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Non-local multi-class traffic flow models / Chiarello, F.A., Goatin, P.. - In: NETWORKS AND HETEROGENEOUS MEDIA. - ISSN 1556-1801. - 14:2(2019), pp. 371-387. [10.3934/nhm.2019015]

*Availability:*

This version is available at: 11583/2814518 since: 2020-04-21T17:04:55Z

*Publisher:*

American Institute of Mathematical Sciences

*Published*

DOI:10.3934/nhm.2019015

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# Non-local multi-class traffic flow models

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August 2, 2018

## Abstract

We prove the existence for small times of weak solutions for a class of non-local systems in one space dimension, arising in traffic modeling. We approximate the problem by a Godunov type numerical scheme and we provide uniform  $L^1$  and BV estimates for the sequence of approximate solutions. We finally present some numerical simulations illustrating the behavior of different classes of vehicles and we analyze two cost functionals measuring the dependence of congestion on traffic composition.

## 1 Introduction

We consider the following class of non-local systems of  $M$  conservation laws in one space dimension:

$$\partial_t \rho_i(t, x) + \partial_x (\rho_i(t, x) v_i((r \leftarrow l_i)(t, x))) = 0, \quad i = 1, \dots, M, \quad (1.1)$$

where

$$r(t, x) := \sum_{i=1}^M \rho_i(t, x), \quad (1.2)$$

$$v_i(\rho) := v_i^{\max}(\rho) \quad (1.3)$$

$$(r \leftarrow l_i)(t, x) := \int_x^{x+\frac{r}{v_i}} r(t, y) l_i(y-x) dy, \quad (1.4)$$

and we assume:

**(H1)** The convolution kernels  $l_i \in \mathbf{C}^1([0, \frac{r}{v_i}; \mathbb{R}^+)$ ,  $\frac{r}{v_i} > 0$ , are non-increasing functions such that  $\int_0^{\frac{r}{v_i}} l_i(y) dy = J_i$ . We set  $W_0 := \max_{i=1, \dots, M} l_i(0)$ .

**(H2)**  $v_i^{\max}$  are the maximal velocities, with  $0 < v_1^{\max} \leq v_2^{\max} \leq \dots \leq v_M^{\max}$ .

**(H3)**  $l: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a smooth non-increasing function such that  $l(0)=1$  and  $l(r)=0$  for  $r \geq 1$  (for simplicity, we can consider the function  $l(r)=\max\{1-r, 0\}$ ).

We couple (1.1) with an initial datum

$$\rho_i(0, x) = \rho_i^0(x), \quad i = 1, \dots, M. \quad (1.5)$$

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Model (1.1) is obtained generalizing the  $n$ -populations model for traffic flow described in [3] and it is a multi-class version of the one dimensional scalar conservation law with non-local flux proposed in [4] where  $\rho_i$  is the density of vehicles belonging to the  $i$ -th class,  $\mathcal{H}$  is proportional to the look-ahead distance and  $J$  is the interaction strength. In our setting, the non-local dependence of the speed functions  $v$  describes the reaction of drivers that adapt their velocity to the downstream traffic, assigning greater importance to closer vehicles, see also [6, 8]. We consider different anisotropic discontinuous kernels for each equation of the system, therefore the results in [1] cannot be applied. The model takes into account the distribution of heterogeneous drivers and vehicles characterized by their maximal speeds and look-ahead visibility in a traffic stream. One of the limitations of the standard LWR traffic flow model [10, 11] is the first in first out rule, conversely in multi-class dynamic faster vehicles can overtake slower ones and slower vehicles slow down the faster ones.

Due to the possible presence of jump discontinuities, solutions to (1.1), (1.5) are intended in the following weak sense.

**Definition 1.** A function  $\rho = (\rho_1, \dots, \rho_M) \in (\mathbf{L}^1 \setminus \mathbf{L}^1)([0, T[ \rightarrow \mathbb{R}; \mathbb{R}^+)$ ,  $T > 0$ , is a weak solution of (1.1), (1.5) if

$$\int_0^T \int_{\mathbb{R}} \rho_i \partial_t \varphi + \int_{\mathbb{R}} \rho_i v_i(r \leftarrow l_i) \partial_x \varphi (t, x) dx dt + \int_{\mathbb{R}} \rho_i^0(x) \varphi(0, x) dx = 0$$

for all  $\varphi \in \mathcal{C}^1_c([0, T[ \rightarrow \mathbb{R}; \mathbb{R})$ ,  $i = 1, \dots, M$ .

The main result of this paper is the proof of existence of weak solutions to (1.1), (1.5), locally in time.

**Theorem 1.** Let  $\rho_i^0(x) \in (\mathbf{BV} \setminus \mathbf{L}^1)(\mathbb{R}; \mathbb{R}^+)$ , for  $i = 1, \dots, M$ , and assumptions (H1) - (H3) hold. Then the Cauchy problem (1.1), (1.5) admits a weak solution on  $[0, T[ \rightarrow \mathbb{R}$ , for some  $T > 0$  sufficiently small.

The paper is organized as follows. Section 2 is devoted to prove uniform  $\mathbf{L}^1$  and BV estimates on the approximate solutions obtained through an approximation argument based on a Godunov type numerical scheme, see [7]. We have to point out that these estimates heavily rely on the monotonicity properties of the kernel functions  $!_i$ . In Section 3 we prove the existence in finite time of weak solutions applying Helly's theorem and a Lax-Wendroff type argument, see [9]. In Section 4 we present some numerical simulations for  $M = 2$ . In particular, we consider the case of a mixed flow of cars and trucks on a stretch of road, and the flow of mixed autonomous and non-autonomous vehicles on a circular road. In this latter case, we analyze two cost functionals measuring the traffic congestion, depending on the penetration ratio of autonomous vehicles. The final Appendix contains an alternative proof of Theorem 1, based on approximate solutions constructed via a Lax-Friedrichs type scheme, which is commonly used in the framework of non-local equations, see [12, 14].

## 2 Godunov type approximate solutions

First of all, we extend  $!_i(x) = 0$  for  $x > \mathcal{H}_i$ . For  $j \in \mathbb{Z}$  and  $n \in \mathbb{N}$ , let  $x_{j+1/2} = j \Delta x$  be the cells interfaces,  $x_j = (j - 1/2) \Delta x$  the cells centers and  $t^n = n \Delta t$  the time mesh. We aim at constructing a finite volume approximate solution  $\rho^x = (\rho_1^x, \dots, \rho_M^x)$ , with

$\vec{u}_i^x(t, x) = \vec{u}_{ij}^n$  for  $(t, x) \in C_j^n = [t^n, t^{n+1}] \times [x_{j-1/2}, x_{j+1/2}]$  and  $i = 1, \dots, M$ .

To this end, we approximate the initial datum  $\vec{u}_i^0$  for  $i = 1, \dots, M$  with a piecewise constant function

$$\vec{u}_{ij}^0 = \frac{1}{X} \int_{x_{j-1/2}}^{x_{j+1/2}} \vec{u}_i^0(x) dx, \quad j \in \mathbb{Z}.$$

Similarly, for the kernel, we set

$$I_i^k := \frac{1}{X} \int_{x_{j-1/2}}^{x_{j+1/2}} I_i^0(x) dx, \quad k \in \mathbb{N},$$

so that  $\sum_{k=0}^{N_i-1} I_i^k = \int_{\mathbb{R}} I_i(x) dx = J_i$  (the sum is indeed finite since  $I_i^k = 0$  for  $k > N_i$  sufficiently large). Moreover, we set  $I_{j+k}^n = \sum_{i=1}^M \vec{u}_{ij+k}^n$  for  $k \in \mathbb{N}$  and

$$V_{ij}^n := V_i^{\max} \sum_{k=0}^{N_i-1} I_i^k r_{j+k}^n A, \quad i = 1, \dots, M, \quad j \in \mathbb{Z}. \quad (2.1)$$

We consider the following Godunov-type scheme adapted to (1.1), which was introduced in [7] in the scalar case:

$$\vec{u}_{ij}^{n+1} = \vec{u}_{ij}^n \downarrow \vec{u}_{ij}^n V_{ij+1}^n \oplus \vec{u}_{ij-1}^n V_{ij}^n \quad (2.2)$$

where we have set  $\delta = \frac{t}{X}$ .

## 2.1 Compactness estimates

We provide here the necessary estimates to prove the convergence of the sequence of approximate solutions constructed via the Godunov scheme (2.2).

**Lemma 1. (Positivity)** For any  $T > 0$ , under the CFL condition

$$\delta \frac{1}{V_M^{\max} k_1} < 1, \quad (2.3)$$

the scheme (2.2) is positivity preserving on  $[0, T] \times \mathbb{R}$ .

**Proof.** Let us assume that  $\vec{u}_{ij}^n \geq 0$  for all  $j \in \mathbb{Z}$  and  $i \in \{1, \dots, M\}$ . It suffices to prove that  $\vec{u}_{ij}^{n+1}$  in (2.2) is non-negative. We compute

$$\vec{u}_{ij}^{n+1} = \vec{u}_{ij}^n \downarrow \vec{u}_{ij}^n V_{ij+1}^n \oplus \vec{u}_{ij-1}^n V_{ij}^n \geq 0 \quad (2.4)$$

under assumption (2.3). ←

**Corollary 2. (L<sup>1</sup>-bound)** For any  $n \in \mathbb{N}$ , under the CFL condition (2.3) the approximate solutions constructed via the scheme (2.2) satisfy

$$\| \bar{x}_i^n - \bar{x}_i^0 \|_1 = \rightarrow 0, \quad i = 1, \dots, M, \quad (2.5)$$

where  $\| \bar{x}_i^n - \bar{x}_i^0 \|_1 := \sum_j x_j^P \| \bar{x}_{i,j}^n - \bar{x}_{i,j}^0 \|_1$  denotes the L<sup>1</sup> norm of the i-th component of  $\bar{x}$ .

**Proof.** Thanks to Lemma 1, for all  $i \in \{1, \dots, M\}$  we have

$$\bar{x}_i^{n+1} = \sum_j x_j^X \bar{x}_{i,j}^{n+1} = \sum_j x_j^X \left( \bar{x}_{i,j}^n - V_{i,j+1}^n + \bar{x}_{i,j}^n V_{i,j}^n \right) = \sum_j x_j^X \bar{x}_{i,j}^n,$$

proving (2.5). ◀

**Lemma 2. (L<sup>1</sup>-bound)** If  $\bar{x}_{i,j}^0 = 0$  for all  $j \in \mathbb{Z}$  and  $i = 1, \dots, M$ , and (2.3) holds, then the approximate solution  $\bar{x}$  constructed by the algorithm (2.2) is uniformly bounded on  $[0, T] \rightarrow \mathbb{R}$  for any  $T$  such that

$$T < \frac{1}{M} \frac{1}{v_M^{\max}} \frac{1}{W_0}.$$

**Proof.** Let  $\bar{\delta} = \max\{ \bar{x}_{i,j}^n, \bar{x}_{i,j}^0 \}$ . Then we get

$$\bar{x}_{i,j}^{n+1} = \bar{x}_{i,j}^n - V_{i,j+1}^n + \bar{x}_{i,j}^n V_{i,j}^n \bar{\delta} + V_{i,j}^n V_{i,j+1}^n \bar{\delta} \quad (2.6)$$

and

$$\begin{aligned} V_{i,j}^n - V_{i,j+1}^n &= v_i^{\max} \sum_{k=0}^1 x_j^k r_{j+k}^n - v_i^{\max} \sum_{k=0}^1 x_{j+1}^k r_{j+k+1}^n \\ &= v_i^{\max} \sum_{k=0}^1 x_j^k (r_{j+k+1}^n - r_{j+k}^n) \\ &= v_i^{\max} \sum_{k=1}^1 x_j^k r_j^n + \sum_{k=1}^1 (x_j^{k-1} - x_j^k) r_{j+k}^n \\ &= v_i^{\max} \sum_{k=1}^1 x_j^k r_j^n + \sum_{k=1}^1 (x_j^{k-1} - x_j^k) r_{j+k}^n \end{aligned} \quad (2.7)$$

where  $x_j^k = (x_1, \dots, x_M)_1 = \max_{i,j} x_{i,j}$ . So, until  $k \leq k_1 \bar{\delta} K$ , for some  $K \rightarrow 0$ , we get

$$\bar{x}_i^{n+1} \leq \bar{\delta} k_1 \bar{\delta} + MK v_M^{\max} W_0 \bar{\delta} t,$$

which implies

$$k_1 \bar{\delta} \leq e^{Cn} t,$$

with  $C = MK v_M^{\max} W_0$ . Therefore we get that  $\bar{x}(t, \cdot) \leq K$  for

$$t \leq \frac{1}{MK v_M^{\max} k_1 W_0} \ln \frac{K}{\bar{\delta}} \leq \frac{1}{Me^{-\bar{\delta}} v_M^{\max} k_1 W_0},$$

where the maximum is attained for  $K = e^{-\delta t} W_0$ .

Iterating the procedure, at time  $t^m$ ,  $m \geq 1$  we set  $K = e^{-\delta t^m} W_0$  and we get that the solution is bounded by  $K$  until  $t^{m+1}$  such that

$$t^{m+1} \leq t^m + \frac{m}{M e^{-\delta t^m} v_M^{\max} k_1 W_0}.$$

Therefore, the approximate solution remains bounded, uniformly in  $x$ , at least for  $t \leq T$  with

$$T \leq \frac{1}{M e^{-\delta t^0} v_M^{\max} k_1 W_0} \sum_{m=1}^M \frac{m}{e^{\delta t^m}} \leq \frac{1}{M e^{-\delta t^0} v_M^{\max} k_1 W_0}.$$

←

**Remark 1.** Unlike the classical multi-population model [3], the simplex

$$S := \left\{ \begin{matrix} \sum_{i=1}^M x_i = 1 \\ x_i \geq 0 \end{matrix} \right\} \subset \mathbb{R}^M$$

is not an invariant domain for (1.1), see Figure 1 for a numerical example.

Indeed, let us consider the system

$$\partial_t x_i(t, x) + \partial_x x_i(t, x) v_i(r(t, x)) = 0, \quad i = 1, \dots, M, \quad (2.8)$$

where  $r$  and  $v_i$  are as in (1.2) and (1.3), respectively. We have the following:

**Lemma 3.** Under the CFL condition

$$\delta \frac{1}{v_M^{\max} k_1 + k_1} \leq h,$$

for any initial datum  $x^0 \in S$  the approximate solutions to (2.8) computed by the upwind scheme

$$x_j^{n+1} = x_j^n + h \mathbf{F}(x_j^n, x_{j+1}^n) - h \mathbf{F}(x_{j-1}^n, x_j^n), \quad (2.9)$$

with  $\mathbf{F}(x_j^n, x_{j+1}^n) = x_j^n (r_{j+1}^n)$ , satisfy the following uniform bounds:

$$x_j^n \in S, \quad x_j^{n+1} \in S, \quad n \in \mathbb{N}.$$

**Proof.** Assuming that  $x_j^n \in S$  for all  $j \in \mathbb{Z}$ , we want to prove that  $x_j^{n+1} \in S$ . Rewriting (2.9), we get

$$x_{ij}^{n+1} = x_{ij}^n + h v_i^{\max}(x_{ij}^n) (r_{j+1}^n) - h v_i^{\max}(x_{i,j-1}^n) (r_j^n).$$

Summing on the index  $i = 1, \dots, M$ , gives

$$\begin{aligned} r_j^{n+1} &= \sum_{i=1}^M x_{ij}^{n+1} = \sum_{i=1}^M x_{ij}^n + h \sum_{i=1}^M v_i^{\max}(x_{ij}^n) (r_{j+1}^n) - h \sum_{i=1}^M v_i^{\max}(x_{i,j-1}^n) (r_j^n) \\ &= r_j^n + (r_{j+1}^n) \sum_{i=1}^M v_i^{\max}(x_{ij}^n) - (r_j^n) \sum_{i=1}^M v_i^{\max}(x_{i,j-1}^n). \end{aligned}$$

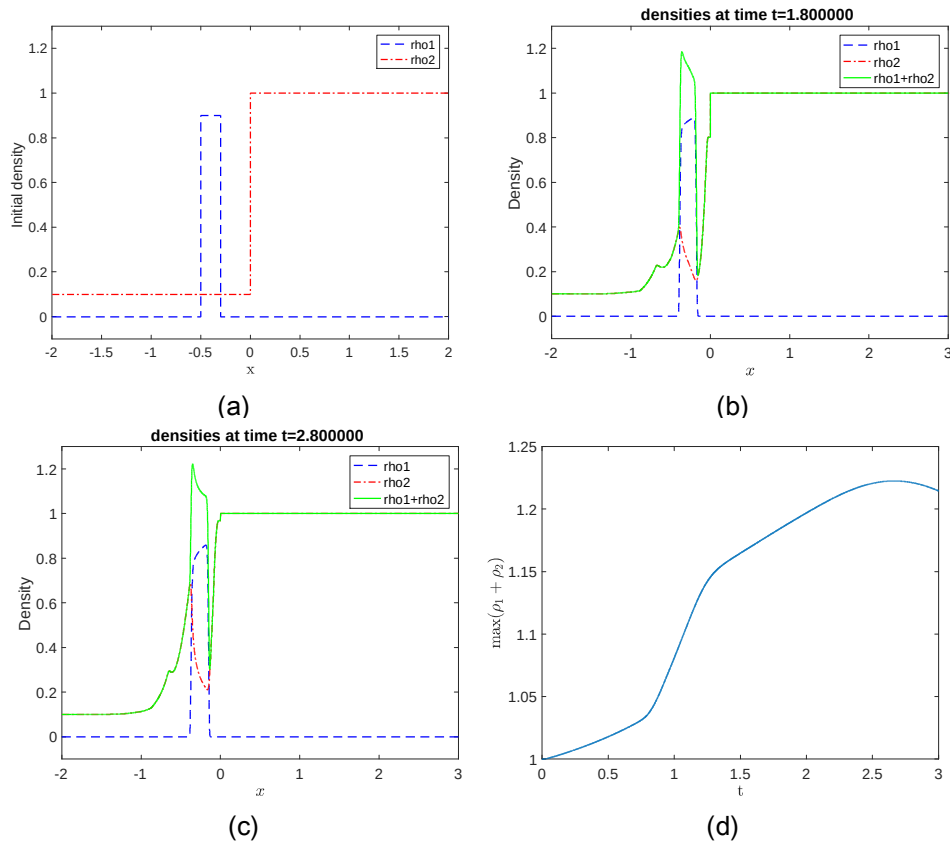


Figure 1: Numerical simulation illustrating that the simplex  $S$  is not an invariant domain for (1.1). We take  $M = 2$  and we consider the initial conditions  $\rho_1(0, x) = 0.9 \chi_{[0.5, 0.3]}$  and  $\rho_2(0, x) = 0.1 \chi_{[0, 1]}$  depicted in (a), the constant kernels  $k_1(x) = k_2(x) = 1/\mathcal{H}$ ,  $\mathcal{H} = 0.5$ , and the speed functions given by  $v_1^{max} = 0.2$ ,  $v_2^{max} = 1$ ,  $\phi = \max\{1 - \phi, 0\}$  for  $\phi \in [0, 1]$ . The space and time discretization steps are  $\Delta x = 0.001$  and  $\Delta t = 0.4 \Delta x$ . Plots (b) and (c) show the density profiles of  $\rho_1$ ,  $\rho_2$  and their sum  $r$  at times  $t = 1.8$ ,  $2.8$ . The function  $\max_{x \in \mathbb{R}} r(t, x)$  is plotted in (d), showing that  $r$  can take values greater than 1, even if  $r(0, x) = \rho_1(0, x) + \rho_2(0, x) \leq 1$ .

Defining the following function of  $\rho_j^n$

$$(\rho_{1,j}^n, \dots, \rho_{M,j}^n) = r_j^n + (r_j^n)^{\sum_{i=1}^M v_i^{max} \rho_{i,j}^n} \chi_{[0, 1]} \quad (r_{j+1}^n)^{\sum_{i=1}^M v_i^{max} \rho_{i,j}^n},$$

we observe that

$$(0, \dots, 0) = (0) \quad \sum_{i=1}^M v_i^{max} \rho_{i,j}^n \chi_{[0, 1]} \delta k k \chi_{[0, 1]} v_M^{max} \delta 1$$

if  $\delta 1/v_M^{max} k k \chi_{[0, 1]}$  and

$$(\rho_{1,j}^n, \dots, \rho_{M,j}^n) = 1 \quad (r_{j+1}^n)^{\sum_{i=1}^M v_i^{max} \rho_{i,j}^n} \delta 1$$

for  $\delta \geq 2S$  such that  $r_j^n = \prod_{i=1}^M r_{ij}^n = 1$ . Moreover

$$\frac{\partial}{\partial r_{ij}^n} (r_j^n) = 1 + \sum_{i=1}^M (r_j^n)^{i-1} v_{ij}^{\max} r_{ij}^n - 1 (r_{j+1}^n) v_i^{\max} = 0$$

if  $\delta \geq 1/\nu_M^{\max} k_1 + \sum_{i=1}^M v_i^{\max}$ . This proves that  $r_j^{n+1} \geq \delta$ .

To prove the positivity of (2.9), we observe that

$$r_{ij}^{n+1} = r_{ij}^n + \sum_{i=1}^M v_i^{\max} (r_{j+1}^n) + v_i^{\max} r_{ij}^n - 1 (r_j^n) = 0$$

if  $\delta \geq 1/\nu_M^{\max} k_1$ .

**Lemma 4. (Spatial BV-bound)** Let  $\delta \geq 2(\mathbf{BV} \setminus \mathbf{L}^1)(R, R^+)$  for all  $i = 1, \dots, M$ . If (2.3) holds, then the approximate solution  $\tilde{x}(t, \cdot)$  constructed by the algorithm (2.2) has uniformly bounded total variation for  $t \in [0, T]$ , for any  $T$  such that

$$T \delta \min_{i=1, \dots, M} \frac{1}{H \text{TV}(\tilde{\rho}_i) + 1}, \quad (2.10)$$

where  $H = k_1 \sum_{i=1}^M v_i^{\max} W_0 M + \sum_{i=1}^M v_i^{\max} k_1$ .

**Proof.** Subtracting the identities

$$r_{i,j+1}^{n+1} = r_{i,j+1}^n + \sum_{i=1}^M v_{i,j+1}^n V_{i,j+2}^n - r_{i,j}^n V_{i,j+1}^n, \quad (2.11)$$

$$r_{i,j}^{n+1} = r_{i,j}^n + \sum_{i=1}^M v_{i,j}^n V_{i,j+1}^n - r_{i,j-1}^n V_{i,j}^n, \quad (2.12)$$

and setting  $r_{i,j+1/2}^n = r_{i,j+1}^n - r_{i,j}^n$ , we get

$$r_{i,j+1/2}^{n+1} = r_{i,j+1/2}^n + \sum_{i=1}^M v_{i,j+1}^n V_{i,j+2}^n - 2 \sum_{i=1}^M v_{i,j}^n V_{i,j+1}^n + r_{i,j-1}^n V_{i,j}^n.$$

Now, we can write

$$r_{i,j+1/2}^{n+1} = 1 + \sum_{i=1}^M v_{i,j+2}^n r_{i,j+1}^n \quad (2.13)$$

$$+ \sum_{i=1}^M v_{i,j}^n r_{i,j-1/2}^n - \sum_{i=1}^M v_{i,j}^n V_{i,j+2}^n - 2 \sum_{i=1}^M v_{i,j+1}^n + \sum_{i=1}^M v_{i,j}^n. \quad (2.14)$$

Observe that assumption (2.3) guarantees the positivity of (2.13). The term (2.14) can be estimated as

$$\begin{aligned} & V_{i,j+2}^n - 2V_{i,j+1}^n + V_{i,j}^n = \\ & = \sum_{k=0}^{\infty} \binom{B}{k} x^k \sum_{i=1}^M v_{i,j+k+2}^n - 2 \sum_{k=0}^{\infty} \binom{A}{k} x^k \sum_{i=1}^M v_{i,j+k+1}^n + \sum_{k=0}^{\infty} \binom{C}{k} x^k \sum_{i=1}^M v_{i,j+k}^n \end{aligned}$$

$$\begin{aligned}
&= v_i^{\max} \binom{0}{\leftarrow_{j+1}} x @ \sum_{k=0}^{\infty} \binom{k}{i} r_{j+k+2}^n \sum_{k=0}^{\infty} \binom{k}{i} r_{j+k+1}^n A \\
&\quad + v_i^{\max} \binom{0}{\leftarrow_j} x @ \sum_{k=0}^{\infty} \binom{k}{i} r_{j+k}^n \sum_{k=0}^{\infty} \binom{k}{i} r_{j+k+1}^n A \\
&= v_i^{\max} \binom{0}{\leftarrow_{j+1}} x @ \sum_{k=1}^{\infty} \binom{k}{i} \binom{k-1}{i} r_{j+k+1}^n + \binom{0}{i} r_{j+1}^n A \\
&\quad + v_i^{\max} \binom{0}{\leftarrow_j} x @ \sum_{k=1}^{\infty} \binom{k}{i} \binom{k-1}{i} r_{j+k}^n + \binom{0}{i} r_j^n A \\
&= v_i^{\max} \left( \binom{0}{\leftarrow_{j+1}} \binom{0}{\leftarrow_j} \right) x @ \sum_{k=1}^{\infty} \binom{k}{i} \binom{k-1}{i} (r_{j+k+1}^n r_{j+k}^n) + \binom{0}{i} (r_j^n r_{j+1}^n) A \\
&\quad + v_i^{\max} \binom{0}{\leftarrow_j} x @ \sum_{k=1}^{\infty} \binom{k}{i} \binom{k-1}{i} (r_{j+k+1}^n r_{j+k}^n) + \binom{0}{i} (r_j^n r_{j+1}^n) A \\
&= v_i^{\max} \binom{0}{\leftarrow_{j+1/2}} \binom{0}{\leftarrow_{j+1}} \binom{0}{\leftarrow_j} x @ \sum_{k=1}^{\infty} \binom{k}{i} \binom{k-1}{i} r_{j+k+3/2}^n A \\
&\quad + v_i^{\max} \binom{0}{\leftarrow_j} x @ \sum_{k=1}^{\infty} \binom{k}{i} \binom{k-1}{i} r_{j+k+1/2}^n + \binom{0}{i} r_{j+1/2}^n A,
\end{aligned}$$

with  $\leftarrow_j \geq \leftarrow_{j+1/2} \geq \leftarrow_{j+1}$  and  $\leftarrow_{j+1/2} \geq \leftarrow_j \geq \leftarrow_{j+1}$ , where we set  $l(a, b) = \min\{a, b\}, \max\{a, b\}$ . For some  $\#, \mu \in [0, 1]$ , we compute

$$\begin{aligned}
\leftarrow_{j+1} \leftarrow_j &= \# \sum_{k=0}^{\infty} \binom{k}{i} r_{j+k+2}^n + (1 - \#) \sum_{k=0}^{\infty} \binom{k}{i} r_{j+k+1}^n \\
&\quad + \mu \sum_{k=0}^{\infty} \binom{k}{i} r_{j+k+1}^n + (1 - \mu) \sum_{k=0}^{\infty} \binom{k}{i} r_{j+k}^n \\
&= \# \sum_{k=1}^{\infty} \binom{k}{i} \binom{k-1}{i} r_{j+k+1}^n + (1 - \#) \sum_{k=0}^{\infty} \binom{k}{i} r_{j+k+1}^n \\
&\quad + \mu \sum_{k=0}^{\infty} \binom{k}{i} r_{j+k+1}^n + (1 - \mu) \sum_{k=1}^{\infty} \binom{k-1}{i} r_{j+k+1}^n \\
&= \sum_{k=1}^{\infty} \left[ \# \binom{k}{i} \binom{k-1}{i} + (1 - \#) \binom{k}{i} \mu \binom{k}{i} + (1 - \mu) \binom{k-1}{i} \right] r_{j+k+1}^n \\
&\quad + (1 - \#) \binom{0}{i} r_{j+1}^n + \mu \binom{0}{i} r_{j+1}^n
\end{aligned}$$

$$(1 - \mu) \sum_{i=0}^n \binom{n}{i} \mu^i (1 - \mu)^{n-i} A.$$

By monotonicity of  $\binom{n}{i}$  we have

$$\binom{n}{i} \mu^i + (1 - \mu) \binom{n}{i+1} \mu^{i+1} \leq \binom{n}{i} \mu^i + (1 - \mu) \binom{n}{i} \mu^i = 2 \binom{n}{i} \mu^i (1 - \mu)^{n-i}.$$

Taking the absolute values we get

$$\begin{aligned} \delta_{j+1} &\leq \sum_{k=2}^8 \binom{n}{k} \mu^k + (1 - \mu) \sum_{k=2}^8 \binom{n}{k+1} \mu^{k+1} + 4 \binom{n}{0} \mu^0 \\ &\leq \sum_{k=2}^8 \binom{n}{k} \mu^k + (1 - \mu) \sum_{k=2}^8 \binom{n}{k+1} \mu^{k+1} + 4 \binom{n}{0} \mu^0 \\ &\leq 6 \sum_{k=2}^8 \binom{n}{k} \mu^k. \end{aligned}$$

Until  $\binom{n}{j} \mu^j \leq K_1$  for  $j = 1, \dots, M$  for some  $K_1 = \binom{n}{j} \mu^j$ , taking the absolute values and rearranging the indexes, we have

$$\sum_j \binom{n+1}{i_{j+1/2}} \delta \sum_j \binom{n}{i_{j+1/2}} \mu^{i_{j+1/2}} \leq \sum_j \binom{n}{i_{j+1/2}} \mu^{i_{j+1/2}} + t H K_1,$$

where  $H = \sum_j \binom{n}{i_{j+1/2}} \mu^{i_{j+1/2}} \leq \sum_j \binom{n}{i_{j+1/2}} \mu^{i_{j+1/2}}$ . Therefore, by (2.7) we get

$$\sum_j \binom{n+1}{i_{j+1/2}} \delta \sum_j \binom{n}{i_{j+1/2}} \mu^{i_{j+1/2}} \leq (1 + t G) \sum_j \binom{n}{i_{j+1/2}} \mu^{i_{j+1/2}} + t H K_1,$$

with  $G = \sum_j \binom{n}{i_{j+1/2}} \mu^{i_{j+1/2}}$ . We thus obtain

$$\sum_j \binom{n}{i_{j+1/2}} \delta e^{Gn} \sum_j \binom{n}{i_{j+1/2}} \mu^{i_{j+1/2}} \leq e^{HK_1 n} \sum_j \binom{n}{i_{j+1/2}} \mu^{i_{j+1/2}},$$

that we can rewrite as

$$\sum_j \binom{n}{i_{j+1/2}} \delta e^{Gn} \sum_j \binom{n}{i_{j+1/2}} \mu^{i_{j+1/2}} \leq e^{HK_1 n} \sum_j \binom{n}{i_{j+1/2}} \mu^{i_{j+1/2}},$$

since  $H \leq G$  and it is not restrictive to assume  $K_1 \leq 1$ . Therefore, we have that  $\sum_j \binom{n}{i_{j+1/2}} \mu^{i_{j+1/2}} \leq K_1$  for

$$t \delta \frac{1}{HK_1} \ln \frac{K_1 + 1}{\sum_j \binom{n}{i_{j+1/2}} \mu^{i_{j+1/2}} + 1},$$

where the maximum is attained for some  $K_1 < e^{TV(\varphi_i^0) + 1} - 1$  such that

$$\ln \frac{K_1 + 1}{TV(\varphi_i^0) + 1} = \frac{K_1}{K_1 + 1}.$$

Therefore the total variation is uniformly bounded for

$$t \leq \frac{1}{He^{TV(\varphi_i^0) + 1}}.$$

Iterating the procedure, at time  $t^m$ ,  $m \geq 1$  we set  $K_1 = e^{m \cdot TV(\varphi_i^0) + 1} - 1$  and we get that the solution is bounded by  $K_1$  until  $t^{m+1}$  such that

$$t^{m+1} \leq t^m + \frac{m}{He^{m \cdot TV(\varphi_i^0) + 1}}. \quad (2.15)$$

Therefore, the approximate solution has bounded total variation for  $t \leq T$  with

$$T \leq \frac{1}{H \cdot TV(\varphi_i^0) + 1}.$$

←

**Corollary 3.** Let  $\varphi_i^0 \in (\mathbf{BV} \setminus \mathbf{L}^1)(\mathbb{R}; \mathbb{R}^+)$ . If (2.3) holds, then the approximate solution  $\varphi_i^X$  constructed by the algorithm (2.2) has uniformly bounded total variation on  $[0, T] \rightarrow \mathbb{R}$ , for any  $T$  satisfying (2.10)

**Proof.** If  $T \leq t$ , then  $TV(\varphi_i^X; [0, T] \rightarrow \mathbb{R}) \leq T \cdot TV(\varphi_i^0)$ . Let us assume now that  $T > t$ . Let  $n \in \mathbb{N} \setminus \{0\}$  such that  $n \cdot t < T \leq (n+1) \cdot t$ . Then

$$\begin{aligned} & TV(\varphi_i^X; [0, T] \rightarrow \mathbb{R}) \\ &= \sum_{n=0}^{\infty} \sum_{j \in \mathbb{Z}} \left( \int_{t \cdot j}^{t \cdot (j+1)} |\varphi_{i,j}^n - \varphi_{i,j}^{n+1}| + (T - n \cdot t) \sum_{j \in \mathbb{Z}} |\varphi_{i,j}^{n \cdot t} - \varphi_{i,j}^{(n+1) \cdot t}| \right) \\ & \leq \underbrace{\sum_{n=0}^{\infty} \sum_{j \in \mathbb{Z}} \int_{t \cdot j}^{t \cdot (j+1)} |\varphi_{i,j}^n - \varphi_{i,j}^{n+1}|}_{\delta T \sup_{t \in [0, T]} TV(\varphi_i^X)(t, \cdot)} + \sum_{n=0}^{\infty} \sum_{j \in \mathbb{Z}} \int_{t \cdot j}^{t \cdot (j+1)} |\varphi_{i,j}^n - \varphi_{i,j}^{n+1}|. \end{aligned}$$

We then need to bound the term

$$\sum_{n=0}^{\infty} \sum_{j \in \mathbb{Z}} \int_{t \cdot j}^{t \cdot (j+1)} |\varphi_{i,j}^n - \varphi_{i,j}^{n+1}|.$$

From the definition of the numerical scheme (2.2), we obtain

$$\begin{aligned} \varphi_{i,j}^{n+1} - \varphi_{i,j}^n &= \int_{t \cdot j}^{t \cdot (j+1)} \left( \int_{t \cdot (j-1)}^{t \cdot j} V_{i,j-1}^n \varphi_{i,j-1}^n - \int_{t \cdot j}^{t \cdot (j+1)} V_{i,j}^n \varphi_{i,j}^n \right) \\ &= \int_{t \cdot j}^{t \cdot (j+1)} \left( \int_{t \cdot (j-1)}^{t \cdot j} V_{i,j-1}^n - \int_{t \cdot j}^{t \cdot (j+1)} V_{i,j}^n \right) \varphi_{i,j}^n + \int_{t \cdot j}^{t \cdot (j+1)} V_{i,j+1}^n \varphi_{i,j+1}^n - \int_{t \cdot j}^{t \cdot (j+1)} V_{i,j}^n \varphi_{i,j}^n. \end{aligned}$$

Taking the absolute values and using (2.7) we obtain

$$|\varphi_{i,j}^{n+1} - \varphi_{i,j}^n| \leq \int_{t \cdot j}^{t \cdot (j+1)} \left( \int_{t \cdot (j-1)}^{t \cdot j} |V_{i,j-1}^n| + \int_{t \cdot j}^{t \cdot (j+1)} |V_{i,j}^n| + |V_{i,j+1}^n| + |V_{i,j}^n| \right) |\varphi_{i,j}^n| + \int_{t \cdot j}^{t \cdot (j+1)} |V_{i,j+1}^n| |\varphi_{i,j+1}^n| + \int_{t \cdot j}^{t \cdot (j+1)} |V_{i,j}^n| |\varphi_{i,j}^n|.$$

Summing on  $j$ , we get

$$\sum_{j \in \mathbb{Z}} \left( \sum_{i=1}^n x_{i,j}^{n+1} - \sum_{i=1}^n x_{i,j}^n \right) = v_i^{\max} \sum_{i=1}^n M k_1 \int_0^t x_{i,j}^n dt + v_i^{\max} k_1 \sum_{j \in \mathbb{Z}} \left( \sum_{i=1}^n x_{i,j}^n - \sum_{i=1}^n x_{i,j}^{n-1} \right),$$

which yields

$$\sum_{n=0}^{\infty} \sum_{j \in \mathbb{Z}} \left( \sum_{i=1}^n x_{i,j}^{n+1} - \sum_{i=1}^n x_{i,j}^n \right) \leq v_M^{\max} k_1 T \sup_{t \in [0, T]} \text{TV}(\vec{x}(t, \cdot)) + v_M^{\max} \sum_{i=1}^n M W_0 T \sup_{t \in [0, T]} \left( \sum_{i=1}^n x_{i,j}^n(t, \cdot) - \sum_{i=1}^n x_{i,j}^{n-1}(t, \cdot) \right)$$

that is bounded by Corollary 2, Lemma 2 and Lemma 4. ←

### 3 Proof of Theorem 1

To complete the proof of the existence of solutions to the problem (1.1), (1.5), we follow a Lax-Wendroff type argument as in [4], see also [9], to show that the approximate solutions constructed by scheme (2.2) converge to a weak solution of (1.1). By Lemma 2, Lemma 4 and Corollary 3 we can apply Helly's theorem, stating that for  $i = 1, \dots, M$ , there exists a subsequence still denoted by  $\vec{x}^n$ , which converges to some  $\vec{x} \in \mathbf{L}^1_{loc}(\mathbf{BV})([0, T] \rightarrow \mathbb{R}; \mathbb{R}^+)$  in the  $\mathbf{L}^1_{loc}$ -norm. Let us fix  $i \in \{1, \dots, M\}$ . Let  $\tau \in \mathbf{C}^1([0, T] \rightarrow \mathbb{R})$  and multiply (2.2) by  $\tau(t^n, x_j)$ . Summing over  $j \in \mathbb{Z}$  and  $n \in \{0, \dots, n_T\}$  we get

$$\begin{aligned} & \sum_{n=0}^{n_T} \sum_j \left( \tau(t^n, x_j) \left( \sum_{i=1}^{n+1} x_{i,j}^{n+1} - \sum_{i=1}^n x_{i,j}^n \right) \right) \\ &= \sum_{n=0}^{n_T} \sum_j \left( \tau(t^n, x_j) \left( \sum_{i=1}^n V_{i,j+1}^n - \sum_{i=1}^n V_{i,j-1}^n \right) \right). \end{aligned}$$

Summing by parts we obtain

$$\begin{aligned} & \sum_j \left( \tau((n_T - 1) t, x_j) \sum_{i=1}^{n_T} x_{i,j}^{n_T} + \sum_j \tau(0, x_j) \sum_{i=1}^0 x_{i,j}^0 \right) + \sum_{n=1}^{n_T} \sum_j \left( \tau(t^n, x_j) \sum_{i=1}^n x_{i,j}^n - \tau(t^{n-1}, x_j) \sum_{i=1}^{n-1} x_{i,j}^{n-1} \right) \\ &+ \sum_{n=0}^{n_T} \sum_j \left( \tau(t^n, x_{j+1}) \sum_{i=1}^n V_{i,j+1}^n - \tau(t^n, x_j) \sum_{i=1}^n V_{i,j-1}^n \right) = 0. \end{aligned} \tag{3.1}$$

Multiplying by  $x$  we get

$$\sum_j \left( \tau((n_T - 1) t, x_j) \sum_{i=1}^{n_T} x_{i,j}^{n_T} + \sum_j \tau(0, x_j) \sum_{i=1}^0 x_{i,j}^0 \right) \tag{3.2}$$

$$+ \sum_{n=1}^{n_T} \sum_j \frac{\tau(t^n, x_j) \sum_{i=1}^n x_{i,j}^n - \tau(t^{n-1}, x_j) \sum_{i=1}^{n-1} x_{i,j}^{n-1}}{t} \tag{3.3}$$

$$+ \sum_{n=0}^{\infty} \sum_j \frac{(t^n, x_{j+1}) - (t^n, x_j)}{x} V_{ij+1}^n - V_{ij}^n = 0. \quad (3.4)$$

By  $\mathbf{L}_{loc}^1$  convergence of  $r^x \rightarrow i$ , it is straightforward to see that the terms in (3.2), (3.3) converge to

$$\int_R^Z \psi(x)(0, x) - \psi(T, x)(T, x) dx + \int_0^Z \int_R^Z \psi(t, x) \psi'(t, x) dx dt, \quad (3.5)$$

as  $x \rightarrow 0$ . Concerning the last term (3.4), we can rewrite

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_j \frac{(t^n, x_{j+1}) - (t^n, x_j)}{x} V_{ij+1}^n - V_{ij}^n \\ = & \sum_{n=0}^{\infty} \sum_j \frac{(t^n, x_{j+1}) - (t^n, x_j)}{x} \downarrow V_{ij}^n V_{ij+1}^n - V_{ij}^n V_{ij}^n \\ & + \sum_{n=0}^{\infty} \sum_j \frac{(t^n, x_{j+1}) - (t^n, x_j)}{x} V_{ij}^n V_{ij}^n. \end{aligned} \quad (3.6)$$

By (2.7) we get the estimate

$$V_{ij+1}^n - V_{ij}^n \leq \delta V_i^{\max} \leq C \|x\|^{-\frac{1}{2}} \|i\|.$$

Set  $R > 0$  such that  $\psi(t, x) = 0$  for  $|x| > R$  and  $j = 0, 1, 2 \in \mathbb{Z}$  such that  $R/2 \in [x_{j_0 - \frac{1}{2}}, x_{j_0 + \frac{1}{2}}]$  and  $R/2 \in [x_{j_1 - \frac{1}{2}}, x_{j_1 + \frac{1}{2}}]$ , then

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_j \frac{(t^n, x_{j+1}) - (t^n, x_j)}{x} (V_{ij+1}^n - V_{ij}^n) \\ \leq & C \|x\|^{-\frac{1}{2}} \sum_{n=0}^{\infty} \sum_{j=j_0}^{\infty} V_i^{\max} \leq C \|x\|^{-\frac{1}{2}} \|i\| \\ \leq & C \|x\|^{-\frac{1}{2}} V_i^{\max} \leq C \|x\|^{-\frac{1}{2}} \|i\| \leq C R^{-\frac{1}{2}} \|i\|, \end{aligned}$$

which goes to zero as  $x \rightarrow 0$ .

Finally, again by the  $\mathbf{L}_{loc}^1$  convergence of  $r^x \rightarrow i$ , we have that

$$\sum_{n=0}^{\infty} \sum_j \frac{(t^n, x_{j+1}) - (t^n, x_j)}{x} V_{ij}^n V_{ij}^n \leq \int_0^Z \int_R^Z \psi(t, x) \psi'(t, x) V_i(t, x) dx dt.$$

## 4 Numerical tests

In this section we perform some numerical simulations to illustrate the behaviour of solutions to (1.1) for  $M = 2$  modeling two different scenarios. In the following, the space mesh is set to  $\Delta x = 0.001$ .

## 4.1 Cars and trucks mixed traffic

In this example, we consider a stretch of road populated by cars and trucks. The space domain is given by the interval  $[2, 3]$  and we impose absorbing conditions at the boundaries, adding  $N_1 = \frac{H_1}{x}$  ghost cells for the first population and  $N_2 = \frac{H_2}{x}$  for the second one at the right boundary, and just one ghost cell for both populations at the left boundary, where we extend the solution constantly equal to the last value inside the domain. The dynamics is described by the following 2 → 2 system

$$\begin{cases} \partial_t \rho_1(t, x) + \partial_x \rho_1(t, x) v_1^{\max} (\rho_1(t, x)) = 0, \\ \partial_t \rho_2(t, x) + \partial_x \rho_2(t, x) v_2^{\max} (\rho_2(t, x)) = 0, \end{cases} \quad (4.1)$$

with

$$\begin{aligned} \rho_1(x) &= \frac{2}{H_1} \sqrt{1 - \frac{x}{H_1}}, & H_1 &= 0.3, \\ \rho_2(x) &= \frac{2}{H_2} \sqrt{1 - \frac{x}{H_2}}, & H_2 &= 0.1, \\ v_1^{\max} &= \max\{1 - \rho_1, 0\}, & v_2^{\max} &= 0, \\ v_1^{\max} &= 0.8, & v_2^{\max} &= 1.3. \end{aligned}$$

In this setting,  $\rho_1$  represents the density of trucks and  $\rho_2$  is the density of cars on the road. Trucks moves at lower maximal speed than cars and have greater view horizon, but of the same order of magnitude. Figure 2 describes the evolution in time of the two population densities, correspondent to the initial configuration

$$\begin{cases} \rho_1(0, x) = 0.5 \quad [1.1, 1.6], \\ \rho_2(0, x) = 0.5 \quad [1.6, 1.9] \end{cases}$$

in which a platoon of trucks precedes a group of cars. Due to their higher speed, cars overtake trucks, in accordance with what observed in the local case [3].

## 4.2 Impact of connected autonomous vehicles

The aim of this test is to study the possible impact of the presence of Connected Autonomous Vehicles (CAVs) on road traffic performances. Let us consider a circular road modeled by the space interval  $[-1, 1]$  with periodic boundary conditions at  $x = \pm 1$ . In this case, we assume that autonomous and non-autonomous vehicles have the same maximal speed, but the interaction radius of CAVs is two orders of magnitude greater than the one of human-driven cars. Moreover, we assume CAVs have constant convolution kernel, modeling the fact that they have the same degree of accuracy on information about surrounding traffic, independent from the distance. In this case, model (1.1) reads

$$\begin{cases} \partial_t \rho_i(t, x) + \partial_x \rho_i(t, x) v_i^{\max} (\rho_i(t, x)) = 0, & i = 1, \dots, M, \\ \rho_1(0, x) = (0.5 + 0.3 \sin(5\pi x)), \\ \rho_2(0, x) = (1 - 0.5 \sin(5\pi x)), \end{cases} \quad (4.2)$$

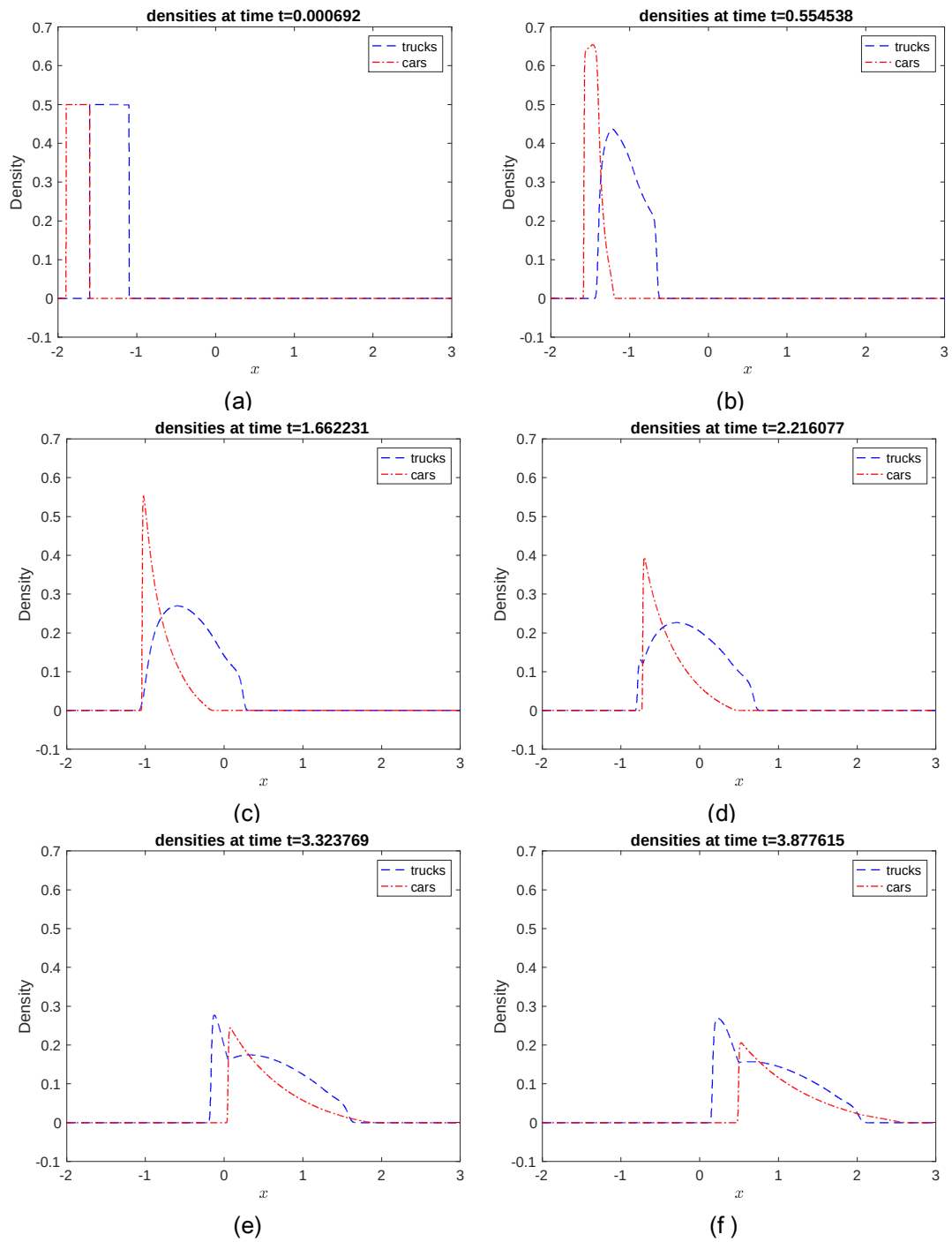


Figure 2: Density profiles of cars and trucks at increasing times corresponding to the non-local model (4.1).

with

$$f_1(x) = \frac{1}{\mathcal{H}} \quad \mathcal{H} = 1,$$

$$\rho_2(x) = \frac{2}{\rho_2} \sqrt{1 - \frac{x}{\rho_2}}, \quad \rho_2 = 0.01,$$

$$v_1 = \max\{1 - \rho_2, 0\}, \quad v_2 = 0,$$

$$v_1^{\max} = v_2^{\max} = 1.$$

Above  $\rho_1$  represents the density of autonomous vehicles,  $\rho_2$  the density of non-autonomous vehicles and  $\alpha \in [0, 1]$  is the penetration rate of autonomous vehicle. Figure 3 displays the traffic dynamics in the case  $\alpha = 0.9$ .

As a metric of traffic congestion, given a time horizon  $T > 0$ , we consider the two following functionals:

$$J(\rho) = \int_0^T \int_{\mathbb{R}^+} |r(t, x) - \rho| dx dt, \quad (4.3)$$

$$F(\rho) = \int_0^T \int_{\mathbb{R}^+} \rho_1(t, \bar{x}) v_1^{\max}((r - \rho_1)(t, \bar{x})) + \rho_2(t, \bar{x}) v_2^{\max}((r - \rho_2)(t, \bar{x})) dx dt, \quad (4.4)$$

where  $\bar{x} = x_0 \neq 0$ . The functional  $J$  measures the integral with respect to time of the spatial total variation of the total traffic density, see [5]. Instead, the functional  $F$  measures the integral with respect to time of the traffic flow at a given point  $\bar{x}$ , corresponding to the number of cars that have passed through  $\bar{x}$  in the studied time interval. Figure 4 displays the values of the functionals  $J$  and  $F$  for different values of  $\alpha = 0, 0.1, 0.2, \dots, 1$ . We can notice that the functionals are not monotone and present minimum and maximum values. The traffic evolution patterns corresponding to these stationary values are reported in Figure 5, showing the  $(t, x)$ -plots of the total traffic density  $r(t, x)$  corresponding to these values of  $\alpha$ .

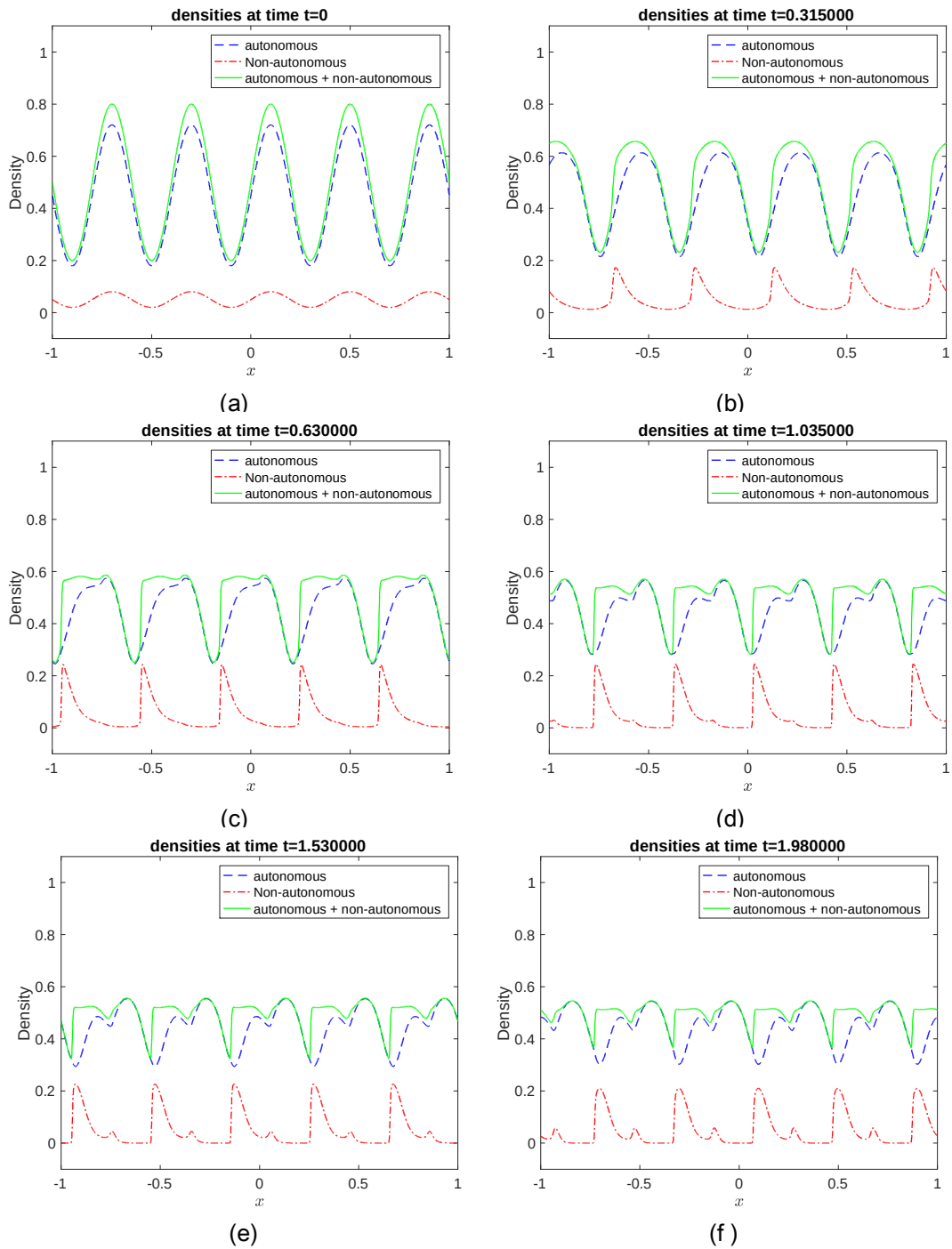


Figure 3: Density profiles corresponding to the non-local problem (4.2) with  $\alpha = 0.9$  at different times.

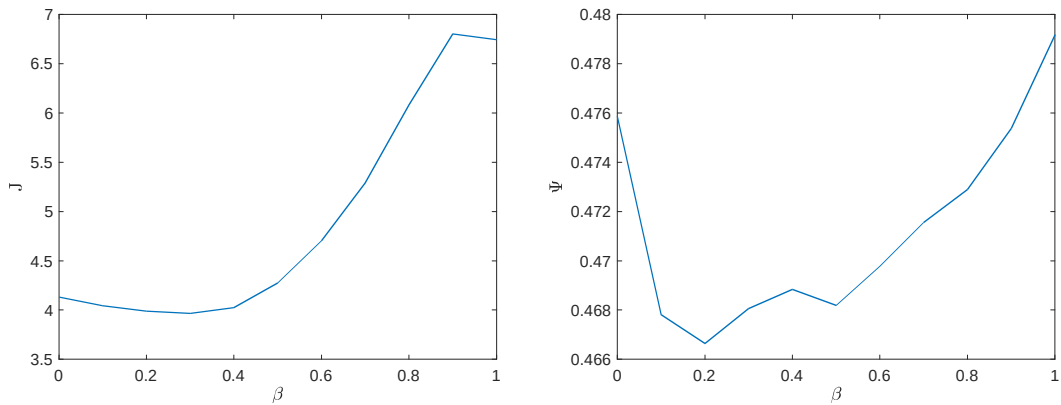


Figure 4: Functional  $J$  (left) and  $\psi$  (right)

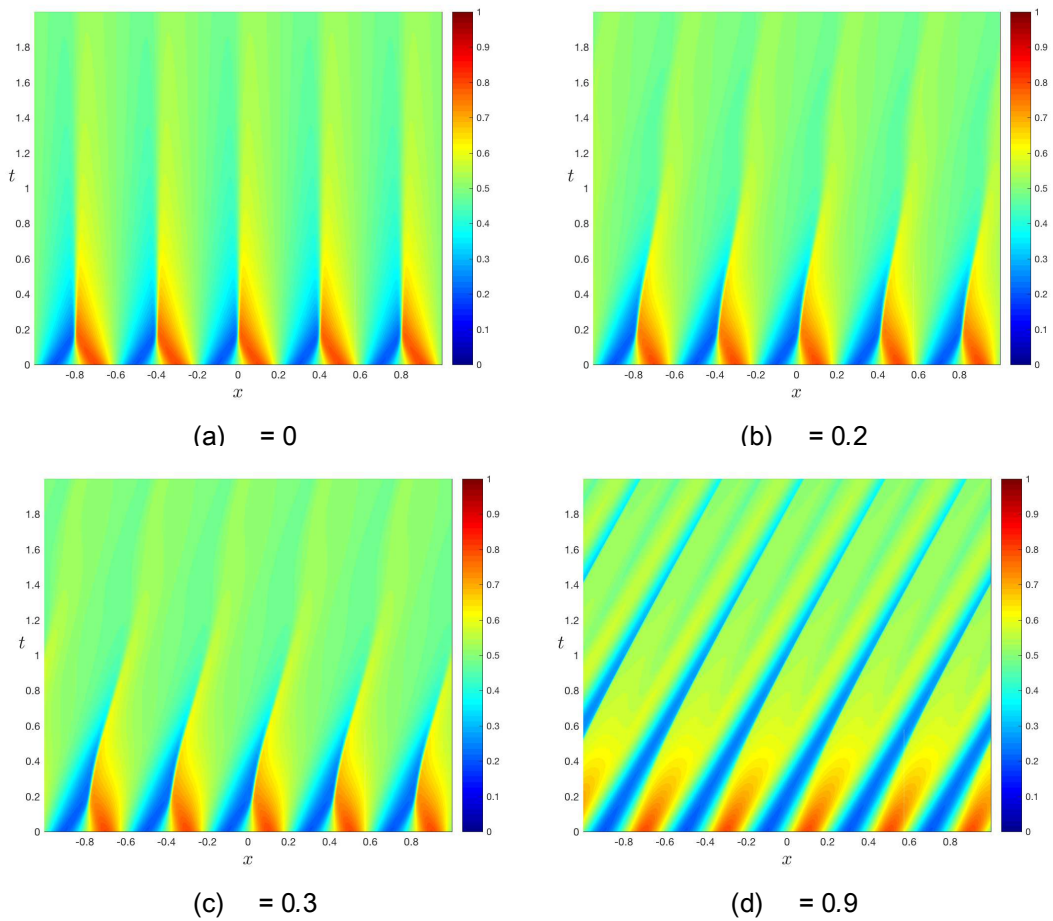


Figure 5:  $(t, x)$ -plots of the total traffic density  $r(t, x) = \rho_1(t, x) + \rho_2(t, x)$  in (4.2) corresponding to different values of  $\beta$ : (a) no autonomous vehicles are present; (b) point of minimum for  $\psi$ ; (c) point of minimum for  $J$ ; (d) point of maximum for  $J$ .

## Acknowledgements

The authors are grateful to Luis M. Villada for suggesting the non-local multi-class traffic model studied in this paper.

## References

- [1] A. Aggarwal, R. M. Colombo, and P. Goatin. Nonlocal systems of conservation laws in several space dimensions. *SIAM J. Numer. Anal.*, 53(2):963–983, 2015.
- [2] P. Amorim, R. Colombo, and A. Teixeira. On the numerical integration of scalar nonlocal conservation laws. *ESAIM M2AN*, 49(1):19–37, 2015.
- [3] S. Benzoni-Gavage and R. M. Colombo. An  $n$ -populations model for traffic flow. *European J. Appl. Math.*, 14(5):587–612, 2003.
- [4] S. Blandin and P. Goatin. Well-posedness of a conservation law with non-local flux arising in traffic flow modeling. *Numer. Math.*, 132(2):217–241, 2016.
- [5] R. M. Colombo and A. Grolli. Minimising stop and go waves to optimise traffic flow. *Appl. Math. Lett.*, 17(6):697–701, 2004.
- [6] F. A. Chiarello and P. Goatin. Global entropy weak solutions for general non-local traffic flow models with anisotropic kernel. *ESAIM: M2AN*, 52(1):163–180, 2018.
- [7] J. Friedrich, O. Kolb, and S. Gottlich. A Godunov type scheme for a class of scalar conservation laws with non-local flux. *ArXiv e-prints*, Feb. 2018.
- [8] P. Goatin and S. Scialanga. Well-posedness and finite volume approximations of the LWR traffic flow model with non-local velocity. *Netw. Heterog. Media*, 11(1):107–121, 2016.
- [9] R. J. LeVeque. *Finite volume methods for hyperbolic problems*. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 2002.
- [10] M. J. Lighthill and G. B. Whitham. On kinematic waves. II. A theory of traffic flow on long crowded roads. *Proc. Roy. Soc. London. Ser. A.*, 229:317–345, 1955.
- [11] P. I. Richards. Shock waves on the highway. *Operations Res.*, 4:42–51, 1956.

## Appendix A Lax-Friedrichs numerical scheme

We provide here an alternative existence proof for (1.1), based on approximate solutions constructed via the following adapted Lax-Friedrichs scheme:

$$\vec{u}_{i,j}^{n+1} = \vec{u}_{i,j}^n - \frac{\Delta t}{\Delta x} \left( F_{i,j+1/2}^n - F_{i,j-1/2}^n \right), \quad (\text{A.1})$$

with

$$F_{i,j+1/2}^n := \frac{1}{2} \vec{u}_{i,j}^n V_{i,j}^n + \frac{1}{2} \vec{u}_{i,j+1}^n V_{i,j+1}^n + \frac{\epsilon}{2} \vec{u}_{i,j}^n - \vec{u}_{i,j+1}^n, \quad (\text{A.2})$$

where  $\epsilon > 0$  is the viscosity coefficient and  $\Delta t = \frac{t}{X}$ .

**Lemma 5.** For any  $T > 0$ , under the CFL conditions

$$\epsilon < 1, \quad (\text{A.3})$$

$$\epsilon < v_M^{\max} k_1, \quad (\text{A.4})$$

the scheme (A.2)-(A.1) is positivity preserving on  $[0, T] \rightarrow \mathbb{R}$ .

**Proof.** Let us assume that  $\vec{u}_{ij}^n = 0$  for all  $j \in \mathbb{Z}$  and  $i \in \{1, \dots, M\}$ . It suffices to prove that  $\vec{u}_{ij}^{n+1}$  in (A.1) is non-negative. Compute

$$\vec{u}_{ij}^{n+1} = \vec{u}_{ij}^n + \frac{\delta}{2} \left( \vec{u}_{i,j+1}^n - 2\vec{u}_{ij}^n + \vec{u}_{i,j-1}^n \right) + \frac{\delta}{2} \vec{u}_{ij}^n V_{ij-1}^n - \vec{u}_{i,j+1}^n V_{ij+1}^n \quad (\text{A.5})$$

$$= \vec{u}_{ij}^n \left( 1 - \frac{\delta}{2} \left( V_{ij-1}^n + V_{ij+1}^n \right) \right) + \frac{\delta}{2} \left( \vec{u}_{i,j-1}^n - \vec{u}_{i,j+1}^n \right) V_{ij}^n, \quad (\text{A.6})$$

under assumptions (A.3) and (A.4), we obtain that  $\vec{u}_{ij}^{n+1}$  is positive.  $\leftarrow$

**Corollary 4. (L<sup>1</sup> bound)** For any  $T > 0$ , under the CFL conditions (A.3)-(A.4) the scheme (A.2)-(A.1) preserves the L<sup>1</sup> norm of the  $i$ -th component of  $\vec{u}^x$ .

**Proof.** See proof of Corollary 2.  $\leftarrow$

**Lemma 6. (L<sup>1</sup> -bound)** If  $\vec{u}_{ij}^0 = 0$  for all  $j \in \mathbb{Z}$  and  $i = 1, \dots, M$ , and the CFL conditions (A.3)-(A.4) hold, the approximate solution  $\vec{u}^x$  constructed by the algorithm (A.2)-(A.1) is uniformly bounded on  $[0, T] \rightarrow \mathbb{R}$  for any  $T$  such that

$$T < \frac{1}{M} \frac{1}{v_M^{\max}} \frac{1}{W_0}. \quad (\text{A.7})$$

**Proof.** From (A.6) we can define

$$\vec{u}_{ij}^{n+1} = \frac{\delta}{2} \left( \vec{u}_{i,j-1}^n + \vec{u}_{i,j+1}^n \right) + \left( 1 - \frac{\delta}{2} \left( V_{ij-1}^n + V_{ij+1}^n \right) \right) \vec{u}_{ij}^n$$

Let  $\vec{u}^n = \max_{i,j} \vec{u}_{ij}^n$ . Then we get

$$\vec{u}_{ij}^{n+1} \leq \vec{u}^n \left( 1 + \frac{\delta}{2} \left( V_{ij-1}^n + V_{ij+1}^n \right) \right)$$

and by (2.7)

$$V_{ij-1}^n + V_{ij+1}^n \leq 2v_M^{\max} \frac{1}{\delta} \vec{u}^n. \quad (\text{A.8})$$

Therefore, until  $k \rightarrow^n k_1 \leq K$ , for some  $K = \frac{1}{2v_M^{\max} \delta}$ , we get

$$\vec{u}_{ij}^{n+1} \leq \vec{u}^n \left( 1 + MKv_M^{\max} \frac{1}{\delta} \vec{u}^n \right),$$

and we can reason as in the proof of Lemma 2.  $\leftarrow$

**Lemma 7. (BV estimates)** Let  $\rho \in C^1(\mathbb{R}, \mathbb{R}^+)$  for all  $i = 1, \dots, M$ . If (A.4) holds and

$$t \partial_x \frac{2}{2 \leftarrow x k \leftarrow k_1 W_0 v_M^{\max} k \leftarrow k_1} x, \quad (\text{A.9})$$

then the solution constructed by the algorithm (A.2)-(A.1) has uniformly bounded total variation for any  $T$  such that

$$T \partial \min_{i=1, \dots, M} \frac{1}{D \text{TV}(\rho) + 1}, \quad (\text{A.10})$$

where  $D = k \leftarrow k_1 v_M^{\max} W_0 M \leftarrow 3 M J_0 k \leftarrow k_1 \leftarrow 0 \leftarrow 1 + 2 \leftarrow 0 \leftarrow 1$ .

**Proof.** Subtracting the following expressions

$$\begin{aligned} \rho_{i,j+1}^{n+1} &= \rho_{i,j+1}^n + \frac{1}{2} \left( \rho_{i,j}^n - 2 \rho_{i,j+1}^n + \rho_{i,j+2}^n \right) + \frac{1}{2} \left( \rho_{i,j}^n V_{i,j}^n - \rho_{i,j+2}^n V_{i,j+2}^n \right), \\ \rho_{i,j}^{n+1} &= \rho_{i,j}^n + \frac{1}{2} \left( \rho_{i,j-1}^n - 2 \rho_{i,j}^n + \rho_{i,j+1}^n \right) + \frac{1}{2} \left( \rho_{i,j-1}^n V_{i,j-1}^n - \rho_{i,j+1}^n V_{i,j+1}^n \right), \end{aligned}$$

we get

$$\begin{aligned} \rho_{i,j+1/2}^{n+1} &= \frac{1}{2} \left( \rho_{i,j}^{n+1} + \rho_{i,j+1}^{n+1} \right) + \frac{1}{2} \left( \rho_{i,j+1/2}^n + \rho_{i,j+3/2}^n \right) \\ &+ \frac{1}{2} \left( V_{i,j}^n \rho_{i,j+1/2}^n + \rho_{i,j+1}^n V_{i,j+1}^n - V_{i,j+1}^n \rho_{i,j+3/2}^n - \rho_{i,j+1}^n V_{i,j+2}^n \right). \end{aligned}$$

Now, we can write

$$\begin{aligned} V_{i,j}^n - V_{i,j+1}^n &= v_i^{\max} \left( \rho_{i,j+1/2}^n - \rho_{i,j+3/2}^n \right) \sum_{k=0}^{\infty} \binom{k}{i} \rho_{i,j+k}^n - \rho_{i,j+k+1}^n \\ &= v_i^{\max} \left( \rho_{i,j+1/2}^n - \rho_{i,j+3/2}^n \right) \sum_{k=0}^{\infty} \binom{k}{i} \rho_{i,j+k+1/2}^n \\ &= v_i^{\max} \left( \rho_{i,j+1/2}^n - \rho_{i,j+3/2}^n \right) \sum_{k=0}^{\infty} \binom{k}{i} \left( \rho_{i,j+k}^n - \rho_{i,j+k+1}^n \right) = \rho_{i,j+1}^n A, \end{aligned}$$

and

$$\begin{aligned} V_{i,j+2}^n - V_{i,j+1}^n &= v_i^{\max} \left( \rho_{i,j+3/2}^n - \rho_{i,j+5/2}^n \right) \sum_{k=0}^{\infty} \binom{k}{i} \rho_{i,j+k+2}^n - \rho_{i,j+k+1}^n \\ &= v_i^{\max} \left( \rho_{i,j+3/2}^n - \rho_{i,j+5/2}^n \right) \sum_{k=0}^{\infty} \binom{k}{i} \rho_{i,j+k+3/2}^n \\ &= v_i^{\max} \left( \rho_{i,j+3/2}^n - \rho_{i,j+5/2}^n \right) \sum_{k=1}^{\infty} \binom{k}{i} \left( \rho_{i,j+k+1}^n - \rho_{i,j+k+2}^n \right) = \rho_{i,j+1}^n A. \end{aligned}$$



$$\begin{aligned}
& + v_i^{\max} \binom{0}{\epsilon_{j+3/2}} x^{\otimes} \sum_{k=0}^{\infty} \binom{k+1}{i} \binom{k}{i} x^{\otimes} \downarrow \binom{n}{j+k+1/2} + \binom{n}{j+k+3/2} \\
& + \binom{0}{i} x^{\otimes} \downarrow \binom{n}{j-1/2} + \binom{n}{j+1/2} \frac{1}{5}.
\end{aligned}$$

where  $\tilde{\epsilon}_{j+1} = 2 \lfloor (\epsilon_{j+1/2}, \epsilon_{j+3/2}) \rfloor$ . For some  $\#, \mu \in [0, 1]$ , we compute

$$\begin{aligned}
\binom{\tilde{\epsilon}_{j-1/2}}{i} \binom{\tilde{\epsilon}_{j+3/2}}{i} &= \# \sum_{k=0}^{\infty} \binom{k}{i} \binom{k}{i} x^{\otimes} \downarrow \binom{n}{j+k} + (1 - \#) \sum_{k=0}^{\infty} \binom{k}{i} \binom{k}{i} x^{\otimes} \downarrow \binom{n}{j+k-1} \\
&\quad \mu \sum_{k=0}^{\infty} \binom{k}{i} \binom{k}{i} x^{\otimes} \downarrow \binom{n}{j+k+2} + (1 - \mu) \sum_{k=0}^{\infty} \binom{k}{i} \binom{k}{i} x^{\otimes} \downarrow \binom{n}{j+k+1} \\
&= \# \sum_{k=0}^{\infty} \binom{k}{i} \binom{k}{i} x^{\otimes} \downarrow \binom{n}{j+k} + (1 - \#) \sum_{k=1}^{\infty} \binom{k+1}{i} \binom{k}{i} x^{\otimes} \downarrow \binom{n}{j+k} \\
&\quad \mu \sum_{k=2}^{\infty} \binom{k}{i} \binom{k-2}{i} x^{\otimes} \downarrow \binom{n}{j+k} + (1 - \mu) \sum_{k=1}^{\infty} \binom{k}{i} \binom{k-1}{i} x^{\otimes} \downarrow \binom{n}{j+k} \\
&= \sum_{k=2}^{\infty} \binom{k}{i} \binom{k}{i} + (1 - \#) \sum_{k=1}^{\infty} \binom{k+1}{i} \binom{k}{i} \mu \binom{k-2}{i} + (1 - \mu) \sum_{k=1}^{\infty} \binom{k}{i} \binom{k-1}{i} x^{\otimes} \downarrow \binom{n}{j+k} \\
&\quad + \# \binom{0}{i} x^{\otimes} \downarrow \binom{n}{j} + \binom{1}{i} x^{\otimes} \downarrow \binom{n}{j+1} \\
&\quad + (1 - \#) \binom{0}{i} x^{\otimes} \downarrow \binom{n}{j-1} + \binom{1}{i} x^{\otimes} \downarrow \binom{n}{j} + \binom{2}{i} x^{\otimes} \downarrow \binom{n}{j+1} \\
&\quad (1 - \mu) \binom{0}{i} x^{\otimes} \downarrow \binom{n}{j+1}.
\end{aligned}$$

By monotonicity of  $\binom{\cdot}{i}$  we have

$$\binom{k}{i} + (1 - \#) \binom{k+1}{i} \geq \mu \binom{k-2}{i} + (1 - \mu) \binom{k-1}{i} \geq 0.$$

Taking the absolute values we get

$$\begin{aligned}
\binom{\tilde{\epsilon}_{j-1/2}}{i} \binom{\tilde{\epsilon}_{j+3/2}}{i} &\leq \sum_{k=2}^{\infty} \binom{k}{i} \binom{k}{i} + (1 - \mu) \binom{k-1}{i} \binom{k}{i} + \binom{k}{i} \binom{k-1}{i} + 3 \binom{i}{0}, \quad Mk \rightarrow {}^n k_1 \\
&\leq \sum_{k=2}^{\infty} \binom{k}{i} \binom{k-1}{i} + 3 \binom{i}{0}, \quad Mk \rightarrow {}^n k_1 \\
&\leq 6 \binom{i}{0} Mk \rightarrow {}^n k_1.
\end{aligned}$$

Observe that assumption (A.4) guarantees the positivity of (A.11) and (A.13). Similarly, (A.9) ensures the positivity of (A.12).

Until  $\sum_{j=1}^M \delta_j^{n_j} K_1$  for  $j = 1, \dots, M$  for some  $K_1 = P_j^0$ , taking the absolute values and rearranging the indexes, we have

$$\sum_j X_{i,j+1/2}^{n+1} \delta_j X_{i,j+1/2}^n \left(1 + \frac{1}{2} V_{ij} + V_{ij+1}\right) + t DK_1$$

where  $D = k_1 \sum_{j=1}^M W_0 M \sum_{j=1}^M k_1 \sum_{j=1}^M \left(1 + \frac{1}{2} V_{ij} + V_{ij+1}\right)$ . Therefore, by (A.8) we get

$$\sum_j X_{i,j+1/2}^{n+1} \delta_j X_{i,j+1/2}^n (1 + tC) + t DK_1$$

with  $C = \sum_{j=1}^M W_0 M k_1$ . In this way we obtain

$$\sum_j X_{i,j+1/2}^{n+1} \delta_j e^{Cn} X_{i,j+1/2}^n + e^{DK_1 n} = 1,$$

that we can rewrite as

$$\text{TV}(\rightarrow_i, x)(n, t, \cdot) \delta_j e^{Cn} \text{TV}(\rightarrow_i^0) + e^{DK_1 n} = 1$$

$$\delta_j e^{DK_1 n} \text{TV}(\rightarrow_i^0) + 1 = 1, \tag{A.14}$$

since  $D \geq 2C$  and it is not restrictive to assume  $K_1 \leq \frac{1}{2}$ . Therefore we have that  $\text{TV}(\rightarrow_i, x) \delta_j K_1$  for

$$t \delta_j \frac{1}{DK_1} \ln \frac{K_1 + 1}{\text{TV}(\rightarrow_i^0) + 1},$$

where the maximum is attained for some  $K_1 \delta_j e^{\text{TV}(\rightarrow_i^0) + 1} = 1$  such that

$$\ln \frac{K_1 + 1}{\text{TV}(\rightarrow_i^0) + 1} = \frac{K_1}{K_1 + 1}.$$

Therefore the total variation is uniformly bounded for

$$t \delta_j \frac{1}{De^{\text{TV}(\rightarrow_i^0) + 1}}.$$

Iterating the procedure, at time  $t^m$ ,  $m \geq 1$  we set  $K_1 = e^m \text{TV}(\rightarrow_i^0) + 1 = 1$  and we get that the solution is bounded by  $K_1$  until  $t^{m+1}$  such that

$$t^{m+1} \delta_j t^m + \frac{m}{De^m \text{TV}(\rightarrow_i^0) + 1}. \tag{A.15}$$

Therefore, the approximate solution satisfies the bound (A.14) for  $t \delta_j T$  with

$$T \delta_j \frac{1}{D \text{TV}(\rightarrow_i^0) + 1}.$$

←

**Corollary 5.** Let  $\varphi \in \mathbf{BV}(\mathbb{R}; [0, 1])$ . If (A.3)-(A.4) holds, then the approximate solution  $\vec{x}$  constructed by the algorithm (A.2)-(A.1) has uniformly bounded total variation on  $[0, T] \rightarrow \mathbb{R}$ , for any  $T$  satisfying (A.10).

**Proof.** Let us fix  $T \in \mathbb{R}^+$  such that (A.10) and (A.7) hold. If  $T \leq t$ , then  $\text{TV}(\vec{x}; \mathbb{R} \rightarrow [0, T]) \leq T \text{TV}(\vec{x}; 0)$ . Let us assume now that  $T > t$ . Let  $M \in \mathbb{N} \setminus \{0\}$  such that  $n = \lfloor T/t \rfloor < T \leq (n+1)t$ . Then

$$\text{TV}(\vec{x}; \mathbb{R} \rightarrow [0, T]) \tag{A.16}$$

$$= \sum_{n=0}^{\lfloor T/t \rfloor} \sum_{j \in \mathbb{Z}} t \left( \vec{x}_{i,j+1}^n - \vec{x}_{i,j}^n \right) + (T - nT) \sum_{j \in \mathbb{Z}} \left( \vec{x}_{i,j+1}^{nT} - \vec{x}_{i,j}^{nT} \right) + \sum_{n=0}^{\lfloor T/t \rfloor} \sum_{j \in \mathbb{Z}} \left( \vec{x}_{i,j}^{n+1} - \vec{x}_{i,j}^n \right) \tag{A.17}$$

$\underbrace{\hspace{15em}}_{\delta T \sup_{t \in [0, T]} \text{TV}(\vec{x})(t, \cdot)}$

We then need to bound the term

$$\sum_{n=0}^{\lfloor T/t \rfloor} \sum_{j \in \mathbb{Z}} \left( \vec{x}_{i,j}^{n+1} - \vec{x}_{i,j}^n \right) \tag{A.18}$$

Let us make use of the definition of the numerical scheme (A.2)-(A.1), we obtain

$$\begin{aligned} \vec{x}_{i,j}^{n+1} - \vec{x}_{i,j}^n &= \frac{1}{2} \left( \vec{x}_{i,j+1}^n + V_{i,j+1} \right) - \left( \vec{x}_{i,j-1}^n + \vec{x}_{i,j}^n \right) - \frac{1}{2} \left( \vec{x}_{i,j}^n + V_{i,j+1} \right) - \left( \vec{x}_{i,j}^n + \vec{x}_{i,j+1}^n \right) \\ &+ \frac{1}{2} \left( \vec{x}_{i,j-1}^n + V_{i,j-1} \right) - V_{i,j+1} \end{aligned}$$

If (A.4) holds, we can take the absolute value

$$\begin{aligned} \left| \vec{x}_{i,j}^{n+1} - \vec{x}_{i,j}^n \right| &= \frac{1}{2} \left| \vec{x}_{i,j+1}^n + V_{i,j+1} - \vec{x}_{i,j-1}^n - \vec{x}_{i,j}^n \right| - \frac{1}{2} \left| \vec{x}_{i,j}^n + V_{i,j+1} - \vec{x}_{i,j}^n - \vec{x}_{i,j+1}^n \right| \\ &+ \frac{1}{2} \left| \vec{x}_{i,j-1}^n + V_{i,j-1} - V_{i,j+1} \right| \end{aligned}$$

Summing on  $j$  and rearranging the indexes we get

$$\begin{aligned} \sum_{j \in \mathbb{Z}} \left( \vec{x}_{i,j}^{n+1} - \vec{x}_{i,j}^n \right) &= \frac{t}{2} \sum_{j \in \mathbb{Z}} \left( \vec{x}_{i,j+1}^n - \vec{x}_{i,j}^n \right) - \sum_{j \in \mathbb{Z}} \left( \vec{x}_{i,j+1}^n + V_{i,j+1} \right) \\ &+ \frac{t}{2} \sum_{j \in \mathbb{Z}} \left( \vec{x}_{i,j-1}^n + V_{i,j-1} - V_{i,j+1} \right) \\ &= \frac{t}{2} \sum_{j \in \mathbb{Z}} \left( \vec{x}_{i,j+1}^n - \vec{x}_{i,j}^n \right) - \sum_{j \in \mathbb{Z}} \left( \vec{x}_{i,j+1}^n + V_{i,j+1} \right) \end{aligned}$$

$$+ \sum_{j \in \mathbb{Z}} t^{\rightarrow_{ij} - 1} x \vee_M^{\max} \circ_1 W_0 M k \rightarrow k_1$$

which yields

$$\sum_{n=0}^{\infty} \sum_{j \in \mathbb{Z}} x^{\rightarrow_{ij}^{n+1}} \rightarrow_{ij}^n \quad (\text{A.19})$$

$$\delta T e^{C_1 n} t^{\downarrow} \text{TV}(\rightarrow^0) + 1 \llcorner + \frac{1}{2} \vee_M^{\max} \circ_1 x W_0 M k \rightarrow k_1 \quad (\text{A.20})$$

$$+ T k \rightarrow k_1 \vee_M^{\max} \circ_1 W_0 M k \rightarrow k_1. \quad (\text{A.21})$$

**Proof of Theorem 1.** Let us define

$$g(\rightarrow_{ij}^n, \dots, \rightarrow_{ij+N}^n) := \frac{1}{2} \rightarrow_{ij}^n \vee_{ij}^n + \frac{1}{2} \rightarrow_{ij+1}^n \vee_{ij+1}^n + \frac{\llcorner \downarrow}{2} \rightarrow_{ij}^n \rightarrow_{ij+1}^n.$$

Fix  $i \in \{1, \dots, M\}$ . Let  $\gamma \in \mathcal{C}^1([0, T] \rightarrow \mathbb{R})$  and multiply (A.1) by  $(t^n, x_j)$ . Summing over  $j \in \mathbb{Z}$  and  $n \in \{0, 1, \dots, n_T\}$  we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_j (t^n, x_j) \rightarrow_{ij}^{n+1} \rightarrow_{ij}^n \\ &= \sum_{n=0}^{\infty} \sum_j (t^n, x_j) g(\rightarrow_{ij}^n, \dots, \rightarrow_{ij+N}^n) - g(\rightarrow_{ij-1}^n, \dots, \rightarrow_{ij+N-1}^n). \end{aligned}$$

Summing by parts we obtain

$$\begin{aligned} & \sum_j ((n_T - 1), t, x_j) \rightarrow_{ij}^{n_T} + \sum_j (0, x_j) \rightarrow_{ij}^0 + \sum_{n=1}^{\infty} \sum_j (t^n, x_j) - (t^{n-1}, x_j) \rightarrow_{ij}^n \\ & + \sum_{n=0}^{\infty} \sum_j (t^n, x_{j+1}) - (t^n, x_j) g(\rightarrow_{ij}^n, \dots, \rightarrow_{ij+N}^n) = 0. \end{aligned} \quad (\text{A.22})$$

Multiplying by  $x$

$$\sum_j x ((n_T - 1), t, x_j) \rightarrow_{ij}^{n_T} + \sum_j x (0, x_j) \rightarrow_{ij}^0 + \sum_{n=1}^{\infty} \sum_j x t^n \frac{(t^n, x_j) - (t^{n-1}, x_j)}{t} \rightarrow_{ij}^n \quad (\text{A.23})$$

$$+ \sum_{n=0}^{\infty} \sum_j x t^n \frac{(t^n, x_{j+1}) - (t^n, x_j)}{x} g(\rightarrow_{ij}^n, \dots, \rightarrow_{ij+N}^n) = 0. \quad (\text{A.24})$$

By  $\mathbf{L}_{loc}^1$  convergence of  $\gamma_i, x \rightarrow \gamma, x$ , it is straightforward to see that the first two terms in (A.23) converge to

$$\int_{\mathbb{R}} (\gamma^0(x))(0, x) - \gamma(T, x)(T, x) dx + \int_0^T \int_{\mathbb{R}} \gamma(t, x) @ (t, x) dx dt \quad (\text{A.25})$$

as  $x \neq 0$ . Concerning the last term, we can observe that

$$\begin{aligned} & g(\vec{r}_{i,j}^n, \dots, \vec{r}_{i,j+N}^n) \vec{r}_{i,j}^n V_{i,j}^n \\ & \partial \frac{\leftarrow + V_M^{\max} k k_1}{2} \vec{r}_{i,j+1}^n \vec{r}_{i,j}^n + \frac{1}{2} (\vec{r}_{i,j+1}^n \vec{r}_{i,j}^n) V_{i,j+1}^n + \vec{r}_{i,j}^n \downarrow V_{i,j+1}^n V_{i,j}^n \\ & \partial \frac{\leftarrow + V_M^{\max} k k_1}{2} \vec{r}_{i,j+1}^n \vec{r}_{i,j}^n + \frac{1}{2} W_0 \times TV(\vec{r}_{i,j}, x(t^n, \cdot)) V_M^{\max} \quad 0 \quad 1 \\ & \partial \frac{\leftarrow + V_M^{\max} k k_1}{2} \vec{r}_{i,j+1}^n \vec{r}_{i,j}^n + J \quad x. \end{aligned}$$

where  $J = \frac{1}{2} V_M^{\max} \quad 0 \quad 1 \quad W_0 \times TV(\vec{r}_{i,j}, x(T, \cdot))$ . Therefore, the last term in (A.22) can be rewritten as

$$\begin{aligned} & x \quad t \quad \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{(t^n, X_{j+1}) (t^n, X_j)}{x} g(\vec{r}_{i,j}^n, \dots, \vec{r}_{i,j+N}^n) \vec{r}_{i,j}^n V_{i,j}^n \\ & = x \quad t \quad \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{(t^n, X_{j+1}) (t^n, X_j)}{x} \vec{r}_{i,j}^n V_{i,j}^n \\ & + x \quad t \quad \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{(t^n, X_{j+1}) (t^n, X_j)}{x} (g(\vec{r}_{i,j}^n, \dots, \vec{r}_{i,j+N}^n) \vec{r}_{i,j}^n V_{i,j}^n). \end{aligned}$$

By  $\mathbf{L}_{loc}^1$  convergence of  $\vec{r}_{i,j}^n \rightarrow \vec{r}_{i,j}$  and boundedness of  $t$ , the first term in the above decomposition converges to

$$\int_0^T \int_{\mathbb{R}} \vec{r}_{i,j}(t, x) v(r \leftarrow \vec{r}_{i,j}) \otimes \vec{r}_{i,j}(t, x) \, dx \, dt.$$

Set  $R > 0$  such that  $\vec{r}_{i,j}(t, x) = 0$  for  $|x| > R$  and  $j \in \mathbb{Z}$  such that  $R/2 \leq X_{j_0-1/2}, X_{j_0+1/2} \leq R$  and  $R/2 \leq X_{j_1-1/2}, X_{j_1+1/2} \leq R$ , then

$$\begin{aligned} & x \quad t \quad \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{(t^n, X_{j+1}) (t^n, X_j)}{x} (g(\vec{r}_{i,j}^n, \dots, \vec{r}_{i,j+N}^n) \vec{r}_{i,j}^n V_{i,j}^n) \\ & \partial x \quad t \quad \sum_{n=0}^{\infty} \sum_{j=j_0}^{\infty} \frac{\leftarrow + V_M^{\max} k k_1}{2} \vec{r}_{i,j+1}^n \vec{r}_{i,j}^n + J \quad x \\ & = \frac{\leftarrow + V_M^{\max} k k_1}{2} k \otimes k_1 \quad x \quad t \quad \sum_{n=0}^{\infty} \sum_{j=j_0}^{\infty} \vec{r}_{i,j+1}^n \vec{r}_{i,j}^n + k \otimes k_1 J \quad x \geq R \\ & \partial \frac{\leftarrow + V_M^{\max} k k_1}{2} k \otimes k_1 \quad TV(\vec{r}_{i,j}, x(T, \cdot)) \quad x + k \otimes k_1 J \quad x \geq R \end{aligned}$$

which goes to zero when  $x \neq 0$ . Finally, again by the  $\mathbf{L}_{loc}^1$  convergence of  $\vec{r}_{i,j}^n \rightarrow \vec{r}_{i,j}$ , we have that

$$x \quad t \quad \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{(t^n, X_{j+1}) (t^n, X_j)}{x} \vec{r}_{i,j}^n V_{i,j}^n \rightarrow \int_0^T \int_{\mathbb{R}} \vec{r}_{i,j}(t, x) v(r \leftarrow \vec{r}_{i,j}) \otimes \vec{r}_{i,j}(t, x) \, dx \, dt.$$