor of the second kind* for the multiple eigenvalue λ if

- $\eta_\lambda|_0 = 0$;
- $e^{-\delta_\lambda}(T_\lambda - \eta_\lambda L_\lambda) : \mathbb{R} \setminus \widetilde{D}_\lambda \to \mathbb{C}^6$ extends to a real-analytic map $\mathbb{R} \to \mathbb{C}^6$.

Remark 11. The integrating factor of the second kind vanishes at the points of D_λ and tends to $\pm\infty$ when u tends to a point of $\widehat{D}_\lambda$ (see Figure 4). The functions $\eta_\lambda L_\lambda$ and $T_\lambda - \eta_\lambda L_\lambda$ behave in a similar way. Multiplying $T_\lambda - \eta_\lambda L_\lambda$ by $e^{-\delta_\lambda}$, the zeroes of one factor compensate the singularities of the other so that the product is a regular analytic function (see Figures 5). The formula expressing the integrating factor of the second kind is given in (6.11). Despite the apparent opacity, the formulas of the integrating factors can be easily made operative using standard symbolic computation programs such as MATHEMATICA 11.

Proposition 8. *If $\lambda \in \mathcal{S}_{\mathfrak{m}}$ is an eigenvalue of the momentum, then*

$$e^{-\delta_\lambda} \sum_{j=0}^{5} L^j_\lambda B_j = \mathbf{A}_\lambda, \tag{4.29}$$

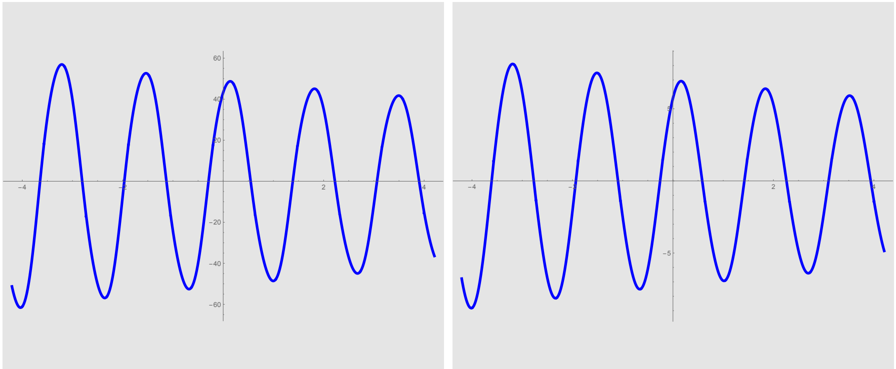

FIGURE 5. The graphs of the functions $e^{-\delta_\lambda}(T^0_\lambda - \eta_\lambda L^0_\lambda)$ (on the left) and $e^{-\delta_\lambda}(T^3_\lambda - \eta_\lambda L^3_\lambda)$ (on the right) when $\lambda \in \mathbb{R}$ is a multiple root of $P_{\mathfrak{m}}$.

where $\mathbf{A}_\lambda \in \mathbb{C}^6$ *is an* $\mathfrak{m}$*-eigenvector of the eigenvalue* λ*. If* λ *is a multiple root of* $P_{\mathfrak{m}}$*, then*

$$e^{-\delta_\lambda} \sum_{j=0}^{5} (T^j_\lambda - \eta_\lambda L^j_\lambda) B_j = \mathbf{C}_\lambda, \tag{4.30}$$

where $\mathbf{C}_\lambda \in \mathbb{C}^6$ *is a nonzero vector, such that*

$$\mathbf{C}_\lambda \wedge \mathbf{A}_\lambda \neq 0, \quad \mathfrak{m}(\mathbf{C}_\lambda) = \lambda \mathbf{C}_\lambda + \mathbf{A}_\lambda.$$

We call $\mathbf{A}_\lambda$ *the* principal vector *of the eigenvalue* λ *and* $\mathbf{C}_\lambda$ *the* secondary principal vector *of the multiple eigenvalue* λ*.*

Proof. Let $\mathbf{L}_\lambda : \mathbb{R} \to \mathbb{C}^6$ be the map

$$\mathbf{L}_\lambda = \sum_{i=0}^{5} L^i_\lambda(u) B_i|_u. \tag{4.31}$$

Let $\mathbb{V}_\lambda$ be the 1-dimensional $\mathfrak{m}$-eigenspace of the eigenvalue λ. Then, (4.18) implies that $\mathbf{L}_\lambda|_u \in \mathbb{V}_\lambda$, for every $u \in \mathbb{R}$. Therefore, there exists a unique real-analytic map $\widetilde{r}_\lambda : \mathbb{R} \setminus D_\lambda \to \mathbb{C}$, such that

$$\dot{\mathbf{L}}_\lambda|_u = \widetilde{r}_\lambda|_u \mathbf{L}_\lambda|_u \quad \forall\, u \in \mathbb{R} \setminus D_\lambda. \tag{4.32}$$

From $\dot{B} = B\,\mathcal{K}$ and by (4.2), we have

$$\dot{\mathbf{L}}_\lambda \equiv \frac{k_2 \dot{k}_2 + \lambda}{k_2^2 - \lambda_j^2} \mathbf{L}_\lambda \mod(B_0, B_1, B_2, B_3, B_4). \tag{4.33}$$

From (4.32) and (4.33), we have $r_\lambda = \widetilde{r}_\lambda$. Using (4.32), it follows that that $e^{-\delta_\lambda}\mathbf{L}_\lambda$ is constant on $\mathbb{R} \setminus D_\lambda$. On the other hand, $e^{-\delta_\lambda} L_\lambda$ extends smoothly across D_λ and hence also $e^{-\delta_\lambda}\mathbf{L}_\lambda$ extends to a real-analytic map $\mathbb{R} \to \mathbb{C}^6$. This implies that $e^{-\delta_\lambda}\mathbf{L}_\lambda = \mathbf{A}_\lambda$, for some $\mathbf{A}_\lambda \in \mathbb{V}_\lambda$. This proves the first part of the statement.

Let λ be a multiple root of $P_{\mathfrak{m}}$. The map T_λ satisfies

$$\mathcal{H}|_u \cdot T_\lambda|_u = \lambda T_\lambda|_u + L_\lambda|_u, \quad T_\lambda|_u \wedge L_\lambda|_u \neq 0, \quad \forall\, u \in \mathbb{R} \setminus D_\lambda. \tag{4.34}$$

Let

$$\mathbf{T}_\lambda = e^{-\delta_\lambda} \sum_{j=0}^{5} T_\lambda^j B_j : \mathbb{R} \setminus D_\lambda \to \mathbb{C}^6.$$

From (4.34), we obtain

$$\mathfrak{m}(\mathbf{T}_\lambda) = \lambda \mathbf{T}_\lambda + \mathbf{A}_\lambda, \quad \mathbf{T}_\lambda \wedge \mathbf{A}_\lambda \neq 0. \tag{4.35}$$

Differentiating the first equation in (4.35), we get $\mathfrak{m}(\mathbf{T}'_\lambda) = \lambda \mathbf{T}'_\lambda$. Thus, there exists a unique real-analytic function $\widetilde{s}_\lambda : \mathbb{R} \setminus D_\lambda \to \mathbb{C}$ such that $\mathbf{T}'_\lambda = \widetilde{s}_\lambda \mathbf{A}_\lambda$. Using $\dot{B} = B\,\mathcal{K}$, we obtain

$$\dot{\mathbf{T}}_\lambda \equiv \frac{\lambda^2 + k_2^2 - 2\lambda k_2 \dot{k}_2}{(\lambda - k_2^2)^2} \mathbf{A}_\lambda, \quad \mathrm{mod}(B_0, B_1, B_2, B_3, B_5).$$

Then, $\widetilde{s}_\lambda = s_\lambda$. This implies that $\mathbf{T}_\lambda - \eta_\lambda \mathbf{A}_\lambda$ is constant on $\mathbb{R} \setminus D_\lambda$. Since $e^{-\delta_\lambda}(T_\lambda - \eta_\lambda L_\lambda)$ extends smoothly across $\widetilde{D}_\lambda$, also $\mathbf{T}_\lambda - \eta_\lambda \mathbf{A}_\lambda$ extends to a smooth (real-analytic) map $\mathbb{R} \to \mathbb{C}^6$. Hence there exists $\mathbf{C}_\lambda \in \mathbb{C}^6$ such that $\mathbf{T}_\lambda - \eta_\lambda A_\lambda = \mathbf{C}_\lambda$. Using (4.35), it follows that $\mathfrak{m}(\mathbf{C}_\lambda) = \lambda \mathbf{C}_\lambda + \mathbf{A}_\lambda$ and that $\mathbf{C}_\lambda \wedge \mathbf{A}_\lambda \neq 0$. This proves the result. ☐

Remark 12. Since γ is a standard configuration, then $B_{|_0} = I_6$ and hence

$$\mathbf{A}_\lambda = L_\lambda|_0, \quad \mathbf{C}_\lambda = T_\lambda|_0.$$

Thus, the principal vectors can be explicitly computed in terms of the phase parameters e_1, e_2 and e_3.

5. Integrability by quadratures

5.1. Integrability by quadratures of the regular conformal worldlines.

Let $\gamma : \mathbb{R} \to \mathcal{E}^{1,3}$ be the standard configuration of a regular linearly full conformal worldline. Its momentum operator has six simple roots $\lambda_0, \ldots, \lambda_5$, ordered as in (4.21). Let δ_j be the integrating factor of the first kind and $\mathbf{A}_j$ be the principal vector of λ_j, respectively. Let $\mathbf{A} \in \mathbb{C}(6,6)$ denote the matrix with column vectors $\mathbf{A}_0, \ldots, \mathbf{A}_5$ and define the real-analytic maps

$$\Delta,\, \Lambda : \mathbb{R} \setminus D_\lambda \to \mathbb{C}(6,6), \quad \mathbf{X} : \mathbb{R} \to \mathbb{C}(6,6)$$

by[13]

$$\begin{cases} \Delta = (e^{-\delta_0}\varepsilon_0, \ldots, e^{-\delta_5}\varepsilon_5), \\ \Lambda = (L_{\lambda_0}, \ldots, L_{\lambda_5}), \\ \mathbf{X} = \Delta\, \Lambda. \end{cases}$$

We have the following.

Theorem C. *Let $\gamma \subset \mathcal{E}^{1,3}$ be the standard configuration of a regular linearly full conformal worldline. Then*

$$\gamma = \left[\mathtt{m}\, {}^t(\mathbf{A}^{-1})\, \mathbf{X}\, \mathtt{m}\, E_0\right],$$

where $\mathtt{m}$ is the matrix representing the scalar product $\langle\, ,\rangle$ with respect to the standard light-cone basis $(E_0, \ldots, E_5)$ of $\mathbb{R}^{2,4}$.

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[13]Here $\varepsilon_j = {}^t(\delta_j^0, \ldots, \delta_j^5)$, $j = 0, \ldots, 5$, denote the column vectors of the canonical basis of $\mathbb{R}^6$.

Proof. Since $\mathbf{A}_j = e^{-\delta_j} B \cdot L_{\lambda_j}$, $j = 0, \dots, 5$, we have $\mathbf{A} = B\,\Lambda\,\Delta$, that is,

$$B = \mathbf{A}\,\Delta^{-1}\,\Lambda^{-1}. \tag{5.1}$$

From (5.1), using the fact that ${}^tB\,\mathtt{m}\,B = \mathtt{m}$, we get

$$\Lambda = \mathtt{m}\,\mathbf{X}^{-1}\,{}^t\mathbf{A}\,\mathtt{m}\,\mathbf{A}\,\Delta^{-1}. \tag{5.2}$$

Substituting (5.2) into (5.1), we obtain

$$B = \mathtt{m}\,{}^t(\mathbf{A}^{-1})\,\mathbf{X}\,\mathtt{m},$$

which implies the result. □

5.2. **Integrability by quadratures of the exceptional conformal worldlines.** Let $\gamma : \mathbb{R} \to \mathcal{E}^{1,3}$ be the standard configuration of an exceptional linearly full conformal worldline. Its momentum operator has four distinct roots $\lambda_0, \dots, \lambda_3$, ordered as in (4.21). Then, λ_0 and λ_1 are simple purely imaginary roots and λ_2, λ_3 are real double roots of $P_{\mathtt{m}}$. For each eigenvalue λ_j, $j = 0, \dots, 3$, let δ_j be its integrating factor of the first kind and $\mathbf{A}_j$ be the corresponding principal vector. For each double root λ_j, $j = 2, 3$, let η_j be the integrating factor of the second kind and $\mathbf{C}_j$ be the secondary principal vector. Let $\widetilde{\mathbf{A}}$ denote the matrix $\widetilde{\mathbf{A}} = (\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{C}_2, \mathbf{C}_3)$. Let

$$\widetilde{\Delta},\ \widetilde{\Lambda} : \mathbb{R} \setminus D_\lambda \to \mathbb{C}(6,6), \quad \widetilde{\mathbf{X}} : \mathbb{R} \to \mathbb{C}(6,6)$$

be the real-analytic maps defined by

$$\begin{cases} \widetilde{\Delta} = (e^{-\delta_0}\varepsilon_0, e^{-\delta_1}\varepsilon_1, e^{-\delta_2}\varepsilon_2, e^{-\delta_3}\varepsilon_3, e^{-\delta_2}\varepsilon_4, e^{-\delta_3}\varepsilon_5), \\ \widetilde{\Lambda} = (L_{\lambda_0}, L_{\lambda_1}, L_{\lambda_2}, L_{\lambda_3}, T_{\lambda_2} - \eta_2 L_{\lambda_2}, T_{\lambda_3} - \eta_3 L\lambda_3), \\ \widetilde{\mathbf{X}} = \widetilde{\Delta}\,\widetilde{\Lambda}. \end{cases}$$

We have the following.

Theorem D. *Let $\gamma \subset \mathcal{E}^{1,3}$ be the standard configuration of an exceptional linearly full conformal worldline. Then*

$$\gamma = \left[\mathtt{m}\,{}^t(\widetilde{\mathbf{A}}^{-1})\,\widetilde{\mathbf{X}}\,\mathtt{m}\,E_0\right].$$

Proof. From Proposition 4.4, it follows that the canonical frame field, the principal vectors and the integrating factors satisfy

$$e^{-\delta_0}B \cdot L_{\lambda_0} = \mathbf{A}_0, \quad e^{-\delta_1}B \cdot L_{\lambda_1} = \mathbf{A}_1, \quad e^{-\delta_2}B \cdot L_{\lambda_2} = \mathbf{A}_2, \quad e^{-\delta_3}B \cdot L_3 = \mathbf{A}_3$$

and

$$e^{-\delta_2}B \cdot (T_{\lambda_2} - \eta_2 L_{\lambda_w}) = \mathbf{C}_2 \quad e^{-\delta_3}B \cdot (T_{\lambda_3} - \eta_3 L_{\lambda_3}) = \mathbf{C}_3.$$

We then have

$$B = \widetilde{\mathbf{A}}\,\widetilde{\Delta}^{-1}\,\widetilde{\Lambda}^{-1}. \tag{5.3}$$

Combining (5.3) with ${}^tB\,\mathtt{m}\,B = \mathtt{m}$, we obtain

$$\widetilde{\Lambda}^{-1} = \widetilde{\Delta}\,\widetilde{\mathbf{A}}^{-1}\,\mathtt{m}\,{}^t(\widetilde{\mathbf{A}}^{-1})\,\widetilde{\mathbf{X}}\,\mathtt{m}. \tag{5.4}$$

Substituting (5.4) into (5.3), we obtain $B = \mathtt{m}\,{}^t(\widetilde{\mathbf{A}}^{-1})\,\widetilde{\mathbf{X}}\,\mathtt{m}$, from which the result follows. □

5.3. **Final comments and remarks.** The theoretical explanation of the integrability by quadratures lies in the Arnold–Liouville integrability of the Euler–Lagrange differential system. This means the following. Let $\mathfrak{m} \in \mathfrak{a}(2,4)$ be the momentum of a linearly full conformal worldline γ with nonconstant curvatures. We know that $\mathfrak{m}$ is either regular or exceptional. The stabilizer $A_+^{\uparrow}(2,4)_{\mathfrak{m}}$ of $\mathfrak{m}$ is a 3-dimensional closed subgroup, diffeomorphic to $S^1 \times \mathbb{R}^2$. The inverse image of $\mathfrak{m}^\natural$ by the momentum map is a 4-dimensional submanifold $Z_{\mathfrak{m}} \subset Z$ and the characteristic vector field X_ζ is tangent to $Z_{\mathfrak{m}}$. The worldlines with momentum $\mathfrak{m}$ are originated by the integral curves of $X_\zeta|_{Z_{\mathfrak{m}}}$. The stabilizer $A_+^{\uparrow}(2,4)_{\mathfrak{m}}$ acts freely on $Z_{\mathfrak{m}}$ and the quotient space $Z_{\mathfrak{m}}/A_+^{\uparrow}(2,4)_{\mathfrak{m}}$ is a circle. This implies that $Z_{\mathfrak{m}} \subset Z$ is diffeomorphic to the Cartesian product of $\mathbb{R}^2$ with a 2-dimensional torus T^2. In principle, the integration by quadratures can be achieved by a diffeomorphism $\Psi_{\mathfrak{m}} : Z_{\mathfrak{m}} \to \mathbb{R}^2 \times T^2$ such that $\Psi_*(X_\zeta)$ is a linear vector field. Since the stabilizer of the momentum operator is not compact, then the trajectory of γ cannot be closed. Instead, if γ is trapped in a 3-dimensional Einstein universe, the stabilizer of the momentum operator can be a maximal compact abelian subgroup of $A_+^{\uparrow}(2,3)$. Thus, in this case, there are countably many closed worldlines with nonconstant curvatures, as it has been shown in [8].

The periodicity of the conformal curvatures implies that the trajectory of γ is left unchanged by the action of the infinite cyclic subgroup generated by $B(\omega)\,B(0)^{-1} \in A_+^{\uparrow}(2,4)$.

At last, we note that if the phase parameters e_1, e_2, e_3 are known, then all steps of the integration procedure can be implemented and computed using Mathematica.

6. Appendix: The integrating factors

6.1. **Integrating factors of the first kind.** Given $\lambda \in \mathcal{S}_{\mathfrak{m}}$, let

$$a = \ell_1 - \lambda^2\ell_3, \quad b = \ell_2 - \lambda\ell_4, \quad c = \frac{\lambda\ell_4}{\ell_2 - \lambda^2\ell_4} \tag{6.1}$$

and

$$d = \frac{\lambda(\ell_2\ell_3 - \ell_1\ell_4)}{(\ell_2 - \lambda^2\ell_4)(\ell_1 - \lambda^2\ell_3)}, \tag{6.2}$$

where ℓ_1, ℓ_2, ℓ_3 and ℓ_4 are constants as in (4.12). From (4.13) and (4.27), we obtain

$$r_\lambda(u) = \frac{1}{2}\frac{d}{du}\left(\ln\frac{a - b\,\mathrm{sn}^2(\sqrt{\ell_3}u, m)}{\ell_3 - \ell_4\,\mathrm{sn}^2(\sqrt{\ell_3}u, m)}\right) + c + \frac{d}{1 - \alpha^2\mathrm{sn}^2(\sqrt{\ell_3}u, m)}, \tag{6.3}$$

where the parameter m is as in (4.12) and α is as in (4.26).

6.1.1. *The integrating factor of the first kind of a non-real eigenvalue.* If λ is a non-real eigenvalue, then

$$1 - \alpha^2\mathrm{sn}^2(\sqrt{\ell_3}u, m) \neq 0, \quad \forall\, u \in \mathbb{R}$$

and

$$\int \frac{du}{1 - \alpha^2\mathrm{sn}^2(u, m)} = \Pi(\alpha^2, \mathrm{am}_m(u), m),$$

where $\Pi(n, \phi, m)$ is the incomplete integral of the third kind and $\mathrm{am}_m(-)$ is the Jacobi amplitude with parameter m. Note that in this case, the restriction of the incomplete integral of the third kind on the real axis is a regular real-analytic function. Since

$$\frac{a - b\,\mathrm{sn}^2(\sqrt{\ell_3}u, m)}{\ell_3 - \ell_4\,\mathrm{sn}^2(\sqrt{\ell_3}u, m)} \notin \mathbb{R}_-, \quad \forall\, u \in \mathbb{R}$$

we can evaluate the logarithm[14] of the function on the left hand side in the above formula. Thus, the integrating factor of the eigenvalue is given by

$$\delta_\lambda(u) = \frac{1}{2}\ln\left(\frac{a - b\,\mathrm{sn}^2(\sqrt{\ell_3}u, m)}{\ell_3 - \ell_4\,\mathrm{sn}^2(\sqrt{\ell_3}u, m)}\right) + cu + \frac{d}{\sqrt{\ell_3}}\Pi(\alpha^2, \mathrm{am}_m(\sqrt{\ell_3}u), m). \tag{6.4}$$

6.1.2. *The integrating factor of the first kind of a real eigenvalue.* If λ is a real eigenvalue, the function r_λ is singular and the evaluation of the integrating factor requires some caution. Let w and v be the real constants

$$w = \frac{\alpha}{\sqrt{(\alpha^2 - m)(\alpha^2 - 1)}},$$

and

$$v = \frac{E(m)}{K(m)} - E(p, m) - \mathrm{cs}(p, m)\mathrm{dn}(p, m) - \frac{\sqrt{(\alpha^2 - m)(\alpha^2 - 1)}}{\alpha},$$

where $p = p_\lambda$ is as in (4.26) and $E(m)$, $E(-, m)$ are the complete and incomplete elliptic integrals of the second kind respectively. Let f_λ be the periodic extension, with period 2ω, of the locally constant function

$$f_\lambda(u) = \begin{cases} -\frac{\pi}{2}, & u \in [p - \omega, p), \\ -\frac{3\pi}{2}, & u \in [p, \omega - p), \\ \frac{\pi}{2}, & u \in [\omega - p, \omega + p), \end{cases}$$

if $\lambda < 0$, and

$$f_\lambda(u) = \begin{cases} -\frac{\pi}{2}, & u \in [p - \omega, p), \\ \frac{\pi}{2}, & u \in [p, \omega + p), \end{cases}$$

if $\lambda > 0$. Denote by $\vartheta_1(-, q_m)$ the first Jacobi theta function with nome

$$q_m = \exp(-\pi K(1 - m)/K(m)).$$

Proceeding as in [22], page 71, we see that

$$g_{\lambda,1}(u) = \frac{w}{2\sqrt{\ell_3}}\ln\left(\frac{\vartheta_1(\frac{\pi}{2K(m)}(p - \frac{u}{\sqrt{\ell_3}}), q_m)}{\vartheta_1(\frac{\pi}{2K(m)}(p + \frac{u}{\sqrt{\ell_3}}), q_m)}\right) + wvu \tag{6.5}$$

is a real-valued primitive of $(1 - \alpha^2\mathrm{sn}^2(\sqrt{\ell_3}u, m))^{-1}$. We take

$$\begin{aligned} g_{\lambda,2}(u) = \frac{1}{2}\ln(a) + cu - \frac{1}{2}\ln(\ell_3 - \ell_4\mathrm{sn}(\sqrt{\ell_3}u, m) + \\ + \frac{1}{2}\ln(1 - \alpha^2\mathrm{sn}^2(\sqrt{\ell_3}u, m)) \end{aligned} \tag{6.6}$$

as a real-valued primitive of

$$\frac{1}{2}\frac{d}{du}\left(\ln\frac{a - b\mathrm{sn}^2(\sqrt{\ell_3}u, m)}{\ell_3 - \ell_4\mathrm{sn}^2(\sqrt{\ell_3}u, m)}\right) + c.$$

[14] We use the standard determination of the natural logarithm, with a branch cut discontinuity in the complex plane running from $-\infty$ to 0.

Then,

$$\delta_\lambda = dg_{\lambda,1} + g_{\lambda,2} + if_\lambda, \tag{6.7}$$

is the integrating factor for the real eigenvalue λ.

6.2. **Integrating factors of the second kind.** Let λ be a multiple root of $P_{\mathfrak{m}}$. Note that λ is necessarily real. From

$$s_\lambda = \frac{\lambda^2 + k_2^2 - 2\lambda k_2 \dot{k}_2}{(\lambda^2 - k_2^2)^2} = -2\lambda \frac{k_2 \dot{k}_2}{(\lambda^2 - k_2^2)^2} + \frac{\lambda^2 + k_2^2}{k_2^2 - \lambda^2},$$

it follows that

$$\int s_\lambda du = \eta_{1,\lambda} + \eta_{2,\lambda},$$

where

$$\begin{aligned} \eta_{1,\lambda} &= -2\lambda \int \frac{k_2 \dot{k}_2}{(\lambda^2 - k_2^2)^2} du = \frac{\lambda}{k_2^2 - \lambda^2} \\ &= \lambda \frac{\ell_3 - \ell_4 \mathrm{sn}^2(\sqrt{\ell_3}u, m)}{(\ell_1 - \lambda^2 \ell_3) - (\ell_2 - \lambda^2 \ell_4)\mathrm{sn}^2(\sqrt{\ell_3}u, m)} \\ &= \frac{\lambda}{\ell_1 - \lambda^2 \ell_3} \frac{\ell_3 - \ell_4 \mathrm{sn}^2(\sqrt{\ell_3}u, m)}{1 - \alpha^2 \mathrm{sn}^2(\sqrt{\ell_3}u, m)} \end{aligned} \tag{6.8}$$

and

$$\begin{aligned} \eta_{2,\lambda} = \int \frac{\lambda^2 + k_2^2}{(\lambda^2 - k_2^2)^2} du &= \mathsf{A} \int \frac{\mathrm{cn}^2(\sqrt{\ell_3}u, m)}{1 - \alpha^2 \mathrm{sn}^2(\sqrt{\ell_3}u, m)} du + \\ + \mathsf{B} \int \frac{\mathrm{cn}^2(\sqrt{\ell_3}u, m)\mathrm{sn}^2(\sqrt{\ell_3}u, m)}{1 - \alpha^2 \mathrm{sn}^2(\sqrt{\ell_3}u, m)} du &+ \mathsf{C} \int \frac{\mathrm{sn}^2(\sqrt{\ell_3}u, m)}{1 - \alpha^2 \mathrm{sn}^2(\sqrt{\ell_3}u, m)} du, \end{aligned} \tag{6.9}$$

where

$$\mathsf{A} = \frac{\ell_3(\ell_1 + \lambda^2 \ell_3)}{(\ell_1 - \lambda^2 \ell_3)^2}, \quad \mathsf{B} = -\frac{\ell_4(\ell_2 + \lambda_2 \ell_4)}{(\ell_1 - \lambda^2 \ell_3)^2},$$

and

$$\mathsf{C} = \frac{(\ell_3 - \ell_4)((\ell_1 - \ell_2) + \lambda^2(\ell_3 - \ell_4))}{(\ell_1 - \lambda^2 \ell_3)^2}.$$

The integrals in the right hand side of (6.9) can be evaluated as in [3], page 218, and, as a result, we obtain

$$\begin{aligned} \eta_{2,\lambda} = &\frac{\mathsf{M}}{\sqrt{\ell_3}} E(\sqrt{\ell_3}u, m) + \mathsf{N}u + \mathsf{P}(g_{\lambda,1} - i\frac{\pi}{2}\widehat{f}_\lambda) + \\ &+ \frac{\mathsf{Q}}{\sqrt{\ell_3}} \frac{\mathrm{sn}(\sqrt{\ell_3}u, m)\mathrm{cn}(\sqrt{\ell_3}u, m)\mathrm{dn}(\sqrt{\ell_3}u, m)}{1 - \alpha^2 \mathrm{sn}^2(\sqrt{\ell_3}u, m)}, \end{aligned} \tag{6.10}$$

where $g_{\lambda,1}$ is as in (6.5) and $\widehat{f}_\lambda$ is the periodic extension. with period ω, of the locally constant function

$$f(u) = \begin{cases} 0, & u \in [0, p), \\ 1, & u \in [p, \omega), \end{cases}$$

and M, N, P, Q are the constants

$$\begin{cases} \mathsf{M} = \frac{\alpha^2}{2\alpha^2(m-\alpha^2)}\left(\mathsf{A} + \frac{1}{\alpha^2}\mathsf{B} + \frac{1}{\alpha^2-1}\mathsf{C}\right), \\ \mathsf{N} = \frac{1}{2\alpha^2}\left(\mathsf{A} + \mathsf{C} - \frac{1}{\alpha^2}\mathsf{B}\right), \\ \mathsf{P} = \frac{1}{2\alpha^2(m-\alpha^2)}\left((2m\alpha^2 - \alpha^4 - m)\mathsf{A} + (\alpha^4 - m)\mathsf{C} + (\alpha^4 - 2\alpha^2 + m)\mathsf{B}\right), \\ \mathsf{Q} = -\frac{\alpha^2}{2(m-\alpha^2)}\left(\mathsf{A} + \frac{1}{\alpha^2-1}\mathsf{C} + \frac{1}{\alpha^2}\mathsf{B}\right). \end{cases}$$

Summarizing: the integrating factor of the second kind of a multiple root is given by

$$\eta_\lambda = \eta_{1,\lambda} + \eta_{2,\lambda}, \tag{6.11}$$

where $\eta_{1,\lambda}$ and $\eta_{2,\lambda}$ are defined as in (6.8) and (6.10).

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