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ON THE L.C.M. OF RANDOM TERMS OF BINARY RECURRENCE SEQUENCES

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ABSTRACT. For every positive integer n and every $\delta \in [0, 1]$, let $B(n, \delta)$ denote the probabilistic model in which a random set $A \subseteq \{1, \dots, n\}$ is constructed by choosing independently every element of $\{1, \dots, n\}$ with probability δ . Moreover, let $(u_k)_{k \geq 0}$ be an integer sequence satisfying $u_k = a_1 u_{k-1} + a_2 u_{k-2}$, for every integer $k \geq 2$, where $u_0 = 0$, $u_1 \neq 0$, and a_1, a_2 are fixed nonzero integers; and let α and β , with $|\alpha| \geq |\beta|$, be the two roots of the polynomial $X^2 - a_1 X - a_2$. Also, assume that α/β is not a root of unity.

We prove that, as $\delta n / \log n \rightarrow +\infty$, for every A in $B(n, \delta)$ we have

$$\log \text{lcm}(u_a : a \in A) \sim \frac{\delta \text{Li}_2(1 - \delta)}{1 - \delta} \cdot \frac{3 \log |\alpha / \sqrt{(a_1^2, a_2)}|}{\pi^2} \cdot n^2$$

with probability $1 - o(1)$, where lcm denotes the lowest common multiple, Li_2 is the dilogarithm, and the factor involving δ is meant to be equal to 1 when $\delta = 1$.

This extends previous results of Akiyama, Tropic, Matiyasevich, Guy, Kiss and Mátyás, who studied the deterministic case $\delta = 1$, and is motivated by an asymptotic formula for $\text{lcm}(A)$ due to Cilleruelo, Rué, Šarka, and Zumalacárregui.

1. INTRODUCTION

It is well known that the Prime Number Theorem is equivalent to the asymptotic formula

$$(1) \quad \log \text{lcm}(1, 2, \dots, n) \sim n,$$

as $n \rightarrow +\infty$, where lcm denotes the lowest common multiple.

For every positive integer n and every $\delta \in [0, 1]$, let $B(n, \delta)$ denote the probabilistic model in which a random set $A \subseteq \{1, \dots, n\}$ is constructed by choosing independently every element of $\{1, \dots, n\}$ with probability δ . Motivated by (1), Cilleruelo, Rué, Šarka, and Zumalacárregui [8] proved the following result (see also [5] for a more precise version, and [6, 7, 12] for others results of similar flavor).

Theorem 1.1. *Let A be a random set in $B(n, \delta)$. Then, as $\delta n \rightarrow +\infty$, we have*

$$\log \text{lcm}(A) \sim \frac{\delta \log(1/\delta)}{1 - \delta} \cdot n,$$

with probability $1 - o(1)$, where the factor involving δ is meant to be equal to 1 for $\delta = 1$.

Let $(u_k)_{k \geq 0}$ be an integer sequence satisfying $u_k = a_1 u_{k-1} + a_2 u_{k-2}$, for every integer $k \geq 2$, where $u_0 = 0$, $u_1 \neq 0$, and a_1, a_2 are two fixed nonzero integers. Moreover, let α and β , with $|\alpha| \geq |\beta|$, be the two roots of the polynomial $X^2 - a_1 X - a_2$. We assume that α/β is not a root of unity, which is a necessary and sufficient condition to have $u_k \neq 0$ for all integers $k \geq 1$.

Akiyama [1] and, independently, Tropic [15] proved the following analog of (1) for the sequence $(u_k)_{k \geq 1}$.

Theorem 1.2. *We have*

$$\log \text{lcm}(u_1, u_2, \dots, u_n) \sim \frac{3 \log |\alpha / \sqrt{(a_1^2, a_2)}|}{\pi^2} \cdot n^2,$$

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as $n \rightarrow +\infty$.

Special cases of Theorem 1.2 were previously proved by Matiyasevich, Guy [11], Kiss and Mátyás [10]. Furthermore, Akiyama [2, 3] generalized Theorem 1.2 to sequences having some special divisibility properties, while Akiyama and Luca [4] studied $\text{lcm}(u_{f(1)}, \dots, u_{f(n)})$ when f is a polynomial, $f = \varphi$ (the Euler's totient function), $f = \sigma$ (the sum of divisors function), or f is a binary recurrence sequence.

Motivated by Theorem 1.1, we give the following generalization of Theorem 1.2.

Theorem 1.3. *Let A be a random set in $B(n, \delta)$. Then, as $\delta n / \log n \rightarrow +\infty$, we have*

$$(2) \quad \text{lcm}(u_a : a \in A) \sim \frac{\delta \text{Li}_2(1 - \delta)}{1 - \delta} \cdot \frac{3 \log |\alpha / \sqrt{(a_1^2, a_2)}|}{\pi^2} \cdot n^2,$$

with probability $1 - o(1)$, where $\text{Li}_2(z) := \sum_{k=1}^{\infty} z^k / k^2$ is the dilogarithm and the factor involving δ is meant to be equal to 1 when $\delta = 1$.

When $\delta = 1/2$ all the subsets $A \subseteq \{1, \dots, n\}$ are chosen by $B(n, \delta)$ with the same probability. Hence, Theorem 1.3 together with the identity $\text{Li}_2(\frac{1}{2}) = (\pi^2 - 6(\log 2)^2)/12$ (see, e.g., [16]) give the following result.

Corollary 1.1. *As $n \rightarrow +\infty$, we have*

$$\text{lcm}(u_a : a \in A) \sim \frac{1}{4} \left(1 - \frac{6(\log 2)^2}{\pi^2} \right) \cdot \log \left| \frac{\alpha}{\sqrt{(a_1^2, a_2)}} \right| \cdot n^2,$$

uniformly for all sets $A \subseteq \{1, \dots, n\}$, but at most $o(2^n)$ exceptions.

2. NOTATION

We employ the Landau–Bachmann “Big Oh” and “little oh” notations O and o , as well as the associated Vinogradov symbols \ll and \gg , with their usual meanings. Any dependence of the implied constants is explicitly stated or indicated with subscripts. For real random variables X and Y , we say that “ $X \sim Y$ with probability $1 - o(1)$ ” if $\mathbb{P}(|X - Y| \geq \varepsilon |Y|) = o_\varepsilon(1)$ for every $\varepsilon > 0$. We write $\text{lcm}(S)$ for the lowest common multiple of the elements of $S \subseteq \mathbb{Z}$, with the convention $\text{lcm}(\emptyset) := 1$. We also let $[a, b]$ and (a, b) denote the lowest common multiple and the greatest common divisor, respectively, of two integers a and b . Throughout, the letters p is reserved for prime numbers, and ν_p denotes the p -adic valuation. As usual, we write $\Lambda(n)$, $\varphi(n)$, $\tau(n)$, and $\mu(n)$, for the von Mangoldt function, the Euler's totient function, the number of divisors, and the Möbius function of a positive integer n , respectively.

3. PRELIMINARIES ON LEHMER SEQUENCES

Let ζ and η be complex numbers such that $c_1 := (\zeta + \eta)^2$ and $c_2 := \zeta\eta$ are nonzero coprime integers and ζ/η is not a root of unity. Also, assume $|\zeta| \geq |\eta|$. The *Lehmer sequence* $(\tilde{u}_k)_{k \geq 0}$ associated to ζ and η is defined by

$$(3) \quad \tilde{u}_k := \begin{cases} (\zeta^k - \eta^k) / (\zeta - \eta) & \text{if } k \text{ is odd,} \\ (\zeta^k - \eta^k) / (\zeta^2 - \eta^2) & \text{if } k \text{ is even,} \end{cases}$$

for every integer $k \geq 0$. It is known that $(\tilde{u}_k)_{k \geq 1}$ is an integer sequence. For every positive integer m coprime with c_2 , let $\varrho(m)$ be the *rank of appearance* of m in the Lehmer sequence $(\tilde{u}_k)_{k \geq 0}$, that is, the smallest positive integer k such that $m \mid \tilde{u}_k$. It is known that $\varrho(m)$ exists. Moreover, for every prime number p not dividing c_2 , put $\kappa(p) := \nu_p(\tilde{u}_{\varrho(p)})$.

We need the following properties of the rank of appearance.

Lemma 3.1. *We have:*

- (i) $m \mid \tilde{u}_k$ if and only if $(m, c_2) = 1$ and $\varrho(m) \mid k$, for all integers $m, k \geq 1$.
- (ii) $\varrho(p^k) = p^{\max(k - \kappa(p), 0)} \varrho(p)$, for all primes p not dividing $2c_2$ and all integers $k \geq 1$.

(iii) $\varrho(2^k) = 2^{\max(k - \nu_2(\tilde{u}_{\varrho(4)}), 0)} \varrho(4)$, for all integers $k \geq 2$.

Proof. (i) We have $(\tilde{u}_k, c_2) = 1$ for all integers $k \geq 1$ [13, Lemma 1]. Also, $(\tilde{u}_k, \tilde{u}_h) = \tilde{u}_{(k,h)}$ for all integers $k, h \geq 1$ [13, Lemma 3]. Hence, on the one hand, if $m \mid \tilde{u}_k$ then $(m, c_2) = 1$ and $m \mid (\tilde{u}_k, \tilde{u}_{\varrho(m)}) = \tilde{u}_{(k, \varrho(m))}$, which in turn implies that $\varrho(m) \mid k$, by the minimality of $\varrho(m)$. On the other hand, if $(c_2, m) = 1$ and $\varrho(m) \mid k$ then $m \mid \tilde{u}_{\varrho(m)} = \tilde{u}_{(k, \varrho(m))} = (\tilde{u}_k, \tilde{u}_{\varrho(m)})$, so that $m \mid \tilde{u}_k$.

(ii) If $p \mid \tilde{u}_m$, for some positive integer m , then $p \parallel \tilde{u}_{pm}/\tilde{u}_m$ [13, Lemma 5]. Hence, it follows by induction on h that $\nu_p(\tilde{u}_{p^h \varrho(p)}) = \kappa(p) + h$, for every integer $h \geq 0$. At this point, the claim follows easily from (i).

(iii) If $4 \mid \tilde{u}_m$, for some positive integer m , then $2 \parallel \tilde{u}_{pm}/\tilde{u}_m$ [13, Lemma 5]. The proof proceeds similarly to the previous point. \square

Hereafter, in light of Lemma 3.1(i), in subscripts of sums and products the argument of ϱ is always tacitly assumed to be coprime with c_2 .

Let us define the cyclotomic numbers $(\phi_k)_{k \geq 1}$ associated to ζ and η by

$$(4) \quad \phi_k := \prod_{\substack{1 \leq h \leq k \\ (h, k) = 1}} \left(\zeta - e^{\frac{2\pi i h}{k}} \eta \right),$$

for every integer $k \geq 0$. It can be proved that $\phi_k \in \mathbb{Z}$ for every integer $k \geq 3$. Moreover, from (4) it follows easily that

$$\zeta^k - \eta^k = \prod_{d \mid k} \phi_d,$$

which in turn, applying Möbius inversion formula and taking into account (3), gives

$$(5) \quad \phi_k = \prod_{d \mid k} \left(\zeta^d - \eta^d \right)^{\mu(k/d)} = \prod_{d \mid k} \tilde{u}_d^{\mu(k/d)},$$

for all integers $k \geq 3$. We need the following result about ϕ_k .

Lemma 3.2. *For every integer $k \geq 13$, we have*

$$|\phi_k| = \lambda_k \cdot \prod_{\varrho(p) = k} p^{\kappa(p)},$$

where λ_k is equal to 1 or to the greatest prime factor of $k/(k, 3)$.

Proof. Let p be a prime number not dividing c_2 . By the definition of $\varrho(p)$, we have that $p \nmid \tilde{u}_h$ for each positive integer $h < \varrho(p)$. Hence, by (5), we obtain that $\nu_p(\phi_{\varrho(p)}) = \nu_p(\tilde{u}_{\varrho(p)}) = \kappa(p)$. In particular, $p \mid \phi_{\varrho(p)}$. Let $k \geq 3$ be an integer and suppose that p is a prime factor of ϕ_k . On the one hand, if $\varrho(p) = k$ then, by the previous consideration, $\nu_p(\phi_k) = \kappa(p)$. On the other hand, if $\varrho(p) \neq k$ then $p \mid (\phi_{\varrho(p)}, \phi_k)$. Finally, for $k \geq 13$ and for every integer $h \geq 3$ with $h \neq k$, we have that (ϕ_h, ϕ_k) divides the greatest prime factor of $k/(k, 3)$ [13, Lemma 7]. \square

We conclude this section with a formula for a sum involving the von Mangoldt function.

Lemma 3.3. *We have*

$$(6) \quad \sum_{\varrho(m) = r} \Lambda(m) = \varphi(r) \log |\zeta| + O_{\zeta, \eta}(\tau(r) \log(r+1)),$$

and, in particular,

$$(7) \quad \sum_{\varrho(m) = r} \Lambda(m) \ll_{\zeta, \eta} \varphi(r),$$

for every positive integer r .

Proof. Clearly, we can assume $r \geq 13$. Write $m = p^k$, where p is a prime number not dividing c_2 and k is a positive integer. First, suppose that $p > 2$. By Lemma 3.1(ii), we have that $\varrho(m) = p^{\max(k-\kappa(p),0)} \varrho(p)$. Hence, $\varrho(m) = r$ if and only if $k \leq \kappa(p)$ and $\varrho(p) = r$, or $k > \kappa(p)$ and $p^{k-\kappa(p)} \varrho(p) = r$. In the first case, the contribution to the sum in (6) is exactly $\kappa(p) \log p$. In the second case, $p \mid r$ and, since k is determined by p and r , the contribution to the sum in (6) is $\log p$. Using Lemma 3.1(iii), the case $p = 2$ can be handled similarly. Therefore,

$$(8) \quad \sum_{\varrho(m)=r} \Lambda(m) = \sum_{\varrho(p)=r} \kappa(p) \log p + O\left(\sum_{p \mid r} \log p\right) = \log |\phi_r| + O(\log r),$$

where we used Lemma 3.2. Furthermore, from (5) and the identity $\sum_{d \mid r} \mu(r/d) d = \varphi(r)$, it follows that

$$\log |\phi_r| = \varphi(r) \log |\zeta| + O\left(\sum_{d \mid r} \log \left|1 - \left(\frac{\eta}{\zeta}\right)^d\right|\right).$$

If $|\eta/\zeta| < 1$ then $\log |1 - (\eta/\zeta)^d| = O_{\zeta,\eta}(1)$. If $|\eta/\zeta| = 1$ then, since η/ζ is an algebraic number that is not a root of unity, it follows from classic bounds on linear forms in logarithms (see, e.g., [9, Lemma 3]) that $\log |1 - (\eta/\zeta)^d| = O_{\zeta,\eta}(\log(d+1))$. Consequently,

$$(9) \quad \log |\phi_r| = \varphi(r) \log |\zeta| + O_{\zeta,\eta}(\tau(r) \log(r+1)).$$

Putting together (8) and (9), we get (6). Finally, the upper bound (7) follows since $\tau(k) \leq k^\varepsilon$ and $\varphi(k) \geq k^{1-\varepsilon}$, for all $\varepsilon > 0$ and every integer $k \gg_\varepsilon 1$ [14, Ch. I.5, Corollary 1.1 and Eq. 12]. \square

4. FURTHER PRELIMINARIES

We need two estimates involving the Euler's totient function. Define

$$\Phi(x) := \sum_{n \leq x} \varphi(n),$$

for every $x \geq 1$.

Lemma 4.1. *We have*

$$\Phi(x) = \frac{3}{\pi^2} x^2 + O(x \log x) \quad \text{and} \quad \sum_{n \leq x} \frac{\varphi(n)}{n} \ll x,$$

for every $x \geq 2$.

Proof. The first formula is well known [14, Ch. I.3, Thm. 4] and implies

$$\sum_{n \leq x} \frac{\varphi(n)}{n} \leq \sum_{n \leq x/2} 1 + \sum_{x/2 < n \leq x} \frac{\varphi(n)}{x/2} \ll x,$$

as desired. \square

The following lemma is an easy inequality that will be useful later.

Lemma 4.2. *It holds $1 - (1-x)^k \leq kx$, for all $x \in [0, 1]$ and all integers $k \geq 0$.*

Proof. The claim is $(1 + (-x))^k \geq 1 + k(-x)$, which follows from Bernoulli's inequality. \square

5. PROOF OF THEOREM 1.3

Henceforth, all the implied constants may depend by a_1 , a_2 , and u_1 . It is well known that the generalized Binet's formula

$$(10) \quad u_k = \frac{\alpha^k - \beta^k}{\alpha - \beta} u_1,$$

holds for every integer $k \geq 0$. We put $\zeta := \alpha/\sqrt{b}$ and $\eta := \beta/\sqrt{b}$, where $b := (a_1^2, a_2)$. Note that indeed $c_1 = a_1^2/b$ and $c_2 = -a_2/b$ are nonzero relatively prime integers, $\zeta/\eta = \alpha/\beta$ is not a root of unity, and $|\zeta| \geq |\eta|$. Moreover, from (3) and (10), it follows easily that

$$u_k = \begin{cases} b^{(k-1)/2} u_1 \tilde{u}_k & \text{if } k \text{ is odd,} \\ a_1 b^{k/2-1} u_1 \tilde{u}_k & \text{if } k \text{ is even,} \end{cases}$$

for every integer $k \geq 0$. Therefore, for every $A \subseteq \{1, \dots, n\}$, we have

$$\log \text{lcm}(u_a : a \in A) = \log \text{lcm}(\tilde{u}_a : a \in A) + O(n).$$

Note that $O(n)$ is a ‘‘little oh’’ of the right-hand side of (2), as $\delta n / \log n \rightarrow +\infty$. Hence, it is enough to prove Theorem 1.3 with $\log \text{lcm}(\tilde{u}_a : a \in A)$ in place of $\log \text{lcm}(u_a : a \in A)$, and this will be indeed our strategy.

Hereafter, let A be a random set in $B(n, \delta)$, and put $L := \text{lcm}(\tilde{u}_a : a \in A)$ and $X := \log L$. For every positive integer m coprime with c_2 , let us define

$$I_A(m) := \begin{cases} 1 & \text{if } \varrho(m) \mid a \text{ for some } a \in A, \\ 0 & \text{otherwise.} \end{cases}$$

The following lemma gives an expression for X in terms of I_A and the von Mangoldt function.

Lemma 5.1. *We have*

$$X = \sum_{\varrho(m) \leq n} \Lambda(m) I_A(m).$$

Proof. For every prime power p^k with $p \nmid c_2$, we know from Lemma 3.1(i) that $p^k \mid L$ if and only if $\varrho(p^k) \mid a$ for some $a \in A$ and, in particular, $\varrho(p^k) \leq n$. Hence,

$$X = \sum_{p^k \mid L} \log p = \sum_{\varrho(p^k) \leq n} (\log p) I_A(p^k) = \sum_{\varrho(m) \leq n} \Lambda(m) I_A(m),$$

as claimed. \square

The next lemma provides two expected values involving I_A and needed in later arguments.

Lemma 5.2. *We have*

$$(11) \quad \mathbb{E}(I_A(m)) = 1 - (1 - \delta)^{\lfloor n/\varrho(m) \rfloor}$$

and

$$\begin{aligned} \mathbb{E}(I_A(m)I_A(\ell)) &= 1 - (1 - \delta)^{\lfloor n/\varrho(m) \rfloor} - (1 - \delta)^{\lfloor n/\varrho(\ell) \rfloor} \\ &\quad + (1 - \delta)^{\lfloor n/\varrho(m) \rfloor + \lfloor n/\varrho(\ell) \rfloor - \lfloor n/[\varrho(m), \varrho(\ell)] \rfloor}, \end{aligned}$$

for all positive integers m and ℓ with $(m\ell, c_2) = 1$.

Proof. By the definition of I_A , we have

$$\mathbb{E}(I_A(m)) = \mathbb{P}(\exists a \in A : \varrho(m) \mid a) = 1 - \mathbb{P}\left(\bigwedge_{t \leq n/\varrho(m)} (\varrho(m)t \notin A)\right) = 1 - (1 - \delta)^{\lfloor n/\varrho(m) \rfloor},$$

which is the first claim. On the one hand, by linearity of expectation and by (11), we obtain

$$\begin{aligned} \mathbb{E}(I_A(m)I_A(\ell)) &= \mathbb{E}(I_A(m) + I_A(\ell) - 1 + (1 - I_A(m))(1 - I_A(\ell))) \\ &= \mathbb{E}(I_A(m)) + \mathbb{E}(I_A(\ell)) - 1 + \mathbb{E}((1 - I_A(m))(1 - I_A(\ell))) \end{aligned}$$

$$= 1 - (1 - \delta)^{\lfloor n/\varrho(m) \rfloor} - (1 - \delta)^{\lfloor n/\varrho(\ell) \rfloor} + \mathbb{E}((1 - I_A(m))(1 - I_A(\ell))).$$

On the other hand, by the definition of I_A ,

$$\begin{aligned} \mathbb{E}((1 - I_A(m))(1 - I_A(\ell))) &= \mathbb{P}(\forall a \in A : \varrho(m) \nmid a \text{ and } \varrho(\ell) \nmid a) \\ &= \mathbb{P}\left(\bigwedge_{\substack{k \leq n \\ \varrho(m) \mid k \text{ or } \varrho(\ell) \mid k}} (k \notin A)\right) = (1 - \delta)^{\lfloor n/\varrho(m) \rfloor + \lfloor n/\varrho(\ell) \rfloor - \lfloor n/[\varrho(m), \varrho(\ell)] \rfloor}, \end{aligned}$$

and the second claim follows too. \square

Now we give an asymptotic formula for the expected value of X .

Lemma 5.3. *We have*

$$\mathbb{E}(X) = \frac{\delta \operatorname{Li}_2(1 - \delta)}{1 - \delta} \cdot \frac{3 \log |\zeta|}{\pi^2} \cdot n^2 + O(\delta n (\log n)^3),$$

for all integers $n \geq 2$. In particular,

$$\mathbb{E}(X) \sim \frac{\delta \operatorname{Li}_2(1 - \delta)}{1 - \delta} \cdot \frac{3 \log |\zeta|}{\pi^2} \cdot n^2,$$

as $n \rightarrow +\infty$, uniformly for $\delta \in (0, 1]$.

Proof. From Lemma 5.1 and Lemma 5.2, it follows that

$$\begin{aligned} \mathbb{E}(X) &= \sum_{\varrho(m) \leq n} \Lambda(m) \mathbb{E}(I_A(m)) \\ &= \sum_{\varrho(m) \leq n} \Lambda(m) (1 - (1 - \delta)^{\lfloor n/\varrho(m) \rfloor}) \\ &= \sum_{r \leq n} (1 - (1 - \delta)^{\lfloor n/r \rfloor}) \sum_{\varrho(m) = r} \Lambda(m). \end{aligned}$$

Consequently, thanks to Lemma 3.3 and Lemma 4.2, we obtain

$$\begin{aligned} (12) \quad \mathbb{E}(X) &= \sum_{r \leq n} (1 - (1 - \delta)^{\lfloor n/r \rfloor}) \varphi(r) \log |\zeta| + O\left(\delta n \sum_{r \leq n} \frac{\tau(r) \log(r+1)}{r}\right) \\ &= \sum_{r \leq n} (1 - (1 - \delta)^{\lfloor n/r \rfloor}) \varphi(r) \log |\zeta| + O(\delta n (\log n)^3), \end{aligned}$$

where we used the fact that

$$\sum_{r \leq n} \frac{\tau(r)}{r} \leq \left(\sum_{s \leq n} \frac{1}{s}\right)^2 \ll (\log n)^2.$$

Note that $\lfloor n/r \rfloor = j$ if and only if $r \in (n/(j+1), n/j]$. Hence,

$$\begin{aligned} (13) \quad \sum_{r \leq n} (1 - (1 - \delta)^{\lfloor n/r \rfloor}) \varphi(r) &= \sum_{j \leq n} (1 - (1 - \delta)^j) \sum_{n/(j+1) < r \leq n/j} \varphi(r) \\ &= \sum_{j \leq n} (1 - (1 - \delta)^j) \left(\Phi\left(\frac{n}{j}\right) - \Phi\left(\frac{n}{j+1}\right)\right) \\ &= \delta \sum_{j \leq n} (1 - \delta)^{j-1} \Phi\left(\frac{n}{j}\right) \\ &= \delta \sum_{j \leq n} \frac{(1 - \delta)^{j-1}}{j^2} \cdot \frac{3}{\pi^2} \cdot n^2 + O\left(\delta \sum_{j \leq n} \frac{n}{j} \log\left(\frac{n}{j}\right)\right) \end{aligned}$$

$$= \frac{\delta \operatorname{Li}_2(1 - \delta)}{1 - \delta} \cdot \frac{3}{\pi^2} \cdot n^2 + O(\delta n (\log n)^2),$$

where we used Lemma 4.1. Finally, putting together (12) and (13), we get the desired claim. \square

The next lemma is an upper bound for the variance of X .

Lemma 5.4. *We have*

$$\mathbb{V}(X) \ll \delta n^3 \log n,$$

for all integers $n \geq 2$.

Proof. On the one hand, by Lemma 5.1, we have

$$\begin{aligned} \mathbb{V}(X) &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 \\ &= \sum_{\varrho(m), \varrho(\ell) \leq n} \Lambda(m) \Lambda(\ell) (\mathbb{E}(I_A(m) I_A(\ell)) - \mathbb{E}(I_A(m)) \mathbb{E}(I_A(\ell))). \end{aligned}$$

On the other hand, from Lemma 5.2 and Lemma 4.2, it follows that

$$\begin{aligned} &\mathbb{E}(I_A(m) I_A(\ell)) - \mathbb{E}(I_A(m)) \mathbb{E}(I_A(\ell)) \\ &= (1 - \delta)^{\lfloor n/\varrho(m) \rfloor + \lfloor n/\varrho(\ell) \rfloor - \lfloor n/[\varrho(m), \varrho(\ell)] \rfloor} (1 - (1 - \delta)^{\lfloor n/[\varrho(m), \varrho(\ell)] \rfloor}) \leq \frac{\delta n}{[\varrho(m), \varrho(\ell)]}. \end{aligned}$$

Therefore,

$$\begin{aligned} (14) \quad \mathbb{V}(X) &\leq \delta n \sum_{\varrho(m), \varrho(\ell) \leq n} \frac{\Lambda(m) \Lambda(\ell)}{[\varrho(m), \varrho(\ell)]} = \delta n \sum_{r, s \leq n} \frac{1}{[r, s]} \sum_{\varrho(m)=r} \Lambda(m) \sum_{\varrho(\ell)=s} \Lambda(\ell) \\ &\ll \delta n \sum_{r, s \leq n} \frac{\varphi(r) \varphi(s)}{[r, s]} = \delta n \sum_{r, s \leq n} (r, s) \frac{\varphi(r) \varphi(s)}{rs}, \end{aligned}$$

where we used Lemma 3.3 and the identity $[r, s] = rs/(r, s)$. At this point, writing $r = dr'$ and $s = ds'$, where $d := (r, s)$, we obtain

$$\begin{aligned} (15) \quad \sum_{r, s \leq n} (r, s) \frac{\varphi(r) \varphi(s)}{rs} &= \sum_{d \leq n} d \sum_{\substack{r', s' \leq n/d \\ (r', s')=1}} \frac{\varphi(dr') \varphi(ds')}{d^2 r' s'} \leq \sum_{d \leq n} d \left(\sum_{t \leq n/d} \frac{\varphi(t)}{t} \right)^2 \\ &\ll \sum_{d \leq n} d \left(\frac{n}{d} \right)^2 \ll n^2 \log n, \end{aligned}$$

where we used Lemma 4.1 and the inequality $\varphi(dm) \leq d\varphi(m)$, holding for every integer $m \geq 1$. Finally, putting together (14) and (15), we get the desired claim. \square

Proof of Theorem 1.3. By Chebyshev's inequality, Lemma 5.3, and Lemma 5.4, we have

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq \varepsilon \mathbb{E}(X)) \leq \frac{\mathbb{V}(X)}{(\varepsilon \mathbb{E}(X))^2} \ll \frac{\log n}{\varepsilon^2 \delta n} = o_\varepsilon(1),$$

as $\delta n / \log n \rightarrow +\infty$. Hence, again by Lemma 5.3, we have

$$X \sim \frac{\delta \operatorname{Li}_2(1 - \delta)}{1 - \delta} \cdot \frac{3 \log |\zeta|}{\pi^2} \cdot n^2,$$

with probability $1 - o(1)$, as desired. \square

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