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On the Cauchy–Riemann geometry of transversal curves in the 3-sphere

Emilio Musso, Lorenzo Nicolodi, and Filippo Salis

Let S^3 be the unit sphere of $\mathbb{C}^2$ with its standard Cauchy–Riemann (CR) structure. This paper investigates the CR geometry of curves in S^3 which are transversal to the contact distribution, using the local CR invariants of S^3. More specifically, the focus is on the CR geometry of transversal knots. Four global invariants of transversal knots are considered: the phase anomaly, the CR spin, the Maslov index, and the CR self-linking number. The interplay between these invariants and the Bennequin number of a knot are discussed. Next, the simplest CR invariant variational problem for generic transversal curves is considered and its closed critical curves are studied.

Key words: CR geometry of the 3-sphere, contact geometry, transversal curves, CR invariants of transversal knots, self-linking number, Bennequin number, the strain functional for transversal curves, critical knots.

Mathematical Subject Classification 2010: 53C50; 53C42; 53A10.

1. Introduction

Legendrian and transversal knots of orientable 3-dimensional contact manifolds have become very popular in contact topology since the seminal work of Bennequin [3] and nowadays represent an active and independent field of research. We refer to [16] for a comprehensive useful survey of results. The basic notion is that of contact isotopy: two transversal knots $\mathcal{K}$ and $\hat{\mathcal{K}}$ in a contact 3-manifold are *contact isotopic* if there exists an isotopy $\{\mathcal{K}_t\}_{t\in[0,1]}$ between $\mathcal{K}$ and $\hat{\mathcal{K}}$ all of whose intermediate curves are transversal. Contact isotopy is stronger than the usual notion of *topological isotopy of knots*. Thus, it is natural to search for global invariants with the aim of distinguishing the contact isotopy classes of a transversal knot within a topological class. The basic invariant is the *Bennequin number*, i.e., the self-linking number of a transversal knot with its push-off in the direction of a nowhere vanishing cross-section of the contact distribution. One of the main results in Bennequin's paper is the determination of an upper bound for the Bennequin number in terms of the Euler characteristic of a Seifert surface of the knot. In 1993, Eliashberg [14] proved that two topologically trivial transversal knots in $\mathbb{R}^3$ (and, more generally, in any tight contact 3-manifold) with the same Bennequin numbers are contact isotopic. In 1997, Fuchs and Tabachnikov [20]

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That is, knots (embedded S^1) that are always tangent, respectively, transverse to the contact plane distribution.

conjectured that topologically isotopic transversal knots in $\mathbb{R}^3$ with equal Bennequin numbers are contact isotopic. An analogue conjecture for Legendrian knots was disproved by Chekanov [9] in 2002. In 1999, Etnyre [15] proved that two transversal positive torus knots in $\mathbb{R}^3$ with the same Bennequin number are contact isotopic. In addition, Etnyre proved that the Bennequin number of a positive torus knot of type (p, q) is bounded above by $pq - p - q$.

In dimension three, every compact oriented contact manifold admits an adapted *Cauchy–Riemann (CR) structure*, namely a complex structure J on the contact distribution, such that the exterior derivative of the contact 1-form restricted to the contact distribution is a positive $(1, 1)$-form [10, 34]. Well known examples of 3-manifolds with a CR structure include the unit sphere S^3 in $\mathbb{C}^2$ and, more generally, the strictly pseudoconvex real hypersurfaces of $\mathbb{C}^2$ investigated by E. E. Levi at the beginning of the 19th century [31, 32]. According to a celebrated example of H. Lewy [33], not all CR three-manifolds arise as strictly pseudoconvex real hypersurfaces.

One important feature of CR geometry is the existence of *local differential invariants*. This fact, conjectured by Poincaré in 1907 [42], was proved by E. Cartan in 1932 for strictly pseudoconvex real hypersurfaces in $\mathbb{C}^2$ [7, 8] and by Tanaka [45] for real hypersurfaces in $\mathbb{C}^n$, $n > 3$. This result was extended to abstract CR manifolds of hypersurface type by Chern and Moser in a celebrated paper published in 1974 [11]. The local geometry of an abstract CR manifold M is encoded in a canonical principal fiber-bundle, the *Chern–Moser bundle*, equipped with a preferred connection, the *Chern–Moser connection*.

E. Cartan also defined a distinguished family of transversal curves on a CR manifold M, called *chains* (cf. Cartan [7, 8], Fefferman [19], Kock [29], Burns–Diederich–Shnider [4], Jacobowitz [26]). Through every point of M there exists a unique chain with assigned transversal direction. An intriguing description of chains can be given in terms of the *Fefferman fibration* (cf. Fefferman [19], Burns–Diederich–Shnider [4], Farrris [18], Lee [30]). For a 3-dimensional (strictly pseudoconvex) CR manifold M, the Fefferman fibration is a trivial circle bundle $N \to M$ over M equipped with a canonical conformal Lorentz structure. The fibers are null geodesics that originate a *shear-free congruence of rays* (cf. Robinson–Trautman [43, 44] and Musso [35]). Therefore, the chains of M are the projections of the null geodesics of N intersecting transversally the fibers. For an updated and complete account on the geometry of the Fefferman fibration, the reader is referred to the recent survey by Barletta and Dragomir [2].

The purpose of this paper is to investigate the CR geometry of transversal curves in S^3, where S^3 is thought of with its natural CR structure induced from $\mathbb{C}^2$. More specifically, we will focus on the CR geometry of transversal knots. In this case, the total space of the Chern–Moser bundle is a Lie group G isomorphic to $\mathrm{SU}(2, 1)$ and the Chern–Moser connection is given by the Maurer–Cartan form of G.

As in the study of knots in $\mathbb{R}^3$ (cf. Banchoff [1], Călugăreanu [6], DeTurck–

Such a CR structure is also called *strictly pseudoconvex* (cf. Section 2.1).

Gluck [12], Fuller [21], Gluck–Pan [22], Pohl [41], White [46]), a differential-geometric approach to the study of transversal knots in S^3 requires some genericity conditions. This can be formulated as follows: given a transversal knot $\mathcal{K} \subset \mathcal{S}$ and a point $p \in \mathcal{K}$, there exists a unique chain tangent to $\mathcal{K}$ at p, called the *osculating chain*. If the order of contact of the knot with its osculating chain at p is strictly larger that one, we say that p is a CR *inflection point*. A transversal knot (curve) is *generic* if it has no CR inflection points. In the framework of CR geometry, the appropriate notion of isotopy can be defined as follows: two generic transversal knots $\mathcal{K}$ and $\hat{\mathcal{K}}$ are said to be CR isotopic if there exists a contact isotopy $\{\mathcal{K}_t\}_{t\in[0,1]}$ between $\mathcal{K}$ and $\hat{\mathcal{K}}$ all of whose intermediate curves are generic. This definition is reminiscent of the notion of curvature-isotopy for knots in Euclidean 3-space. In addition, two knots $\mathcal{K}$ and $\hat{\mathcal{K}}$ are said to be *CR congruent* to each other if there exists a CR automorphism Φ of the 3-sphere, such that $\Phi(\hat{\mathcal{K}}) = \mathcal{K}$. Since the group of CR automorphisms of the 3-sphere is connected, congruent transversal knots are CR isotopic.

1.1. Description of results and organization of the paper Using the method of moving frames, to any parametrized generic transversal curve γ in S^3 we associate a canonical lift to the Lie group G, called the *Wilczynski frame field* along γ. This fact has two main consequences. The first is the existence of a canonical line element (the *infinitesimal strain*) which, in turn, can be used to distinguish preferred parametrizations (*natural parametrizations*) inducing a special affine structure on the curve. The second consequence is the existence of two basic local CR differential invariants for γ, namely the *bending* κ and the *twist* τ. The natural parametrization, the bending, and the twist completely characterize the CR congruence class of a generic transversal curve. The existence of canonical frames is also used to define various *global invariants*, such as the *phase anomaly*, the *CR spin*, the *Maslov index*, and the *CR self-linking number*.

Among all generic transversal knots, those with constant bending and twist are called *isoparametric knots*. There are two families of isoparametric knots. The CR congruence classes of the members of each family depend on a rational parameter $r \in (-2, -1/2)$, the *spectral ratio*, and a real (continuous) parameter $\rho \in (0, \sqrt{2})$, the *Clifford parameter*. The isoparametric knots of the first family are negative torus knots of type (p, q), where $\frac{p}{q} = \frac{2+r}{1+2r}$. The isoparametric knots of the second family are positive torus knots of type (p, q), where $\frac{p}{q} = \frac{2+r}{1-r}$. For both families, we compute the phase anomaly, the CR spin, and the Maslov index. Some numerical experiments are also performed to compute the Bennequin number and the CR self-linking number of isoparametric knots. These computations rely on the numerical evaluation of the appropriate Gaussian linking integrals and give convincing experimental support for supposing that the Bennequin number of an isoparametric knot is $pq - (p + q)$ and that the self-linking number is equal to pq. This experimental evidence shows that the isoparametric knots of the second family have maximal Bennequin number and is reminiscent of a result of Banchoff [1] on the self-linking number of a knot lying in a flat torus of the 3-sphere.

In the last part of the paper, we study the *total strain functional*, defined by the integral of the infinitesimal strain. We prove that the closed critical curves are isoparametric knots of the second family with spectral ratio $r = \frac{p-2q}{p+q}$ and Clifford parameter

$$\rho(r) = \sqrt{\frac{6 + 6r + 4\sqrt{3(1+r+r^2)} - \sqrt[4]{5 + 8r + 5r^2 + 3(1+r)\sqrt{3(1+r+r^2)}}}{1-r}}. \quad (1.1)$$

More specifically, the material and the results are organized as follows.

Section 2 will collect the necessary preliminary material and will set up the basic notation. We recall the notion of a 3-dimensional CR manifold and describe the standard CR structure of the 3-sphere, viewed as the strictly pseudoconvex real hypersurface $\mathcal{S} \subset \mathbb{C}P^2$ of all isotropic complex lines of $\mathbb{C}^{2,1}$, that is, $\mathbb{C}^3$ with a Hermitian scalar product of signature $(2,1)$. We write the Maurer–Cartan equation of the CR transformation group G. We then describe the Chern–Moser fibration $G \to \mathcal{S}$, the Heisenberg chart and the Heisenberg projection $\mathcal{S} \setminus \{P_\infty\} \to \mathbb{R}^3$ of the 3-sphere minus the point at infinity onto Euclidean 3-space. The latter is the analogue of the stereographic projection in Möbius geometry. Next, we consider the Fefferman fibration of the 3-sphere. The total space, equipped with its canonical Lorentz conformal structure, can be identified with a compact form of the 4-dimensional Einstein static universe. We then describe the maximal abelian subgroups of G and their orbits in the 3-sphere. The regular orbits are 2-dimensional tori, referred to as Heisenberg cyclides. They are the analogues of the regular cyclides of Dupin in classical Möbius geometry. We conclude the first section with some remarks about the chains of the 3-sphere.

Section 3 is devoted to the CR geometry of transversal curves and knots in S^3. We define the notion of CR inflection point and give a characterization of a CR inflection point in terms of the analytic contact with the osculating chain (cf. Proposition 3.14). For a generic transversal curve, that is, a transversal curve with no CR inflection points, we define the infinitesimal strain and the natural parametrization, and introduce the two main local CR invariants, namely the bending and the twist. We then construct the Wilczynski frame field along a generic transversal curve (cf. Proposition 3.18) and prove the existence and uniqueness theorem for generic transversal curves with assigned bending and twist (cf. Theorem 3.20). A geometric interpretation of the bending and the twist in terms of the Fefferman lift and of the dual curve is given. We then define the CR invariant moving trihedron and the CR normal vector field along a generic transversal curve. The existence of canonical moving frames is then reformulated in the language of exterior differential systems. We build the configuration space $Y = G \times \mathbb{R}^2$ and a Pfaffian differential system $(\mathbb{J}, \eta)$ on Y, with independence condition $\eta \in \Omega^1(Y)$, whose integral curves account for the prolongations of generic transversal curves, i.e., the Wilczynski frames, the bending, and the twist of generic transversal curves. For later use, we also consider the phase space of the Pfaffian system $(\mathbb{J}, \eta)$, its Cartan–Poincaré form, and the associated Cartan

system. In the last part of Section 3, we define the main global invariants of a generic closed transversal curve: the phase anomaly, the CR spin, the Maslov index, and the CR self-linking number of a generic transversal knot. The latter is the linking number of the knot with its push-off in the direction of the CR normal vector field.

Section 4 deals with the class of isoparametric curves, that is, generic transversal curves with constant bending and twist. To any isoparametric curve we associate a self-adjoint endomorphism H of $\mathbb{C}^{2,1}$, called the Hamiltonian of the curve, and prove that an isoparametric curve can be closed if and only if H has three real eigenvalues $e_1 < e_2 < e_3$, such that $e_1 + e_2 + e_3 = 0$ (cf. Proposition 4.4). We show that such an isoparametric curve is closed (isoparametric string) if and only if $r := e_1/e_3$ (the spectral ratio) is a rational number such that $-2 < r < -1/2$ (cf. Proposition 4.6). We then divide isoparametric strings into two classes, depending on whether $3\kappa < |\tau|^{1/2}$ (the first class), or $3\kappa > |\tau|^{1/2}$ (the second class). Proposition 4.8 proves a technical result on the causal characters of the eigenspaces of the Hamilltonian of an isoparametric string. Next, we construct explicit examples of isoparametric strings of the first class, referred to as the *symmetrical configurations of the first kind*, which depend on two parameters (r, ρ), a rational parameter $r \in (-2, -1/2)$, the spectral parameter, and a real parameter $\rho \in (0, \sqrt{2})$, the Clifford parameter. Proposition 4.10 proves that the trajectory $\mathcal{K}_{r,\rho}$ of a symmetrical configuration of the first kind is a negative torus knot of type (p, q), where the integers $p > 0$ and $q < 0$ are, respectively, the numerator and the denominator of $\frac{2+r}{1+2r}$. In Propositions 4.12 and 4.13, we compute the phase anomaly, the spin and the Maslov index of $\mathcal{K}_{r,\rho}$. In Proposition 4.16, we show that any isoparametric string of the first class is indeed congruent to a symmetrical configuration of the first kind. Next, we construct explicit examples of isoparametric strings of the second class, referred to as the *symmetrical configurations of the second kind*. They also depend on a rational parameter $r \in (-2, -1/2)$ and a real parameter $\rho \in (0, \sqrt{2})$. Proposition 4.18 proves that the trajectory $\mathcal{E}_{r,\rho}$ of a symmetrical configuration of the second kind is a positive torus knot of type (p, q), where the positive integers p and q are, respectively, the numerator and the denominator of $\frac{2+r}{1-r}$. In Propositions 4.20 and 4.21 we compute the phase anomaly, the CR spin, and the Maslov index of $\mathcal{E}_{r,\rho}$. Finally, Proposition 4.24 shows that any isoparametric string of the second class is congruent to a symmetrical configuration of the second kind.

Section 5 is devoted to the study of the total strain functional on the space of closed generic transversal curves. This is the functional defined by the integral of the infinitesimal strain. Following Griffiths' approach to the calculus of variation (cf. [23, 24]), we use the phase space of the Pfaffian system $(\mathbb{J}, \eta)$, and the related formalism, to prove that a critical curve of the total strain functional is congruent to a symmetrical configuration of the second kind $\eta_{r,\rho}$, with positive knot type (p, q), where $\frac{p}{q} \in (0, 1)$, $r = \frac{p-2q}{p+q}$, and $\rho(r)$ as in (1.1). For the study of related variational problems, the reader is referred to [13, 27, 36–39].

2. Preliminaries

2.1. Cauchy–Riemann three-manifolds

Definition 2.1. A Cauchy–Riemann (CR) 3-manifold is an orientable smooth 3-dimensional manifold M, equipped with a contact form ζ and a complex structure J on the contact distribution $\mathcal{D}$ of ζ, such that

$$d\zeta(X, JX) > 0, \tag{2.1}$$

for all nonzero $X \in \mathcal{D}$ (i.e., $d\zeta$ restricted to $\mathcal{D}$ is a positive $(1,1)$ form). The latter condition is still fulfilled if ζ is replaced with $\tilde{\zeta} = r\zeta$, where r is any strictly positive smooth function on M.

Example 2.2. Standard examples of CR 3-manifolds are provided by *strictly pseudoconvex* (or *Levi definite*) real hypersurfaces of a complex surface. If M is a smooth real hypersurface of a complex surface N, then locally M is the zero locus of a real-valued function $F(x_1, y_1, x_2, y_2)$, where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are holomorphic coordinates on N. Let ζ be the restriction to $T(M)$ of the 1-form

$$\sum_{j=1}^{2} \partial_{y_j} F dx^j - \sum_{j=1}^{2} \partial_{x_j} F dy^j.$$

If M is strictly pseudoconvex, then ζ is a contact form. If $\mathcal{D} \subset T(M)$ is the contact distribution, then $\mathcal{D}|_p$ is a 1-dimensional complex subspace of $T_p(N)$, for every $p \in M$. Consequently, $\mathcal{D}$ inherits a complex structure tamed by $d\zeta|_{\mathcal{D}\times\mathcal{D}}$. It is a classical result of Eugenio Elia Levi [31, 32] that the (smooth) boundary of a domain of holomorphy is pseudoconvex.

Definition 2.3. Let M be a 3-dimensional CR manifold and $p \in M$. A basis (v_1, v_2, v_3) of the tangent space $T_p(M)$ is said to be *adapted* to M if

$$v_2,\ v_3 = J(v_2) \in \mathcal{D} \quad \text{and} \quad d\zeta(v_2, v_3) = \zeta(v_1).$$

Definition 2.4. Let K be the closed subgroup of $\mathrm{GL}(3, \mathbb{R})$ consisting of all 3×3 matrices of the form

$$Y(\rho, \phi, p, q) = \begin{pmatrix} \frac{1}{\rho^2} & 0 & 0 \\ \frac{p}{\rho} & \frac{\cos 3\phi}{\rho} & \frac{\sin 3\phi}{\rho} \\ \frac{q}{\rho} & -\frac{\sin 3\phi}{\rho} & \frac{\cos 3\phi}{\rho} \end{pmatrix}, \tag{2.2}$$

where $\rho > 0$ and ϕ, p, $q \in \mathbb{R}$.

It is not difficult to prove the following.

Proposition 2.5. *The totality $P(M)$ of all adapted basis of M is a reduction of the linear frame bundle $L(M)$ of M with structure group K. We call $P(M)$ the structure bundle of the CR manifold M.*

If (2.1) is satisfied, the CR manifold M is said to be *strictly pseudoconvex* (cf. [29, 30]).

Definition 2.6. An *automorphism* of a 3-dimensional CR manifold M is a contactomorphism $\Phi : (M, \zeta) \to (M, \zeta)$, such that the restriction to the contact distribution of its differential Φ_* commutes with J.

Contrary to what happens for contact structures, one interesting feature of CR structures is the existence of local differential invariants. This implies that the automorphism group of a 3-dimensional CR manifold M is a Lie transformation group of dimension less or equal than 8. The equality is attained when M is the 3-dimensional sphere equipped with its standard CR structure described below (cf. [7, 8, 11, 42]).

2.2. The standard CR structure on the 3-sphere Let $\mathbb{C}^{2,1}$ denote $\mathbb{C}^3$ with the indefinite Hermitian scalar product of signature $(2, 1)$ given by

$$\langle \mathbf{z}, \mathbf{w} \rangle = {}^t\overline{\mathbf{z}}\,\mathbf{h}\,\mathbf{w}, \quad \mathbf{h} = (h_{ij}) = \begin{pmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{pmatrix}. \tag{2.3}$$

Let G be the special pseudounitary group of (2.3), i.e., the group of unimodular complex 3×3 matrices preserving the indefinite Hermitian scalar product (2.3),

$$G = \{A \in \mathrm{SL}(3, \mathbb{C}) \mid {}^t\bar{A}\mathbf{h}A = \mathbf{h}\}. \tag{2.4}$$

Let $\mathfrak{g}$ be the Lie algebra of G,

$$\mathfrak{g} = \{X \in \mathfrak{sl}(3, \mathbb{C}) \mid {}^t\bar{X}\mathbf{h} + \mathbf{h}X = 0\}. \tag{2.5}$$

The center of G is $Z = \{\varepsilon I_3 \mid \varepsilon \in \mathbb{C},\ \varepsilon^3 = 1\} \cong \mathbb{Z}_3$, where I_3 denotes the 3×3 identity matrix. Let $[G]$ denote the quotient Lie group G/Z. Given any $A \in G$, its equivalence class in $[G]$ is denoted by $[A]$. Thus $[A] = [B]$ if and only if $B = \epsilon A$, for some cube root of unity ε.

For any $A \in G$, the column vectors (A_1, A_2, A_3) of A form a basis of $\mathbb{C}^{2,1}$ satisfying $\langle A_i, A_j \rangle = h_{ij}$ and $\det(A_1, A_2, A_3) = 1$. Such a basis is referred to as a *light-cone basis*. On the other hand, a basis (U_1, U_2, U_3) of $\mathbb{C}^{2,1}$ such that $\det(U_1, U_2, U_3) = 1$ and $\langle U_i, U_j \rangle = \delta_{ij}\epsilon_j$, where $\epsilon_1 = \epsilon_2 = 1$, $\epsilon_3 = -1$, is referred to as a *pseudo-unitary basis*.

We adhere to the terminology commonly used in relativity and say that a nonzero vector $\mathbf{z} \in \mathbb{C}^{2,1}$ is *spacelike*, *timelike* or *lightlike*, depending on whether $\langle \mathbf{z}, \mathbf{z} \rangle$ is positive, negative or zero. The cone of all lightlike vectors will be denoted by $\mathcal{N}$.

The most effective way of describing the standard CR structure of the 3-sphere $\mathrm{S}^3 \subset \mathbb{C}^2$ and the group of its CR automorphisms is to realize the 3-sphere as the real hypersurface $\mathcal{S}$ of $\mathbb{C}P^2$, defined by

$$\mathcal{S} = \big\{[\mathbf{z}] \in \mathbb{C}P^2 \mid \langle \mathbf{z}, \mathbf{z} \rangle = i(\overline{z}_1 z_3 - \overline{z}_3 z_1) + \overline{z}_2 z_2 = 0\big\} \tag{2.6}$$

where $\mathbf{z} = {}^t(z_1, z_2, z_3)$ are homogeneous coordinates in $\mathbb{C}P^2$. The restriction of the affine chart

$$s : \mathbb{C}^2 \ni (z_1, z_2) \mapsto \left[{}^t\Big(\frac{1+z_1}{2}, i\frac{z_2}{\sqrt{2}}, i\frac{1-z_1}{2} \Big) \right] \in \mathcal{S} \subset \mathbb{C}P^2 \tag{2.7}$$

to the unit sphere S^3 of $\mathbb{C}^2$ defines a smooth diffeomorphism between S^3 and $\mathcal{S}$. For each $p = [\mathbf{z}] \in \mathcal{S}$, the differential $(1,0)$-form

$$\tilde{\zeta}\big|_p = -\frac{i\langle \mathbf{z}, \mathbf{dz}\rangle}{\overline{\mathbf{z}}\,{}^t\mathbf{z}}\Big|_p \in \Omega^{1,0}(\mathbb{C}P^2)\big|_p \tag{2.8}$$

is well defined. In addition, the null space of the imaginary part of $\tilde{\zeta}|_p$ is $T(\mathcal{S})|_p$, namely the tangent space of $\mathcal{S}$ at p. Thus, the restriction of $\tilde{\zeta}$ to $T(\mathcal{S})$ is a real-valued 1-form $\zeta \in \Omega^1(\mathcal{S})$. Since the pullback of ζ by the diffeomorphism $s : \mathrm{S}^3 \to \mathcal{S}$ is the standard contact form $i\overline{\mathbf{z}} \cdot d\mathbf{z}|_{\mathrm{S}^3}$ of S^3, then ζ is a contact form whose contact distribution $\mathcal{D}$ is, by construction, a complex sub-bundle of $T(\mathbb{C}P^2)|_{\mathcal{S}}$. Therefore, $\mathcal{D}$ inherits from $T(\mathbb{C}P^2)\big|_{\mathcal{S}}$ a complex structure J.

Using the standard identification $T(\mathbb{C}P^2)|_{[\mathbf{z}]} \cong \mathbb{C}^3/[\mathbf{z}]$, the contact distribution $\mathcal{D}|_{[\mathbf{z}]}$ is the complex subspace $[\mathbf{z}]^\perp/[\mathbf{z}]$, where $[\mathbf{z}]^\perp$ is the orthogonal complement of the complex line $[\mathbf{z}]$, with respect to the pseudohermitian scalar product $\langle\,,\rangle$. Since $[\mathbf{z}]$ is an isotropic line, $\langle\,,\rangle$ induces a positive definite Hermitian scalar product on $[\mathbf{z}]^\perp/[\mathbf{z}]$. Note that the complex structure on $\mathcal{D}|_{[\mathbf{z}]} \cong [\mathbf{z}]^\perp/[\mathbf{z}]$ is just the multiplication by i. Using these identifications, we have

$$d\zeta\big|_{[\mathbf{z}]}(U, JV) = \langle U, V\rangle,\ \forall\, U, V \in [\mathbf{z}]^\perp/[\mathbf{z}]. \tag{2.9}$$

Then, $d\zeta|_{\mathcal{D}\times\mathcal{D}}$ is tamed by J and $\mathcal{S}$, equipped with the contact form ζ and the complex structure J on the contact distribution $\mathcal{D}$, is a CR manifold.

2.3. The structure equations Let α_1^1, β_1^1, α_1^2, β_1^2, α_1^3, α_3^2, β_3^2, α_3^1 be the entries of the Maurer–Cartan form

$$A^{-1}dA = \begin{pmatrix} \alpha_1^1 + i\beta_1^1 & -i\alpha_3^2 - \beta_3^2 & \alpha_3^1 \\ \alpha_1^2 + i\beta_1^2 & -2i\beta_1^1 & \alpha_3^2 + i\beta_3^2 \\ \alpha_1^3 & i\alpha_1^2 + \beta_1^2 & -\alpha_1^1 + i\beta_1^1 \end{pmatrix} \tag{2.10}$$

of the special unitary group G of the pseudohermitian scalar product (2.3). Then

$$(\alpha_1^1, \beta_1^1, \alpha_1^2, \beta_1^2, \alpha_1^3, \alpha_3^2, \beta_3^2, \alpha_3^1)$$

is a basis of the dual space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G satisfying the following Maurer–Cartan equations:

$$\begin{cases} d\alpha_1^1 &= -\alpha_3^2\wedge\beta_1^2+\beta_3^2\wedge\alpha_1^2-\alpha_3^1\wedge\alpha_1^3\\ d\beta_1^1 &= \alpha_3^2\wedge\alpha_1^2+\beta_3^2\wedge\beta_1^2\\ d\alpha_1^2 &= \alpha_1^1\wedge\alpha_1^2-3\beta_1^1\wedge\beta_1^2-\alpha_3^2\wedge\alpha_1^3\\ d\beta_1^2 &= 3\beta_1^1\wedge\alpha_1^2+\alpha_1^1\wedge\beta_1^2-\beta_3^2\wedge\alpha_1^3\\ d\alpha_1^3 &= 2\alpha_1^1\wedge\alpha_1^3+2\alpha_1^2\wedge\beta_1^2\\ d\alpha_3^2 &= -\alpha_1^2\wedge\alpha_3^1-\alpha_1^1\wedge\alpha_3^2-3\beta_1^1\wedge\beta_3^2\\ d\beta_3^2 &= -\beta_1^2\wedge\alpha_3^1+3\beta_1^1\wedge\alpha_3^2-\alpha_1^1\wedge\beta_3^2\\ d\alpha_3^1 &= -2\alpha_1^1\wedge\alpha_3^1-2\alpha_3^2\wedge\beta_3^2. \end{cases} \tag{2.11}$$

2.4. The Chern–Moser fibration The group G acts transitively and almost effectively on $\mathcal{S}$ by

$$A[\mathbf{z}]=[A\mathbf{z}],\quad \forall\ A\in G,\ \forall\ [\mathbf{z}]\in\mathcal{S}. \tag{2.12}$$

This action descends to an effective action of $[G]=G/Z$ on $\mathcal{S}$. It is a classical result of E. Cartan [7,8,11] that $[G]$ is the group of CR automorphisms of $\mathcal{S}$.

The natural projection

$$\pi_{\mathcal{S}}: G\ni A\mapsto [A_1]\in\mathcal{S} \tag{2.13}$$

makes G into a (trivial) principal fiber-bundle with structure group

$$G_0=\left\{A\in G\mid A[{}^t(1,0,0)]=[{}^t(1,0,0)]\right\}. \tag{2.14}$$

The elements of G_0 consist of all 3×3 unimodular matrices of the form

$$\mathbf{X}(\rho,\theta,v,r)=\begin{pmatrix}\rho e^{i\theta} & -i\rho e^{-i\theta}\overline{v} & e^{i\theta}(r-\frac{i}{2}\rho|v|^2)\\ 0 & e^{-2i\theta} & v\\ 0 & 0 & \rho^{-1}e^{i\theta}\end{pmatrix}, \tag{2.15}$$

where $v\in\mathbb{C}$, $r\in\mathbb{R}$, $0\le\theta<2\pi$, and $\rho>0$.

Definition 2.7. The principal bundle $\pi_{\mathcal{S}}: G\to\mathcal{S}$ is the *Chern–Moser bundle* of $\mathcal{S}$.

Remark 2.8. The left-invariant 1-forms α_1^2, β_1^2, α_1^3 are semi-basic and linearly independent. So, if $s: U\subseteq\mathcal{S}\to G$ is a local cross section of $\pi_{\mathcal{S}}$, then

$$\left(s^*\alpha_1^3, s^*\alpha_1^2, s^*\beta_1^2\right)$$

defines a coframe on U and $s^*\alpha_1^3$ is a positive contact form. Let s_1, s_2, and s_3 be the parallelization of U dual to the coframe. Then $\mathbf{s}=(s_1,s_2,s_3)$ is a local cross-section of the structure bundle $P(\mathcal{S})$. If $\tilde{s}: U\subseteq\mathcal{S}\to G$ is another cross section of the canonical bundle and

$$\tilde{s}=sX(\rho,\theta,v,r),$$

then

$$\tilde{s}^*\begin{pmatrix}\alpha_1^3\\ \alpha_1^2\\ \beta_1^2\end{pmatrix}\begin{pmatrix}\rho^2 & 0 & 0\\ \rho^2[-p\cos 3\theta+q\sin 3\theta] & \rho\cos 3\theta & -\rho\sin 3\theta\\ \rho^2[-p\sin 3\theta-q\cos 3\theta] & \rho\sin 3\theta & \rho\cos 3\theta\end{pmatrix}s^*\begin{pmatrix}\alpha_1^3\\ \alpha_1^2\\ \beta_1^2\end{pmatrix} \tag{2.16}$$

where p and q are the real and the imaginary parts of v. This implies

$$\tilde{\mathbf{s}}=\mathbf{s}\begin{pmatrix}\frac{1}{\rho^2} & 0 & 0\\ \frac{p}{\rho} & \frac{\cos 3\theta}{\rho} & \frac{\sin 3\theta}{\rho}\\ \frac{q}{\rho} & -\frac{\sin 3\theta}{\rho} & \frac{\cos 3\theta}{\rho}\end{pmatrix} \tag{2.17}$$

Thus, the adapted linear frame $\tilde{\mathbf{s}}$ at a point $[\mathbf{z}]\in U$ only depends on the value of s at $[\mathbf{z}]$. Therefore, there exists a unique map

$$\Lambda : G\to P(\mathcal{S}) \tag{2.18}$$

such that $\Lambda(A)=\mathbf{s}([A_1])$, where $\mathbf{s}$ is the adapted linear frame field induced by any local cross section $s : U\to G$ defined in an open neighborhood U of $[A_1]$ such that $s([A_1])=A$. Let $\varphi : G_0\to K$ be the 3-dimensional representation of G_0 onto the subgroup $K\subset \mathrm{GL}_+(3,\mathbb{R})$ (cf. Definition 2.4) defined by

$$\varphi : G_0\ni X(\rho,\theta,p+iq,r)\mapsto\begin{pmatrix}\frac{1}{\rho^2} & 0 & 0\\ \frac{p}{\rho} & \frac{\cos 3\theta}{\rho} & \frac{\sin 3\theta}{\rho}\\ \frac{q}{\rho} & -\frac{\sin 3\theta}{\rho} & \frac{\cos 3\theta}{\rho}\end{pmatrix}\in \mathrm{GL}_+(3,\mathbb{R}), \tag{2.19}$$

then

$$\Lambda(AX)=\Lambda(A)\varphi(X),\qquad \forall\, A\in G,\ \forall\, X\in G_0. \tag{2.20}$$

From this, it follows that the map Λ gives G the structure of a principal fiber bundle over $P(\mathcal{S})$ with structure group

$$\left\{\begin{pmatrix}\varepsilon & 0 & r\varepsilon\\ 0 & \varepsilon & 0\\ 0 & 0 & \varepsilon\end{pmatrix}\mid \varepsilon\in\mathbb{C},\ \varepsilon^3=1,\ r\in\mathbb{R}\right\}. \tag{2.21}$$

In addition, the tangent bundle $T(\mathcal{S})$ can be canonically identified with $G\times_\varphi\mathbb{R}^3$, that is, $T(\mathcal{S})$ is the quotient manifold of $G\times\mathbb{R}^3$ by the free action of G_0 on $G\times\mathbb{R}^3$, given by

$$(A,\mathbf{u})*Y=(AY,\varphi(Y)^{-1}\mathbf{u}). \tag{2.22}$$

2.5. The Heisenberg group and the Heisenberg projection Consider $P_0={}^t(1,0,0)]\in\mathcal{S}$ and $P_\infty=[{}^t(0,0,1)]\in\mathcal{S}$ as the origin and the point at infinity of $\mathcal{S}$. Then, $\dot{\mathcal{S}}:=\mathcal{S}\setminus\{P_\infty\}$ can be identified with Euclidean 3-space with its standard contact structure $dz-ydx+xdy$ by means of the *Heisenberg projection*

$$p_H : \dot{\mathcal{S}}\ni[\mathbf{z}]\mapsto {}^t(\Re(z_2/z_1),\Im(z_2/z_1),\Re(z_3/z_1))\in\mathbb{R}^3.$$

This map is the analogue of the stereographic projection in Möbius (conformal) geometry.

The inverse of the Heisenberg projection is the *Heisenberg chart*

$$j_H : \mathbb{R}^3 \ni {}^t(x, y, z) \mapsto \left[{}^t\big(1, x + iy, z + \frac{i}{2}(x^2 + y^2)\big)\right] \in \dot{\mathcal{S}}.$$

The Heisenberg chart j_H can be lifted to a map

$$\mathcal{J}_H : \mathbb{R}^3 \ni {}^t(x, y, z) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ x + iy & 1 & 0 \\ z + \frac{i(x+iy)}{2} & ix + y & 1 \end{pmatrix} \in G \tag{2.23}$$

whose image is a 3-dimensional closed subgroup $\mathbb{H}^3$ of G, which is isomorphic to the 3-dimensional *Heisenberg group*.

Remark 2.9. Under the identification $\dot{\mathcal{S}} \cong \mathbb{R}^3$, $\mathcal{J}_H$ originates the cross section

$$s_H : \dot{\mathcal{S}} \ni [\mathbf{v}]_{\mathbb{C}} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ v_1^{-1}v_2 & 1 & 0 \\ v_1^{-1}v_3 & \overline{iv_1^{-1}v_2} & 1 \end{pmatrix} \in G \tag{2.24}$$

which, in turns, gives rise to the adapted linear frame field $\mathbf{s}_H : \dot{\mathcal{S}} \to P(\mathcal{S})$. By construction,

$$s_H{}^*(\alpha_1^3) = dz + xdy - ydx, \quad s_H{}^*(\alpha_1^2) = dx, \quad s_H{}^*(\beta_1^2) = dy,$$

where x, y and z are the Heisenberg coordinates. This implies that

$$\mathbf{s}_H = (\partial_z, \partial_x + y\partial_z, \partial_y - x\partial_z).$$

If $s : U \subseteq \dot{\mathcal{S}} \to G$ is any other local cross-section, then $s_H = sY(\rho, \theta, p + iq, r)$, where $Y(\rho, \theta, p + iq, r) : U \to G_0$ is a smooth map. The linear frames $\mathbf{s}$, $\mathbf{s}_H : U \to P(\mathcal{S})$ are related by

$$\mathbf{s} = \mathbf{s}_H \varphi(Y)^{-1}.$$

From this we obtain

$$\begin{cases} s_1 = \frac{1}{\rho^2}\partial_z + \frac{p}{\rho}(\partial_x + y\partial_z) + \frac{q}{\rho}(\partial_y - x\partial_z), \\ s_2 = \frac{\cos 3\theta}{\rho}(\partial_x + y\partial_z) - \frac{\sin 3\theta}{\rho}(\partial_y - x\partial_z), \\ s_3 = \frac{\sin 3\theta}{\rho}(\partial_x + y\partial_z) + \frac{\cos 3\theta}{\rho}(\partial_y - x\partial_z). \end{cases} \tag{2.25}$$

2.6. The Fefferman fibration Let $\mathcal{E} \subset \mathbb{R}P^5$ be the nondegenerate smooth hyperquadric of $\mathbb{R}P^5$ consisting of all lightlike real lines of $\mathbb{C}^{2,1} \cong \mathbb{R}^{2,4}$. Let $x_1 + iy_1$, $x_2 + iy_2$, $x_3 + iy_3$ be the coordinates of $\mathbb{C}^{2,1}$, with respect to the pseudo-unitary basis (U_1, U_2, U_3) given by

$$U_1 = {}^t\big(\frac{1}{\sqrt{2}}, 0, \frac{i}{\sqrt{2}}\big), \quad U_2 = {}^t\big(0, 1, 0\big), \quad U_3 = {}^t\big(\frac{i}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\big).$$

In these coordinates, the equation defining $\mathcal{E}$ takes the form

$$-x_1^2 - y_1^2 + x_2^2 + y_2^2 + x_3^2 + y_3^2 = 0.$$

The restriction to $\mathcal{E}$ of the quadratic form

$$g_{\mathcal{E}} = 2\frac{-dx_1^2 - dy_1^2 + dx_2^2 + dy_2^2 + dx_3^2 + dy_3^2}{x_1^2 + y_1^2 + x_2^2 + y_2^2 + x_3^2 + y_3^2}$$

defines a Lorentz pseudo-metric on $\mathcal{E}$ and the action of G on $\mathcal{E}$ is by conformal transformations. The map $\varphi : \mathrm{S}^1 \times \mathrm{S}^3 \subset \mathbb{R}^2 \times \mathbb{R}^4 \to \mathcal{E}$,

$$(x_1 + iy_1, x_2 + iy_2, x_3 + iy_3) \mapsto \Big[\sum_{j=1}^{3}(x_j + iy_j)U_j\Big]_{\mathbb{R}}, \tag{2.26}$$

is a smooth diffeomorphism such that

$$\varphi^*(g_{\mathcal{E}}) = -dx_1^2 - dy_1^2 + dx_2^2 + dy_2^2 + dx_3^2 + dy_3^2.$$

Thus, $\mathcal{E}$ can be identified with the compact form $(\mathrm{S}^1 \times \mathrm{S}^3, \varphi^*(g_{\mathcal{E}}))$ of the Einstein static universe [13]. The map

$$\Psi : \mathcal{E} \ni [V]_{\mathbb{R}} \mapsto [V]_{\mathbb{C}} \in \mathcal{S}$$

makes $\mathcal{E}$ into a trivial circle bundle over $\mathcal{S}$. This is called the *Fefferman fibration*. The action of the structure group on $\mathcal{E}$ is given by

$$\Big[\sum_{j=1}^{3}(x_j + iy_j)U_j\Big]_{\mathbb{R}} * e^{it} = \Big[\sum_{j=1}^{3}(x_j + iy_j)U_j\Big]_{\mathbb{R}}$$

and the map

$$\mathcal{S} \ni \left[{}^t\Big(\frac{1+w_1}{2}, \frac{iw_2}{\sqrt{2}}, i\frac{1-w_1}{2}\Big)\right]_{\mathbb{C}} \mapsto \left[{}^t\Big(\frac{1+w_1}{2}, \frac{iw_2}{\sqrt{2}}, i\frac{1-w_1}{2}\Big)\right]_{\mathbb{R}} \in \mathcal{E}$$

defines a global cross section of the Fefferman fibration.

Remark 2.10. The fibers of the Fefferman fibration are lightlike geodesics of the Einstein static pseudo-metric and originates a share-free congruence of rays [2, 35, 43, 44]. The group G can be viewed as the group of orientation and time-orientation preserving conformal transformations of $\mathcal{E}$ which in addition preserves the shear-free congruence of rays.

2.7. Maximal compact abelian subgroups The maximal compact abelian subgroups of G are conjugate to the 2-dimensional torus

$$\mathrm{T}^2 = \{R(\theta_1, \theta_2) \mid \theta_1, \theta_2 \in \mathbb{R}\} \subset G, \tag{2.27}$$

where

$$R(\theta_1, \theta_2) = \begin{pmatrix} e^{-\frac{i}{6}(\theta_1+2\theta_2)}\cos\frac{\theta_1}{2} & 0 & e^{-\frac{i}{6}(\theta_1+2\theta_2)}\sin\frac{\theta_1}{2} \\ 0 & e^{\frac{i}{3}(\theta_1+2\theta_2)} & 0 \\ -e^{-\frac{i}{6}(\theta_1+2\theta_2)}\sin\frac{\theta_1}{2} & 0 & e^{-\frac{i}{6}(\theta_1+2\theta_2)}\cos\frac{\theta_1}{2} \end{pmatrix}. \tag{2.28}$$

Definition 2.11. We call θ_1 and θ_2 the *Clifford angle-variables* of T^2.

Note that $R(\theta_1, \theta_2)$ can be factorized as

$$R(\theta_1, \theta_2) = R(\theta_1, 0)R(0, \theta_2). \tag{2.29}$$

Correspondingly, let K' and K'' be the subgroups

$$K' = \{R(\theta, 0) \mid \theta \in \mathbb{R}\}, \qquad K'' = \{R(0, \theta) \mid \theta \in \mathbb{R}\}. \tag{2.30}$$

Within the Heisenberg model, the action of $R(\theta_1, \theta_2)$ as a pseudo-group of contact transformations is given by

$$R(\theta_1, \theta_2)V = p_H \left(R(\theta_1, \theta_2) j_H(V)\right). \tag{2.31}$$

It follows that $R(0, \theta_2)$ is the counterclockwise rotation of an angle θ_2 around the positive oriented z-axis and $R(\theta_1, 0)$ is a "counterclockwise elliptical toroidal rotation" of angle θ_1 around the Clifford circle: $x^2 + y^2 = 2$, $z = 0$.

The arc

$$\Sigma = \left\{ P(\rho) = \left[{}^t\!\left(1, \rho, \frac{i}{2}\rho^2\right) \right] \mid 0 \leq \rho \leq \sqrt{2} \right\}$$

is a slice for the action of T^2 on $\mathcal{S}$. Let $\mathcal{T}_\rho$ denote the orbit of T^2 through the point $P(\rho)$. The singular orbits are $\mathcal{T}_0$ and $\mathcal{T}_{\sqrt{2}}$. The Heisenberg projection of $\mathcal{T}_0$ is the z-axis, while the Heisenberg projection of $\mathcal{T}_{\sqrt{2}}$ is the the Clifford circle. The regular orbits of T^2 are the *standard Heisenberg cyclides* with parameter $\rho \in (0, \sqrt{2})$, that is, the rotationally invariant 2-dimensional tori $\mathcal{T}_\rho$, whose parametric equations in the Heisenberg coordinates take the form $F_\rho(\theta_1, \theta_2) = R_z(\theta_2)\eta_\rho(\theta_1)$. Here $R_z(\theta_2)$ denotes the rotation through angle θ_2 around the z-axis and $\eta_\rho(\theta_1) = {}^t\!(x_\rho(\theta_1), y_\rho(\theta_1), z_\rho(\theta_1))$ is defined by

$$\begin{cases} x_\rho(\theta_1) = 2\rho \left(\frac{2+\rho^2+(2-\rho^2)\cos\theta_1}{4+\rho^4+(4-\rho^4)\cos\theta_1} \right), \\ y_\rho(\theta_1) = -\frac{2\rho(\rho^2-2)\sin\theta_1}{4+\rho^4+(4-\rho^4)\cos\theta_1}, \\ z_\rho(\theta_1) = \frac{(\rho^4-4)\sin\theta_1}{4+\rho^4+(4-\rho^4)\cos\theta_1}. \end{cases} \tag{2.32}$$

Definition 2.12. Let $\hat{\mathrm{T}}^2$ be any maximal compact abelian subgroup of G. Then, $\hat{\mathrm{T}}^2 = B\mathrm{T}^2B^{-1}$, and hence $\mathcal{O}_1 = B\mathcal{T}_0B^{-1}$ and $\mathcal{O}_2 = B\mathcal{T}_{\sqrt{2}}B^{-1}$ are the two singular orbits of the action of $\hat{\mathrm{T}}^2$ on $\mathcal{S}$. We call $\mathcal{O}_1$ and $\mathcal{O}_2$ the *axes of symmetry* of the maximal torus $\hat{\mathrm{T}}^2$. The regular orbits of the action of $\hat{\mathrm{T}}^2$ on $\mathcal{S}$ are 2-dimensional tori, referred to as *Heisenberg cyclides*. They can be obtained via the action of G on standard ones.

The terminology "Heisenberg cyclide" is used is analogy with the classical notion of *cyclides of Dupin* in Möbius geometry (cf. [28] and the literature therein).

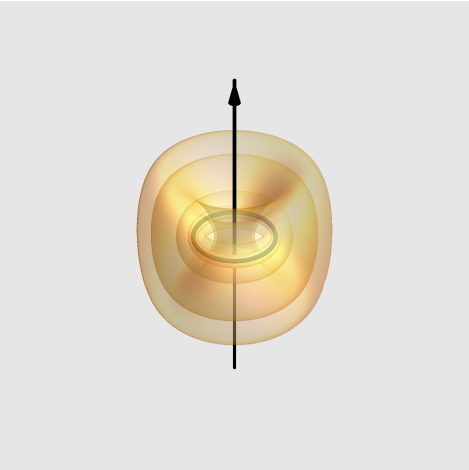

Fig. 2.1: Standard Heisenberg cyclides with parameters $\rho = 0.6, 0.7, 1.6$, and the two singular orbits (the z-axis and the Clifford circle).

2.8. Chains We start by recalling the following.

Definition 2.13 ([7, 8, 11, 19, 29])**.** A curve of $\mathcal{S}$ consisting of the lightlike lines orthogonal to a fixed spacelike vector is said to be a *chain*. For a given spacelike vector $\mathbf{S} = (S_1, S_2, S_3)$, the corresponding chain is denoted by

$$\mathcal{C}_{[\mathbf{S}]} = \{[\mathbf{z}] \in \mathcal{S} \mid \langle \mathbf{S}, \mathbf{z} \rangle = 0\}.$$

Remark 2.14. The totality of all chains can be identified with the open domain $\Omega_+ \subset \mathbb{P}(\mathbb{C}^{2,1})$ defined by

$$-iz_1\overline{z}_3 + iz_3\overline{z}_1 + z_2\overline{z}_2 > 0.$$

Note that Ω_+ is naturally equipped with a G-invariant Kähler structure of type $(1, 1)$.

Example 2.15. Let (E_1, E_2, E_3) be the standard basis of $\mathbb{C}^{2,1}$. The singular orbit $\mathcal{T}_0 \subset \mathcal{S}$ of the maximal torus T^2 is the chain of all lightlike lines of $\mathbb{P}(\mathbb{C}^{2,1})$ orthogonal to the spacelike vector $U_+ = E_1 + iE_3$. The other singular orbit $\mathcal{T}_{\sqrt{2}} \subset \mathcal{S}$ is the chain of all lightlike lines of $\mathbb{P}(\mathbb{C}^{2,1})$ orthogonal to the spacelike vector $U_- = iE_1 + E_3$. Since G acts transitively on chains, the chains are the singular orbits of a maximal compact abelian subgroup of G.

In the Heisenberg model, excluding the lines parallel to the z-axis (that is, the chains passing through the point at infinity), all other chains are ellipses (possibly circles) whose projections onto the xy-plane are circles. More precisely, if $\mathbf{S} = (S_1, S_2, S_3)$ is a spacelike vector such that $S_1 \neq 0$, then the Heisenberg projection

of the chain $\mathcal{C}_{[\mathbf{S}]}$ is the ellipse parameterized by

$$\begin{cases} x(t) = \frac{\|\mathbf{S}\|}{|S_1|}\cos t + \Re(\frac{S_2}{S_1}), \\ y(t) = \frac{\|\mathbf{S}\|}{|S_1|}\sin t + \Im(\frac{S_2}{S_1}), \\ z(t) = \frac{\|\mathbf{S}\|}{|S_1|}\left(\Im(\frac{S_2}{S_1})\cos t - \Re(\frac{S_2}{S_1})\sin t\right) + \Re(\frac{S_3}{S_1}). \end{cases} \tag{2.33}$$

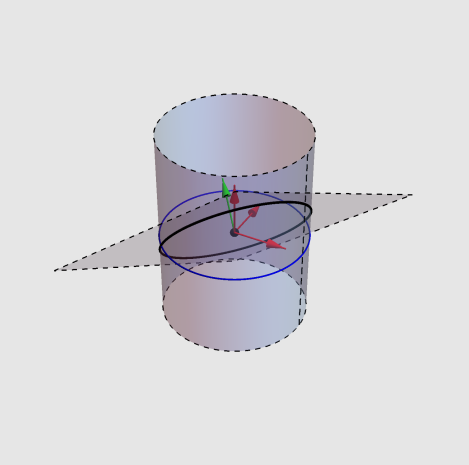

Fig. 2.2: The Heisenberg model of the chain $\mathcal{C}_{[\mathbf{S}]}$, where $\mathbf{S} = {}^t(0.9, e^{i\frac{\pi}{4}}, -i)$.

We have the following.

Proposition 2.16. *The chains are transversal to the contact distribution. Moreover, for each point $p \in \mathcal{S}$ and for each tangent vector $\mathbf{v} \in T_p(\mathcal{S})$, with $\mathbf{v} \notin \mathcal{D}_p$, there exists a unique chain passing through p and tangent to $\mathbf{v}$ at p.*

Remark 2.17. In the Heisenberg model, the chain passing through the point (x_0, y_0, z_0) and tangent to the vector $\mathbf{v} = (x_1, y_1, z_1)$, such that $(x_1)^2 + (y_1)^2 \neq 0$, is the smooth curve parametrized by

$$\begin{cases} x(t) = \frac{1}{2c_1}\big(y_1 + 2c_1x_0 - y_1\cos(2c_1t) + x_1\sin(2c_1t)\big), \\ y(t) = \frac{1}{2c_1}\big(-x_1 + 2c_1y_0 + x_1\cos(2c_1t) + y_1\sin(2c_1t)\big), \\ z(t) = z_0 + \frac{(x_0x_1+y_0y_1)}{2c_1} - \frac{x_0x_1+y_0y_1}{2c_1}\cos(2c_1t) + \frac{x_1^2+2c_1(x_0y_1-x_1y_0)+y_1^2}{4c_1^2}\sin(2c_1t), \end{cases}$$

where $c_1 = \frac{x_1^2+y_1^2}{2(x_1y-xy_1-z_1)}$.

3. CR geometry of transversal curves

3.1. Transversal curves, lifts, CR inflection points, and CR isotopies

Definition 3.1. Let $\gamma : J \to \mathcal{S}$ be a smooth curve. We say that γ is *transversal* (to the contact distribution $\mathcal{D}$) if the tangent vector $\dot\gamma(t) \notin \mathcal{D}|_{\gamma(t)}$, for every

$t \in J$. The parametrization γ is said to be *positive* if $\zeta(\dot\gamma(t)) > 0$, for every t and for every positive contact form compatible with the CR structure. From now on, we will assume that the parametrization of a transversal curve is positive.

Definition 3.2. Let $\gamma : J \to \mathcal{S}$ be a smooth curve. A *lift* of γ is a map $\Gamma : J \to \mathcal{N}$ into the null-cone $\mathcal{N} \subset \mathbb{C}^{2,1}$, such that $\gamma(t) = [\Gamma(t)]$, for every $t \in J$.

If Γ is a lift, any other lift is given by $r\Gamma$, where r is a smooth complex-valued function, such that $r(t) \neq 0$, for every $t \in J$. From the definition of the contact distribution, we have the following.

Proposition 3.3. *A parameterized curve $\gamma : J \to \mathcal{S}$ is transversal and positively oriented if and only if $i\langle\Gamma, \Gamma'\rangle|_t > 0$, for every $t \in J$ and every lift Γ.*

Definition 3.4. A *frame field along* γ is a lift $A : J \to G$ of γ to the total space of the Chern–Moser fibration $\pi_{\mathcal{S}} : A \in G \to [A_1] \in \mathcal{S}$. Since the canonical fibration is trivial, frame fields do exist for every transversal curve. Note that if A is a frame field along γ, the first column vector A_1 of A is a lift of γ.

Let A be a frame field along γ. Then

$$A^{-1}A' = \begin{pmatrix} a_1^1 + ib_1^1 & -ia_3^2 - b_3^2 & a_3^1 \\ a_1^2 + ib_1^2 & -2ib_1^1 & a_3^2 + ib_3^2 \\ a_1^3 & ia_1^2 + b_1^2 & -a_1^1 + ib_1^1 \end{pmatrix},$$

where a_1^3 is a strictly positive real-value function. Any other frame field along γ is given by $\tilde A = AX$, where $X : J \to G_0$ is a smooth map with values in the structural group of the canonical fibration. We then write

$$X(t) = X(\rho(t), \varphi(t), v(t), r(t))$$

where ρ, r, φ are real-valued functions, with $\rho > 0$, and v is complex-valued. If we let

$$\tilde A^{-1}\tilde A' = \begin{pmatrix} \tilde a_1^1 + i\tilde b_1^1 & -i\tilde a_3^2 - \tilde b_3^2 & \tilde a_3^1 \\ \tilde a_1^2 + i\tilde b_1^2 & -2i\tilde b_1^1 & \tilde a_3^2 + i\tilde b_3^2 \\ \tilde a_1^3 & i\tilde a_1^2 + \tilde b_1^2 & -\tilde a_1^1 + i\tilde b_1^1 \end{pmatrix},$$

then

$$\tilde A^{-1}\tilde A' = X^{-1}A^{-1}A'X + X^{-1}X', \tag{3.1}$$

which implies

$$\tilde a_1^3 = \rho^2 a_1^3 \tag{3.2}$$

and

$$\tilde a_1^2 + i\tilde b_1^2 = e^{2i\varphi}\rho(e^{i\varphi}(a_1^2 + ib_1^2) - \rho(p + iq)a_1^3). \tag{3.3}$$

Therefore, along any parametrized transversal curve there exists a frame field for which

$$a_1^3 = 1, \quad a_1^2 + ib_1^2 = 0. \tag{3.4}$$

Definition 3.5. A frame field A along γ is said to be of *first order* if conditions (3.4) are satisfied.

Definition 3.6. Let Γ be a lift of a transversal curve γ. If

$$\det(\Gamma, \Gamma', \Gamma'')\big|_{t_0} = 0,$$

for some $t_0 \in J$, then $\gamma(t_0)$ is called a *CR inflection point*. The notion of CR inflection point is independent of the lift Γ. A transversal curve with no CR inflection points is said to be *generic*. The notion of a CR inflection point is invariant under reparametrizations of the curve and under the action of the group of CR automorphisms.

Definition 3.7. If γ is transversal and Γ is one of its lifts, then the complex plane $[\Gamma \wedge \Gamma']_t$ is of type $(1,1)$ and the set of null complex lines contained in $[\Gamma \wedge \Gamma']_t$ is a chain, which do not depend on the choice of the lift Γ. This chain is denoted by $\mathcal{C}_\gamma|_t$ and is called the *osculating chain* of γ at $\gamma(t)$. By construction, $\mathcal{C}_\gamma|_t$ is the unique chain passing through $\gamma(t)$ and tangent to γ at the contact point $\gamma(t)$.

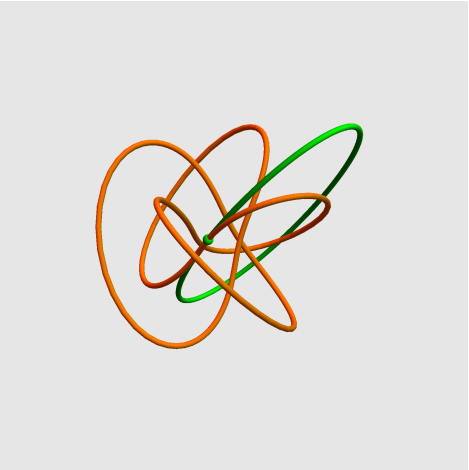

Fig. 3.1: A transversal torus knot (orange) and one of its osculating chains (green).

Remark 3.8. If $\det(\Gamma, \Gamma', \Gamma'') = 0$, then $[\Gamma \wedge \Gamma']$ is constant and hence the curve is a chain. Thus, chains can be characterized as those transversal curves all of whose points are CR inflection points.

Definition 3.9. Let $\gamma : J \to \mathcal{S}$ be a transversal curve. Consider the manifold $\Omega_+ \subset \mathbb{P}(\mathbb{C}^{2,1})$ of all chains. Then,

$$\mathcal{C}_\gamma : J \ni t \mapsto \mathcal{C}_\gamma|_t \in \Omega_+$$

is a smooth curve. We call $\mathcal{C}_\gamma$ the *osculating curve* of γ.

We now prove the following.

Proposition 3.10. *Let γ be a transversal curve. The following are equivalent:*

1. *$\gamma(t_0)$ is a CR inflection point;*
2. *$d\mathcal{C}_\gamma|_{t_0} = 0$;*
3. *the order of contact of γ with its osculating chain $\mathcal{C}_\gamma|_{t_0}$ at the contact point $\gamma(t_0)$ is strictly bigger than 1.*

Proof. First, we observe the following two facts.

Fact 1. Let β, $\gamma : J \subseteq \mathbb{R} \to \mathbb{C}P^3$ be two parametrized curves, such that $\gamma(t_0) = \beta(t_0)$, and let B, $\Gamma : J \to \mathcal{N}$ be the lifts of β and γ, respectively. Then, β and γ have a contact of order greater than 1 at the contact point $\gamma(t_0) = \beta(t_0)$ if and only if there exist complex numbers $\rho_0 \neq 0$, ρ_1, ρ_2 and real numbers $h_1 \neq 0$, h_2, such that

1. $\Gamma|_{t_0} = \rho_0 B|_{t_0}$,
2. $\Gamma'|_{t_0} = \rho_1 B|_{t_0} + \rho_0 h_1 B'|_{t_0}$,
3. $\Gamma''|_{t_0} = \rho_2 B|_{t_0} + (2\rho_1 h_1 + \rho_0 h_2) B'|_{t_0} + \rho_0 (h_1)^2 B''|_{t_0}$.

Fact 2. $d\gamma|_{t_0} = 0$ if and only if $\Gamma|_{t_0} \wedge \Gamma'|_{t_0} = 0$.

Next, we prove the following.

Lemma 3.11. *Let $\gamma : J \to \mathcal{S}$ be a transversal curve. Then there exists a first order frame field $\tilde{A}$ along γ, such that*

$$\tilde{A}^{-1}\tilde{A}' = \begin{pmatrix} 0 & -i(\tilde{a}_3^2 - i\tilde{b}_3^2) & \tilde{a}_3^1 \\ 0 & 0 & \tilde{a}_3^2 + i\tilde{b}_3^2 \\ 1 & 0 & 0 \end{pmatrix}. \tag{3.5}$$

Proof of the Lemma. Let $A : J \to G$ be any first order frame field along γ. Then, (3.2) and (3.3) imply that any other first order frame field is given by

$$\tilde{A} = AX\big(1, \varphi(t), 0, r(t)\big).$$

From (3.1), we get

$$\tilde{a}_1^1 = a_1^1 - r$$

and

$$\tilde{b}_1^1 = b_1^1 + \varphi'.$$

Then, by taking $r = a_1^1$ and $\varphi = -\int b_1^1 dt$, we get a first order frame field satisfying (3.5), as claimed. □

If $\tilde{A}$ is a first order frame field satisfying (3.5), then $\tilde{A}_1$ is a lift of γ, such that

$$\tilde{A}_1 \wedge \tilde{A}_1' \wedge \tilde{A}_1'' = -(\tilde{a}_3^2 + i\tilde{b}_3^2)\tilde{A}_1 \wedge \tilde{A}_2 \wedge \tilde{A}_3.$$

Therefore, $\gamma(t_0)$ is a CR inflection point if and only if $\tilde{a}_3^2|_{t_0} = \tilde{b}_3^2|_{t_0} = 0$. The osculating curve of γ is given by

$$\mathcal{C}_\gamma : J \ni t \mapsto \left[\tilde{A}_2\big|_t\right] \in \Omega_+.$$

Then, $\tilde{A}_2$ is a lift of $\mathcal{C}_\gamma$ to $\mathbb{C}^{2,1}$, such that

$$\tilde{A}_2 \wedge \tilde{A}_2' = i(\tilde{a}_3^2 + i\tilde{b}_3^2)\tilde{A}_1 \wedge \tilde{A}_2.$$

From this and by Fact 2, it follows that $\gamma(t_0)$ is a CR inflection point if and only if $d\mathcal{C}_\gamma|_{t_0} = 0$. Next, let $t_0 \in J$. Then

$$\beta : J \ni t \mapsto \left[\tilde{A}_1\big|_{t_0} + (t - t_0)\tilde{A}_3\big|_{t_0}\right] \in \mathcal{S}$$

is a parametric equation of the osculating chain of γ at $\gamma(t_0)$ and

$$B : J \ni t \mapsto \tilde{A}_1\big|_{t_0} + (t - t_0)\tilde{A}_3\big|_{t_0} \in \mathbb{C}^3$$

is a lift of β.

Now, suppose that $\gamma(t_0)$ is a CR inflection point, i.e., $\tilde{a}_3^2|_{t_0} = \tilde{b}_3^2|_{t_0} = 0$. Then

$$\begin{cases} \tilde{A}_1\big|_{t_0} = B|_{t_0}, \\ \tilde{A}_1'\big|_{t_0} = \tilde{A}_3\big|_{t_0} = B'|_{t_0}, \\ \tilde{A}_1''\big|_{t_0} = \tilde{a}_3^1\tilde{A}_1\big|_{t_0} = \tilde{a}_3^1 B|_{t_0}. \end{cases}$$

Hence, putting $\rho_0 \neq 1$, $\rho_1 = 0$, $\rho_2 = \tilde{a}_3^1|_{t_0}$, $h_1 = 1$ and $h_2 = 0$, the lifts $\tilde{A}_1$ and B of γ and β satisfy (3.5). This implies that β and γ have an analytic contact of order strictly bigger that 1 at $\gamma(t_0)$.

Suppose now that β and γ have an analytic contact of order strictly bigger that 1 at $\gamma(t_0)$. By Fact 1, there exist complex numbers $\rho_0 \neq 0$, ρ_1, ρ_2, and real numbers $h_1 \neq 0$, h_2, such that

$$\begin{cases} \tilde{A}_1\big|_{t_0} = \rho_0 B|_{t_0}, \\ \tilde{A}_1'\big|_{t_0} = \rho_1 B\big|_{t_0} + \rho_0 h_1 B'|_{t_0}, \\ \tilde{A}_1''\big|_{t_0} = \rho_2 B\big|_{t_0} + (2\rho_1 h_1 + \rho_0 h_2)B'\big|_{t_0} + \rho_0 h_1^2 B''\big|_{t_0}. \end{cases}$$

Now, by construction, we have

$$\begin{cases} \tilde{A}_1'\big|_{t_0} = \tilde{A}_3\big|_{t_0}, \\ \tilde{A}_1''\big|_{t_0} = \tilde{a}_3^1\tilde{A}_1\big|_{t_0} + (\tilde{a}_3^2 + i\tilde{b}_3^2)\tilde{A}_2\big|_{t_0}, \end{cases} \qquad \begin{cases} B|_{t_0} = \tilde{A}_1'\big|_{t_0}, \\ B'|_{t_0} = \tilde{A}_3\big|_{t_0}. \\ B''|_{t_0} = 0. \end{cases}$$

Accordingly, it follows that $\tilde{a}_3^2|_{t_0} = \tilde{b}_3^2|_{t_0} = 0$, namely $\gamma(t_0)$ is a CR inflection point, which concludes the proof of Proposition 3.10. □

Definition 3.12. A *generic transversal knot* $\mathcal{K} \subset \mathcal{S}$ is the image of a periodic generic transversal curve $\gamma : \mathbb{R} \to \mathcal{S}$, with minimal period ω, such that the restriction of γ to the interval $[0, \omega)$ is one-to-one. Two generic transversal knots $\mathcal{K}$ and $\hat{\mathcal{K}}$ are said to be *CR isotopic* if there exists a smooth 1-parameter family $\{\mathcal{K}_t\}_{t\in[0,1]}$ of generic transversal knots, i.e., a *CR isotopy*, such that $\mathcal{K}_0 = \mathcal{K}$ and $\mathcal{K}_1 = \hat{\mathcal{K}}$.

3.2. Local CR invariants for transversal curves: the equivalence problem From now on, we will consider generic transversal curves.

Definition 3.13. Let γ be generic transversal curve. A lift Γ of γ, such that

$$\det(\Gamma, \Gamma', \Gamma'') = -1,$$

is said to be a *Wilczynski lift* (W-lift) of γ. If Γ is a Wilczynski lift, any other is given by $\varepsilon\Gamma$, where $\varepsilon \in \mathbb{C}$ is a cube root of the unity. The function

$$a_\gamma = i\langle \Gamma, \Gamma' \rangle^{-1}$$

is smooth, real-valued, and independent of the choice of Γ. We call a_γ the *strain density* of the parameterized transversal curve γ. The linear differential form $ds = a_\gamma dt$ is called the *infinitesimal strain*.

Proposition 3.14. *The strain density and the infinitesimal strain are invariant under the action of the CR transformation group. In addition, if $h : I \to J$ is a change of parameter, then the infinitesimal strains ds and $d\tilde{s}$ of γ and $\tilde{\gamma} = \gamma \circ h$, respectively, are related by $d\tilde{s} = h^*(ds)$.*

Proof. If $A \in G$ and if Γ is a Wilczynski lift of γ, then $\hat{\Gamma} = A\Gamma$ is a Wilczynski lift of $\hat{\gamma} = A\gamma$. This implies that $a_\gamma = a_{\hat{\gamma}}$. Next, consider a reparametrization $\tilde{\gamma} = \gamma \circ h$ of γ. Then, $\Gamma^* = \Gamma \circ h$ is a lift of $\tilde{\gamma}$, such that

$$\det\left(\Gamma^*, (\Gamma^*)', (\Gamma^*)''\right) = -(h')^3.$$

This implies that $\tilde{\Gamma} = (h')^{-1}\Gamma^*$ is a Wilczynski lift of $\tilde{\gamma}$. Hence

$$\langle (\Gamma^*)', (\Gamma^*)' \rangle = (h')^{-1}\langle \Gamma, \Gamma' \rangle \circ h.$$

Therefore, the strain densities of γ and $\tilde{\gamma}$ are related by

$$a_{\tilde{\gamma}} = h' a_\gamma \circ h.$$

Consequently, we have

$$h^*(ds) = h' a_\gamma \circ h \; dt = d\tilde{s}.$$

□

As a straightforward consequence of Proposition 3.14, we have the following.

Corollary 3.15. *A generic transversal curve γ can be parametrized so that $a_\gamma = 1$.*

Definition 3.16. If $a_\gamma = 1$, we say that $\gamma : J \to \mathcal{S}$ is a *natural parametrization*, or a parametrization by the *pseudoconformal strain* or *pseudoconformal parameter*.

In the following, we will use the symbol s to denote the *natural parameter*.

Definition 3.17. Let $\gamma : J \to \mathcal{S}$ be a natural parametrization of a transversal curve and $\Gamma : \mathbb{R} \to \mathcal{N}$ be a W-lift of γ. The *pseudoconformal bending* κ and the *pseudoconformal twist* τ of γ are the smooth real-valued functions defined, respectively, by

$$\kappa := \frac{1}{2}\langle \Gamma', \Gamma' \rangle \tag{3.6}$$

and

$$\tau := \Im\left(\langle \Gamma'', \Gamma' \rangle\right) + 3\kappa^2. \tag{3.7}$$

We can state the following.

Proposition 3.18. *Let $\gamma : J \to \mathcal{S}$ be a natural parametrization of a generic transversal curve. Then there exists a first order frame field $\mathcal{F} = (F_1, F_2, F_3) : J \to G$ along γ, such that F_1 is a W-lift and*

$$\mathcal{F}^{-1}\mathcal{F}' = \begin{pmatrix} i\kappa & -i & \tau \\ 0 & -2i\kappa & 1 \\ 1 & 0 & i\kappa \end{pmatrix}. \tag{3.8}$$

The frame field $\mathcal{F}$ is called a Wilczynski frame. If $\mathcal{F}$ is a Wilczynski frame, any other is given by $\varepsilon\mathcal{F}$, where ε is a cube root of the unity.

Proof. First, we construct the the Wilczynski frame. Let Γ be a W-lift of γ and set

$$F_1 = \Gamma, \tag{3.9}$$

$$F_3 = \Gamma' - i\kappa\Gamma. \tag{3.10}$$

Then, $\langle F_1, F_1 \rangle = \langle F_3, F_3 \rangle = 0$ and $\langle F_1, F_3 \rangle = i$. Consequently, there exists a spacelike vector field $F_2 : J \to \mathbb{C}^{2,1}$, such that $(F_1, F_2, F_3)|_s$ is a (unimodular) light-cone basis of $\mathbb{C}^{2,1}$, for every $s \in J$. We claim that $\mathcal{F} = (F_1, F_2, F_3)$ is a Wilczynski frame. To this end, we prove that

$$F_2 = \Gamma'' - 2i\kappa\Gamma' - (\tau + \kappa^2 + i\kappa')\Gamma. \tag{3.11}$$

Let us write

$$\Gamma'' = aF_1 + bF_2 + cF_3.$$

Since $\det(\Gamma, \Gamma', \Gamma'') = -1$, $b = 1$. Moreover, since $\langle \Gamma, \Gamma'' \rangle = -2\kappa$, $c = 2i\kappa$. Then,

$$\Gamma'' = (a + 2\kappa^2)\Gamma + F_2 + 2i\kappa\Gamma'.$$

Since $\langle \Gamma'', \Gamma' \rangle = \kappa' + i(\tau - 3\kappa^2)$, taking into account that $\langle \Gamma, \Gamma' \rangle = i$ and $\langle \Gamma', \Gamma' \rangle = 2\kappa$, we have $a = \tau - \kappa^2 + i\kappa'$, and hence (3.11).

By construction, we have

$$\Gamma' = F_3 + i\kappa\Gamma. \tag{3.12}$$

Differentiating (3.12), using (3.11), (3.10) and (3.9), yields

$$F_3' = \tau F_1 + F_2 + i\kappa F_3. \tag{3.13}$$

Let

$$F_2' = uF_1 + vF_2 + wF_3.$$

Then, using (3.12) and (3.13), we obtain

$$\begin{cases} 0 = \langle F_3 + i\kappa F_1, F_2 \rangle = \langle F_1', F_2 \rangle = -\langle F_1, F_2' \rangle = -iw \\ 1 = \langle \tau F_1 + F_2 + i\kappa F_3, F_2 \rangle = \langle F_3', F_2 \rangle = -\langle F_3, F_2' \rangle = iu \\ 0 = \det(F_1', F_2, F_3) + \det(F_1, F_2', F_3) + \det(F_1, F_2, F_3') = 2i\kappa + v \end{cases}$$

Therefore,

$$F_2' = -iF_1 + -2i\kappa F_2. \tag{3.14}$$

Equations (3.12), (3.13) and (3.14) imply that $\mathcal{F}$ is a Wilczynski frame field along γ.

Let $\tilde{\mathcal{F}} = (\tilde{F}_1, \tilde{F}_2, \tilde{F}_3)$ be any other Wilczynski frame field along γ. Since F_1 and $\tilde{F}_1$ are W-lifts of γ, then $\tilde{F}_1 = \varepsilon F_1$, where ε is a cube root of the unity. This implies

$$\tilde{F}_3 = \tilde{F}_1' - i\kappa\tilde{F}_1 = \varepsilon(F_1' - i\kappa F_1) = \varepsilon F_3.$$

Taking into account that $\tilde{\mathcal{F}}$ and $\mathcal{F}$ are both unimodular light-cone basis of $\mathbb{C}^{2,1}$, $\tilde{F}_1 = \varepsilon F_1$ and $\tilde{F}_3 = \varepsilon F_3$ imply that $\tilde{F}_2 = \varepsilon F_2$. This concludes the proof. □

Remark 3.19. Let $\mathcal{K} \subset \mathcal{S}$ be an embedded generic transversal curve and $\gamma : J \to \mathcal{K}$ be a natural parametrization of $\mathcal{K}$ with bending κ and twist τ. Then $\mathfrak{k} = \kappa \circ \gamma^{-1}$ and $\mathfrak{t} = \tau \circ \gamma^{-1}$ do not depend on the choice of γ. Thus, the bending and the twist can be viewed as smooth real-valued functions defined on $\mathcal{K}$.

As a consequence of Proposition 3.18, we have the following.

Theorem 3.20. *Let $\mathcal{K}$, $\tilde{\mathcal{K}} \subset \mathcal{S}$ be two embedded generic transversal curves. Then, $\mathcal{K}$ and $\tilde{\mathcal{K}}$ are congruent to each other if and only if there exists an orientation-preserving diffeomorphism $h : \mathcal{K} \to \tilde{\mathcal{K}}$, such that $\tilde{\mathfrak{k}} \circ h = \mathfrak{k}$ and $\tilde{\mathfrak{t}} \circ h = \mathfrak{t}$. In addition, given two smooth functions κ, $\tau : J \to \mathbb{R}$, there exists a generic transversal curve $\gamma : J \to \mathcal{S}$, parametrized by the natural parameter, with bending κ and twist τ.*

Proof. Suppose that $\mathcal{K}$ and $\tilde{\mathcal{K}}$ are congruent to each other. Then there exists $A \in G$, such that $\tilde{\mathcal{K}} = A\mathcal{K}$. Denote by $h_A : \mathcal{K} \to \tilde{\mathcal{K}}$ the diffeomorphism induced

by A and let γ be a natural parametrization of $\mathcal{K}$. Then, $\tilde{\gamma} = A\gamma$ is a natural parametrization of $\tilde{\mathcal{K}}$, such that $\tilde{\kappa} = \kappa$ and $\tilde{\tau} = \tau$. Since $\tilde{\gamma} = h_A \circ \gamma$, we have

$$\tilde{\mathfrak{k}} = \kappa \circ \tilde{\gamma}^{-1} = \kappa \circ \gamma^{-1} \circ (h_A)^{-1} = \mathfrak{k} \circ (h_A)^{-1}$$

and

$$\tilde{\mathfrak{t}} = \tau \circ \tilde{\gamma}^{-1} = \tau \circ \gamma^{-1} \circ (h_A)^{-1} = \mathfrak{t} \circ (h_A)^{-1}.$$

Conversely, suppose that $h : \mathcal{K} \to \tilde{\mathcal{K}}$ is an orientation-preserving diffeomorphism, such that $\tilde{\mathfrak{k}} \circ h = \mathfrak{k}$ and $\tilde{\mathfrak{t}} \circ h = \mathfrak{t}$. Let $\tilde{\gamma}$ be a natural parametrization of $\tilde{\mathcal{K}}$. Then $\gamma = \tilde{\gamma} \circ h$ is a natural parametrization of $\mathcal{K}$. Since $\tilde{\mathfrak{k}} \circ h = \mathfrak{k}$ and $\tilde{\mathfrak{t}} \circ h = \mathfrak{t}$, the parameterizations γ and $\tilde{\gamma}$ have the same bending and twist. Let $\mathcal{F}$ and $\tilde{\mathcal{F}}$ be two W-frame fields along γ and $\tilde{\gamma}$, respectively. Then, $\mathcal{F}^{-1}\mathcal{F}' = \tilde{\mathcal{F}}^{-1}\tilde{\mathcal{F}}'$. Consequently, by the Cartan–Darboux rigidity theorem, there exists $A \in G$ such that $\tilde{\mathcal{F}} = A\mathcal{F}$. This implies that $\tilde{\gamma} = A\gamma$, and hence $\tilde{\mathcal{K}} = A\mathcal{K}$.

The second part of the theorem follows from the global existence theorem for linear systems of first-order ODEs. In fact, given κ, $\tau : J \to \mathbb{R}$, consider the $\mathfrak{g}$-valued smooth function given by

$$K = \begin{pmatrix} i\kappa & -i & \tau \\ 0 & -2i\kappa & 1 \\ 1 & 0 & i\kappa \end{pmatrix}.$$

Consider the linear system of first-order ODEs

$$\begin{pmatrix} \rho_1' \\ \rho_2' \\ \rho_3' \end{pmatrix} = K \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix},$$

where ${}^t\rho_1(s)$, ${}^t\rho_2(s)$, ${}^t\rho_3(s) \in \mathbb{C}^3$, $s \in J$, are unknown vectors of $\mathbb{C}^3$. Let $R_1(s)$, $R_2(s)$, $R_3(s)$ be the solution of the system, satisfying the initial conditions $R_1(s_0) = (1,0,0)$, $R_2(s_0) = (0,1,0)$, $R_3(s_0) = (0,0,1)$, for $s_0 \in J$, Let $\mathcal{F} : J \to \mathbb{C}(3,3)$ denote the matrix-valued smooth function with row vectors R_1, R_2 and R_3. Then $\mathcal{F}$ is a solution of the Cauchy problem

$$\begin{cases} \mathcal{F}' = K\mathcal{F} \\ \mathcal{F}(s_0) = I_3, \end{cases} \tag{3.15}$$

where I_3 is the 3×3 identity matrix. Accordingly, since K is $\mathfrak{g}$-valued (cf. (2.5)), it follows that ${}^t\overline{\mathcal{F}}\mathbf{h}\mathcal{F}$ and $\det(\mathcal{F})$ are constant, which implies that $\mathcal{F}$ is G-valued (cf. (2.4)). Let F_1 be the first column vector of $\mathcal{F}$ and $\gamma : J \to \mathcal{S}$ be defined by $\gamma(s) = [F_1(s)]$, for every $s \in J$. Then, (3.15) implies that γ is a generic transversal curve, such that $\mathcal{F}$ is one of its Wilczynski frame fields and that κ and τ are, respectively, the bending and the twist of γ. □

Remark 3.21. Let $\gamma : J \to \mathcal{S}$ be a natural parametrization of a generic transversal curve and let Γ be a W-lift of γ. Then,

$$\eta : J \ni s \mapsto [\Gamma(s)]_{\mathbb{R}} \in \mathcal{E}$$

is an immersed curve in the Einstein static universe, the *Fefferman lift* of γ. It is a computational matter to check that

$$\kappa = \frac{{}^t\overline{\Gamma}(s)\Gamma(s)}{2} g_{\mathcal{E}}(\eta, \eta').$$

Therefore, the bending can be viewed as a measure of how much the Fefferman lift of γ differs from being a lightlike curve.

Remark 3.22. Let $\gamma : J \to \mathcal{S}$ be as above and $\mathcal{F} = (F_1, F_2, F_3) : J \to G$ be a Wilczynski frame field along γ. Then,

$$\gamma^{\#} : J \ni s \mapsto [F_3(s)]_{\mathbb{C}} \in \mathcal{S}$$

is an immersed curve, called the *dual* of γ. The dual curve is Legendrian (i.e., tangent to the contact distribution) if and only if $\tau = 0$. Consequently, the twist can be viewed as a measure of how the dual curve differs from being a Legendrian curve.

Consider the CR prolongation $\Lambda : G \to P(\mathcal{S})$ of the structure bundle of $\mathcal{S}$. Let $\gamma : J \to \mathcal{S}$ be a natural parametrization of a generic transversal curve and $\mathcal{F} = J \to G$ be a Wilczynski frame field along γ. Then, $\mathbf{F} = (\mathbf{F_1}, \mathbf{F_2}, \mathbf{F_3}) = \Lambda \circ \mathcal{F} : J \to P(\mathcal{S})$ does not depend on the choice of the Wilczynski frame field.

Definition 3.23. We call $\mathbf{F}$ the *CR moving trihedron* along γ. By construction, $\mathbf{F_1}|_t$ is tangent to γ at $\gamma(t)$ and $\mathbf{F_2}|_t$, $\mathbf{F_3}|_t$ belong to the contact plane $\mathcal{D}|_{\gamma(t)}$. In analogy with the elementary differential geometry of space curves, we call $\mathbf{F_1}|_t$ and $\mathbf{F_2}|_t$ the *CR tangent vector field* and the *CR normal vector field* of γ at $\gamma(t)$. They will be denoted by $\vec{T}$ and $\vec{N}$, respectively. Observe that $\mathbf{F_3} = J(\vec{N})$.

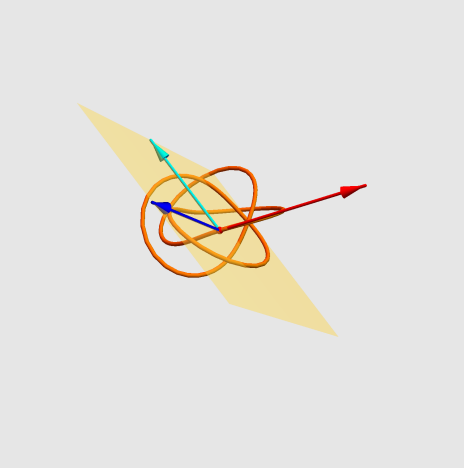

Fig. 3.2: CR trihedron along a generic transversal torus knot of type $(3, 4)$ and the contact plane; the cyan vector is the CR normal vector.

Remark 3.24. Let γ be a generic transversal curve whose trajectory is contained in $\dot{\mathcal{S}} := \mathcal{S} \setminus \{P_\infty\} \cong \mathbb{R}^3$. Suppose the analytic expression of a W-lift $\Gamma : J \to \mathcal{N}$ of γ is known. Then, using Remark 2.9, the CR trihedron can be explicitly constructed as follows. Firstly, we compute the Wilczynski frame along γ by the formula

$$\mathcal{F} = (\Gamma, \Gamma'' - 2i\kappa\Gamma' - (\tau + \kappa^2 + i\kappa')\Gamma, \Gamma' - i\kappa\Gamma) : J \to G,$$

where $\kappa = \langle \Gamma', \Gamma' \rangle / 2$ and $\tau = \Im(\langle \Gamma'', \Gamma' \rangle) + 3\kappa^2$ are, respectively, the bending and the twist of γ. Secondly, we compute the map $\mathbf{S} : J \to G$ defined by

$$\mathbf{S} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\Gamma_2}{\Gamma_1} & 1 & 0 \\ \frac{\Gamma_3}{\Gamma_1} & i\frac{\overline{\Gamma}_2}{\overline{\Gamma}_1} & 1 \end{pmatrix}.$$

Then, the CR trihedron is given by

$$\mathbf{F} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & y(s) & -x(s) \end{pmatrix} \varphi(J\, {}^t\bar{\mathcal{F}} J \mathbf{S}),$$

where $\varphi : G \to K$ is the 3-dimensional representation defined by (2.19) and $x(s)$, $y(s)$ are the real and imaginary parts of the complex-valued function $\Gamma_2(s)/\Gamma_1(s)$.

3.3. The Cartan system of prolongations of generic transversal curves

Definition 3.25. Let γ be a generic transversal curve parametrized by the natural parameter (strain) and let $\mathcal{F}$ be a Wilczynski frame field along γ. Let $[\mathcal{F}] : \mathbb{R} \to [G]$ be the unique lift originated by $\mathcal{F}$. The map

$$\mathfrak{F} : \mathbb{R} \ni s \mapsto \big([\mathcal{F}(s)], \kappa(s), \tau(s)\big) \in [G] \times \mathbb{R}^2$$

is called the *prolongation* of γ. The Cartesian product $Y := [G] \times \mathbb{R}^2$ is referred to as the *configuration space*. The coordinates on $\mathbb{R}^2$ will be denoted by (κ, τ).

With some abuse of notation, we use α_1^1, β_1^1, α_1^2, β_1^2, α_3^1, α_3^2, β_3^2, α_3^1 to denote the entries of the Maurer–Cartan form of $[G]$ and their pull-backs on the configuration space Y. By Proposition 3.18, the prolongations are the integral curves of the Pfaffian differential system $(\mathbb{J}, \eta)$ on Y generated by the 1-forms

$$\mu^1 = \alpha_1^2, \quad \mu^2 = \beta_1^2, \quad \mu^3 = \alpha_3^2 - \alpha_3^1,$$

$$\mu^4 = \beta_3^2, \quad \mu^5 = \alpha_1^1, \quad \mu^6 = \beta_1^1 - \kappa\alpha_1^3, \quad \mu^7 = \alpha_3^1 - \tau\alpha_1^3,$$

with the independence condition $\eta = \alpha_1^3$. If we put

$$\pi^1 = d\kappa, \quad \pi^2 = d\tau,$$

the 1-forms $(\eta, \mu^1, \ldots, \mu^7, \pi^1, \pi^2)$ define an absolute parallelism on Y.

Remark 3.26. By construction, the integral curves of $(\mathbb{J}, \eta)$ are the prolongations of generic transversal curves of $\mathcal{S}$. The set of the closed integral curves of the differential system is denoted by $\mathcal{V}(\mathbb{J}, \eta)$.

The Maurer–Cartan equations of G imply that the coframe $(\eta, \mu^1, \dots, \mu^7, \pi^1, \pi^2)$ satisfies the following structure equations:

$$\begin{cases} d\eta = 2\mu^1 \wedge \mu^2 + 2\mu^5 \wedge \eta, \\ d\pi^1 = d\pi^2 = 0, \end{cases}$$

$$\begin{cases} d\mu^1 = -\mu^1 \wedge \mu^5 + 3\mu^2 \wedge \mu^6 + (3\kappa\mu^2 - \mu^3) \wedge \eta, \\ d\mu^2 = -3\mu^1 \wedge \mu^6 - \mu^2 \wedge \mu^3 - (3\kappa\mu^1 + \mu^4) \wedge \eta, \\ d\mu^3 = -2\mu^\wedge\mu^2 - \mu^1 \wedge \mu^7 + \mu^3 \wedge \mu^5 + 3\mu^4 \wedge \mu^6 - (\tau\mu^1 - 3\kappa\mu^4 + 3\mu^5) \wedge \eta, \\ d\mu^4 = -\mu^2 \wedge \mu^7 - 3\mu^3 \wedge \mu^6 + \mu^4 \wedge \mu^5 - (\tau\mu^2 + 3\kappa\mu^3 - 3\mu^6) \wedge \eta, \\ d\mu^5 = -\mu^1 \wedge \mu^4 + \mu^2 \wedge \mu^3 + (\mu^2 - \mu^7) \wedge \eta, \\ d\mu^6 = -2\kappa\mu^1 \wedge \mu^2 - \mu^1 \wedge \mu^3 - \mu^2 \wedge \mu^4 - (\mu^1 + 2\kappa\mu^5) \wedge \eta - \pi^1 \wedge \eta, \\ d\mu^7 = -2\tau\mu^1 \wedge \mu^2 - 2\mu^3 \wedge \mu^4 - 2\mu^5 \wedge \mu^7 + (2\mu^4 - 2\tau\mu^5) \wedge \eta - \pi^2 \wedge \eta. \end{cases}$$

Remark 3.27. It follows from this that the derived flag of $\mathbb{J}$ is given by $\mathbb{J}_{(4)} \subset \mathbb{J}_{(3)} \subset \mathbb{J}_{(2)} \subset \mathbb{J}_{(1)}$, where $\mathbb{J}_{(4)} = \{0\}$, $\mathbb{J}_{(3)} = \operatorname{span}\{\mu^1\}$, $\mathbb{J}_{(2)} = \operatorname{span}\{\mu^1, \mu^2, \mu^3\}$, $\mathbb{J}_{(1)} = \operatorname{span}\{\mu^1, \mu^2, \mu^3, \mu^4, \mu^5\}$. Thus, all the derived systems have constant rank.

The 1-forms $\mu^1, \dots, \mu^7$ and the independence condition η generate an affine subbundle $\mathcal{Z}$ of $T^*(Y)$, namely

$$\mathcal{Z} = \eta + \operatorname{span}\{\mu^1, \dots, \mu^7\}.$$

Definition 3.28. Following [24], we call $\mathcal{Z}$ the *phase space* of the Pfaffian differential system $\mathbb{J}$ with independence condition η. The restriction to $\mathbb{J}$ of the Liouville form of $T^*(Y)$ is denoted by ξ. The exterior differential 2-form $\Xi = d\xi$ is said the Cartan–Poincaré form of $(\mathbb{J}, \eta)$.

With some abuse of notation, we use the same symbols to denote the exterior differential forms on Y and their pull-backs on $\mathcal{Z}$. Let $p_1, \dots, p_7$ be the fiber coordinates of the bundle map $\mathcal{Z} \to Y$, with respect to the trivialization of $\mathcal{Z}$ determined by η and the 1-forms $\mu^1, \dots, \mu^7$. Then,

$$\xi = \eta + p_1\mu^1 + \cdots + p_7\mu^7.$$

Using the structure equations, we get

$$\begin{aligned} \Xi \equiv \sum_{j=1}^{7} dp_j \wedge \mu^j + 2\mu^5 \wedge \eta + p_1(3\kappa\mu^2 - \mu^3) \wedge \eta - p_2(3\kappa\mu^1 + \mu^4) \wedge \eta \\ - p_3(\tau\mu^1 - 3\kappa\mu^4 + 3\mu^5) \wedge \eta - p_4(\tau\mu^2 + 3\kappa\mu^3 - 3\mu^6) \wedge \eta \\ + p_5(\mu^2 - \mu^7) \wedge \eta - p_6(\pi^1 + \mu^1 + 2\kappa\mu^5) \wedge \eta - p_7(\pi^2 - 2\mu^4 + 4\tau\mu^5) \wedge \eta, \end{aligned}$$

where the sign '$\equiv$' denotes equality modulo the span of $\{\mu^i \wedge \mu^j\}_{i,j=1,\dots,7}$.

On $\mathcal{Z}$, consider the coframe $(\eta, \mu^1, \dots, \mu^7, \pi^1, \pi^2, dp_1, \dots, dp_7)$ and the corresponding dual parallelization $(\partial_\eta, \partial_{\mu^1}, \dots, \partial_{\mu^7}, \partial_{\pi^1}, \partial_{\pi^2}, \partial_{p_1}, \dots, \partial_{p_7})$ of $T(\mathcal{Z})$. Contracting the 2-form Ξ with the elements of this parallelization yields

$$\begin{cases} \partial_{p_j} \lrcorner\, \Xi \equiv \mu^j, j = 1, \dots, 7, \\ -\partial_{\pi^1} \lrcorner\, \Xi \equiv p_6 \eta =: \dot{\pi}_1, \\ -\partial_{\pi^2} \lrcorner\, \Xi \equiv p_7 \eta =: \dot{\pi}_2, \\ -\partial_{\mu^1} \lrcorner\, \Xi \equiv dp_1 + (3\kappa p_2 + \tau p_3 + p_6)\eta =: \dot{\mu}^1, \\ -\partial_{\mu^2} \lrcorner\, \Xi \equiv dp_2 - (3\kappa p_1 - \tau p_4 + p_5)\eta =: \dot{\mu}^2, \\ -\partial_{\mu^3} \lrcorner\, \Xi \equiv dp_3 + (p_1 + 3\kappa p_4)\eta =: \dot{\mu}^3, \\ -\partial_{\mu^4} \lrcorner\, \Xi \equiv dp_4 + (p_2 - 3\kappa p_3 - 2p_7)\eta =: \dot{\mu}^4, \\ -\partial_{\mu^5} \lrcorner\, \Xi \equiv dp_5 - (2 - 3p_3 - 2\kappa p_6 + 4\tau p_7)\eta =: \dot{\mu}^5, \\ -\partial_{\mu^6} \lrcorner\, \Xi \equiv dp_6 - 3p_4 \eta =: \dot{\mu}^6, \\ -\partial_{\mu^7} \lrcorner\, \Xi \equiv dp_7 + 3p_5 \eta =: \dot{\mu}^7, \\ -\partial_{\eta} \lrcorner\, \Xi \equiv \pi_6 \pi^1 + p_7 \pi^2 =: \dot{\eta}. \end{cases} \tag{3.16}$$

The *Cartan system* associated to the closed 2-form Ξ is the Pfaffian differential system $C(\Xi)$ generated by the set of 1-forms $\{\mu_1, \dots, \mu_7, \dot{\pi}_1, \dot{\pi}_2, \dot{\mu}_1, \dots, \dot{\mu}_7, \dot{\eta}\}$.

3.4. Global invariants of a generic transversal knot The main classical invariant of a transversal knot of $\mathbb{R}^3$ is the *Bennequin number*, which is defined as follows. Consider a nowhere vanishing cross section ξ of the contact distribution $\mathcal{D}$. If $\mathcal{K}$ is a transversal knot parametrized by a periodic transversal curve $\gamma : \mathbb{R} \to \mathbb{R}^3$ and if $\epsilon \ll 1$ is a sufficiently small positive real number, then the map

$$\gamma_\epsilon : \mathbb{R} \ni t \mapsto \gamma(t) + \epsilon \xi|_{\gamma(t)} \in \mathbb{R}^3$$

parametrizes a transversal knot $\mathcal{K}_\epsilon$, disjoint from $\mathcal{K}$, called the *contact push-off* of $\mathcal{K}$ in the direction of ξ. The linking number $\mathrm{Lk}(\mathcal{K}, \mathcal{K}_\epsilon)$ is independent of the choice of ξ and is invariant by contact isotopies.

Definition 3.29. The *Bennequin number* of a transversal knot $\mathcal{K}$ is the integer given by the linking number $\mathrm{Lk}(\mathcal{K}, \mathcal{K}_\epsilon)$. The Bennequin number is denoted by $\beta(\mathcal{K})$.

Remark 3.30. The Bennequin number played an important role in proving that certain contact structures of $\mathbb{R}^3$ were not equivalent to the standard one. In 1997, Fuchs and Tabachnikov [20] conjectured that two transversal knots of $\mathbb{R}^3$ with the same knot type and the same Bennequin number are contact isotopic. To our knowledge, the conjecture is still open. In 2002, a similar conjecture for Legendrian knots was disproved by Chekanov [9]. In 1993, Eliashberg proved the following.

Theorem 3.31 ([14])**.** *Let $\mathcal{K}$ and $\hat{\mathcal{K}}$ be two topologically trivial transversal knots in $\mathbb{R}^3$ (or more generally in any tight contact 3-manifold). If $\mathcal{K}$ and $\hat{\mathcal{K}}$ have the same Bennequin number, they are contact isotopic.*

In 1999, Etnyre proved the following.

Theorem 3.32 ([15]). *Let $\mathcal{K}$ and $\hat{\mathcal{K}}$ be two positive torus knots of type (p, q) in $\mathbb{R}^3$. If $\mathcal{K}$ and $\hat{\mathcal{K}}$ have the same Bennequin number, then they are contact isotopic. In addition, the Bennequin number $\beta(\mathcal{K})$ satisfy the inequality $\beta(\mathcal{K}) \leq pq - p - q$.*

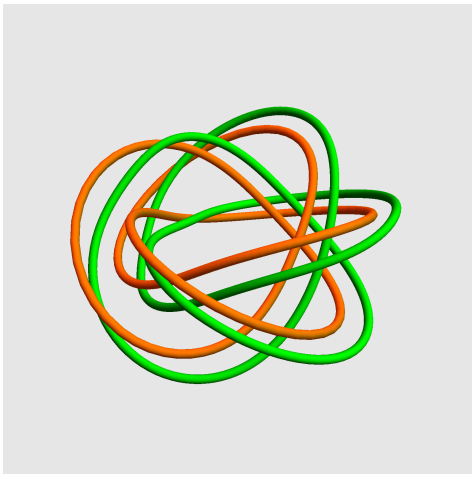

Fig. 3.3: A generic transversal positive torus knot of type $(3, 4)$ (orange) and its contact push-off (green) in the direction of the vector field $\xi = \frac{1}{\sqrt{1+y^2}}(\partial_x + y\partial_z)$ at distance 0.4. The Bennequin number of this knot attains the maximum possible value, i.e., $\beta = 5$.

Definition 3.33. Let $\mathcal{K}$ be a generic transversal knot, with natural parametrization $\gamma : \mathbb{R} \to \mathcal{S}$. Let ω denote the minimal period of γ and let $\Gamma : \mathbb{R} \to \mathbb{C}^{2,1}$ be a W-lift of γ. Then $\Gamma(\omega) = \varepsilon_\gamma \Gamma(0)$, where ε_γ is a cube root of the unity which is independent of the choice of the lift. The phase exponent of ε_γ in the interval $[0, 2\pi)$ is called the *phase anomaly* of $\mathcal{K}$. If $\varepsilon_\gamma \neq 1$, the minimal period of Γ is 3ω. In this case, we say that $\mathcal{K}$ is a *generic transversal knot* with *CR spin* $1/3$. Otherwise, the *CR spin* of $\mathcal{K}$ is 1.

Remark 3.34. The CR anomaly and the CR spin are invariant under CR isotopies.

Since $\Gamma : \mathbb{R} \to \mathbb{C}^{2,1}$ takes values in the null-cone of $\mathbb{C}^{2,1}$, $\Gamma_1 - i\Gamma_3$ is a nowhere vanishing $\mathbb{C}$-valued map of period $\sigma_\gamma \omega$, where σ_γ is the CR spin of γ. Consider the map

$$\chi = \Gamma_1 - i\Gamma_3 : \frac{\mathbb{R}}{\sigma_\gamma \omega \mathbb{Z}} \cong \mathrm{S}^1 \to \mathbb{C} \setminus \{0\}$$

and let $\mu_\mathcal{K} \in \mathbb{Z}$ be the degree of χ, that is,

$$\mu_\mathcal{K} = \frac{i}{2\pi} \int_0^{\sigma_\gamma \omega} \chi^{-1} d\chi.$$

It follows that $\mu_{\mathcal{K}}$ is independent of Γ and, by construction, is invariant under CR isotopies.

Definition 3.35. The integer $\mu_{\mathcal{K}}$ is called the *Maslov index* (or *rotation number*) of $\mathcal{K}$. The rotation number is well defined for any closed generic transversal curve.

To define the last invariant we use the Heisenberg model, so that $\mathbb{R}^3$ with its standard contact structure is identified with $\mathcal{S} \setminus \{P_\infty\}$ via the Heisenberg chart. Let $\mathcal{K} \subset \mathbb{R}^3$ be a generic transversal knot and let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be a natural parametrization of $\mathcal{K}$, with minimal period ω. Consider the CR trihedron $(\vec{T}, \vec{N}, J(\vec{N}))$ along γ. Then, for small values of the parameter ϵ, the map

$$\tilde{\gamma}_\epsilon : \mathbb{R} \ni s \mapsto \gamma(s) + \epsilon \frac{\vec{N}(\gamma(t))}{\|\vec{N}(\gamma(t))\|} \in \mathbb{R}^3$$

parametrizes a generic transversal knot $\tilde{\mathcal{K}}_\epsilon$ disjoint from $\mathcal{K}$, called the *CR push-off* of $\mathcal{K}$.

Definition 3.36. The linking number $\mathrm{Lk}(\mathcal{K}, \tilde{\mathcal{K}}_\epsilon)$, denoted by $\mathrm{SL}(\mathcal{K})$, is called the *CR self-linking number* of $\mathcal{K}$. By construction, $\mathrm{SL}(\mathcal{K})$ is invariant under CR isotopies.

Remark 3.37. The CR self-linking number is the analogue of the self-linking number of a knot in $\mathbb{R}^3$ with no ordinary inflection points (cf. [1,6,12,22,41,46]). The CR self-linking number can be evaluated via the Gaussian linking integral, that is,

$$\mathrm{SL}(\mathcal{K}) = \int_0^\omega \int_0^\omega \frac{\det\left(\gamma(t) - \tilde{\gamma}_\epsilon(s), \gamma'(t), \tilde{\gamma}'_\epsilon(s)\right)}{\|\gamma(t) - \tilde{\gamma}_\epsilon(s)\|^3} dt ds.$$

If the CR normal vector field $\vec{N}$ along $\mathcal{K}$ can be extended to a nowhere vanishing global cross section of the contact distribution $\mathcal{D}$, then the CR self-linking number and the Bennequin number do coincide.

4. Isoparametric strings and knots

4.1. Isoparametric curves

Definition 4.1. A generic transversal curve is called *isoparametric* if its bending κ and twist τ are constant. A closed isoparametric curve is referred to as an *isoparametric string*. An isoparametric string which is a knot is referred to as an *isoparametric knot*.

Let $\gamma : \mathbb{R} \to \mathcal{S}$ be a natural parametrization of an isoparametric curve with bending κ and twist τ. Let $\mathrm{K}_{\kappa,\tau}$ be the element of the Lie algebra $\mathfrak{g}$ defined by

$$\mathrm{K}_{\kappa,\tau} = \begin{pmatrix} i\kappa & -i & \tau \\ 0 & -2i\kappa & 1 \\ 1 & 0 & i\kappa \end{pmatrix}$$

Then, γ is congruent to the orbit of the 1-parameter group of CR automorphisms

$$\mathcal{G}_{\kappa,\tau} : \mathbb{R} \ni s \mapsto \mathrm{Exp}(s\mathrm{K}_{\kappa,\tau}) \in G$$

passing through the origin $P_0 = [{}^t(1,0,0)] \in \mathcal{S}$. The purposes of this section are twofold. The first is to investigate for which values of the parameters κ, τ and for which orbits of $\mathcal{G}_{\kappa,\tau}$ the corresponding isoparametric curve is a knot. The second purpose is to analyze the global invariants of isoparametric knots.

Definition 4.2. The *Hamiltonian* of an isoparametric curve γ, with bending κ and twist τ, is the traceless self-adjoint endomorphism of $\mathbb{C}^{2,1}$ defined by $\mathrm{H}_{\kappa,\tau} = i\mathrm{K}_{\kappa,\tau}$.

Remark 4.3. The characteristic polynomial of $\mathrm{H}_{\kappa,\tau}$ is

$$P_{\kappa,\tau}(t) = -t^3 + (3\kappa^2 - \tau)t + 2\kappa^3 + 2\kappa\tau - 1$$

and the discriminant of $P_{\kappa,\tau}(t)$ is given by

$$D_{\kappa,\tau} = -27 + 108\kappa(\kappa^2 + \tau) - 324\kappa^4\tau - 72\kappa^2\tau^2 - 4\tau^3.$$

4.2. Isoparametric strings We prove the following.

Proposition 4.4. *Let γ be an isoparametric string with bending κ and twist τ. Then $D_{\kappa,\tau} > 0$.*

Proof. Let λ be an eigenvalue of $\mathrm{H}_{\kappa,\tau}$. Then, the corresponding eigenspace, denoted by $\mathbb{V}_\lambda$, is generated by the vector

$$v_\lambda = {}^t\Big(\lambda^2 - \kappa\lambda - 2\kappa^2, -1, -2i\kappa + i\lambda\Big).$$

Claim 1. If $D_{\kappa,\tau} < 0$, then γ cannot be periodic.

If $D_{\kappa,\tau} < 0$, the Hamiltonian $\mathrm{H}_{\kappa,\tau}$ has a simple real eigenvalue $-2a$ and two complex conjugate eigenvalues $\lambda = a + ib$, $\overline{\lambda} = a - ib$, with $b > 0$. Since the eigenspaces of λ and $\overline{\lambda}$ are lightlike, there exists a light-cone basis $\mathcal{B}$ of $\mathbb{C}^{2,1}$, such that

$$\mathcal{B}\mathrm{K}\mathcal{B}^{-1} = \begin{pmatrix} i\lambda & 0 & 0 \\ 0 & -2ia & 0 \\ 0 & 0 & i\overline{\lambda} \end{pmatrix} = \begin{pmatrix} ia - b & 0 & 0 \\ 0 & -2ia & 0 \\ 0 & 0 & ia + b \end{pmatrix}.$$

Therefore, by possibly replacing γ with a congruent curve, the map

$$\mathcal{F} = \mathcal{B}^{-1}\,\mathrm{Exp}(s\mathrm{K})\,\mathcal{B} = \mathcal{B}^{-1} \begin{pmatrix} e^{i(a+ib)s} & 0 & 0 \\ 0 & e^{-2ias} & 0 \\ 0 & 0 & e^{(ia+b)s} \end{pmatrix} \mathcal{B}$$

is a Wilczynski frame along γ. Since $\mathcal{F}$ is not periodic, γ cannot be periodic.

Claim 2. If $D_{\kappa,\tau} = 0$, then γ cannot be periodic.

There are two possible cases: (1) either $\mathrm{H}_{\kappa,\tau}$ has a unique eigenvalue with algebraic multiplicity 3; or (2) it has two distinct real eigenvalues with algebraic multiplicity one and two, respectively.

(1) In the first case, the eigenvalue is necessarily 0, the bending 1/2 an the twist 3/4. Thus, by possibly considering a congruent curve, the Wilczynski frame along γ is given by

$$\mathcal{F} = \begin{pmatrix} \frac{1}{4}(4+2is+s^2) & -\frac{1}{4}(4i+s)s & \frac{1}{8}(6-is)s \\ \frac{s^2}{2} & 1-is-\frac{s^2}{2} & s-\frac{is^2}{4} \\ s+\frac{is^2}{2} & -\frac{is^2}{4} & \frac{1}{4}(4+2is+s^2) \end{pmatrix}.$$

Since $\mathcal{F}$ is not periodic, γ cannot be periodic.

(2) In the second case, suppose that $\mathrm{H}_{\kappa,\tau}$ has an eigenvalue a with algebraic multiplicity 2 and a simple eigenvalue, $-2a$, with $a \in \mathbb{R}\setminus\{0\}$. We prove that the eigenspace $\mathbb{V}_{-2a} \subset \mathbb{C}^{2,1}$ cannot be timelike. By contradiction, if $\mathbb{V}_{-2a}$ is timelike, its Hermitian orthogonal complement $\mathbb{V}^{\perp}_{-2a}$ is 2-dimensional, spacelike and H-invariant. Hence, the restriction of H to $\mathbb{V}^{\perp}_{-2a}$ is a self-adjoint endomorphism of a Hermitian vector space. In particular, it is diagonalizable and $\mathbb{V}^{\perp}_{-2a}$ would be the eigenspace of the eigenvalue $-2a$. This contradicts the fact that the eigenspaces of H are 1-dimensional. Now, we show that $\mathbb{V}_{-2a}$ a cannot be lightlike. By contradiction, if $\mathbb{V}_{-2a}$ a is lightlike, then $\mathbb{V}_{-2a} \subset \mathbb{V}^{\perp}_{-2a}$ and $\mathbb{V}_a \subset \mathbb{V}^{\perp}_{-2a}$. Since $\mathbb{V}_a$ is orthogonal to a lightlike vector, it follows that $\mathbb{V}_a$ is spacelike. Thus, $\mathbb{V}^{\perp}_{-2a}$ would be the direct sum of $\mathbb{V}_{-2a}$ and $\mathbb{V}_a$, where $\mathbb{V}_{-2a}$ a is light-like and $\mathbb{V}_a$ is spacelike. Hence, there exist a light-cone basis $\mathcal{B} = (B_1, B_2, B_3)$ of $\mathbb{C}^{2,1}$ such that $B_1 \in \mathbb{V}_{-2a}$ and $B_2 \in \mathbb{V}_a$. Then,

$$\mathcal{B}^{-1}\mathrm{K}\mathcal{B} = \begin{pmatrix} -2a & 0 & ic \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}, \ a, c \in \mathbb{R}. \tag{4.1}$$

Therefore, we would have $-2ia = \langle \mathrm{H}B_1, B_3\rangle = \langle B_1, \mathrm{H}B_3\rangle = ia$, and hence $a = 0$, which is a contradiction. Thus the only possibility is that $\mathbb{V}_{-2a}$ is spacelike and $\mathbb{V}_a$ is lightlike. Then, there exists a light-cone basis $\mathcal{B} = (B_1, B_2, B_3)$, such that the real constant c in (4.1) is different from 0. By possibly replacing γ with a congruent curve, the map

$$\mathcal{F} = \mathcal{B}^{-1}\operatorname{Exp}(s\mathrm{K})\,\mathcal{B} = \mathcal{B}^{-1}\begin{pmatrix} e^{ias} & 0 & ce^{ias} \\ 0 & e^{-2ias} & 0 \\ 0 & 0 & e^{ias} \end{pmatrix}\mathcal{B}.$$

is a Wilczynski frame along the curve. Then, $\mathcal{F}$ is non-periodic and the curve cannot be closed. □

Definition 4.5. If $D_{\kappa,\tau} > 0$, the Hamiltonian $\mathrm{H}_{\kappa,\tau}$ has three distinct real eigenvalues:

$$e_1(\kappa,\tau) < e_2(\kappa,\tau) < e_3(\kappa,\tau).$$

The eigenvalue $e_3(\kappa,\tau)$ is positive, $e_1(\kappa,\tau)$ is negative, and $e_2(\kappa,\tau) = -e_3(\kappa,\tau) - e_1(\kappa,\tau)$. The quotient

$$r(\kappa,\tau) = \frac{e_1(\kappa,\tau)}{e_3(\kappa,\tau)}$$

is said to be the *spectral ratio*. By construction, $r(\kappa,\tau) \in (-2,-1/2)$.

Proposition 4.6. *Let γ be an isoparametric curve with $D_{\kappa,\tau} > 0$. Then, γ is periodic if and only if the spectral ratio $r(\kappa,\tau)$ is rational.*

Proof. Let a be the highest eigenvalue of the Hamiltonian H. The spectrum of H is given by

$$e_1 = ra < e_2 = -(1+r)a < e_3 = a,$$

where r is the spectral ratio. We claim that the eigenspaces cannot be light-like. Let (C_1, C_2, C_3) be a basis whose elements are eigenvectors relative to the eigenvalues e_1, e_2 and $_3$, respectively. If, by contradiction, one of them is light-like, for instance C_1, then C_1, C_2 and C_3 belong to the orthogonal complement of C_1. Consequently, they would be linearly dependent. Then, there exists a pseudo-unitary basis $\mathcal{B}$ of $\mathbb{C}^{2,1}$, such that

$$\mathcal{B}^{-1}\mathrm{K}\mathcal{B} = \begin{pmatrix} -ie_{\sigma(1)} & 0 & 0 \\ 0 & -ie_{\sigma(2)} & 0 \\ 0 & 0 & -ie_{\sigma(3)} \end{pmatrix},$$

where σ is a permutation of $(1,2,3)$. Hence, by possibly replacing γ with a congruent curve, the Wilczynski frame field along γ is given by

$$\mathcal{F} = \mathcal{B}^{-1} \begin{pmatrix} e^{-ie_{\sigma(1)}s} & 0 & 0 \\ 0 & e^{-ie_{\sigma(2)}s} & 0 \\ 0 & 0 & e^{-ie_{\sigma(3)}s} \end{pmatrix} \mathcal{B}.$$

The curve γ is periodic if and only if $\mathcal{F}$ periodic. On the other hand, from the previous formula, it follows that $\mathcal{F}$ is periodic if and only if e_1/e_3 and e_2/e_3 are rational. Since $e_2 = -e_1 - e_3$, we may conclude that γ is closed if and only if r is rational, as claimed. □

The region

$$\mathcal{P} = \big\{(\kappa,\tau) \in \mathbb{R}^2 \mid D(\kappa,\tau) > 0\big\}$$

consists of two connected components,

$$\mathcal{P}_- = \Big\{(\kappa,\tau) \in \mathbb{R}^2 \mid D(\kappa,\tau) > 0,\, 3\kappa < |\tau|^{1/2}\Big\}$$

and

$$\mathcal{P}_+ = \Big\{(\kappa,\tau) \in \mathbb{R}^2 \mid D(\kappa,\tau) > 0,\, 3\kappa > |\tau|^{1/2}\Big\}.$$

Definition 4.7. An isoparametric string with spectral ratio $r = \frac{p}{q} \in \mathbb{Q} \cap (-2,-1/2)$, bending κ and twist τ, is said to be of the *first class* if $(\kappa,\tau) \in \mathcal{P}_+$; of the *second class* if $(\kappa,\tau) \in \mathcal{P}_-$.

Proposition 4.8. *Let γ be an isoparametric curve such that $D_{\kappa,\tau} > 0$. Let $e_1 < e_2 < e_3$ be the eigenvectors of the Hamiltonian $\mathrm{H}_{\kappa,\tau}$ and $\mathbb{V}_1(\kappa,\tau)$, $\mathbb{V}_2(\kappa,\tau)$, $\mathbb{V}_3(\kappa,\tau)$ the corresponding eigenspaces. Then,*

1. *$\mathbb{V}_1(\kappa,\tau)$ is spacelike, for every $(\kappa,\tau) \in \mathcal{P}$;*

2. *if $(\kappa,\tau) \in \mathcal{P}_+$, $\mathbb{V}_2(\kappa,\tau)$ is time-like and $\mathbb{V}_3(\kappa,\tau)$ is spacelike;*

3. *if $(\kappa,\tau) \in \mathcal{P}_-$, $\mathbb{V}_2(\kappa,\tau)$ is space-like and $\mathbb{V}_3(\kappa,\tau)$ is timelike.*

Proof. Given $a, b \in \mathbb{R}$, such that $a > 0$, $b \neq 0$ and $4a^3 - 27b^2 > 0$, let

$$\begin{cases} \lambda_1(a,b) = -2\sqrt{\frac{a}{3}}\cos\left(\frac{1}{3}\arctan\left(-\frac{\sqrt{12a^3-81b^2}}{9b}\right) + \frac{\pi}{6}(1+\operatorname{sign} b)\right), \\ \lambda_2(a,b) = -2\sqrt{\frac{a}{3}}\cos\left(\frac{1}{3}\arctan\left(-\frac{\sqrt{12a^3-81b^2}}{9b}\right) - \frac{\pi}{6}(3-\operatorname{sign} b)\right), \\ \lambda_3(a,b) = 2\sqrt{\frac{a}{3}}\cos\left(\frac{1}{3}\arctan\left(\frac{\sqrt{12a^3-81b^2}}{9b}\right) - \frac{\pi}{6}(1+\operatorname{sign} b)\right) \\ \qquad\quad -2\sqrt{\frac{a}{3}}\sin\left(\frac{1}{3}\arctan\left(\frac{\sqrt{12a^3-81b^2}}{9b}\right) - \frac{\pi}{6}\operatorname{sign} b\right). \end{cases}$$

Then, if $1 - 2\kappa^2 - 2\kappa\tau \neq 0$, the eigenvalues of the Hamiltonian can be written as

$$e_j(\kappa,\tau) = \lambda_j(3\kappa^2 - \tau, 2\kappa^3 + 2\kappa\tau - 1). \tag{4.2}$$

From the proofs of Propositions 4.4 and 4.6, it follows that $\mathbb{V}_j(\kappa,\tau)$, $j = 1,2,3$, cannot be lightlike and that is spanned by the vector

$$V_j(\kappa,\tau) := {}^t\big(-2\kappa^2 - \kappa e_j(\kappa,\tau) + e_j(\kappa,\tau)^2, -1, -2i\kappa + ie_j(\kappa,\tau)\big). \tag{4.3}$$

For all $j = 1,2,3$, consider the functions

$$\Phi_j(\kappa,\tau) := \langle V_j(\kappa,\tau), V_j(\kappa,\tau)\rangle = 1 - 8\kappa^3 + 6\kappa e_j(\kappa,\tau)^2 - 2e_j(\kappa,\tau)^3,$$

which are nowhere zero on the domain $\mathcal{P}$. The half line $\{(\kappa,0) \mid \kappa > 1/\sqrt{2}\}$ is contained in the connected component $\mathcal{P}_+$ of $\mathcal{P}$. From (4.3), we have, for every $\kappa > 1/\sqrt{2}$,

$$\Phi_1(\kappa,0) > 0, \quad \Phi_2(\kappa,0) < 0, \quad \Phi_3(\kappa,0) > 0.$$

Then, $\mathbb{V}_1(\kappa,\tau)$, $\mathbb{V}_3(\kappa,\tau)$ are spacelike and $\mathbb{V}_2(\kappa,\tau)$ is timelike, for every $(\kappa,\tau) \in \mathcal{P}_+$.

The half line $\{(0,\tau) \mid \tau < -(3/2)^{2/3}\}$ is contained in the connected component $\mathcal{P}_-$ of $\mathcal{P}$. From (4.3), we have for every $\tau < -(3/2)^{2/3}$

$$\Phi_1(\kappa,0) > 0, \quad \Phi_2(\kappa,0) > 0, \quad \Phi_3(\kappa,0) < 0.$$

Then, $\mathbb{V}_1(\kappa,\tau)$, $\mathbb{V}_2(\kappa,\tau)$ are spacelike and $\mathbb{V}_3(\kappa,\tau)$ is timelike, for every $(\kappa,\tau) \in \mathcal{P}_-$. □

4.3. Symmetrical configurations of the first kind Let $\mathfrak{A}$ be the domain

$$\mathfrak{A} = \left\{ (r,\rho) \in (-2,-1/2)\times(0,\sqrt{2}) \mid \rho < \sqrt{\frac{2(-3+2\sqrt{2-r-r^2})}{1+2r}} \right\} \tag{4.4}$$

and $\mu_1 : \mathfrak{A} \to (0,1)$ the smooth function defined by

$$\mu_1(r,\rho) = \frac{4+8r+12\rho^2+(1+2r)\rho^4}{4+8r+12\rho^2+(1+2r)\rho^4-2((2+3r-3r^2-2r^3)(-4\rho+\rho^5))^{2/3}}. \tag{4.5}$$

For each $(r,\rho) \in \mathfrak{A}$, consider

$$e_1^1(r,\rho) = r\frac{\mu_1(r,\rho)}{1-\mu_1(r,\rho)} < e_1^2(r,\rho) = -(1+r)\frac{\mu_1(r,\rho)}{1-\mu_1(r,\rho)} < e_1^3(r,\rho) = \frac{\mu_1(r,\rho)}{1-\mu_1(r,\rho)},$$

the diagonal matrix

$$D_1(r,\rho) = \begin{pmatrix} -ie_1^2(r,\rho) & 0 & 0 \\ 0 & -ie_1^3(r,\rho) & 0 \\ 0 & 0 & -ie_1^1(r,\rho) \end{pmatrix}$$

and the curve $\gamma_{r,\rho} : \mathbb{R} \to \mathcal{S}$ defined by

$$\gamma_{r,\rho} : \mathbb{R} \ni s \mapsto \left(\mathcal{U}\operatorname{Exp}(sD_1(r,\rho))\mathcal{U}^{-1}\right) S(\rho) \in \mathcal{S}, \tag{4.6}$$

where

$$\mathcal{U} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{i}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

and

$$S(\rho) = [{}^t(1,\rho,i\rho^2/2)] \in \mathcal{S}.$$

Definition 4.9. If $(r,\rho) \in \mathfrak{A}$ and $r \in \mathbb{Q}$, the curve $\gamma_{r,\rho} : \mathbb{R} \to \mathcal{S}$ defined by (4.6) is called a *symmetrical configuration* of the *first kind* with parameters (r,ρ). The parameter r (respectively, ρ) is referred to as the *spectral* (respectively, *Clifford*) parameter of the symmetrical configuration.

Proposition 4.10. *If $\gamma_{r,\rho}$ is a symmetrical configuration of the first kind, with parameters (r,ρ), then $\gamma_{r,\rho}$ is an isoparametric knot of the first class, with spectral ratio r. Moreover, the trajectory of $\gamma_{r,\rho}$ is a negative torus knot of type (p,q), where $p>0$ and $q<0$ are, respectively, the numerator and the denominator of $\frac{2+r}{1+2r}$. The trajectory of $\gamma_{r,\rho}$ is contained in the standard Heisenberg cyclide $\mathcal{T}_\rho$ with parameter ρ.*

Proof. First, note that $\gamma_{r,\rho}$ is the orbit through $S(\rho) \in \mathcal{S}$ of the 1-parameter group

$$\mathcal{B} : \mathbb{R} \ni s \mapsto \left(\mathcal{U}\operatorname{Exp}(sD_1(r,\rho))\mathcal{U}^{-1}\right) \in G,$$

where $D_1(r,\rho)$ is a diagonal matrix with purely imaginary eigenvalues $-ie_1^1(r,\rho)$, $-ie_1^2(r,\rho)$, and $-ie_1^3(r,\rho)$, such that $e_1^1(r,\rho)/e_1^3(r,\rho)$, $e_1^2(r,\rho)/e_1^3(r,\rho) \in \mathbb{Q}$. This implies that $\mathcal{B}$ is a periodic map. Consider the lift of $\gamma_{r,\rho}$,

$$\Gamma_{r,\rho} : \mathbb{R} \to \mathcal{N} \subset \mathbb{C}^{2,1},$$

defined by

$$\Gamma_{r,\rho}(s) = -\sigma_1(r,\rho)\mathcal{B}(s)S(\rho),$$

where

$$\sigma_1(r,\rho) = \frac{2(1-\mu_1(r,\rho))}{\mu_1(r,\rho)\sqrt[3]{\rho(-2-3r+3r^2+2r^3)(4-\rho^4)}}. \tag{4.7}$$

By elementary calculations, it follows that the components of the lift are given by

$$\begin{cases} \Gamma_1 = -\frac{\sigma_1}{4} e^{i\mu_1 \frac{(1+r)s}{1-\mu_1}} \left(2+\rho^2+e^{-i\mu_1\frac{(1+2r)s}{1-\mu_1}}(2-\rho^2)\right), \\ \Gamma_2 = -\sigma_1\rho e^{-i\frac{\mu_1 s}{1-\mu_1}}, \\ \Gamma_3 = -\frac{\sigma_1}{4} e^{i\mu_1 \frac{(1+r)s}{1-\mu_1}} \left(2+\rho^2-e^{-i\mu_1\frac{(1+2r)s}{1-\mu_1}}(2-\rho^2)\right). \end{cases} \tag{4.8}$$

Thus, $\det(\Gamma_{r,\rho}, \Gamma'_{r,\rho}, \Gamma''_{r,\rho}) = -1$ and $-i\langle \Gamma_{r,\rho}, \Gamma'_{r,\rho}\rangle = 1$, which implies that $\gamma_{r,\rho}$ is a generic transversal curve parametrized by the natural parameter and that $\Gamma_{r,\rho}$ is a W-lift along $\gamma_{r,\rho}$. Since $\gamma_{r,\rho}$ is an orbit of a 1-parameter group of CR transformations, its bending and twist are constants. This implies that $\gamma_{r,\rho}$ is an isoparametric string. Let $\mathcal{F}$ be the Wilczynski frame along $\gamma_{r,\rho}$ with first column vector $\Gamma_{r,\rho}$ and $\tilde{\mathcal{F}} : \mathbb{R} \to \mathrm{G}$ be the frame field along $\gamma_{r,\rho}$ defined by

$$\tilde{\mathcal{F}} : \mathbb{R} \ni s \mapsto \mathcal{B}(s)\mathfrak{s}(\rho) \in G,$$

where

$$\mathfrak{s}(\rho) = \begin{pmatrix} 1 & 0 & 0 \\ \rho & 1 & 0 \\ \frac{i\rho^2}{2} & i\rho & 1 \end{pmatrix} \in G.$$

Since the bending and the twist of $\gamma_{r,\rho}$ are constant, then $\mathcal{F}^{-1}\mathcal{F}'$ is a constant element K of the Lie algebra $\mathfrak{g}$. By construction, $\tilde{\mathcal{F}}^{-1}\tilde{\mathcal{F}}' = \mathfrak{s}(\rho)^{-1}\mathcal{B}(s)\mathfrak{s}(\rho)$ is also constant. Therefore, there exists an element B of the structure group G_0 of the Chern–Moser bundle such that $\mathcal{F} = \tilde{\mathcal{F}}B$. The Hamiltonian of $\gamma_{r,\rho}$ is then given by

$$\mathrm{H}_{r,\rho} = iB^{-1}\mathfrak{s}(\rho)^{-1}\mathcal{U}D_1(r,\rho)\mathcal{U}^{-1}\mathfrak{s}(\rho)B.$$

This implies that $e_1^1(r,\rho) < e_1^2(r,\rho) < e_1^3(r,\rho)$ is the spectrum of $\mathrm{H}_{r,\rho}$, which proves that r is the spectral ratio of $\gamma_{r,\rho}$. In addition, the corresponding eigenspaces are spanned, respectively, by the vectors

$$\begin{cases} V_1 = B^{-1}\mathfrak{s}(\rho)^{-1}\mathcal{U}_3, \\ V_2 = B^{-1}\mathfrak{s}(\rho)^{-1}\mathcal{U}_1, \\ V_3 = B^{-1}\mathfrak{s}(\rho)^{-1}\mathcal{U}_2, \end{cases}$$

where $\mathcal{U}_1$, $\mathcal{U}_2$, and $\mathcal{U}_3$ are the column vectors of a pseudo-unitary basis $\mathcal{U}$. Taking into account that $\mathfrak{s}(\rho)B \in G$, we conclude that V_1 and V_3 are spacelike an that V_2 is timelike. By Proposition 4.8, it follows that $\gamma_{r,\rho}$ is an isoparametric string of the first class.

Consider the parametrization of the standard Heisenberg cyclide $\mathcal{T}_\rho$ in terms of the Clifford angles θ_1 and θ_2 (cf. (2.27)). Then the parametric equations of $\gamma_{r,\rho}$ can be written as

$$\theta_1(s) = \frac{(1+2r)\mu_1(r,\rho)}{1-\mu_1(r,\rho)}s, \quad \theta_2(s) = -\frac{(2+r)\mu_1(r,\rho)}{1-\mu_1(r,\rho)}s.$$

Since the elliptical profile is counterclockwise-oriented, then $\gamma_{r,\rho}$ is a torus knot of type (p,q), where p, q are relatively prime integers, such that

$$\frac{p}{q} = -\frac{\theta_2(s)}{\theta_1(s)} = \frac{2+r}{1+2r}.$$

□

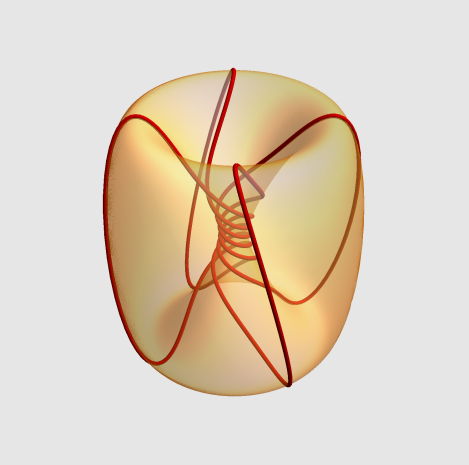

Fig. 4.1: The trajectories of the Heisenberg projections of the symmetrical configuration $\gamma_{r,\rho}$, with $r = -5/6$ and $\rho = 0.47343$. The torus knot has spin 1 and is of type $(7,-4)$.

Remark 4.11. Recalling that $\kappa = 12\langle\Gamma',\Gamma'\rangle$ and $\tau = \Im(\langle\Gamma'',\Gamma'\rangle) + 3\kappa^2$, it follows from (4.8) that the bending and the twist of $\gamma_{r,\rho}$ are given, respectively, by

$$\kappa_{r,\rho} = \frac{-8r^2\rho^2 - (-2+\rho^2)^2 - 2r(2+\rho^2)^2}{4((2+3r-3r^2-2r^3)\rho(-4+\rho^4))^{2/3}},$$

and

$$\tau_{r,\rho} = \frac{9\left((8r^2\rho^2 + (-2+\rho^2)^2 + 2r(2+\rho^2)^2\right)^2 - 4(1+r+r^2)\left(4+12\rho^2+\rho^4+2r(4+\rho^4)\right)^2}{16((2+3r-3r^2-2r^3)\rho(-4+\rho^4))^{4/3}}.$$

The minimal period of the W-lift $\Gamma_{r,\rho}$ is

$$\omega_{r,\rho} = \frac{2\pi(1-\mu_1(r,\rho))}{\mu_1(r,\rho)}\text{denominator}(r).$$

Proposition 4.12. *Let $\mathcal{K}_{r,\rho}$ be the trajectory of $\gamma_{r,\rho}$, a symmetrical configuration of the first kind, with spectral parameter $r = \frac{m}{n} \in (-2,-1/2)$, $\text{gdc}(m,n) = 1$, $m<0$, and Clifford parameter ρ. The following hold true:*

1. *$\mathcal{K}_{r,\rho}$ has spin $1/3$ and phase anomaly $4\pi/3$ if and only if there exist $h,k\in\mathbb{Z}$, $h<0$, $k>0$, such that $m=1+3h$ and $n=1+3k$;*

2. *$\mathcal{K}_{r,\rho}$ has spin $1/3$ and phase anomaly $2\pi/3$ if and only if there exist $h,k\in\mathbb{Z}$, $h<0$, $k>0$, such that $m=2+3h$ and $n=2+3k$.*

Proof. Using the Heisenberg chart, the natural parametrization of $\mathcal{K}_{r,\rho}$ is given by $\gamma_{r,\rho}(s) = (x_{r,\rho}(s), y_{r,\rho}(s), z_{r,\rho}(s))$, where

$$\begin{cases} x_{r,\rho}(s) = 2\rho\frac{(2+\rho^2)\cos(A_{r,\rho}s)-(-2+\rho^2)\cos(B_{r,\rho}s)}{4+\rho^4-(-4+\rho^4)\cos(C_{r,\rho}s)} \\ y_{r,\rho}(s) = 2\rho\frac{(2+\rho^2)\sin(B_{r,\rho}s)-(-2+\rho^2)\sin(A_{r,\rho}s)}{4+\rho^4-(-4+\rho^4)\cos(C_{r,\rho}s)} \\ z_{r,\rho}(s) = \frac{(\rho^4-4)\sin(C_{r,\rho}s)}{4+\rho^4-(-4+\rho^4)\cos(C_{r,\rho}s)} \end{cases} \tag{4.9}$$

and where

$$\begin{cases} A_{r,\rho} = (2+r)\frac{\mu_1(r,\rho)}{1-\mu_1(r,\rho)}, \\ B_{r,\rho} = (1-r)\frac{\mu_1(r,\rho)}{1-\mu_1(r,\rho)}, \\ C_{r,\rho} = (1+2r)\frac{\mu_1(r,\rho)}{1-\mu_1(r,\rho)}. \end{cases}$$

Let $r=m/n$, where $m,-n\in\mathbb{Z}_+$, $\text{gdc}(m,n)=1$ and $-2<m/n<-1/2$. Then, $\gamma_{r,\rho}(0) = (\rho,0,0)$ and the coordinates of $\gamma_{r,\rho}(\omega_{r,\rho}/3)$ are

$$\begin{cases} x_{r,\rho}(\omega_{r,\rho}/3) = -2\rho\frac{(-2+\rho^2)\cos(2(m-n)\pi/3)-(2+\rho^2)\cos(2(m+2n)\pi/3)}{4+\rho^4-(-4+\rho^4)\cos(2(2m+n)\pi/3)} \\ y_{r,\rho}(\omega_{r,\rho}/3) = -2\rho\frac{(2+\rho^2)\sin(2(m+2n)\pi/3)+(-2+\rho^2)\sin(2(m-n)\pi/3)}{4+\rho^4-(-4+\rho^4)\cos(2(2m+n)\pi/3)} \\ z_{r,\rho}(\omega_{r,\rho}/3) = \frac{(\rho^4-4)\sin(2(2m+n)\pi/3)}{4+\rho^4-(-4+\rho^4)\cos(2(2m+n)\pi/3)} \end{cases}$$

Suppose that $\gamma_{r,\rho}$ has spin $1/3$ (i.e., $\gamma_{r,\rho}$ has minimal period $\omega_{r,\rho}/3$). Then $\gamma_{r,\rho}(0) = \gamma_{r,\rho}(\omega_{r,\rho}/3)$, which implies

$$\sin\left(\frac{2}{3}(2m+n)\pi\right) = 0.$$

Thus, either $n = 3\tilde{k}-2m>0$, or $n = 3/2+3\tilde{k}-2m$, for some integer $\tilde{k}$. In the latter case, $\gamma_{r,\rho}(\omega_{r,\rho}/3) = (2/\rho,0,0) \neq \gamma_{r,\rho}(0)$. Therefore, only the first case may occur. Hence

$$n = 3\tilde{k}-2m.$$

By using (4.8), we can compute a W-lift $\Gamma_{r,\rho}$ of $\gamma_{r,\rho}$. It follows that

$$\Gamma_{r,\rho}(\omega_{r,\rho}/3) = e^{i\frac{4m\pi}{3}}\Gamma_{r,\rho}(0).$$

Suppose that the phase anomaly of $\mathcal{K}_{r,\rho}$ is $4\pi/3$. Then there exists an integer $\tilde{h}$, such that $m = 1 + 3\tilde{h}/2$. From this it follows that $\tilde{h}$ is even and negative. Putting $\tilde{h} = 2h < 0$, we get $m = 1 + 3h$ and $n = 1 + 3k$, where $k = \tilde{k} - 2h - 1$.

If the phase anomaly of $\mathcal{K}_{r,\rho}$ is $2\pi/3$, then there exists an integer $\tilde{h}$, such that $m = 1/2 + 3\tilde{h}/2$. From this it follows that $\tilde{h}$ is odd and negative. Putting $\tilde{h} = 2h + 1$, we get $m = 2 + 3h$ and $n = 2 + 3k$, where $k = \tilde{k} - 2h - 2$.

Suppose that $r = \frac{1+3h}{1+3k}$. From (4.9), it follows that $\gamma_{r,\rho}$ has minimal period $\omega_{r,\rho}/3$, i.e., has spin $1/3$. Moreover, by using (4.8), we get that the phase anomaly of $\mathcal{K}_{r,\rho}$ is $4\pi/3$.

Suppose $r = \frac{2+3h}{2+3k}$. From (4.9), it follows that $\gamma_{r,\rho}$ has minimal period $\omega_{r,\rho}/3$, i.e., has spin $1/3$. In addition, by using (4.8) we get that the phase anomaly of $\mathcal{K}_{r,\rho}$ is $2\pi/3$. □

Proposition 4.13. *Let $\mathcal{K}_{r,\rho}$ be the trajectory of a symmetrical configuration of the first kind, with spectral parameter $r = m/n \in (-2, -1/2)$, $\gcd(m, n) = 1$, and $m < 0$. Then, its Maslov index is $-(n + m)$.*

Proof. According to Definition 3.35, the Maslov index of a generic transversal knot is equal to

$$\mu = \frac{i}{2\pi} \int_0^\omega \frac{d\chi}{\chi},$$

where $\chi = \Gamma_1 - i\Gamma_3$ and ω is the minimal period of a W-lift $\Gamma_{r,\rho}$. From (4.8), it follows that

$$\frac{i}{2\pi} \frac{d\chi}{\chi} = \frac{(1 + r)\mu_1(r, \rho)}{2\pi(\mu_1(r, \rho) - 1)}.$$

Using the expression of the minimal period ω given in Remark 4.11, we have

$$\mu = -(1 + r)\text{denominator}(r) = -(n + m),$$

as claimed. □

Remark 4.14. The Bennequin number of $\mathcal{K}_{r,\rho}$ can be estimated via a numerical evaluation of the Gaussian linking integral of $\gamma_{r,\rho}$ with the contact push-off $\gamma_{r,\rho} + \epsilon\xi \circ \gamma_{r,\rho}$ of $\mathcal{K}_{r,\rho}$ in the direction of the vector field $\xi = \partial_x + y\partial_z$. The numerical experiments give convincing support for supposing that

$$\beta(\mathcal{K}_{r,\rho}) = pq + p + q.$$

Remark 4.15. The CR invariant moving trihedron of $\mathcal{K}_{r,\rho}$ can be explicitly computed via the procedure explained in Remark 3.24. The self-linking number of $\mathcal{K}_{r,\rho}$ can be estimated via a numerical evaluation of the Gaussian linking integral of $\gamma_{r,\rho}$ with the push-off of $\gamma_{r,\rho}$ in the direction of the CR normal vector field $\vec{N}$ along $\gamma_{r,\rho}$. The numerical experiments give support for supposing that

$$\text{SL}(\mathcal{K}_{r,\rho}) = pq.$$

We now prove the following.

Proposition 4.16. *Any isoparametric string of the first class is congruent to a symmetrical configuration of the first kind.*

Proof. Let $\gamma : \mathbb{R} \to \mathcal{S}$ be an isoparametric string of the first class, with curvature κ and torsion τ. Let $\mathcal{F} : \mathbb{R} \to G$ be a Wilczynski frame along γ. Without loss of generality, we may assume that $\mathcal{F}(0) = I_3$. Let $e_1 < e_2 < e_3$ be the spectrum of the Hamiltonian H and write

$$e_3 = \frac{m}{m-1},\ e_2 = -(1+r)\frac{m}{m-1},\ e_1 = r\frac{m}{m-1},$$

where $m \in (0,1)$ and $r \in (-2,-1/2) \cap \mathbb{Q}$ is the spectral ratio. Let $\|V\|^2 = \langle V, V\rangle$ be the pseudo-norm of $\mathbb{C}^{2,1}$ and let $V_j(\kappa,\tau)$ the eigenvector of H relative to the eigenvalues e_j defined by (4.3). We recall that $V_1(\kappa,\tau)$ and $V_3(\kappa,\tau)$ are spacelike, while $V_2(\kappa,\tau)$ is timelike. Then,

$$\mathcal{W}(\kappa,\tau) = \left(\frac{V_2(\kappa,\tau)}{\|V_2(\kappa,\tau)\|}, i\frac{V_3(\kappa,\tau)}{\|V_3(\kappa,\tau)\|}, \frac{V_1(\kappa,\tau)}{\|V_1(\kappa,\tau)\|}\right)$$

is a unimodular pseudo-unitary basis of $\mathbb{C}^{2,1}$, such that

$$\mathrm{K}_{\kappa,\tau} = \mathcal{W}(\kappa,\tau)\Delta_1(m,r)\mathcal{W}(\kappa,\tau)^{-1},$$

where

$$\mathrm{K}_{\kappa,\tau} = \begin{pmatrix} i\kappa & -i & \tau \\ 0 & -2i\kappa & 1 \\ 1 & 0 & i\kappa \end{pmatrix}$$

and

$$\Delta_1(m,r) = \begin{pmatrix} -ie_2 & 0 & 0 \\ 0 & -ie_3 & 0 \\ 0 & 0 & -ie_1 \end{pmatrix}$$

Let $\mathcal{B} \in G$ the light-cone basis of $\mathbb{C}^{2,1}$ such that

$$\mathcal{W}(\kappa,\tau) = \mathcal{B}\mathcal{U},$$

where

$$\mathcal{U} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{i}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Then,

$$\mathcal{F}(s) = \mathcal{B}\mathcal{U}\,\mathrm{Exp}(s\Delta_1(m,r))\,\mathcal{U}^{-1}\mathcal{B}^{-1}.$$

The curve $\tilde{\gamma} = \mathcal{B}^{-1}\gamma$ is congruent to γ and

$$\tilde{\Gamma} : \mathbb{R} \ni s \mapsto \mathcal{U}\,\mathrm{Exp}(s\Delta_1(m,r))\,\mathcal{U}^{-1}\mathcal{B}^{-1}E_1 \in \mathcal{N} \subset \mathbb{C}^{2,1} \tag{4.10}$$

is a W-lift of γ. By construction, $\mathcal{U}\,\mathrm{Exp}(s\Delta_1(m,r))\,\mathcal{U}^{-1}$ belongs to the standard torus $\mathrm{T}^2 \subset G$. Therefore, there exists a unique $\rho \in [0,2]$ and unique $R \in \mathrm{T}^2$ such that

$$[\mathcal{B}^{-1}E_1] = R[S(\rho)]. \tag{4.11}$$

Note that $\rho \neq 0, \sqrt{2}$. Otherwise, $\tilde{\gamma}$ would be one of the special orbits for the action of T^2 on $\mathcal{S}$. But this would imply that $\tilde{\gamma}$ is a chain, in contradiction with the hypothesis that the curve is generic. By possibly replacing $\tilde{\gamma}$ with $R\tilde{\gamma}$, it follows from (4.10) and (4.11) that the Wilczynsky lifts of $\tilde{\gamma}$ can be written as

$$\tilde{\Gamma}(s) = \zeta \operatorname{Exp}(s\Delta_1(m,r))\, S(\rho),$$

where ζ is a nonzero complex number. From $\det(\tilde{\Gamma}, \tilde{\Gamma}', \tilde{\Gamma}'')$, we get

$$\zeta = \frac{2(1-m)}{\sqrt[3]{\rho(4-\rho^4)(2r^3+3r^2-3r-2)}}.$$

By requiring that $\langle \tilde{\Gamma}, \tilde{\Gamma}' \rangle = i$, it follows that $m = \mu_1(r,\rho)$ (cf. (4.5)) and $\zeta = -\sigma_1(r,\rho)$ (cf. (4.7)). This implies that $\tilde{\gamma}$ is the natural parametrization of the symmetrical configuration with spectral ratio r and Clifford parameter ρ. □

4.4. Symmetrical configurations of the second kind Let $\mathfrak{B}$ be the rectangular domain $(-2,-1/2) \times (0,\sqrt{2})$ and let $\mu_2 : \mathfrak{B} \to (0,1)$ be the smooth function

$$\mu_2(r,\rho) = \frac{-4+4r-12\rho^2(1+2r)+(r-1)\rho^4}{-4+4r-12\rho^2(1+2r)+(r-1)\rho^4-2((2+3r-3r^2-2r^3)(-4\rho+\rho^5))^{2/3}}. \tag{4.12}$$

For each $(r,\rho) \in \mathfrak{B}$, consider

$$e_2^1(r,\rho) = r\frac{\mu_2(r,\rho)}{1-\mu_2(r,\rho)} < e_2^2(r,\rho) = -(1+r)\frac{\mu_2(r,\rho)}{1-\mu_2(r,\rho)} < e_2^3(r,\rho) = \frac{\mu_2(r,\rho)}{1-\mu_2(r,\rho)},$$

the diagonal matrix

$$D_2(r,\rho) = \begin{pmatrix} -ie_2^3(r,\rho) & 0 & 0 \\ 0 & -ie_2^2(r,\rho) & 0 \\ 0 & 0 & -ie_2^1(r,\rho) \end{pmatrix},$$

and the curve $\eta_{r,\rho} : \mathbb{R} \to \mathcal{S}$ defined by

$$\eta_{r,\rho} : \mathbb{R} \ni s \mapsto \left(\mathcal{U} \operatorname{Exp}(sD_2(r,\rho))\, \mathcal{U}^{-1}\right) S(\rho) \in \mathcal{S}, \tag{4.13}$$

where, as above, $\mathcal{U} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{i}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$.

Definition 4.17. If (r,ρ) is a point of $\mathfrak{B}$, with $r \in \mathbb{Q}$, the curve $\eta_{r,\rho} : \mathbb{R} \to \mathcal{S}$ defined by (4.13) is called a *symmetrical configuration* of the *second kind*, with parameters (r,ρ). The parameter r (respectively, ρ) is referred to as the *spectral* (respectively, *Clifford*) parameter of the symmetrical configuration.

Proposition 4.18. *If $\eta_{r,\rho}$ is a symmetrical configuration of the second kind, with parameters (r,ρ), then $\eta_{r,\rho}$ is an isoparametric knot of the second class, with spectral ratio r. The trajectory $\mathcal{E}_{r,\rho}$ of $\eta_{r,\rho}$ is a positive torus knot of type (p,q), where the positive integers p and q are, respectively, the numerator and the denominator of $\frac{2+r}{1-r}$. The trajectory $\mathcal{E}_{r,\rho}$ is contained in the standard Heisenberg cyclide $\mathcal{T}_\rho$ with parameter ρ.*

Proof. The proof is rather similar to that of Proposition 4.10. First, note that $\eta_{r,\rho}$ is the orbit through the point $S(\rho) \in \mathcal{S}$ of the 1-parameter group

$$\mathcal{B} : s \in \mathbb{R} \to \mathcal{U}\,\mathrm{Exp}(sD_2(r,\rho))\,\mathcal{U}^{-1} \in G$$

where $D_2(r,\rho)$ is a diagonal matrix with purely imaginary eigenvalues $-ie_2^1(r,\rho)$, $ie_2^2(r,\rho)$, and $ie_2^3(r,\rho)$, such that $e_1^1(r,\rho)/e_1^3(r,\rho)$, $e_1^2(r,\rho)/e_1^3(r,\rho) \in \mathbb{Q}$. This implies that $\mathcal{B}$ is a periodic map. Consider the lift

$$\Gamma_{r,\rho} : \mathbb{R} \to \mathcal{N} \subset \mathbb{C}^{2,1}$$

of $\eta_{r,\rho}$ defined by

$$\Gamma_{r,\rho}(s) = -\sigma_2(r,\rho)\mathcal{B}(s)S(\rho),$$

where

$$\sigma_2(r,\rho) = \frac{2(1-\mu_2(r,\rho))}{\mu_2(r,\rho)\sqrt[3]{\rho(-2-3r+3r^2+2r^3)(4-\rho^4)}} \tag{4.14}$$

By elementary calculations, it follows that the components of the lift are given by

$$\begin{cases} \Gamma_1 = \frac{\sigma_2}{4}\left(e^{-i\frac{\mu_2 rs}{1-\mu_2}}(2-\rho^2) + e^{-i\frac{\mu_2 s}{1-\mu_2}}(2+\rho^2)\right), \\ \Gamma_2 = \sigma_2\rho e^{i\frac{\mu_2(1+r)s}{1-\mu_2}}, \\ \Gamma_3 = i\frac{\sigma_2}{4}\left(e^{-i\frac{\mu_2 s}{1-\mu_2}}(2+\rho^2) - e^{-i\frac{\mu_2 rs}{1-\mu_2}}(2-\rho^2)\right). \end{cases} \tag{4.15}$$

Thus, $\det(\Gamma_{r,\rho}, \Gamma'_{r,\rho}, \Gamma''_{r,\rho}) = -1$ and $-i\langle \Gamma_{r,\rho}, \Gamma'_{r,\rho}\rangle = 1$. From this we deduce that $\eta_{r,\rho}$ is a generic transversal curve parameterized by the natural parameter and that $\Gamma_{r,\rho}$ is a W-lift along $\eta_{r,\rho}$. Since $\eta_{r,\rho}$ is an orbit of a 1-parameter group of CR transformations, its bending and torsion are constant. This implies that $\eta_{r,\rho}$ is an isoparametric string. Let $\mathcal{F}$ be the Wilczynski frame field along $\eta_{r,\rho}$ with first column vector $\Gamma_{r,\rho}$ and let $\tilde{\mathcal{F}} : \mathbb{R} \to \mathrm{G}$ be the frame field along $\eta_{r,\rho}$ defined by

$$\tilde{\mathcal{F}} : \mathbb{R} \ni s \mapsto \mathcal{B}(s)\mathfrak{s}(\rho) \in G,$$

where

$$\mathfrak{s}(\rho) = \begin{pmatrix} 1 & 0 & 0 \\ \rho & 1 & 0 \\ \frac{i\rho^2}{2} & i\rho & 1 \end{pmatrix} \in G.$$

Since the bending and the twist of $\eta_{r,\rho}$ are constant, then $\mathcal{F}^{-1}\mathcal{F}'$ is a constant element K of the Lie algebra $\mathfrak{g}$. By construction, $\tilde{\mathcal{F}}^{-1}\tilde{\mathcal{F}}' = \mathfrak{s}(\rho)^{-1}\mathcal{B}(s)\mathfrak{s}(\rho)$ is also constant. Therefore, there exists an element B of the structure group G_0 of the Chern–Moser bundle such that $\mathcal{F} = \tilde{\mathcal{F}}B$. Then, the Hamiltonian of $\eta_{r,\rho}$ is given by

$$\mathrm{H}_{r,\rho} = iB^{-1}\mathfrak{s}(\rho)^{-1}\mathcal{U}D_2(r,\rho)\mathcal{U}^{-1}\mathfrak{s}(\rho)B.$$

This implies that $e_2^1(r,\rho) < e_2^2(r,\rho) < e_2^3(r,\rho)$ is the spectrum of $\mathrm{H}_{r,\rho}$. This proves that r is the spectral ratio of $\eta_{r,\rho}$. In addition, the corresponding eigenspaces are

spanned, respectively, by the vectors

$$\begin{cases} V_1 = B^{-1}\mathfrak{s}(\rho)^{-1}\mathcal{U}_3; \\ V_2 = B^{-1}\mathfrak{s}(\rho)^{-1}\mathcal{U}_1; \\ V_3 = B^{-1}\mathfrak{s}(\rho)^{-1}\mathcal{U}_2; \end{cases}$$

where $\mathcal{U}_1$, $\mathcal{U}_2$ and $\mathcal{U}_3$ are the column vectors of a pseudo-unitary basis $\mathcal{U}$. Taking into account that $\mathfrak{s}(\rho)B \in G$, it follows that V_1 and V_3 are spacelike, while V_2 is timelike. By Proposition 4.8, it follows that $\eta_{r,\rho}$ is an isoparametric string of the second class.

Consider the parametrization of the standard Heisenberg cyclide $\mathcal{T}_\rho$ in terms of the Clifford angles θ_1 and θ_2. Then the parametric equations of $\eta_{r,\rho}$ can be written as

$$\theta_1(s) = -\frac{(1-r)\mu_2(r,\rho)}{1-\mu_2(r,\rho)}s, \quad \theta_2(s) = \frac{(2+r)\mu_2(r,\rho)}{1-\mu_2(r,\rho)}s.$$

Since the elliptical profile is counterclockwise-oriented, then $\eta_{r,\rho}$ is a torus knot of type (p,q), where p, q are relatively prime integers, such that

$$\frac{p}{q} = -\frac{\theta_2(s)}{\theta_1(s)} = \frac{2+r}{1-r}.$$

□

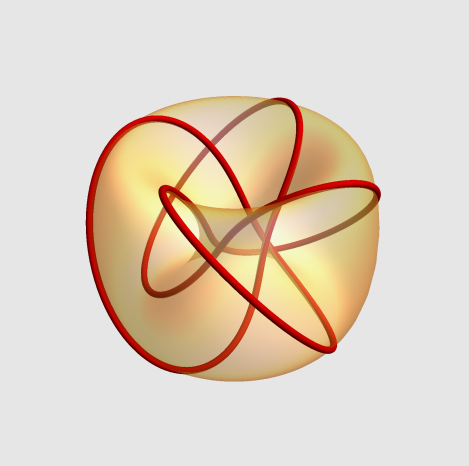

Fig. 4.2: The Heisenberg projection of the symmetrical configuration $\eta_{r,\rho}$, with $r = -5/7$ and $\rho = 0.7$. The torus knot has spin $1/3$, is of type $(3,4)$, and its total strain is ≈ 6.01323.

Remark 4.19. From (4.8) and the fact that $\kappa = \frac{1}{2}\langle \Gamma', \Gamma' \rangle$ and $\tau = \mathrm{Im}(\langle \Gamma'', \Gamma' \rangle) + 3\kappa^2$, we deduce that the bending and the twist of $\eta_{r,\rho}$ are given, respectively, by

$$\kappa_{r,\rho} = \frac{16r\rho^2 - (-2+\rho^2)^2 + r^2(2+\rho^2)^2}{4((2+3r-3r^2-2r^3)\rho(-4+\rho^4))^{2/3}},$$

and

$$\tau_{r,\rho}=\frac{3\Big(16r\rho^2-(-2+\rho^2)^2+r^2(2+\rho^2)^2\Big)^2-4(1+r+r^2)\Big(4+12\rho^2+\rho^4-r(4-12\rho2+\rho^4)\Big)^2}{16((2+3r-3r^2-2r^3)\rho(-4+\rho^4))^{4/3}}.$$

The minimal period of the W-lift $\Gamma_{r,\rho}$ is

$$\omega_{r,\rho}=2\pi\frac{1-\mu_2(r,\rho)}{\mu_2(r,\rho)}\text{denominator}(r).$$

Proposition 4.20. *Let $\mathcal{E}_{r,\rho}$ be the trajectory of $\eta_{r,\rho}$, a symmetrical configuration of the second kind, with spectral parameter $r=\frac{m}{n}\in(-2,-1/2)$, $\gcd(m,n)=1$, $m<0$, and Clifford parameter ρ. The following hold true:*

1. *$\mathcal{E}_{r,\rho}$ has spin $1/3$ and phase anomaly $4\pi/3$ if and only if there exist $h,k\in\mathbb{Z}$, $h<0$, $k>0$, such that $m=1+3h$ and $n=1+3k$;*

2. *$\mathcal{E}_{r,\rho}$ has spin $1/3$ and phase anomaly $2\pi/3$ if and only if there exist $h,k\in\mathbb{Z}$, $h<0$, $k>0$, such that $m=2+3h$ and $n=2+3k$.*

Proof. Using the Heisenberg chart, the natural parametrization of $\mathcal{E}_{r,\rho}$ is given by $\eta_{r,\rho}(s)=(x_{r,\rho}(s),y_{r,\rho}(s),z_{r,\rho}(s))$, where

$$\begin{cases}x_{r,\rho}(s)=2\rho\frac{(2+\rho^2)\cos(A^2_{r,\rho}s)-(-2+\rho^2)\cos(C^2_{r,\rho}s)}{4+\rho^4-(-4+\rho^4)\cos(B^2_{r,\rho}s)}\\ y_{r,\rho}(s)=2\rho\frac{(2+\rho^2)\sin(B^2_{r,\rho}s)-(-2+\rho^2)\sin(C^2_{r,\rho}s)}{4+\rho^4-(-4+\rho^4)\cos(B^2_{r,\rho}s)}\\ z_{r,\rho}(s)=\frac{(\rho^4-4)\sin(B^2_{r,\rho}s)}{4+\rho^4-(-4+\rho^4)\cos(B^2_{r,\rho}s)}\end{cases}$$

and we denote by $A^2_{r,\rho}$, $B^2_{r,\rho}$ and $C^2_{r,\rho}$

$$\begin{cases}A^2_{r,\rho}=(2+r)\frac{\mu_2(r,\rho)}{1-\mu_2(r,\rho)};\\ B^2_{r,\rho}=(1-r)\frac{\mu_2(r,\rho)}{1-\mu_2(r,\rho)};\\ C^2_{r,\rho}=(1+2r)\frac{\mu_2(r,\rho)}{1-\mu_2(r,\rho)}.\end{cases}$$

Let $r=m/n$, where $m,-n\in\mathbb{Z}_+$, $\gcd(m,n)=1$ and $-2<m/n<-1/2$. Then, $\eta_{r,\rho}(0)=(\rho,0,0)$. Taking into account (4.15), the coordinates of $\eta_{r,\rho}(\omega_{r,\rho}/3)$ are

$$\begin{cases}x_{r,\rho}(\omega_{r,\rho}/3)=-2\rho\frac{(-2+\rho^2)\cos(2(m-n)\pi/3)-(2+\rho^2)\cos(2(m+2n)\pi/3)}{4+\rho^4-(-4+\rho^4)\cos(2(2m+n)\pi/3)}\\ y_{r,\rho}(\omega_{r,\rho}/3)=-2\rho\frac{(2+\rho^2)\sin(2(m+2n)\pi/3)+(-2+\rho^2)\sin(2(m-n)\pi/3)}{4+\rho^4-(-4+\rho^4)\cos(2(2m+n)\pi/3)}\\ z_{r,\rho}(\omega_{r,\rho}/3)=\frac{(\rho^4-4)\sin(2(2m+n)\pi/3)}{4+\rho^4-(-4+\rho^4)\cos(2(2m+n)\pi/3)}\end{cases}\tag{4.16}$$

Suppose that $\eta_{r,\rho}$ has spin $1/3$ (i.e., $\eta_{r,\rho}$ has minimal period $\omega_{r,\rho}/3$). Then $\eta_{r,\rho}(0)=\eta_{r,\rho}(\eta_{r,\rho}/3)$. This implies

$$\sin\left(\frac{2}{3}(m-n)\pi\right)=0.$$

Thus, either $m = 3\tilde{k} + n > 0$, or $m = 3\tilde{k} + n + \frac{3}{2}$, for some integer $\tilde{k}$. In the latter case $\eta_{r,\rho}(\omega_{r,\rho}/3) = (-2/\rho, 0, 0) \neq \eta_{r,\rho}(0)$. Therefore, only the first case may occur. Hence

$$m = 3\tilde{k} + m.$$

By using (4.15), we can compute a W-lift $\Gamma_{r,\rho}$ of $\eta_{r,\rho}$. It follows that

$$\Gamma_{r,\rho}(\omega_{r,\rho}/3) = e^{i\frac{4n\pi}{3}}\Gamma_{r,\rho}(0).$$

Suppose that the phase anomaly of $\mathcal{E}_{r,\rho}$ is $4\pi/3$. Then there exists an integer $\tilde{h}$ such that $n = 1 + 3\tilde{h}/2$. From this it follows that $\tilde{h}$ is even. If we put $\tilde{h} = 2h$, we get $n = 1 + 3h$ and $m = 1 + 3k$, where $k = \tilde{k} + h$.

If the phase anomaly of $\mathcal{E}_{r,\rho}$ is $2\pi/3$, then there exists an integer $\tilde{h}$ such that $n = 1/2 + 3\tilde{h}/2$. From this it follows that $\tilde{h}$ is odd. Putting $\tilde{h} = 2h + 1$, we get $n = 2 + 3h$ and $m = 2 + 3k$, where $k = \tilde{k} + h$.

Suppose that $r = \frac{1+3h}{1+3k}$. From (4.16), it follows that $\eta_{r,\rho}$ has minimal period $\omega_{r,\rho}/3$, i.e., has spin $1/3$. Moreover, by using (4.15) we get that the phase anomaly of $\mathcal{E}_{r,\rho}$ is $4\pi/3$.

Suppose that $r = \frac{2+3h}{2+3k}$. From (4.16), it follows that $\eta_{r,\rho}$ has minimal period $\omega_{r,\rho}/3$, i.e., has spin $1/3$. In addition, by using (4.15) we get that the phase anomaly of $\mathcal{E}_{r,\rho}$ is $2\pi/3$. □

Proposition 4.21. *Let $\mathcal{E}_{r,\rho}$ be the trajectory of a symmetrical configuration of the second kind. Then, its Maslov index is $p+q$, where (p, q) is the torus knot type of $\mathcal{E}_{r,\rho}$.*

Proof. The Maslov index of generic transversal knot is equal to $\frac{i}{2\pi}\int_0^\omega \frac{d\chi}{\chi}$, where $\chi = \Gamma_1 - i\Gamma_3$ and ω is the minimal period of a W-lift $\Gamma_{r,\rho}$. The proof follows from (4.15) and the expression of the minimal period ω given in Remark 4.19. □

Remark 4.22. The Bennequin number of $\mathcal{E}_{r,\rho}$ can be estimated via a numerical evaluation of the Gaussian linking integral of $\eta_{r,\rho}$ with the contact push-off of $\mathcal{E}_{r,\rho}$ in the direction of the vector field $\xi = \partial_x + y\partial_z$. The numerical experiments give support for supposing that

$$\beta(\mathcal{E}_{r,\rho}) = pq - (p + q).$$

In view of the Etnyre Theorem, the symmetrical configurations are positive torus knots with maximal Bennequin invariant.

Remark 4.23. The CR invariant moving trihedron of $\mathcal{E}_{r,\rho}$ can be explicitly computed via the procedure explained in Remark 3.24. The self-linking number of $\mathcal{E}_{r,\rho}$ can be estimated via a numerical evaluation of the Gaussian linking integral of $\eta_{r,\rho}$ with the push-off of $\eta_{r,\rho}$ in the direction of CR normal vector field $\vec{N}$ along $\eta_{r,\rho}$. The numerical experiments give support for supposing that

$$\mathrm{SL}(\mathcal{E}_{r,\rho}) = pq.$$

In analogy with Proposition 4.16, we can prove the following.

Proposition 4.24. *Any isoparametric string of the second class is congruent to a symmetrical configuration of the second kind.*

5. The total strain functional

Let $\gamma : J \to \mathcal{S}$ be a generic transversal curve with infinitesimal strain ds and $C \subset J$ be a closed interval. The integral

$$\mathfrak{S}_{\gamma,C} = \int_C ds$$

is the *total strain* of the generic transversal arc $\gamma(C)$.

The main result of this section is the following.

Theorem 5.1. *A closed critical curve of the strain functional is equivalent to a symmetrical configuration of the second kind, with positive knot type* (p, q)*, where* $\frac{p}{q} \in (0, 1)$*, spectral parameter* $r = \frac{p-2q}{p+q}$*, and Clifford parameter*

$$\rho = \sqrt{\frac{6 + 6r + 4\sqrt{3(1 + r + r^2)} - \sqrt[4]{5 + 8r + 5r^2 + 3(1 + r)\sqrt{3(1 + r + r^2)}}}{1 - r}}. \tag{5.1}$$

Proof. According to Section 3.3, generic transversal curves are in one-to-one correspondence with the integral curves of the Pfaffian system $(\mathbb{J}, \eta)$ defined on the configuration space $Y = [G] \times \mathbb{R}^2$. Let $\gamma : J \to \mathcal{S}$ be a generic transversal curve parameterized by the natural parameter and $\mathcal{F} : J \to Y$ its prolongation. Following Griffiths' approach to the calculus of variations [24], if $\mathcal{F}$ admits a lift $\Phi : J \to \mathcal{Z}$ to the phase space (cf. Section 3.3), such that the linear map $\Xi(\Phi'|_s, \cdot) : X \in T_{\Phi(s)}(\mathcal{Z}) \to \mathbb{R}$ is zero, for every s, then γ is a critical curve of the total strain functional with respect to compactly supported variations.

As observed by R. Bryant [5], if the derived systems of $\mathbb{J}$ have constant rank, as in the present case, then the converse is also true. More precisely, let $\Phi : J \to \mathcal{Z}$ be a curve in the phase space such that $\Xi(\Phi'|_s, \cdot) = 0$, for every s, and that $\Phi^*(\xi)$ is nowhere vanishing. Let $\mathrm{pr} : \mathcal{Z} \to \mathcal{S}$ be the natural projection of $\mathcal{Z}$ onto $\mathcal{S}$. Then, $\gamma = \Phi \circ \mathrm{pr} : J \to \mathcal{S}$ is a critical curve for the total strain functional, with respect to compactly supported variations.

We prove that a curve $\Phi : J \to \mathcal{Z}$ satisfies $\Xi(\Phi'|_s, \cdot) = 0$ and $\Phi^*(\xi)|_s \neq 0$, for every s, if and only if the bending and the twist of $\gamma = \Phi \circ \mathrm{pr}$ are constant and satisfy the equation $\tau = -9\kappa^2$. Recall that a curve Φ in the phase space is of the form

$$\Phi(s) = ([A(s)], \kappa(s), \tau(s), p_1(s), \ldots, p_7(s)),$$

where $A : J \to G$ is a smooth map and $\kappa, \tau, p_1, \ldots, p_7$ are smooth functions. The projection $\gamma = \Phi \circ \mathrm{pr}$ is the smooth curve

$$\gamma(s) = [A_1(s)]_{\mathbb{C}}$$

where $A_1 : J \to \mathbb{C}^{2,1}$ is the first column vector of A.

Observe that $\Phi : J \to \mathcal{Z}$ satisfies $\Xi(\Phi'|_s, \cdot) = 0$ and $\Phi^*(\xi)|_s \neq 0$, for every s, if and only if Φ is an integral curve of the Cartan system $C(\Xi)$ associated to the Cartan–Poincaré form Ξ (cf. Definition 3.28), such that $\Phi^*(\eta)|_s \neq 0$. From the first equation of (3.16), it follows

$$\Phi^*(\mu^j) = 0, \quad \forall\, j = 1, \dots, 7,$$

and

$$\Phi^*(\eta) \neq 0.$$

By definition,

$$\Phi^*(\mu^j) = A^*(\mu^j), \quad \forall\, j = 1, \dots, 7,$$

and

$$\Phi^*(\eta) = A^*(\alpha_1^3).$$

This implies that $([A], \kappa, \tau)$ is an integral curve of the Pfaffian differential system $(\mathbb{J}, \eta)$ on the configuration space Y. Hence, γ is a generic transversal curve with bending κ, twist τ, and such that A is a Wilczynski frame field along γ. From the second, the third and the fourth equation of (3.16), we have

$$\begin{cases} \Phi^*(\dot\pi_1) = p_6 \Phi^*(\eta) = 0, \\ \Phi^*(\dot\pi_2) = p_7 \Phi^*(\eta) = 0, \\ \Phi^*(\dot\eta) = p_6 \Phi^*(\pi^1) + p_7 \Phi^*(\pi^2) = 0. \end{cases}$$

Since $\Phi^*(\eta) \neq 0$, it follows that $p_6 = p_7 = 0$. From the last two equations of (3.16), we have

$$\begin{cases} \Phi^*(\dot\mu_6) = dp_6 - 3p_4 \Phi^*(\eta) = 0, \\ \Phi^*(\dot\mu_7) = dp_7 + 3p_5 \Phi^*(\eta) = 0. \end{cases}$$

Since $\Phi^*(\eta) \neq 0$ and $p_6 = p_7 = 0$, it follows that $p_4 = p_5 = 0$. From the eighth and the ninth equation of (3.16), we have

$$\begin{cases} \Phi^*(\dot\mu_4) = dp_4 + (p_2 - 3\kappa p_3 - 2p_7) \Phi^*(\eta) = 0, \\ \Phi^*(\dot\mu_5) = dp_5 - (2 - 3p_3 - 2\kappa p_6 + 4\tau p_7) \Phi^*(\eta) = 0. \end{cases}$$

Since $\Phi^*(\eta) \neq 0$ and $p_4 = p_5 = p_6 = p_7 = 0$, it follows $p_3 = \frac{2}{3}$ and $p_2 = 2\kappa$. From the seventh equation of (3.16), we have

$$\Phi^*(\dot\mu_3) = dp_3 + (p_1 + 3\kappa p_4) \Phi^*(\eta) = 0.$$

Since $\Phi^*(\eta) \neq 0$, $p_3 = 2/3$ and $p_4 = 0$, we obtain $p_1 = 0$. From the fifth equation of (3.16), we have

$$\Phi^*(\dot\mu_1) = dp_1 + (3\kappa p_2 + \tau p_3 + p_6) \Phi^*(\eta) = 0.$$

Since $\Phi^*(\eta) \neq 0$, $p_2 = 2\kappa$, $p_3 = \frac{2}{3}$ and $p_1 = p_6 = 0$, we obtain $\tau = -9\kappa^2$. From the sixth equation of (3.16), we have

$$\Phi^*(\dot{\mu}_2) = dp_2 - (3\kappa p_1 - \tau p_4 + p_5)\Phi^*(\eta) = 0.$$

Since $\Phi^*(\eta) \neq 0$, $p_2 = 2\kappa$, $p_1 = p_4 = p_5 = 0$, we obtain that κ is constant. This proves that γ is an isoparametric curve such that $\tau = -9\kappa^2$.

Conversely, let γ be an isoparametric curve, such that $\tau = -9\kappa^2$, and let $\mathcal{F}$ be a Wilczynski frame along γ. Then,

$$\Phi(s) = \left(\mathcal{F}(s), \kappa, -9\kappa^2, 0, 2\kappa, \frac{2}{3}, 0, 0, 0, 0\right)$$

is a lift of γ to the phase space $\mathcal{Z}$, such that $\Phi^*(\eta) \neq 0$. By construction, Φ is an integral curve of the Cartan system $C(\Xi)$. We have thus proved that the critical curves of the strain functional are isoparametric and satisfy $\tau = -9\kappa^2$, as claimed.

Next, we show that an isoparametric string with $\tau = -9\kappa^2$ must be of the second class. Substituting $\tau = -9\kappa^2$ into the discriminant of the Hamiltonian (cf. Remark 4.3), we get

$$D(\kappa, -9\kappa^2) = -27(1 + 32\kappa^3).$$

Thus, $D(\kappa, -9\kappa^2) > 0$ if and only if $\kappa < -2^{-\frac{5}{3}}$. Since the infimum of the bending of an isoparametric string of the first class is $\frac{1}{2}$ (cf. the formula for $\kappa_{r,\rho}$ in Remark 4.11 and the definition of the domain $\mathfrak{A}$ given by (4.4)), we conclude that isoparametric strings of the first class cannot be critical for the total strain functional.

Next, we prove that a symmetrical configuration of the second kind, with spectral parameter r and Clifford parameter ρ, is a critical point of the total strain functional if and only if (5.1) holds true. The bending and the twist of a standard configuration of the second kind are given in Remark 4.19 as a function of their spectral parameter r and Clifford parameter ρ. By taking into consideration such formulas, it follows that $\tau = -9\kappa^2$ if and only if

$$16 - 96\rho^2 - 40\rho^4 - 24\rho^6 + \rho^8 - r(16 + 96\rho^2 - 40\rho^4 + 24\rho^6 + \rho^8) = 0.$$

Consequently, ρ^2 needs to be equal to one of the following

$$\begin{aligned}
f_1(r) &= \frac{6+6r-4\sqrt{3}\sqrt{1+r+r^2} - \sqrt[4]{5+5r^2-3\sqrt{3}\sqrt{1+r+r^2}+r(8-3\sqrt{3}\sqrt{1+r+r^2})}}{1-r} \\
f_2(r) &= \frac{6+6r-4\sqrt{3}\sqrt{1+r+r^2} + \sqrt[4]{5+5r^2-3\sqrt{3}\sqrt{1+r+r^2}+r(8-3\sqrt{3}\sqrt{1+r+r^2})}}{1-r} \\
f_3(r) &= \frac{6+6r+4\sqrt{3}\sqrt{1+r+r^2} - \sqrt[4]{5+5r^2+3\sqrt{3}\sqrt{1+r+r^2}+r(8+3\sqrt{3}\sqrt{1+r+r^2})}}{1-r} \\
f_4(r) &= \frac{6+6r+4\sqrt{3}\sqrt{1+r+r^2} + \sqrt[4]{5+5r^2+3\sqrt{3}\sqrt{1+r+r^2}+r(8+3\sqrt{3}\sqrt{1+r+r^2})}}{1-r}
\end{aligned}$$

Since $0 < \rho < \sqrt{2}$, taking into account that $f_1(r) < 0$, $f_2(r) < 0$ and $f_4(r) > 2$ and that $0 < 6 - 4\sqrt{2} < f_3(r) < 2$, we obtain the claimed expression (5.1) for ρ.

We conclude the proof recalling that, by Proposition 4.18, the spectral parameter and the torus knot type (p, q) of a standard configuration of the second kind are related by $\frac{2+r}{1-r} = \frac{p}{q}$. Hence, $r = \frac{p-2q}{p+q}$. □

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