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*Original*

Remodelling of biological tissues with fibre recruitment and reorientation in the light of the theory of material uniformity / Hamedzadeh, Amir; Grillo, Alfio; Epstein, Marcelo; Federico, Salvatore. - In: MECHANICS RESEARCH COMMUNICATIONS. - ISSN 0093-6413. - 96:(2019), pp. 56-61. [[10.1016/j.mechrescom.2019.02.001](https://doi.org/10.1016/j.mechrescom.2019.02.001)]

*Availability:*

This version is available at: [11583/2799278](https://doi.org/10.1016/j.mechrescom.2019.02.001) since: 2020-03-02T10:30:05Z

*Publisher:*

Elsevier

*Published*

DOI:[10.1016/j.mechrescom.2019.02.001](https://doi.org/10.1016/j.mechrescom.2019.02.001)

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# Remodelling of Biological Tissues with Fibre Recruitment and Reorientation in the Light of the Theory of Material Uniformity

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## Abstract

This study focusses on the remodelling of biological tissues in the framework of the theory of material uniformity. A constitutive evolution model is introduced, including fibre recruitment and reorientation, and subjected to the entropy inequality, which enforces the Second Principle of Thermodynamics. The model is applied to a numerical example describing a pressurised fibre-reinforced cylinder, roughly representing an artery, and is able to capture the major characteristics of remodelling in arteries, as reported in the literature.

*Keywords:* collagen fibre; recruitment; remodelling; growth; material uniformity

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## 1. Introduction

Growth and remodelling in biological tissues can be studied as *anelastic phenomena*. Anelastic processes, such as plasticity or growth-remodelling, are accompanied by a change in microstructure resulting in *configurational forces* and residual stresses (e.g., Hoger, 1997; Gurtin, 1999). While plasticity occurs at constant mass, biological tissues not only experience a change in microstructure, but also an increase (growth) or decrease (resorption) of mass. Among the first attempts to approach the problem of growth and remodelling from the continuum mechanical perspective are the seminal works by Cowin and Hegedus (1976) and Hegedus and Cowin (1976) on bone remodelling. Rodriguez et al. (1994) studied growth and remodelling in arteries and used the Bilby-Kröner-Lee decomposition of the deformation gradient  $\mathbf{F}$  into a growth part  $\mathbf{F}_g$  and an elastic part  $\mathbf{F}_e$ . In practice, they considered a residually stressed reference configuration which grows into a stress-free intermediate (and generally incompatible) configuration, and finally deforms elastically to the current (and compatible) configuration actually attained by the body. Moreover, the fact that the collagen fibres in a biological tissue may be undulated in the reference configuration, and will thus bear stress only after a certain threshold stretch, has been studied as an additional remodelling parameter for the case of aneurysms (Watton et al., 2004; Watton and Hill, 2009). Here we employ the framework proposed by Epstein and Maugin (2000), in which growth and remodelling are

seen as the two aspects of an evolution process implying a local rearrangement of material inhomogeneities, described in terms of an *implant*, under the light of the *theory of material uniformity*. In this framework, growth and remodelling are governed by the *inhomogeneity rate*,  $\mathbf{L}_P = \dot{\mathbf{P}}\mathbf{P}^{-1}$ , where  $\mathbf{P}^{-1}$  formally corresponds to the growth tensor  $\mathbf{F}_g$  of Rodriguez et al. (1994). Specifically, the trace of  $\mathbf{L}_P$  is often required to be proportional to the source or sink of mass due to growth that features in the local mass balance of the body. Given  $\mathbf{L}_P$ , the implant tensor  $\mathbf{P}$  can be determined by integrating the differential equation  $\dot{\mathbf{P}} = \mathbf{L}_P\mathbf{P}$ . However, the way in which  $\mathbf{L}_P$  is supplied is not unique.

We had previously modelled the effect of the undulation of the individual fibrils in a collagen fibre (Hamedzadeh et al., 2018) and, in this study, we employ the same mechanism for an entire fibre, and in terms of the theory of material uniformity. Therefore, we introduce the proper material implant describing both reorientation and recruitment of the fibres in an artery, and solve the benchmark problem previously studied by Grillo et al. (2015) in order to elucidate our results.

## 2. Theory of Uniformity

We follow the *theory of uniformity*, originally introduced by Noll (1967) and further developed by Epstein and Maugin (1990). A material body  $\mathcal{B}$  is said to be *uniform* if all of its points are made of the *same* material. This implies that the tangent spaces  $T_X\mathcal{B}$  of the points  $X$  of  $\mathcal{B}$  have been modelled on an archetypal vector space  $\mathcal{A} \equiv \mathbb{R}^3$ , called precisely the *archetype*, via an isomorphism

$$\mathbf{P}(X) : \mathcal{A} \rightarrow T_X\mathcal{B}, \quad (1)$$

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at every point  $X$ . In other words, if we look at the microscopic structures surrounding two materially uniform points  $X$  and  $Y$ , we might not see identical pictures, as one might have been distorted or rotated in a different manner than the other. However, we can pass from  $X$  to  $Y$  via  $\mathbf{P}(Y)\mathbf{P}^{-1}(X) : T_X\mathcal{B} \rightarrow T_Y\mathcal{B}$ . For this reason,  $\mathbf{P}$  is called the *material isomorphism*.

Now, suppose to have an elastic material with elastic potential  $W(X, t) = \hat{W}(\mathbf{F}(X, t), X, t)$  depending explicitly on the point  $X$  and time  $t$ . If the body is uniform, then the elastic potential depends on the point  $X$  and time  $t$  only through the (in this case, time-dependent) uniformity field  $\mathbf{P}$ , i.e.,

$$\hat{W}(\mathbf{F}(X, t), X, t) = J_{\mathbf{P}}^{-1}(X, t) \tilde{W}(\mathbf{F}(X, t)\mathbf{P}(X, t)), \quad (2)$$

where  $\tilde{W}$  is the elastic potential in the archetype, and  $J_{\mathbf{P}}^{-1}$  comes from the theorem of the change of variables (Epstein and Maugin, 1990).

### 3. Material Implant for a Single Fibre

The generic fibre is straight with no undulation in the archetype, and the implant  $\mathbf{P}(X, t)$  rotates the fibre, crimps it and maps it into the tangent space  $T_X\mathcal{B}$  at  $X$ , as shown in Figure 1. Note that using the implant  $\mathbf{P}$  is equivalent to assuming the existence of a non-compatible intermediate configuration, which is mapped onto by the *straightening deformation*  $\mathbf{F}_s$  coming from the multiplicative decomposition  $\mathbf{F} = \mathbf{F}_c\mathbf{F}_s$  (Hamedzadeh et al., 2018).

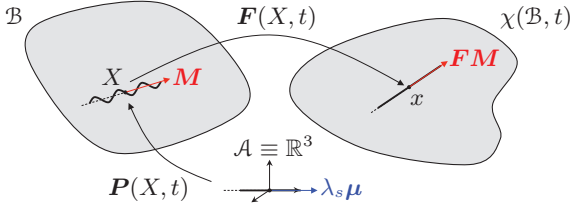


Figure 1: Collagen fibre recruitment seen in terms of the theory of uniformity, with the straightened fibre in the archetype.

The archetypal straightened fibre is represented by the vector  $\lambda_s\boldsymbol{\mu}$ , where  $\boldsymbol{\mu}$  is a unit vector and  $\lambda_s$  is the straightening stretch needed to map a fibre from its referential crimped state back to the archetypal straight state. The uniformity field  $\mathbf{P}$  maps the archetypal vector  $\lambda_s\boldsymbol{\mu}$  into the unit referential vector  $\mathbf{M}$ . Application of the polar decomposition theorem to  $\mathbf{P}$  yields

$$\mathbf{P} = \mathbf{R}\mathbf{U} = \mathbf{R}\hat{\mathbf{U}}(\lambda_s), \quad P^A{}_\beta = R^A{}_\alpha U^\alpha{}_\beta, \quad (3)$$

where  $\mathbf{R}$  rotates and shifts the fibre vector  $\boldsymbol{\mu} \in \mathcal{A}$  from the archetype to the referential vector  $\mathbf{M} \in T_X\mathcal{B}$ , and  $\mathbf{U} = \hat{\mathbf{U}}(\lambda_s)$  is the crimping experienced by the fibre when passing from the straight archetypal configuration to the undulated referential one. In order to find the expressions of  $\mathbf{R}$  and  $\mathbf{U}$ , we need some geometrical preliminaries.

Let  $\mathbf{g}$  be a metric in the archetype  $\mathcal{A}$  and  $\{\mathbf{a}_\alpha\}_{\alpha=1}^3$  a  $\mathbf{g}$ -orthonormal basis of  $\mathcal{A}$ . Since the body  $\mathcal{B}$  is a *trivial* manifold embedded in the affine space  $\mathcal{S} \equiv \mathbb{E}^3$ , we can afford the luxury of choosing Cartesian coordinates  $\{Z^\alpha\}$ , such that the associated basis  $\{\mathbf{I}_\alpha\}_{\alpha=1}^3$  coincides with the archetypal basis  $\{\mathbf{a}_\alpha\}_{\alpha=1}^3$  at every tangent space  $T_X\mathcal{B}$ . We also choose a system of curvilinear coordinates  $\{X^A\}$  in the body  $\mathcal{B}$ , with associated basis  $\{\mathbf{E}_A\}_{A=1}^3$ . The change of basis and the transformation rule for vectors are

$$\mathbf{E}_A = \frac{\partial Z^\alpha}{\partial X^A} \mathbf{I}_\alpha, \quad W^A = \frac{\partial X^A}{\partial Z^\alpha} W^\alpha. \quad (4)$$

Consider the vector  $\tilde{\mathbf{M}} \in \mathcal{A}$  such that its components are equal to the Cartesian components of  $\mathbf{M} \in T_X\mathcal{B}$ , i.e.,  $\tilde{M}^\alpha = M^\alpha$ . The orthogonal tensor  $\mathbf{R}$  is obtained as

$$R^A{}_\beta = \frac{\partial X^A}{\partial Z^\alpha} Q^\alpha{}_\beta, \quad (5)$$

where  $Q^\alpha{}_\beta$  are the components of the archetypal tensor  $\mathbf{Q}$  rotating the archetypal direction  $\boldsymbol{\mu}$  into  $\tilde{\mathbf{M}}$ . The corresponding matrix  $[\mathbf{Q}]$  is found as a function of the unit vector  $\boldsymbol{\omega} = \boldsymbol{\mu} \times \tilde{\mathbf{M}} / \|\boldsymbol{\mu} \times \tilde{\mathbf{M}}\|$ , which describes the axis of rotation, and the amplitude  $\theta = \arccos(\boldsymbol{\mu} \cdot \tilde{\mathbf{M}})$  of the rotation. Then, the rotation matrix  $[\mathbf{Q}]$  can be obtained by exponentiating the skew-symmetric matrix  $[\boldsymbol{\Omega}]$  associated with the vector  $\boldsymbol{\omega}$ , i.e.,

$$[\mathbf{Q}] = e^{[\boldsymbol{\Omega}]\theta}, \quad \Omega^\alpha{}_\gamma = \epsilon^\alpha{}_{\beta\gamma} \omega^\beta, \quad (6)$$

which can be conveniently expressed by Rodriguez' formula (Koks, 2006) as

$$Q^\alpha{}_\gamma = \delta^\alpha{}_\gamma + (\sin \theta) \Omega^\alpha{}_\gamma + (1 - \cos \theta) \Omega^\alpha{}_\beta \Omega^\beta{}_\gamma. \quad (7)$$

The components of the pure stretch  $\mathbf{U}$  are given by

$$U^\alpha{}_\beta = (\lambda_s^{-1} - 1) \mu^\alpha \mu_\beta + \delta^\alpha{}_\beta, \quad (8)$$

where  $\mu_\beta = \mathbf{g}_{\beta\gamma} \mu^\gamma$  are the components of the covector  $\boldsymbol{\mu}^b$  associated with  $\boldsymbol{\mu}$  via the archetypal metric  $\mathbf{g}$ . Finally, the material implant  $\mathbf{P}$  is given by

$$P^A{}_\gamma = \frac{\partial X^A}{\partial Z^\alpha} Q^\alpha{}_\beta [(\lambda_s^{-1} - 1) \mu^\beta \mu_\gamma + \delta^\beta{}_\gamma], \quad (9)$$

which can be simplified into

$$P^A{}_\gamma = (\lambda_s^{-1} - 1) M^A \mu_\gamma + \frac{\partial X^A}{\partial Z^\alpha} Q^\alpha{}_\gamma, \quad (10)$$

since  $Q^\alpha{}_\beta \mu^\beta = \tilde{M}^\alpha$  and  $(\partial X^A / \partial Z^\alpha) \tilde{M}^\alpha = M^A$ . For an *isochoric* implant  $\mathbf{P}$  (i.e., *pure* remodelling, *no* growth, see Epstein and Elzanowski, 2007), the stretch  $\mathbf{U}$  must be changed into

$$U^\alpha{}_\beta = (\lambda_s^{-1} - \lambda_s^{1/2}) \mu^\alpha \mu_\beta + \lambda_s^{1/2} \delta^\alpha{}_\beta, \quad (11)$$

so that we have

$$P^A{}_\gamma = (\lambda_s^{-1} - \lambda_s^{1/2}) M^A \mu_\gamma + \lambda_s^{1/2} \frac{\partial X^A}{\partial Z^\alpha} Q^\alpha{}_\gamma. \quad (12)$$

#### 129 4. Material Implant for a Distribution of Fibres

130 We assume that the fibres in our biological tissue have  
 131 a statistical distribution of orientation. Thus, rather than  
 132 implanting fibres individually, we can implant a whole fam-  
 133 ily of statistically oriented fibres into a material point  $X$ .  
 134 We also assume that the elastic potential  $\hat{W}_f$  of the fi-  
 135 bres is the sum of an isotropic part  $\hat{W}_{\hat{f}_i}$  and an anisotropic  
 136 part  $\hat{W}_{\hat{f}_a}$ . With an abuse of notation, we do not indicate  
 137 the arguments  $(X, t)$  of the tensor fields, and write the  
 138 anisotropic *ensemble* elastic potential of the fibres (Fed-  
 139 erico and Herzog, 2008) as

$$\hat{W}_e(\mathbf{C}, X, t) = \int_{\mathbb{S}_X^2, \mathcal{B}} \hat{W}_{\hat{f}_a}(\hat{I}_4, X, t) \Psi(\mathbf{M}; X, t), \quad (13)$$

140 where  $\hat{I}_4 = \mathbf{C} : (\mathbf{M} \otimes \mathbf{M})$  is the fourth invariant of the  
 141 right Cauchy-Green deformation  $\mathbf{C}$  along the vector  $\mathbf{M}$ ,  
 142 and the probability distribution  $\Psi$  depends explicitly on  $X$   
 143 and  $t$ . Following the definition (2) of material uniformity,  
 144 the fibre elastic potential  $\hat{W}_{\hat{f}_a}$  is related to its archetypical  
 145 counterpart by

$$\hat{W}_{\hat{f}_a}(\hat{I}_4, X, t) = J_{\mathbf{P}}^{-1} \check{W}_{\hat{f}_a}(\check{I}_4), \quad (14)$$

146 where  $\check{I}_4 = \mathbf{P}^T \mathbf{C} \mathbf{P} : \boldsymbol{\mu} \otimes \boldsymbol{\mu}$  is the fourth invariant of  
 147  $\mathbf{P}^T \mathbf{C} \mathbf{P}$  along the vector of  $\boldsymbol{\mu}$ . Thus, Eq. (13) becomes

$$\hat{W}_e(\mathbf{C}, X, t) = J_{\mathbf{P}}^{-1} \int_{\mathbb{S}^2} \check{W}_f(\check{I}_4) \check{\Psi}(\boldsymbol{\mu}), \quad (15)$$

148 where  $\mathbb{S}^2$  denotes the archetypical unit sphere and  $\check{\Psi}$  is the  
 149 archetypal probability distribution.

#### 150 5. Dissipation Inequality and Evolution Law

151 An evolution equation is required as an additional dif-  
 152 ferential equation providing the inhomogeneity rate  $\mathbf{L}_{\mathbf{P}} =$   
 153  $\dot{\mathbf{P}}\mathbf{P}^{-1}$  as a function of all quantities that can act as driving  
 154 forces of the evolution process, i.e.,

$$\mathbf{L}_{\mathbf{P}}(X, t) = \hat{\mathcal{F}}(\mathbf{P}(X, t), \boldsymbol{\mathfrak{A}}(X, t), X), \quad (16)$$

155 where  $\boldsymbol{\mathfrak{A}}$  represents all possible driving force arguments,  
 156 such as Eshelby stress,  $\boldsymbol{\mathfrak{E}} = W \mathbf{I}^T - \mathbf{F}^T \mathbf{T}$ , or Mandel  
 157 stress,  $\boldsymbol{\mathfrak{M}} = \mathbf{F}^T \mathbf{T}$ ,  $\mathbf{T}$  being the first Piola-Kirchhoff stress.  
 158 Note that, here,  $\hat{\mathcal{F}}$  does not depend on time explicitly, i.e.,  
 159 it is *autonomous* with respect to time.

160 As shown by Epstein and Maugin (2000) and Epstein  
 161 and Elzanowski (2007), and mentioned in the Introduc-  
 162 tion, there are some restrictions that are essential for an  
 163 appropriate choice of evolution law. First, the evolution  
 164 law should be invariant with respect to a change of ref-  
 165 erence configuration. Such an evolution law is said to be  
 166 *reduced to the archetypal* and reads

$$\mathbf{L}_{\mathbf{P}} = \dot{\mathbf{P}}\mathbf{P}^{-1} = \check{\mathcal{F}}(J_{\mathbf{P}} \mathbf{P}^T \boldsymbol{\mathfrak{A}} \mathbf{P}^{-T}). \quad (17)$$

Second, the evolution law should satisfy the dissipation in- 167  
 equality, i.e., within a purely mechanical framework and 168  
 for a hyperelastic material, for which the first Piola-Kirchhoff 169  
 stress tensor  $\mathbf{T}$  is given by  $\mathbf{T} = (\partial \hat{W} / \partial \mathbf{F})(\mathbf{F})$ , the dissipa- 170  
 tion  $\mathfrak{D}$  per unit reference volume satisfies (Epstein and 171  
 Elzanowski, 2007) 172

$$\mathfrak{D} = -\dot{W} + \mathbf{T} : \dot{\mathbf{F}} = -\boldsymbol{\mathfrak{M}} : \mathbf{L}_{\mathbf{P}} \geq 0. \quad (18)$$

The same result has been found with the BKL decomposi- 173  
 tion in several works on inelastic processes (see e.g., Simo 174  
 and Hughes, 1986; Simo, 1988; Cleja-Tigoiu and Maugin, 175  
 2000; Imatani and Maugin, 2002; Grillo et al., 2018; Di Ste- 176  
 fano et al., 2018; Crevacore et al., 2018). Here, we assume 177  
 a rate-dependent type of remodelling and reformulate 178  
 $\mathfrak{D} = \check{\mathfrak{D}}(\mathbf{C}, \mathbf{P}, \mathbf{L}_{\mathbf{P}})$  as a quadratic function of  $\boldsymbol{\mathfrak{M}}$  via 179  
 a Legendre transformation on  $\mathbf{L}_{\mathbf{P}}$  and enforcing the Prin- 180  
 ciple of Maximum Dissipation (Hackl and Fischer, 2008). 181  
 Setting  $\mathfrak{D} = \check{\mathfrak{D}}(\mathbf{C}, \mathbf{P}, \boldsymbol{\mathfrak{M}}) = -\boldsymbol{\mathfrak{M}} : \check{\mathbb{K}}(\mathbf{F}, \mathbf{P}) : \boldsymbol{\mathfrak{M}}$ , we have 182

$$\mathbf{L}_{\mathbf{P}} = -\frac{1}{2} \frac{\partial \check{\mathfrak{D}}}{\partial \boldsymbol{\mathfrak{M}}} = -\check{\mathbb{K}}(\mathbf{F}, \mathbf{P}) : \boldsymbol{\mathfrak{M}}, \quad (19)$$

where  $\check{\mathbb{K}}(\mathbf{F}, \mathbf{P})$  is a fourth-order tensor with major symme- 183  
 try only. For the purpose of this work, we define  $\check{\mathbb{K}}(\mathbf{F}, \mathbf{P})$  184  
 as  $\check{\mathbb{K}}(\mathbf{F}, \mathbf{P}) = k \mathbf{b}_{\mathbf{P}} \otimes \mathbf{c}_{\mathbf{P}}$  (with components  $k (\mathbf{b}_{\mathbf{P}})^{AC} (\mathbf{c}_{\mathbf{P}})^{BD}$ ) 185  
 the “tensor-down” product  $\otimes$  is defined in Curnier et al., 186  
 1995), with  $k$  being a positive constant, and  $\mathbf{b}_{\mathbf{P}} = \mathbf{P} \mathbf{g}^{-1} \mathbf{P}^T$  187  
 and  $\mathbf{c}_{\mathbf{P}} = \mathbf{b}_{\mathbf{P}}^{-1}$  being the “left Cauchy-Green tensor” and 188  
 the “Finger tensor” associated with  $\mathbf{P}$ , respectively. More- 189  
 over, in order to enforce a deviatoric  $\mathbf{L}_{\mathbf{P}}$  (no growth), we 190  
 make it function of the deviatoric Mandel stress  $\boldsymbol{\mathfrak{M}}_d =$  191  
 $\boldsymbol{\mathfrak{M}} - \frac{1}{3}(\mathbf{I} : \boldsymbol{\mathfrak{M}})\mathbf{I}^T$ , i.e., 192

$$\mathbf{L}_{\mathbf{P}} = -k \mathbf{b}_{\mathbf{P}} \boldsymbol{\mathfrak{M}}_d \mathbf{c}_{\mathbf{P}}, \quad (20)$$

which can be shown to respect condition (17). 193

#### 194 6. Example: Application to the Arterial Wall

Here, we apply our recruitment-reorientation remodel- 195  
 ling framework to the benchmark problem reported by 196  
 Olsson and Klarbring (2008) and Grillo et al. (2015), with 197  
 a cylinder reinforced by two families of fibres (mimicking 198  
 the arterial wall) under plane strain in the plane orthogo- 199  
 nal to the direction  $X^3 \equiv Z$  of the axis of the cylinder. 200

**Fibre Implant.** At each material point, we implant 201  
 an archetypal distribution with dominant direction  $\boldsymbol{\mu}_0 =$  202  
 $0 \mathbf{a}_1 + 0 \mathbf{a}_2 + 1 \mathbf{a}_3$  into two families of fibres with equal and 203  
 opposite angles,  $\gamma$  and  $-\gamma$ , measured from the  $Z$ -direction 204  
 in the  $\Theta$ - $Z$ -plane and corresponding to the material direc- 205  
 tions  $\mathbf{M}_{0+}$  and  $\mathbf{M}_{0-}$ , as shown in Figure 2. This amounts 206  
 to defining an implant tensor  $\mathbf{P}$  and then adapting its ex- 207  
 pression to the two angles  $\gamma$  and  $-\gamma$ , which gives the im- 208  
 plants  $\mathbf{P}_+$  and  $\mathbf{P}_-$ , respectively. The polar decomposition 209

210  $\mathbf{P} = \mathbf{R}\mathbf{U}$  of the implant (Equation (3)) yields

$$\llbracket \mathbf{U} \rrbracket = \begin{bmatrix} \sqrt{\lambda_s} & 0 & 0 \\ 0 & \sqrt{\lambda_s} & 0 \\ 0 & 0 & \lambda_s^{-1} \end{bmatrix}, \llbracket \mathbf{R} \rrbracket = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \gamma & \sin \gamma \\ 0 & -\sin \gamma & \cos \gamma \end{bmatrix}. \quad (21)$$

211 **Fibre Orientation Probability.** In each family, the  
212 fibre orientation follows a *bivariate* von Mises distribution  
213 (Holzapfel et al., 2015; Gizzi et al., 2018), in which we set  
214 the constants so to normalise it to one, i.e.,

$$\check{\Psi}(\beta, \alpha) = \sqrt{\frac{2b}{\pi}} \frac{\exp(a \cos 2\alpha) \exp(b(1 + \cos 2\beta))}{2\pi I_0(a) \operatorname{erfi}(\sqrt{2b})}, \quad (22)$$

215 where  $\alpha$  and  $\beta$  are the archetypal longitude and co-  
216 latitude angle,  $\operatorname{erfi}$  is the *imaginary error function* and  $I_0$   
217 is the *Bessel function* of zero kind (see Abramowitz and  
218 Stegun, 1964). In this study, we used the values  $a = -1$   
219 and  $b = 5$  of the concentration parameters, to obtain fibres  
220 mostly laying in the  $\Theta$ - $Z$ -plane, as illustrated in Figure 2.

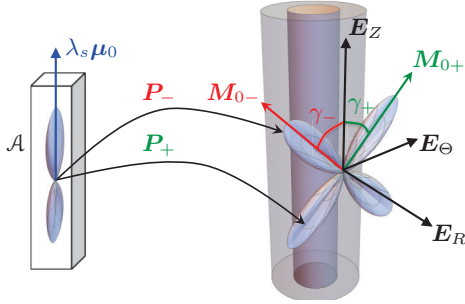


Figure 2: Tensors  $\mathbf{P}_+$  and  $\mathbf{P}_-$ , with identical expressions except for the angles  $\gamma$  and  $-\gamma$ , implant the two fibre families, described by  $\mathbf{M}_{0+}$  and  $\mathbf{M}_{0-}$ , from the archetypal straight state, described by  $\lambda_s \boldsymbol{\mu}_0$ .

221 **Deformation.** We cover the body manifold with a polar  
222 chart, denoted by  $(R, \Theta, Z)$ , in which,  $R \in [R_i, R_o]$ ,  $\Theta \in$   
223  $[0, 2\pi]$ ,  $Z \in [0, L]$ . Here,  $R_i$  and  $R_o$ , are the inner and outer  
224 radii respectively,  $\Theta$  is the referential polar angle and  $L$  is  
225 the length of the cylinder. The current configuration is  
226 obtained under the assumption of pure inflation as:

$$(R, \Theta, Z) \mapsto (r, \theta, z) = (\chi^r(R, t), \Theta, Z). \quad (23)$$

227 For convenience, from this point forward, we write  $\xi \equiv \chi^r$ .  
228 Since  $\xi$  is a function solely of the radial coordinate  $R$  and  
229 time, we denote  $\xi' \equiv \partial \chi^r / \partial R$ . The orthonormal bases for  
230 the tangent spaces of the referential and the current config-  
231 urations are denoted by  $\{\mathbf{E}_R, \mathbf{E}_\Theta, \mathbf{E}_Z\}$  and  $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ ,  
232 respectively. Thus, the deformation gradient  $\mathbf{F}$  reads

$$\mathbf{F}(R, t) = \xi'(R, t) \mathbf{e}_r \otimes \mathbf{E}^R + \frac{\xi(R, t)}{R} \mathbf{e}_\theta \otimes \mathbf{E}^\Theta + \mathbf{e}_z \otimes \mathbf{E}^Z. \quad (24)$$

233 Imposing incompressibility, i.e.,  $J = \det \mathbf{F} = 1$ , we have

$$\xi'(R, t) \xi(R, t) = R. \quad (25)$$

Note that the condition  $J = 1$ , together with the restric-  
234 tion  $J_{\mathbf{P}} = 1$ , amounts to require that also the tensor  $\mathbf{F}\mathbf{P}$   
235 has unitary determinant.  
236

The separable differential equation (25) has solution  
237

$$\xi(R, t) = \sqrt{R^2 + v(t)}, \quad (26)$$

in which the function  $v$  is *independent* of  $R$  and has to be  
238 determined from the boundary conditions. Note that, in  
239 order for  $\xi(R, t)$  to be well defined,  $v(t)$  must be bounded  
240 from below, i.e., it must hold  $v(t) \geq -R_i^2$ , for all  $t$ . Also,  
241 we have  
242

$$\xi'(R, t) = \frac{R}{\sqrt{R^2 + v(t)}} = \frac{R}{\xi(R, t)}, \quad (27)$$

so that the matrix representation of  $\mathbf{F}$  is  
243

$$\llbracket \mathbf{F}(R, t) \rrbracket = \begin{bmatrix} \frac{R}{\xi(R, t)} & 0 & 0 \\ 0 & \frac{\xi(R, t)}{R} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (28)$$

**Constitutive Equations.** Following the premises in  
244 Section 4, the artery is modelled as hyperelastic with an  
245 isotropic matrix contribution  $\hat{W}_m$ , an isotropic fibre con-  
246 tribution  $\hat{W}_{fi}$  and an anisotropic fibre contribution  $\hat{W}_{e\pm}$ ,  
247 integral of the anisotropic fibre contribution  $\hat{W}_{fa\pm}$ , based  
248 on the ensemble potential  $\hat{W}_e$  introduced in (13). Thus,  
249

$$\hat{W}(\mathbf{C}, X) = (1 - \Phi_f) \hat{W}_m(\mathbf{C}) + \Phi_f (\hat{W}_{fi}(\mathbf{C}) + \hat{W}_{e+}(\mathbf{C}, X) + \hat{W}_{e-}(\mathbf{C}, X)), \quad (29)$$

where  $\Phi_f$  is the fibre volumetric fraction, assumed homo-  
250 geneous through the sample, and  
251

$$\hat{W}_m(\mathbf{C}) = \frac{1}{2} k_m [\hat{I}_1 - 3], \quad (30a)$$

$$\hat{W}_{fi}(\mathbf{C}) = \frac{1}{2} k_{fi} [\hat{I}_1 - 3], \quad (30b)$$

$$\hat{W}_{fa\pm}(\mathbf{C}, X) = \frac{1}{4} k_{fa} \mathcal{H}(\hat{I}_{4\pm}(X) - 1) [\hat{I}_{4\pm}(X) - 1]^2, \quad (30c)$$

where  $\hat{I}_1 = \operatorname{tr}(\mathbf{C})$  and the step function  $\mathcal{H}$  is needed to  
252 “switch-off” fibres with stretch smaller than one. The sec-  
253 ond Piola-Kirchhoff stress is obtained as  $\mathbf{S} = 2 \partial \hat{W} / \partial \mathbf{C}$   
254 and, in particular, the anisotropic ensemble contribution  
255 is given by  
256

$$\mathbf{S}_{e\pm} = J_{\mathbf{P}}^{-1} \int_{\mathbb{S}^2} 2 \frac{\partial \hat{W}_{fa\pm}}{\partial \hat{I}_{4\pm}} \frac{\partial \hat{I}_{4\pm}}{\partial \mathbf{C}} \check{\Psi}(\boldsymbol{\mu}), \quad (31)$$

where we used (14) to transform  $\hat{W}_{fa\pm}$  into  $\check{W}_{fa\pm}$  and  
257

$$\frac{\partial \hat{I}_{4\pm}}{\partial \mathbf{C}} = \frac{\partial (\mathbf{P}_\pm^T \mathbf{C} \mathbf{P}_\pm : \boldsymbol{\mu} \otimes \boldsymbol{\mu})}{\partial \mathbf{C}} = \mathbf{P}_\pm^T \otimes \mathbf{P}_\pm^T : \boldsymbol{\mu} \otimes \boldsymbol{\mu}, \quad (32)$$

with components  $(\mathbf{P}_\pm^A)_\alpha (\mathbf{P}_\pm^B)_\beta \mu^\alpha \mu^\beta$  (see Curnier et al.,  
258 1995, for the definition of the “tensor-down” product  $\otimes$ ).  
259

In order to enforce the incompressibility constraint, we  
260 employ the pulled-back deviatoric part (see Federico, 2012)  
261 of the second Piola-Kirchhoff stress,  
262

$$\mathbf{S}_d \equiv \operatorname{Dev}^* \mathbf{S} = \mathbf{S} - \frac{1}{3} (\mathbf{C} : \mathbf{S}) \mathbf{C}^{-1}. \quad (33)$$

263 We emphasise that, since we consider that the elastic po-  
 264 tential of the matrix does not evolve and we have two  
 265 families of fibres with different implants, we only consider  
 266 the fibre part of the deviatoric Mandel stress as the driving  
 267 force of evolution, i.e.,

$$\mathfrak{M}_{ed\pm} = \text{Dev}(\mathbf{C}\mathbf{S}_{e\pm}) = \mathbf{C}\mathbf{S}_{e\pm} - \frac{1}{3}(\mathbf{I} : \mathbf{C}\mathbf{S}_{e\pm})\mathbf{I}^T. \quad (34)$$

### 268 Equilibrium, Boundary Conditions, Integration.

269 The cylinder is under uniform pressure  $\varphi$  on the inner  
 270 boundary  $\partial\mathcal{B}_i$  and zero traction on the outer bound-  
 271 ary  $\partial\mathcal{B}_o$ , and body force and inertial effects are neglected.  
 272 Thus, the evolution of the tissue is governed by the equa-  
 273 tion for  $\mathbf{P}$ , given in (17) and equipped with appropriate  
 274 initial conditions, and by the boundary value problem

$$\text{Div } \mathbf{T} = \mathbf{0}, \quad \text{in } \mathcal{B}. \quad (35a)$$

$$\mathbf{T}\mathbf{N} = -J\varphi\mathbf{F}^{-T}, \quad \text{on } \partial\mathcal{B}_i, \quad (35b)$$

$$\mathbf{T}\mathbf{N} = \mathbf{0}, \quad \text{on } \partial\mathcal{B}_o, \quad (35c)$$

275 where  $\mathbf{N}$  is the normal covector to the boundary  $\partial\mathcal{B}$ , and  
 276 the hypothesis of isochoric deformation implies  $J = 1$ .

277 Since we consider an axisymmetric problem, the first  
 278 Piola-Kirchhoff stress is independent of  $\Theta$  and  $Z$ . Also,  
 279 the boundary conditions ensure that the matrix associated  
 280 with the first Piola-Kirchhoff stress is diagonal, i.e.,  $[\mathbf{T}] =$   
 281  $\text{diag}[T_r^R, T_\theta^\Theta, T_z^Z]$ . The first Piola-Kirchhoff stress can  
 282 be expressed as the sum of its hydrostatic and deviatoric  
 283 components, and in terms of the deviatoric second Piola-  
 284 Kirchhoff stress, as

$$\mathbf{T} = \mathbf{T}_h + \mathbf{T}_d = -Jp\mathbf{F}^{-T} + \mathbf{g}\mathbf{F}\mathbf{S}_d. \quad (36)$$

285 The hydrostatic pressure  $p$  is found from (35) (see Grillo  
 286 et al., 2015).

287 **Evolution Equation.** The evolution equation for each  
 288 of the  $\dot{\mathbf{P}}_\pm$  is obtained from that of  $\mathbf{L}_{\mathbf{P}_\pm} = \dot{\mathbf{P}}_\pm\mathbf{P}_\pm^{-1}$  by  
 289 right-multiplying Equation (20) written for each  $\mathbf{P}_\pm$ , by  
 290 the corresponding  $\mathbf{P}_\pm$

$$\dot{\mathbf{P}}_\pm = -k J_{\mathbf{P}_\pm} \mathbf{P}_\pm \mathbf{g}^{-1} \mathbf{P}_\pm^T \mathfrak{M}_{ed\pm} \mathbf{P}_\pm^{-T} \mathbf{g}. \quad (37)$$

291 In our example, using (3) and (21), solving Equation (37)  
 292 for the deviatoric Mandel stress  $\mathfrak{M}_{ed\pm}$  of each of the two  
 293 fibre families, and then summing to obtain the overall devi-  
 294 atoric Mandel stress of the fibres  $\mathfrak{M}_{ed}$ , yields

$$\frac{\dot{\lambda}_s}{\lambda_s} = k (\mathfrak{M}_{ed})_R^R, \quad (38a)$$

$$\frac{\frac{1}{2}\lambda_s^2(3\cos(2\gamma)-1)\dot{\lambda}_s - (\lambda_s^6-1)\dot{\gamma}\sin(2\gamma)}{\lambda_s^3} = k (\mathfrak{M}_{ed})_\Theta^\Theta. \quad (38b)$$

295 **Numerical Algorithm.** To study the numerical ex-  
 296 ample discussed in the previous sections, a code is devel-  
 297 oped in *Wolfram Mathematica*. The main focus of the  
 298 numerical algorithm in this study is to have high *accuracy*  
 299 and *precision* as we are studying a model with a simple ge-  
 300 ometry (isochoric inflation of a hollow cylinder). Although  
 301 the geometry is simple, the evolution equation (38) makes

Parameter	Value	Symbol
inner radius	1 mm	$R_i$
outer radius	2 mm	$R_o$
internal pressure	0.02 MPa	$\varphi_i$
initial angle	$\pi/4$	$\gamma_0$
initial $\lambda_s$	1.014	$\lambda_{s0}$
matrix stiffness	0.0375 MPa	$k_m$
fibre isotropic stiffness	0.0375 MPa	$k_{fi}$
fibre anisotropic stiffness	0.0375 MPa	$k_{fa}$
remodelling stiffness	$5 \times 10^{-8}$ s/Pa	$k$
fibre volume fraction	0.2	$\Phi_f$

Table 1: Parameters employed in the numerical analysis.

302 the model computationally heavy. In this numerical study,  
 303 we have two types of integrals: the surface integral over  
 304 the unit sphere  $\mathbb{S}^2$ , which describes the fibre distribution,  
 305 and the integral over the interval bounded by the inner  
 306 and the outer radii  $[R_i, R_o]$ . For the surface integral, we  
 307 use the *Lebedev quadrature* (Lebedev, 1977), in which the  
 308 grid points and the corresponding weights are obtained  
 309 from the exact integration of *spherical harmonics* up to  
 310 an arbitrary order. The model parameters are given in  
 311 Table 1.

## 312 7. Numerical Results

313 Figure 3 represents the evolution of the straightening  
 314 stretch  $\lambda_s$ . The behaviour of  $\lambda_s$  is monotonically decreas-  
 315 ing in the radius  $R$  throughout the evolution. The differ-  
 316 ence  $\lambda_s(R_i, t) - \lambda_s(R_o, t)$  increases monotonically with  
 317 time. We note that the  $\lambda_s(R_o)$  evolves due to the fact that  
 318 the radial deviatoric Mandel stress of the fibres,  $\mathfrak{M}_{ed}$  is not  
 319 zero (Equation (38)), although the total Mandel stress  $\mathfrak{M}$   
 320 vanishes due to the boundary conditions.

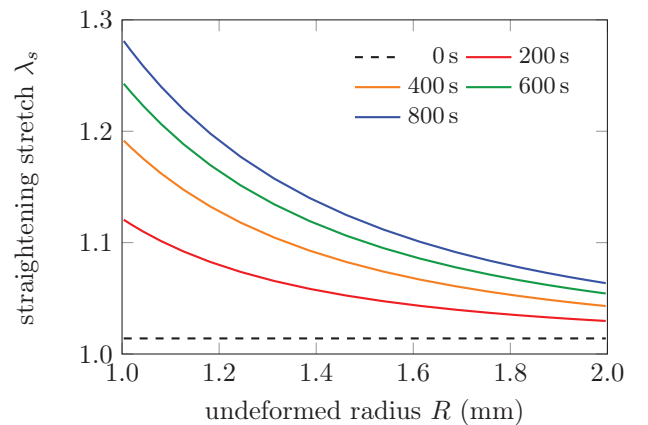


Figure 3: Evolution of the straightening stretch  $\lambda_s$  with time.

321 Figure 4 shows the evolution of the behaviour of the  
 322 angle  $\gamma$  describing the preferred fibres direction with time.  
 323 After remodelling, the maximum and minimum angles occur  
 324 at the inner and outer radii, respectively. The differ-

325 ence  $\gamma(R_i, t) - \gamma(R_o, t)$  is more pronounced in the early  
 326 cycles and then tends to remain constant with time.

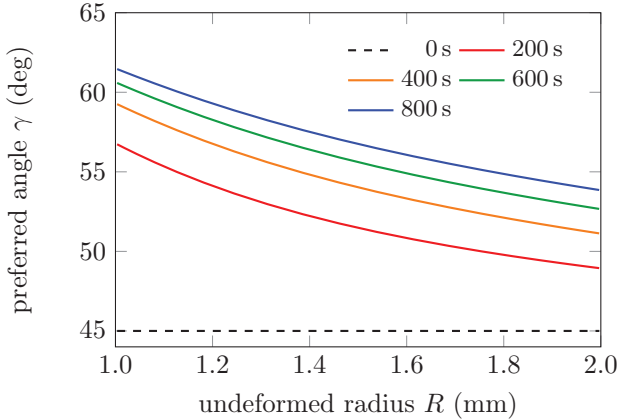


Figure 4: Evolution of the preferred fibre angle  $\gamma$  with time.

327 Figure 5 shows the evolution of the radial first Piola-  
 328 Kirchhoff stress  $T_r^R$  (dashed lines) and circumferential  
 329 first Piola-Kirchhoff stress  $T_\theta^\Theta$  (solid lines) as a function  
 330 of the deformed radius  $r = \xi(R, t)$ . The remodelling makes  
 331 the circumferential stress  $T_\theta^\Theta$  more homogeneous through-  
 332 out the thickness of the tube. The difference  $T_\theta^\Theta(R_i, t) -$   
 333  $T_\theta^\Theta(R_o, t)$  before remodelling is about 23 kPa at  $t = 0$  s  
 334 and it reduces to 16 kPa at  $t = 400$  s and to 14 kPa at  
 335  $t = 800$  s.

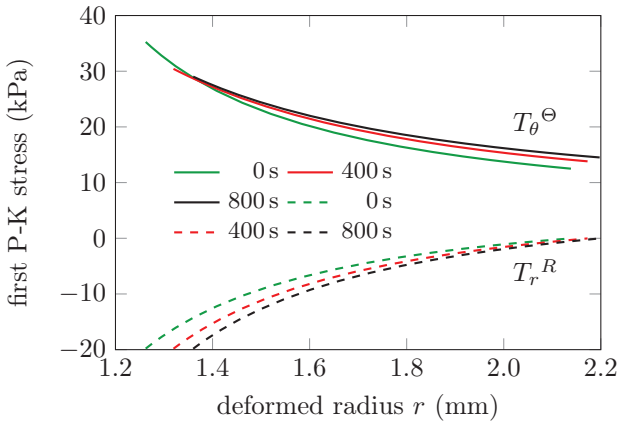


Figure 5: First Piola-Kirchhoff stresses  $T_r^R$  (dashed lines) and  $T_\theta^\Theta$  (solid lines).

336 One of the most prominent mechanical aspects of bi-  
 337 ological tissues is the presence of residual stresses. Fung  
 338 (1983) predicted that the distribution of residual stresses  
 339 in the arteries is such that the residual circumferential  
 340 stress (along  $\Theta$ -axis) is compressive in the interior layers  
 341 and tensile in the outer ones. The residual second Piola-  
 342 Kirchhoff stresses for our benchmark problem is shown in  
 343 Figure 6 as a function of the undeformed radius  $R$ , at time  
 344  $t = 800$  s. All three principal residual stresses increase  
 345 monotonically and the residual circumferential stress  $S^{\Theta\Theta}$ ,

in accordance with Fung (1983), is compressive at the inner  
 346 inner wall and tensile at the outer wall.

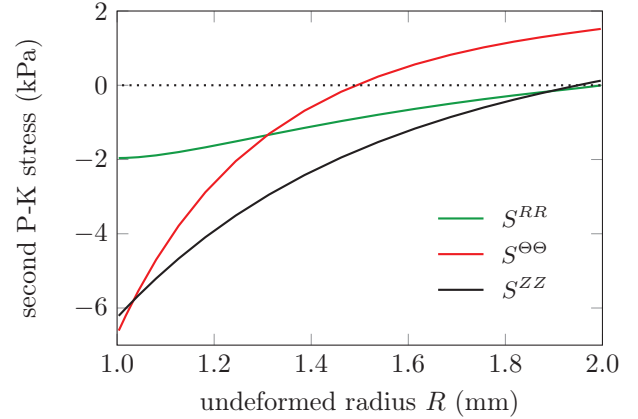


Figure 6: Residual second Piola-Kirchhoff stresses at time  $t = 800$  s.

## 8. Discussion and Conclusions

347 In this work we introduced a thermodynamically ad-  
 348 missible model for pure remodelling of a fibre-reinforced  
 349 material representing the arterial wall tissue. The ap-  
 350 proach is based on the theory of material uniformity, which  
 351 is described by the material implant  $\mathbf{P}$ . We proposed a  
 352 simple evolution law, in which the inhomogeneity rate  $\mathbf{L}_P$   
 353 is linearly related to the deviatoric Mandel stress  $\mathfrak{M}_d$ .

354 Using the evolution law (38), we solved a benchmark  
 355 numerical problem describing a pressurised thick-walled  
 356 cylinder under plane strain conditions, with uniform in-  
 357 ternal pressure, as in the works by Olsson and Klarbring  
 358 (2008) and Grillo et al. (2015). We use the same consti-  
 359 tutive laws as in the work by Grillo et al. (2015) but a  
 360 more realistic fibre orientation probability, with two fami-  
 361 lies of fibres each obeying a bivariate von Mises distribu-  
 362 tion (Holzapfel et al., 2015; Gizzi et al., 2018) (Figure 2).

363 The results for the remodelling angle are qualitatively  
 364 similar to those obtained by Grillo et al. (2015). Both  
 365 models predict that the preferred angle  $\gamma$  increases with  
 366 time, with values at the inner radius  $R_i$  being the largest.  
 367 Moreover, the dependence on radius and time of the ra-  
 368 dial and circumferential stresses  $T_r^R$  and  $T_\theta^\Theta$  in our model  
 369 (Figure 5) is similar to that in the paper by Grillo et al.  
 370 (2015). However, while in Grillo et al. (2015) the cylin-  
 371 der deflates as it becomes stiffer *circumferentially*, in our  
 372 study the cylinder inflates. This is not surprising, as we  
 373 have *two* evolving mechanisms that work simultaneously,  
 374 namely the relaxation of the fibres (increasing straighten-  
 375 ing stretch  $\lambda_s$ ) and the change in fibre angle (increasing  
 376 preferred angle  $\gamma$ ). Indeed, when  $\lambda_s$  increases, it causes a  
 377 *relaxation* of the fibres, and the cylinder needs to inflate  
 378 so that the fibres reach their straightening stretch and are  
 379 able to bear load.

380 Other studies considered a change of undulation of the  
 381 fibres or fibrils and our model is in agreement with these  
 382

findings, despite being fundamentally different in the basic assumptions. Indeed, Humphrey (1999) considers resorption and deposition of new fibres and Watton and Hill (2009) and Watton et al. (2009) consider pre-stretch in  $Z$ -direction. The relaxation effect that our model predicts has been observed by Kamiya and Togawa (1980). In addition, the residual stress is compressive in the inner layer and tensile in the outer layer, in agreement with the behaviour described by Fung (1983).

It is noteworthy that, in our model, we did not prescribe the evolution law in accordance to experimental observations. Rather, we postulated an evolution law solely based on the conditions of reduction to the archetype (17) and of compliance with the dissipation inequality (18). In spite of its relatively simple form, the evolution law could qualitatively reproduce the remodelling behaviour seen in other studies. This indicates that the framework based on the theory of evolution and material uniformity can be a viable and promising paradigm to explore growth and remodelling of biological tissues.

This work followed Epstein and Maugin (2000) and Epstein and Elzanowski (2007), who used the theory of uniformity with a time-dependent implant  $\mathbf{P}$ , which constitutes an *internal variable*. In contrast, Grillo et al. (2015) treated the fibre mean angle as a kinematic variable that satisfies a balance of generalised forces, following the same philosophy used by Di Carlo and Quiligotti (2002). Although different in nature, these two approaches give qualitatively similar results.

The proposed model constitutes a step further in the study of growth and remodelling of fibre-reinforced soft biological tissues, in the framework of material implant theory. Even though the numerical example lacks the necessary details to study specific cases such as *hypertension* and *aneurysms*, the agreements of the results with previous studies make this framework promising.

## Acknowledgements

This work was supported in part by the Natural Sciences and Engineering Research Council of Canada, through the NSERC Discovery Programme [SF,ME], and the Dipartimento di Eccellenza 2018-2022, Politecnico di Torino (Italy), Project no. E11G18000350001 [AG].

## Competing Interests

The authors declare no competing interests.

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