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Effective properties of hierarchical fiber-reinforced composites via a three-scale asymptotic homogenization approach

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Abstract

The study of the properties of multiscale composites is of great interest in engineering and biology. Particularly, hierarchical composite structures can be found in nature and in engineering. During the past decades, the multiscale asymptotic homogenization technique has shown its potential in the description of such composites by taking advantage of their characteristics at the smaller scales, ciphered in the so-called *effective coefficients*. Here, we extend previous works by studying the in-plane and out-of-plane effective properties of hierarchical linear elastic solid composites via a three-scale asymptotic homogenization technique. In particular, the approach is adjusted for a multiscale composite with a square-symmetric arrangement of uniaxially aligned cylindrical fibers, and the formulae for computing its effective properties are provided. Finally, we show the potential of the proposed asymptotic homogenization procedure by modeling the effective properties of musculoskeletal mineralized tissues, and we compare the results with theoretical and experimental data for bone and tendon tissues.

Keywords Hierarchical composites, Three-scale asymptotic homogenization, Fiber-reinforced composites, Musculoskeletal mineralized tissues, Effective coefficients

1 Introduction

Hierarchical solids are multiscale materials made of different phases which themselves exhibit a finer scale structure. Several examples of the existence of hierarchical composite structures can be found in nature such as musculoskeletal mineralized tissues (MMTs), lotus leaves, among many others. Nowadays, the study of the physical properties of multiscale composite materials is of great interest due to its utility, for instance, in the modeling and design of bioinspired and biomimetic hierarchical materials [5, 28, 64]. In particular, MMTs constitute a widely studied class of hierarchical composite materials. For instance, we refer to the compilation of articles edited by Cowin [13] on structural and mechanical properties of bone.

The different homogenization techniques used in the modeling of multiscale composites have the important advantage of decoupling the structural characteristic lengths. In the case of linear elastic composite materials, the scientific literature develops in two main approaches, the asymptotic homogenization and the average field theory (see, e.g., the review paper [26] and references therein). On one hand, average field

28 techniques [22, 33] aim to find the effective elastic properties which relate the fine scale strain and stress
29 averages over a representative volume, characterizing, in an ideal form, the heterogeneity of the material.
30 On the other hand, the asymptotic homogenization technique [6, 7, 10, 55, 3] exploits the scales separation
31 among the characteristic lengths of the local structures and the one of the whole material by employing
32 multiple scale expansions of the fields.

33 The multiscale asymptotic homogenization techniques take advantage of the information available at the
34 smaller scales to obtain an effective description of the medium or phenomenon at its larger scales. In the
35 scientific literature, there exist several works focusing on modeling and simulation of the macroscopic prop-
36 erties of hierarchical composite materials using average field techniques [31, 4, 37, 21], reiterated asymptotic
37 homogenization [7, 30, 2, 14, 58, 29, 53, 16, 61, 35] and hybrid models [41]. For instance, starting from
38 the basic equations of the phases of a composite featuring a heterogeneous structure over several separated
39 scales, [30] achieved to deduce the phenomenological equations of a porous medium and, in the process,
40 the authors also obtained the governing equations for the intermediate scales of the mixture. Afterwards, a
41 rigorous foundation of the technique was given in [2] who focused on the heat equation for composites and
42 in [60], a further generalization of the reiterated homogenization technique was introduced via a three-scale
43 convergence approach providing a groundwork where the asymptotic parameters independently approach
44 zero. Moreover, in [53], the authors adopted an asymptotic homogenization technique to obtain a homoge-
45 nized model for a fluid saturated porous medium containing double porous substructures by considering a
46 hierarchical porous arrangement. In the study conducted by [16], recurrent sequences of local and averaged
47 elasticity problems for a fiber reinforced composite were written through the introduction of a power series
48 expansion for each level. Furthermore, in [61], the authors considered a hierarchical laminated composite
49 with the particularity that the microstructure presented a combination of linear and non-linear generalized
50 periodicity. Therein, the solution of the problem was sought via a multi-step homogenization approach.
51 In addition, a step-by-step approach to study the properties of bone using models of micromechanics and
52 composite laminate theory was followed in [37] and [21]. Finally, the approach proposed by [41] uses a
53 combination of Eshelby based techniques with the asymptotic homogenization to analyze in a bottom-up
54 process the stiffening of old bone tissues. From a computational point of view, the work by [65] proposes
55 a methodology for the development of adaptive methods for hierarchical modeling of elastic heterogeneous
56 bodies.

57 In this work, we exploit the three-scale asymptotic homogenization approach developed in [47, 48] to
58 investigate the effective properties of linear elastic, hierarchical, fiber-reinforced composites. The three-
59 scale homogenization approach permits to individualize each hierarchical level and to investigate how the
60 properties at the lower scales influence the effective ones in a single scheme. In a previous work [47], the
61 three-scale asymptotic technique has been applied to compute the effective shear modulus for hierarchical
62 fiber-reinforced composites. Here, we go further and propose a procedure to compute the effective in-
63 plane elastic coefficients, which involve the solution of coupled elastic problems. Furthermore, we show
64 the potential of the multiscale asymptotic homogenization process by applying it to a biological scenario
65 of interest. Specifically, we are interested in modeling the effective properties of MMTs by performing a
66 parametric analysis of the mineral crystals' volume fraction. Since the goal is to offer a modeling tool for
67 studying hierarchical composites, we conveniently adopt the modeling assumptions made in [59]·[41]. In [59],
68 the authors studied the elastic stiffness tensor of a mineralized turkey leg tendon tissue using a multiscale
69 model based on average fields Eshelby techniques, such as the Mori-Tanaka and the self-consistent schemes.
70 In [41], the approach in [59] was extended to the asymptotic homogenization technique by means of a hybrid
71 hierarchical modeling framework applicable to MMTs, and capable to account for fused mineral structures
72 in the composite tissue. The results of the present framework are consistent with the experimental and
73 theoretical data reported in [59, 41].

74 The manuscript is organized as follows. First, the physical and mathematical framework of the problem
75 is introduced. Next, we present the principal results of the three-scale asymptotic homogenization technique
76 and address the general local problems associated to each hierarchical level. The in-plane and out-of-plane
77 local problems for uniaxially fiber-reinforced hierarchical composites with isotropic constituents are also
78 specified. In addition, the form of the effective coefficients is provided. Furthermore, we compute the

79 effective properties of MMTs and compare the results with experimental and numerical data provided in the
80 scientific literature. Finally, we discuss the current approach and give directions for future developments of
81 the study.

82 2 Formulation of the problem

83 2.1 Geometrical description

84 Let us denote by $\Omega \subset \mathbb{R}^3$ a multiscale composite characterized by three well-separated characteristics lengths
85 (see Fig. 1), namely ℓ_1 , ℓ_2 and L , and introduce the scaling parameters ε_1 and ε_2 as follows,

$$\varepsilon_1 = \frac{\ell_1}{L} \ll 1 \quad \text{and} \quad \varepsilon_2 = \frac{\ell_2}{L} \ll \varepsilon_1. \quad (1)$$

86 We note that in (1), we have amended a typo on the definition of ε_2 in previous works [47, 48]. From relation
87 (1), two formally independent variables are introduced, i.e.

$$\eta = \frac{x}{\varepsilon_1} \quad \text{and} \quad \varsigma = \frac{x}{\varepsilon_2}. \quad (2)$$

88 In what follows, we consider each field and material property Φ^ε to be η - and ς - periodic and we introduce
89 the notation $\Phi^\varepsilon(x) = \Phi(x, \eta, \varsigma)$.

90 At the first hierarchical level, the composite Ω comprises two solid constituents and is partitioned into
91 two sub-domains $\Omega_m^{\varepsilon_1}$ and $\Omega_f^{\varepsilon_1}$. The former denotes the host (or matrix) phase and the latter represents a
92 finite collection of disjoints subphases (e.g. inclusions or fibers). Specifically, $\bar{\Omega} = \bar{\Omega}_m^{\varepsilon_1} \cup \bar{\Omega}_f^{\varepsilon_1}$ with $\bar{\Omega}_m^{\varepsilon_1} \cap \Omega_f^{\varepsilon_1} =$
93 $\Omega_m^{\varepsilon_1} \cap \bar{\Omega}_f^{\varepsilon_1} = \emptyset$ and we denote with Γ^{ε_1} the interface between both constituents $\Omega_m^{\varepsilon_1}$ and $\Omega_f^{\varepsilon_1}$. Furthermore, we
94 denote by \mathcal{Y} the unitary periodic cell containing a portion of the host phase \mathcal{Y}_m and one subphase (or a finite
95 collection of subphases) \mathcal{Y}_f . We enforce that the constituents of each periodic cell satisfy that $\bar{\mathcal{Y}} = \bar{\mathcal{Y}}_m \cup \bar{\mathcal{Y}}_f$
96 with $\bar{\mathcal{Y}}_m \cap \mathcal{Y}_f = \mathcal{Y}_m \cap \bar{\mathcal{Y}}_f = \emptyset$, and we indicate with $\Gamma_{\mathcal{Y}}$ the interface between \mathcal{Y}_m and \mathcal{Y}_f .

97 At the second hierarchical level, we consider that each subphase ${}_i\Omega_f^{\varepsilon_1}$ ($i = 1, \dots, N$) is also a composite
98 material with periodic structure. We suppose that each subphase ${}_i\Omega_f^{\varepsilon_1}$ is composed of a host phase $\Omega_m^{\varepsilon_2}$ with
99 a finite number of subphases denoted by $\Omega_f^{\varepsilon_2}$. In particular, we assume that for each i , ${}_i\bar{\Omega}_f^{\varepsilon_1} = \bar{\Omega}_m^{\varepsilon_2} \cup \bar{\Omega}_f^{\varepsilon_2}$ with
100 $\bar{\Omega}_m^{\varepsilon_2} \cap \Omega_f^{\varepsilon_2} = \Omega_m^{\varepsilon_2} \cap \bar{\Omega}_f^{\varepsilon_2} = \emptyset$ and the interface between $\Omega_m^{\varepsilon_2}$ and $\Omega_f^{\varepsilon_2}$ is denoted with Γ^{ε_2} . At this hierarchical
101 level, \mathcal{Z} stands for the unitary periodic cell containing a portion of the host phase indicated with \mathcal{Z}_m and
102 one subphase (or a finite collection of subphases) \mathcal{Z}_f . Analogously to the upper hierarchical level, we impose
103 that $\bar{\mathcal{Z}} = \bar{\mathcal{Z}}_m \cup \bar{\mathcal{Z}}_f$, with $\bar{\mathcal{Z}}_m \cap \mathcal{Z}_f = \mathcal{Z}_m \cap \bar{\mathcal{Z}}_f = \emptyset$ and we indicate with $\Gamma_{\mathcal{Z}}$ the interface between \mathcal{Z}_m and \mathcal{Z}_f .

104 In Table 1, we resume the symbols used in this work.

Table 1: Description of symbols.

Symbol	Description
Ω	Multiscale composite body
$\Omega_m^{\varepsilon_1}$ ($\Omega_m^{\varepsilon_2}$)	Host (or matrix) phase at the ε_1 (ε_2)-hierarchical level
$\Omega_f^{\varepsilon_1}$ ($\Omega_f^{\varepsilon_2}$)	Finite collection of disjoints subphases at the ε_1 (ε_2)-hierarchical level
Γ^{ε_1} (Γ^{ε_2})	Interface between constituents $\Omega_m^{\varepsilon_1}$ and $\Omega_f^{\varepsilon_1}$ ($\Omega_m^{\varepsilon_2}$ and $\Omega_f^{\varepsilon_2}$)
\mathcal{Y} (\mathcal{Z})	Unitary periodic cell at the ε_1 (ε_2)-hierarchical level
\mathcal{Y}_m (\mathcal{Z}_m)	Portion of $\Omega_m^{\varepsilon_1}$ ($\Omega_m^{\varepsilon_2}$) contained in the unitary cell \mathcal{Y} (\mathcal{Z})
\mathcal{Y}_f (\mathcal{Z}_f)	Finite collection of subphases $\Omega_f^{\varepsilon_1}$ ($\Omega_f^{\varepsilon_2}$) contained in the unitary cell \mathcal{Y} (\mathcal{Z})
$\Gamma_{\mathcal{Y}}$ ($\Gamma_{\mathcal{Z}}$)	Interface between \mathcal{Y}_m and \mathcal{Y}_f (\mathcal{Z}_m and \mathcal{Z}_f)

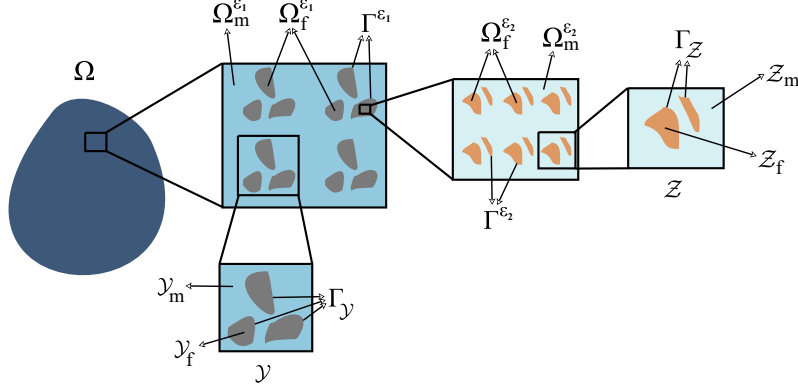


Figure 1: Schematic of the cross-section of a hierarchical periodic composite with three structural levels.

2.2 Formulation of the problem

105

106 We consider that the constitutive response of all the constituents of the hierarchical composite body Ω is
 107 linear elastic. This assumption implies that the constituents' constitutive relationships are all given by the
 108 formula,

$$\boldsymbol{\sigma}^\varepsilon = \mathcal{C}^\varepsilon : \mathbf{E}(\mathbf{u}^\varepsilon), \quad (3)$$

109 where $\mathbf{E}(\mathbf{u}^\varepsilon) := \text{Sym}(\text{Grad}\mathbf{u}^\varepsilon)$ represents the strain tensor under the hypothesis of small displacements \mathbf{u}^ε ,
 110 and \mathcal{C}^ε is the fourth-order, positive definite elasticity tensor with both major and minor symmetries, i.e.,
 111 component-wise, $\mathcal{C}_{ijkl}^\varepsilon = \mathcal{C}_{jikl}^\varepsilon = \mathcal{C}_{ijlk}^\varepsilon = \mathcal{C}_{kl ij}^\varepsilon$ ($i, j, k, l = 1, 2, 3$), which is supposed to be phase-wise smooth.

112 Then, ignoring inertia and volume forces, the differential problem arising from the (local) balance of
 113 linear momentum when equipped, for example, with Dirichlet-Neumann external boundary conditions reads

$$(\mathcal{P}^\varepsilon) \begin{cases} \text{Div}[\mathcal{C}^\varepsilon : \mathbf{E}(\mathbf{u}^\varepsilon)] = \mathbf{0}, & \text{in } \Omega \setminus (\Gamma^{\varepsilon_1} \cup \Gamma^{\varepsilon_2}), \\ \mathbf{u}^\varepsilon = \mathbf{u}^*, & \text{on } \partial\Omega_D, \\ [\mathcal{C}^\varepsilon : \mathbf{E}(\mathbf{u}^\varepsilon)] \cdot \mathbf{N} = \mathbf{S}^*, & \text{on } \partial\Omega_N, \end{cases} \quad (4)$$

114 where \mathbf{N} is the outward unit vector field normal to the boundary $\partial\Omega$ of Ω , \mathbf{u}^* is the displacement field
 115 prescribed on the Dirichlet portion of $\partial\Omega$, i.e. $\partial\Omega_D$, and \mathbf{S}^* is the field of tractions imposed on the Neumann
 116 boundary $\partial\Omega_N$. It holds that $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$, with $\partial\Omega_D \cap \partial\Omega_N = \emptyset$. Furthermore, continuity conditions
 117 for displacements and traction are imposed on both Γ^{ε_1} and Γ^{ε_2} , i.e.

$$\llbracket \mathbf{u}^\varepsilon \rrbracket = \mathbf{0}, \quad \text{on } \Gamma^{\varepsilon_1} \cup \Gamma^{\varepsilon_2}, \quad (5a)$$

$$\llbracket (\mathcal{C}^\varepsilon : \mathbf{E}(\mathbf{u}^\varepsilon)) \cdot \mathbf{N}_y \rrbracket = \mathbf{0}, \quad \text{on } \Gamma^{\varepsilon_1}, \quad (5b)$$

$$\llbracket (\mathcal{C}^\varepsilon : \mathbf{E}(\mathbf{u}^\varepsilon)) \cdot \mathbf{N}_z \rrbracket = \mathbf{0}, \quad \text{on } \Gamma^{\varepsilon_2}, \quad (5c)$$

118 where \mathbf{N}_y and \mathbf{N}_z represent the outward unit vectors normal to the surfaces Γ^{ε_1} and Γ^{ε_2} , respectively. The
 119 operator $\llbracket \Phi^\varepsilon \rrbracket$ denotes the jump of Φ^ε across the interface between two constituents in the same hierarchical
 120 level.

3 Three-scale asymptotic homogenization procedure

121

122 The property of separation of scales together with definition (2), imply that,

$$\text{Grad}\Phi^\varepsilon(x) = \text{Grad}_x\Phi(x, \eta, \varsigma) + \varepsilon_1^{-1}\text{Grad}_\eta\Phi(x, \eta, \varsigma) + \varepsilon_2^{-1}\text{Grad}_\varsigma\Phi(x, \eta, \varsigma), \quad (6)$$

123 where the chain rule has been used, and the sub-indices of the gradient operators on the right-hand-side
 124 indicate that the derivative is performed with respect to x , η , and ς . In addition, the following average
 125 operators over the periodic cells \mathcal{Y} and \mathcal{Z} are introduced,

$$\langle \Phi^\varepsilon(x) \rangle_\eta = \frac{1}{|\mathcal{Y}|} \int_{\mathcal{Y}} \Phi(x, \eta, \varsigma) d\eta, \quad (7a)$$

$$\langle \Phi^\varepsilon(x) \rangle_\varsigma = \frac{1}{|\mathcal{Z}|} \int_{\mathcal{Z}} \Phi(x, \eta, \varsigma) d\varsigma, \quad (7b)$$

126 where $|\mathcal{Y}|$ and $|\mathcal{Z}|$ denote the volume fractions of the periodic cells \mathcal{Y} and \mathcal{Z} , respectively.

127 At this stage, we perform a three-scale asymptotic expansion for the displacement \mathbf{u}^ε in powers of the
 128 scaling parameters ε_1 and ε_2 . Specifically, we impose that

$$\mathbf{u}^\varepsilon(x) = \tilde{\mathbf{u}}^{(0)}(x, \eta, \varsigma) + \sum_{i=1}^{+\infty} \tilde{\mathbf{u}}^{(i)}(x, \eta, \varsigma) \varepsilon_2^i, \quad (8)$$

129 where

$$\tilde{\mathbf{u}}^{(0)}(x, \eta, \varsigma) = \mathbf{u}^{(0)}(x, \eta, \varsigma) + \sum_{j=1}^{+\infty} \mathbf{u}^{(j)}(x, \eta, \varsigma) \varepsilon_1^j. \quad (9)$$

130 Now, we embrace the homogenization process illustrated in [47, 48]. That is, we first substitute the expansion
 131 (8) into the original problem constituted by equations (4) and (5a)-(5c), and then, we equate the resulting
 132 expressions in powers of ε_2 , and subsequently, using (9), in powers of ε_1 .

133 Following this procedure, it can be shown that the term $\mathbf{u}^{(0)}$ is a function of the “slow” variable only,
 134 i.e., $\mathbf{u}^{(0)}(x, \eta, \varsigma) \equiv \mathbf{u}^{(0)}(x)$, and solution of the *homogenized problem*

$$(\mathcal{P}) \begin{cases} \text{Div}_x[\hat{\mathcal{C}} : \mathbf{E}_x(\mathbf{u}^{(0)})] = \mathbf{0}, & \text{in } \Omega_h, \\ \mathbf{u}^{(0)} = \mathbf{u}^*, & \text{on } \partial\Omega_D^h, \\ [\hat{\mathcal{C}} : \mathbf{E}_x(\mathbf{u}^{(0)})] \cdot \mathbf{N} = \mathbf{S}^*, & \text{on } \partial\Omega_N^h, \end{cases} \quad (10)$$

135 where Ω_h represents the homogeneous macro-scale domain in which the homogenized equations are defined.
 136 In (10), $\hat{\mathcal{C}}$ represents the *effective fourth-order elasticity tensor* of the hierarchical composite material, which
 137 is given by the formula

$$\hat{\mathcal{C}} = \langle \mathcal{C}^{\varepsilon_1} + \mathcal{C}^{\varepsilon_1} : \mathbf{T}\mathbf{E}_\eta(\boldsymbol{\omega}) \rangle_\eta, \quad (11)$$

138 where the fourth-order tensor $\mathcal{C}^{\varepsilon_1}$ is given by

$$\mathcal{C}^{\varepsilon_1}(x) = \begin{cases} \mathcal{C}^{m,\eta}(x, \eta), & \eta \in \Omega_m^{\varepsilon_1}, \\ \mathcal{C}^{f,\eta}(x, \eta), & \eta \in \Omega_f^{\varepsilon_1}. \end{cases} \quad (12)$$

139 In (12), $\mathcal{C}^{m,\eta}$ and $\mathcal{C}^{f,\eta}$ represent the elasticity tensors corresponding to the constituents $\Omega_m^{\varepsilon_1}$ and $\Omega_f^{\varepsilon_1}$,
 140 respectively. Furthermore, $\boldsymbol{\omega}$ is a third-order, η -periodic tensor field such that

$$\mathbf{u}^{(1)}(x, \eta, \varsigma) \equiv \mathbf{u}^{(1)}(x, \eta) = \boldsymbol{\omega}(x, \eta) : \mathbf{E}_x[\mathbf{u}^{(0)}(x)], \quad (13)$$

141 with $\mathbf{E}_x[\mathbf{u}^{(0)}(x)] := \text{Sym}[\text{Grad}_x \mathbf{u}^{(0)}(x)]$. Moreover, $\mathbf{T}\mathbf{E}_\beta(\boldsymbol{\omega}) = \frac{1}{2}[\text{TGrad}_\beta \boldsymbol{\omega} + {}^t(\text{TGrad}_\beta \boldsymbol{\omega})]$, with $\beta = x, \eta, \varsigma$
 142 (see [49]). The operation ${}^t(\mathcal{A})$ transposes the fourth-order tensor \mathcal{A} by exchanging the order of its first pair
 143 of indices only, and $\text{TGrad}_\beta \boldsymbol{\omega}$ is the fourth-order tensor defined as

$$\text{TGrad}_\beta \boldsymbol{\omega} = \frac{\partial \omega_{ijkl}}{\partial \beta_j} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l. \quad (14)$$

144 Note that we are not using the covariant formalism in this work, otherwise the partial differentiation on the
 145 right-hand-side of (14) should be substituted with a covariant derivative.

146 Particularly, the third-order tensor field $\boldsymbol{\omega}$ is determined by solving the following auxiliary cell problem

$$(\mathcal{P}_Y) \begin{cases} \text{Div}_\eta[\mathcal{C}^{\varepsilon_1} + \mathcal{C}^{\varepsilon_1} : \mathbf{T}\mathbf{E}_\eta(\boldsymbol{\omega})] = \mathbf{0}, & \text{in } Y \setminus \Gamma_Y, \\ [(\mathcal{C}^{\varepsilon_1} + \mathcal{C}^{\varepsilon_1} : \mathbf{T}\mathbf{E}_\eta(\boldsymbol{\omega})) \cdot \mathbf{N}_Y] = \mathbf{0}, & \text{on } \Gamma_Y, \\ [\boldsymbol{\omega}] = \mathbf{0}, & \text{on } \Gamma_Y, \end{cases} \quad (15)$$

147 where the condition $\langle \boldsymbol{\omega} \rangle_\eta = \mathbf{0}$ is imposed to guarantee uniqueness in the local problem (15). We remark that
 148 the condition of zero average of the third-order tensor $\boldsymbol{\omega}$ is just one particular way, without losing generality,
 149 to close the problem (15).

150 At this point we note that in this formulation (see [47, 48] for more details), the homogenization process
 151 accomplishes to relate the length scales in a cascade mode from the lower to the higher one, so that, the
 152 fourth-order elasticity tensor $\mathcal{C}^{\mathbf{f},\eta}$ in (12), corresponding to the constituent $\Omega_f^{\varepsilon_1}$, is in fact, an effective one,
 153 and is given through the formula

$$\mathcal{C}^{\mathbf{f},\eta} \equiv \check{\mathcal{C}} = \langle \mathcal{C}^{\varepsilon_2} + \mathcal{C}^{\varepsilon_2} : \mathbf{T}\mathbf{E}_\varsigma(\tilde{\boldsymbol{\omega}}) \rangle_\varsigma. \quad (16)$$

154 We denote with $\check{\mathcal{C}}$ the *effective fourth-order elasticity tensor at the ε_1 -hierarchical level* of the composite
 155 material. In particular, for $\eta \in \Omega_f^{\varepsilon_1}$,

$$\mathcal{C}^{\varepsilon_2}(x) = \begin{cases} \mathcal{C}^{\mathbf{m},\varsigma}(x, \eta, \varsigma), & \varsigma \in \Omega_m^{\varepsilon_2}, \\ \mathcal{C}^{\mathbf{f},\varsigma}(x, \eta, \varsigma), & \varsigma \in \Omega_f^{\varepsilon_2}, \end{cases} \quad (17)$$

156 where $\mathcal{C}^{\mathbf{m},\varsigma}$ and $\mathcal{C}^{\mathbf{f},\varsigma}$ denote the elasticity tensors corresponding to the constituents $\Omega_m^{\varepsilon_2}$ and $\Omega_f^{\varepsilon_2}$, respectively.
 157 In (16), $\tilde{\boldsymbol{\omega}}$ is a third-order, ς - and η -periodic tensor field such that

$$\begin{aligned} \tilde{\boldsymbol{\omega}}^{(1)}(x, \eta, \varsigma) &= \tilde{\boldsymbol{\omega}}(x, \eta, \varsigma) : (\mathcal{I} + \mathbf{T}\mathbf{E}_\eta[\boldsymbol{\omega}(x, \eta)]) : \mathbf{E}_x[\mathbf{u}^{(0)}(x)] \\ &\quad + \tilde{\boldsymbol{\omega}}(x, \eta, \varsigma) : \mathbf{T}\mathbf{E}_x[\boldsymbol{\omega}(x, \eta)] : \mathbf{E}_x[\mathbf{u}^{(0)}(x)]_{\varepsilon_1}, \end{aligned} \quad (18)$$

158 where \mathcal{I} is the fourth-order identity tensor, i.e., for every symmetric tensor \mathbf{A} , it holds that $\mathcal{I} : \mathbf{A} = \mathbf{A}$.
 159 Furthermore, the tensor $\tilde{\boldsymbol{\omega}}$ is solution of the cell problem

$$(\mathcal{P}_Z) \begin{cases} \text{Div}_\varsigma[\mathcal{C}^{\varepsilon_2} + \mathcal{C}^{\varepsilon_2} : \mathbf{T}\mathbf{E}_\varsigma(\tilde{\boldsymbol{\omega}})] = \mathbf{0}, & \text{in } Z \setminus \Gamma_Z, \\ [(\mathcal{C}^{\varepsilon_2} + \mathcal{C}^{\varepsilon_2} : \mathbf{T}\mathbf{E}_\varsigma(\tilde{\boldsymbol{\omega}})) \cdot \mathbf{N}_Z] = \mathbf{0}, & \text{on } \Gamma_Z, \\ [\tilde{\boldsymbol{\omega}}] = \mathbf{0}, & \text{on } \Gamma_Z, \end{cases} \quad (19)$$

160 where the condition $\langle \tilde{\boldsymbol{\omega}} \rangle_\varsigma = \mathbf{0}$ is imposed to guarantee uniqueness in the local problem (19).

161 4 Effective properties of hierarchical fiber-reinforced composites

162 In this section, we particularize the results given in the previous section by focusing on a three-scale composite
 163 material with a square-symmetric arrangement of uniaxially aligned cylindrical fibers (see Fig. 2). For this
 164 particular case, the three-dimensional cell problems (15) and (19) can be re-formulated as two-dimensional
 165 local problems defined over the cells' cross-sections corresponding to a square embedding a single circle.

166 Specifically, we assume that at the ε_2 -hierarchical level, both $\mathcal{C}^{\mathbf{m},\varsigma}$ and $\mathcal{C}^{\mathbf{f},\varsigma}$ are piece-wise constant. This
 167 consideration indicates that the dependence of the cell problem \mathcal{P}_Z on η and x is lost, and consequently,
 168 that the auxiliary third-order tensor $\tilde{\boldsymbol{\omega}}$ depends only on ς . Therefore, the effective elasticity tensor at the
 169 ε_1 -hierarchical level, $\check{\mathcal{C}}$, is likewise piece-wise constant. Additionally, considering that $\mathcal{C}^{\mathbf{m},\eta}$ is piece-wise
 170 constant, it can be deduced, in a similar way, that $\boldsymbol{\omega}$ will only depend on η and that the effective elasticity
 171 tensor, $\hat{\mathcal{C}}$, will be piece-wise constant.

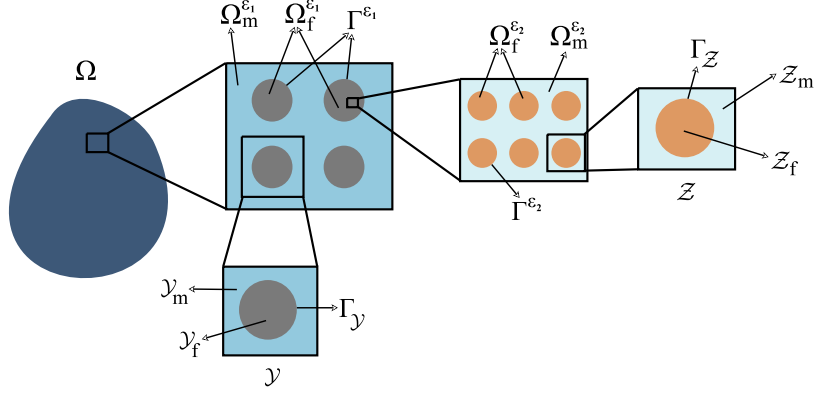


Figure 2: Schematic of the cross-section of a hierarchical fiber-reinforced periodic composite with three structural levels.

172 In like manner, we suppose that all the constituents in Ω are isotropic. This assumption together with the
 173 specified geometrical microstructure at the ε_2 -hierarchical level implies that \mathcal{C} is tetragonal symmetric. This
 174 means that the effective elasticity tensor \mathcal{C} has six independent elastic coefficients. Moreover, the assumption
 175 of isotropy of the constituent $\Omega_m^{\varepsilon_1}$ induces that the effective coefficient \mathcal{C} is at most monoclinic. Therefore,
 176 the cell problems \mathcal{P}_Z and \mathcal{P}_Y uncouple in sets of equations for the in-plane and out-of-plane stresses. That
 177 is, the local problems (15) and (19) rewrite, each one, as four in-plane problems \mathcal{P}_α^{qq} ($q = 1, 2, 3$) and \mathcal{P}_α^{12} ,
 178 with $\alpha = \eta, \varsigma$

$$(\mathcal{P}_\alpha^{qq}) \begin{cases} \frac{\partial \sigma_{11}^{qq\gamma, \alpha}}{\partial \alpha_1} + \frac{\partial \sigma_{12}^{qq\gamma, \alpha}}{\partial \alpha_2} = 0, & \text{in } \tilde{K}_\alpha^\gamma, \\ \frac{\partial \sigma_{21}^{qq\gamma, \alpha}}{\partial \alpha_1} + \frac{\partial \sigma_{22}^{qq\gamma, \alpha}}{\partial \alpha_2} = 0, & \text{in } \tilde{K}_\alpha^\gamma, \\ \llbracket \omega_{1qq}^\alpha \rrbracket = 0, \quad \llbracket \omega_{2qq}^\alpha \rrbracket = 0, & \text{on } \tilde{\Gamma}_\alpha, \\ \llbracket \sigma_{11}^{qq, \alpha} N_1^\alpha + \sigma_{12}^{qq, \alpha} N_2^\alpha \rrbracket = -\llbracket \mathcal{C}_{11qq}^\alpha N_1^\alpha \rrbracket, & \text{on } \tilde{\Gamma}_\alpha, \\ \llbracket \sigma_{21}^{qq, \alpha} N_1^\alpha + \sigma_{22}^{qq, \alpha} N_2^\alpha \rrbracket = -\llbracket \mathcal{C}_{22qq}^\alpha N_2^\alpha \rrbracket, & \text{on } \tilde{\Gamma}_\alpha, \end{cases} \quad (20a)$$

$$(\mathcal{P}_\alpha^{12}) \begin{cases} \frac{\partial \sigma_{11}^{12\gamma, \alpha}}{\partial \alpha_1} + \frac{\partial \sigma_{12}^{12\gamma, \alpha}}{\partial \alpha_2} = 0, & \text{in } \tilde{K}_\alpha^\gamma, \\ \frac{\partial \sigma_{21}^{12\gamma, \alpha}}{\partial \alpha_1} + \frac{\partial \sigma_{22}^{12\gamma, \alpha}}{\partial \alpha_2} = 0, & \text{in } \tilde{K}_\alpha^\gamma, \\ \llbracket \omega_{1qq}^\alpha \rrbracket = 0, \quad \llbracket \omega_{2qq}^\alpha \rrbracket = 0, & \text{on } \tilde{\Gamma}_\alpha, \\ \llbracket \sigma_{11}^{12\alpha} N_1^\alpha + \sigma_{12}^{12\alpha} N_2^\alpha \rrbracket = -\llbracket \mathcal{C}_{1212}^\alpha N_2^\alpha \rrbracket, & \text{on } \tilde{\Gamma}_\alpha, \\ \llbracket \sigma_{21}^{12\alpha} N_1^\alpha + \sigma_{22}^{12\alpha} N_2^\alpha \rrbracket = -\llbracket \mathcal{C}_{1212}^\alpha N_1^\alpha \rrbracket, & \text{on } \tilde{\Gamma}_\alpha, \end{cases} \quad (20b)$$

179 and two anti-plane problems \mathcal{P}_α^{3q} ($q = 1, 2$)

$$(\mathcal{P}_\alpha^{3q}) \begin{cases} \frac{\partial \sigma_{31}^{3q\gamma, \alpha}}{\partial \alpha_1} + \frac{\partial \sigma_{32}^{3q\gamma, \alpha}}{\partial \alpha_2} = 0, & \text{in } \tilde{K}_\alpha^\gamma, \\ \llbracket \omega_{33q}^\alpha \rrbracket = 0, & \text{on } \tilde{\Gamma}_\alpha, \\ \llbracket \sigma_{31}^{3q, \alpha} N_1^\alpha + \sigma_{32}^{3q, \alpha} N_2^\alpha \rrbracket = -\llbracket \mathcal{C}_{3131}^\alpha N_q^\alpha \rrbracket, & \text{on } \tilde{\Gamma}_\alpha, \end{cases} \quad (21)$$

180 where $\gamma = m, f$, and $\tilde{K}_\zeta^\gamma := \tilde{Z}_\gamma$ and $\tilde{K}_\eta^\gamma := \tilde{Y}_\gamma$ denote, respectively, the two-dimensional cross-sections of \mathcal{Z}_γ
 181 and \mathcal{Y}_γ . The interface between the constituents \tilde{Z}_m and \tilde{Z}_f (\tilde{Y}_m and \tilde{Y}_f) is denoted by $\tilde{\Gamma}_Z$ ($\tilde{\Gamma}_Y$).
 182 Additionally, in (20a)–(21)

$$\omega_{kpq}^\alpha := \begin{cases} \tilde{\omega}_{kpq}, & \text{for } \alpha = \varsigma, \\ \omega_{kpq}, & \text{for } \alpha = \eta, \end{cases} \quad (22)$$

183 and

$$\sigma_{ij}^{pq\gamma, \alpha} := \begin{cases} \mathcal{C}_{ijkl}^{\gamma, \varsigma} \frac{\partial \tilde{\omega}_{kpq}}{\partial \varsigma_l}, & \text{for } \alpha = \varsigma, \\ \mathcal{C}_{ijkl}^{\gamma, \eta} \frac{\partial \omega_{kpq}}{\partial \eta_l}, & \text{for } \alpha = \eta. \end{cases} \quad (23)$$

184 In (23), $\mathcal{C}_{ijkl}^{\gamma, \varsigma}$ and $\mathcal{C}_{ijkl}^{\gamma, \eta}$ are the components of the elasticity tensor of the constituent $\gamma = m, f$ at the ε_2 - and
 185 ε_1 -hierarchical levels, respectively.

186 Furthermore, component-wise, the fourth-order effective elasticity tensor at the ε_1 -hierarchical level $\hat{\mathcal{C}}$,
 187 and the fourth-order effective elasticity tensor of the hierarchical composite material $\hat{\mathcal{C}}$, are

$$\hat{\mathcal{C}}_{ijpq} = \langle \mathcal{C}_{ijpq}^{\varepsilon_2} + \mathcal{C}_{ijkl}^{\varepsilon_2} \frac{\partial \tilde{\omega}_{kpq}}{\partial \varsigma_l} \rangle_\varsigma, \quad (24a)$$

$$\hat{\mathcal{C}}_{ijpq} = \langle \mathcal{C}_{ijpq}^{\varepsilon_1} + \mathcal{C}_{ijkl}^{\varepsilon_1} \frac{\partial \omega_{kpq}}{\partial \eta_l} \rangle_\eta, \quad (24b)$$

188 respectively.

189 The theory of analytical functions in [34] applied to the cell problems (20a)–(21) allow us to find the
 190 effective coefficients $\hat{\mathcal{C}}_{ijpq}$ and $\hat{\mathcal{C}}_{ijpq}$ given in (24a) and (24b), respectively. In the present study we follow
 191 the procedure adopted in [45, 52, 54, 8] and we adapt it to the obtained scale-coupled cell problems (see
 192 Appendix). We note that in the previous work [47] we dealt with the solution of the coupled-anti-plane
 193 cell problems, and therefore only the procedure for the coupled-in-plane cell problems is shown here. In
 194 particular, the choice of the microstructure and material symmetry, and the generality of the analytical
 195 approach permit us to focus on the solution of the cell problems in only one hierarchical level. We note that
 196 due to the algebraic complexity of the analytical formulae for the effective coefficients given by relations
 197 (53a)–(53d) and (55), we use Matlab in order to solve the infinite linear systems (49) and (51), truncated
 198 to a fixed order, and, subsequently, to evaluate the results in the corresponding formulae for the effective
 199 coefficients.

200 5 Modeling MMTs' effective properties

201 In the present section we show the potential of the three-scale asymptotic homogenization approach by mod-
 202 eling the effective properties of MMTs. Bones and tendons are examples of MMTs, which are hierarchically
 203 structured materials, and whose principal constituents, organized spanning several length scales, are mineral
 204 crystals, collagen, and water. The principal elements of MMTs are cylindrical mineralized collagen fibrils
 205 consisting in self-assembled collagen molecules that are aligned in staggered arrays [59]. The hydroxyapatite
 206 crystals are distributed in both the intrafibrillar space, reinforcing the collagen fibrils, and in the extrafibrillar
 207 space, which primarily consists of mineral and water (see [59, 62] and references therein).

208 5.1 Geometrical model for MMTs

209 In the present work, we consider an approximated model for MMTs. Specifically, at the ε_2 -hierarchical
 210 level we suppose that \mathcal{Z}_m represents the minerals surrounding a single collagen fiber denoted by \mathcal{Z}_f . The
 211 collection of all collagen fibers at the ε_2 -hierarchical level $\Omega_1^{\varepsilon_2}$, together with the host phase $\Omega_m^{\varepsilon_2}$ (representing

212 the minerals) will constitute the mineralized collagen fiber \mathcal{Y}_f at the ε_1 -hierarchical level. The finite collection
 213 of mineralized collagen fibers $\Omega_f^{\varepsilon_1}$ are supposed to be periodically distributed in the extrafibrillar space $\Omega_m^{\varepsilon_1}$.
 214 The union of the disjoint sets $\Omega_f^{\varepsilon_1}$ with $\Omega_m^{\varepsilon_1}$ will form each one of the mineralized collagen fibril bundles.
 215 Finally, the extrafibrillar space is supposed to be a mixture of water and minerals (see Fig. 3). The situation
 216 just described, where mineralized collagen fibers are unidirectionally aligned, can be, for example, the case
 217 of a mineralized turkey leg tendon, and it can be considered as a simplified model for bones [59].

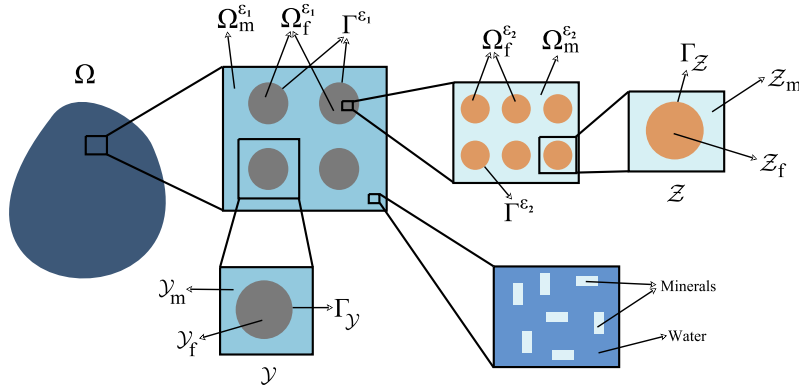


Figure 3: Schematic of the cross-section of MMTs.

218 In order to find the effective properties of the extrafibrillar space we take advantage of Reuss' lower
 219 bound formula [51] to compute the effective properties of the mixture $\Omega_m^{\varepsilon_1}$ as follows

$$\mathcal{C}^{m,\eta} = \langle (\mathcal{C}_{ES})^{-1} \rangle^{-1}, \quad (25)$$

220 where

$$\mathcal{C}_{ES}(x) = \begin{cases} \mathcal{C}^{w,\varsigma}(x, \eta, \varsigma), & \text{if } \varsigma \text{ is in the water phase,} \\ \mathcal{C}^{m,\varsigma}(x, \eta, \varsigma), & \text{if } \varsigma \text{ is in the mineral phase.} \end{cases} \quad (26)$$

221 In (26), $\mathcal{C}^{w,\varsigma}$ and $\mathcal{C}^{m,\varsigma}$ are the elasticity tensors related to the water and mineral phases, respectively. In
 222 particular, and following [59], we replace the material properties of water by those of polymethylmethacrylate
 223 (PMMA).

224 We remark that the present three-scale asymptotic approach can be improved to compute the effective
 225 properties of the composite extrafibrillar space. However, a realistic geometrical description of the structure
 226 of the extrafibrillar space requires numerical simulations in three dimensions for elastic composites (see e.g.
 227 [42, 41, 43]) which are beyond the scope of this work. Here we estimate the effective elastic constants of
 228 the extrafibrillar space by means of the Reuss bounds, thus obtaining a fully semi-analytic computational
 229 framework at each hierarchical level of organization. Reuss's formula (25) permits to obtain a lower bound
 230 for the current model. When we say that we obtain a lower bound for the model, it means that indeed, by
 231 considering the asymptotic homogenization approach instead, effective values above those computed using
 232 Reuss' scheme are expected [43].

233 5.2 Effective properties of MMTs

234 To model the effective properties of MMTs, we conveniently take advantage of some of the modeling as-
 235 sumptions in [59], [41]. Specifically, we consider all constituents of the hierarchical composite material are
 236 isotropic and that correspond to those of a bone tissue [59]. That is, Young's modulus (E) and Poisson's
 237 ratio (ν) of the mineral crystals, collagen fibers and water constituents (individuated by the subscripts m, c
 238 and p, respectively) are given as reported in Table 2.

Table 2: Young’s modulus and Poisson’s ratio of the mineral crystals, collagen fibers and water constituents.

Parameter	Unit	Value
E_M	[GPa]	110
E_c	[GPa]	5.00
E_p	[GPa]	4.96
ν_M	[-]	0.28
ν_c	[-]	0.30
ν_p	[-]	0.37

Moreover, we perform a parametric analysis of the MMTs’ effective properties by increasing the volume fraction of the mineral crystals, denoted by V , in the mineralized collagen fibril bundle from 0.2 to 0.5 [59]. Following [59], we also take into account the mineral distribution parameter ϕ , defined as the ratio of the mineral volume in the mineralized collagen fibril to the total mineral volume in the mineralized collagen fibril bundle. In [1], the mineral distribution parameter was estimated to be less than or equal to 0.7, here we chose $\phi = 0.5$. Specifically, the parameter ϕ is related to the phase volume fractions using the following empirical formula [50, 59]

$$V^{f,\eta} = \phi V + h(V), \quad (27)$$

where $h(V) := \frac{\varpi}{1+\varpi}(1 - V)$ and $\varpi := 0.36 + 0.084 e^{6.7V}$. In (27), the symbol $V^{f,\eta}$ represents the volume fraction of the mineralized collagen fibrils in the mineralized collagen fibril bundle. Therefore, the volume fraction of the extrafibrillar space in the mineralized collagen fibril bundle is given by $V^{m,\eta} = 1 - V^{f,\eta}$. Additionally, the volume fractions of the mineral crystals ($V^{m,s}$) and of collagen ($V^{f,s}$) in the mineralized collagen fibril are given by [59]

$$V^{m,s} = \phi \frac{V}{V^{f,\eta}} \quad \text{and} \quad V^{f,s} = 1 - V^{m,s}. \quad (28)$$

Finally, the volume fractions of the mineral crystals and water phases in the extrafibrillar space are

$$V^{f,ES} = (1 - \phi) \frac{V}{1 - V^{f,\eta}} \quad \text{and} \quad V^{m,ES} = 1 - V^{f,ES}, \quad (29)$$

respectively.

Figure 4 shows the effective coefficients $\hat{\mathcal{C}}_{11}$ (left panel) and $\hat{\mathcal{C}}_{33}$ (right panel), obtained by applying the three-scale homogenization approach, plotted with respect to the degree of mineralization of the tissue. In Fig. 4, we also show a comparison with the theoretical results obtained in [59]. Qualitatively, the results are in agreement with the ones obtained by [59], that is, the effective axial and transverse stiffness coefficients increase with respect to the minerals volume fraction. It is known that the results obtained by the asymptotic homogenization method are closer to those obtained by Reuss formula. Therefore, even in this case, we are positive that using an asymptotic approach for the characterization of the composite extrafibrillar space, the effective elastic coefficients will remain close to those in [59].

It is also known that the asymptotic homogenization technique gives effective properties lying between those computed using Reuss and Voigt formulae (see e.g. [43]). In Fig. 4 (right panel), the results are below those obtained by [59] for $\hat{\mathcal{C}}_{33}$. However, for $\hat{\mathcal{C}}_{11}$, the results lie above those in [59]. Even though we were not quite expecting this, the curve found with the present approach remains closer to that predicted by [59]. Furthermore, we obtain a satisfactory agreement with experimental data, and actually the obtained bounds are tighter than those in [59], as shown by Fig 5. In Fig. 5, we compare the effective axial and transverse stiffness coefficients with the experimental data showed in [59] corresponding to mineralized turkey leg tendon, human femur and mice bone. As commented before, the results fit very well the experimental

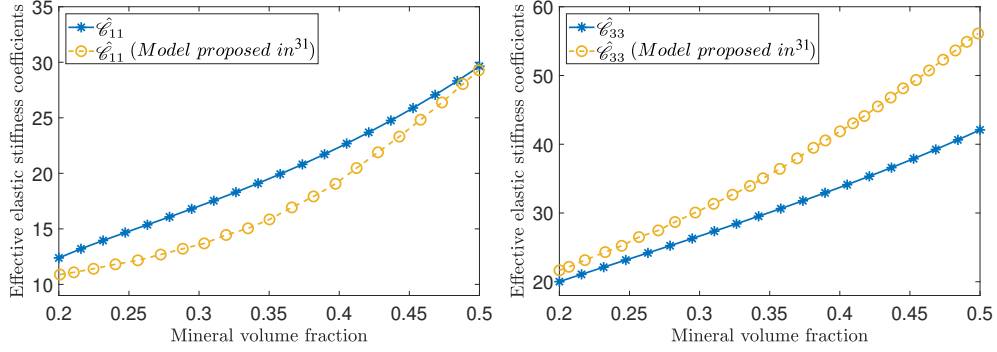


Figure 4: Elastic stiffness coefficients $\hat{\mathcal{C}}_{11}$ (left) and $\hat{\mathcal{C}}_{33}$ (right) with respect to the mineral volume fraction V . A comparison with the theoretical results in [59] are also shown.

269 data. We note that a Voigt formulation for computing the extrafibrillar space’s effective properties is also
 270 plausible. Indeed, we also considered Voigt upper bounds to model the properties of the extrafibrillar space.
 271 However, we preferred not to show them since the results did not match well the experimental and theoretical
 272 data.

273 The results shown in Fig. 5 could be of special interest for clinical applications including, for instance,
 274 tissue reconstruction. Indeed, following the methodology presented in this work, and considering other
 275 internal structures and properties, we could assess, in principle, how well fabricated a composite is by
 276 matching our analytical/computational results with the real properties of a target tissue (see e.g. [23]).
 277 Since the present homogenization approach takes into consideration three spatial scales, with respect to
 278 two-scale methods, it provides a better “microscope” to resolve the internal structure of a composite and to
 279 capture its material properties.

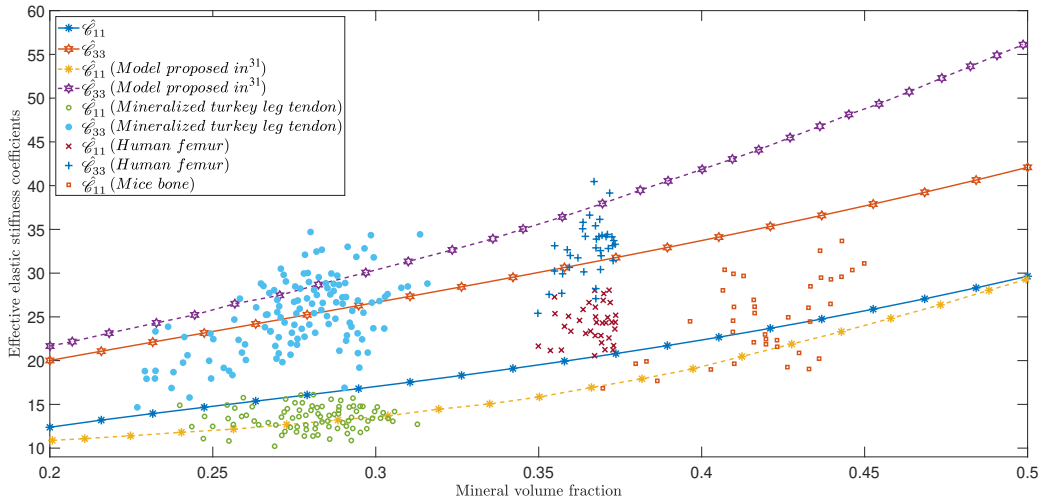


Figure 5: Comparison of the predicted and measured elastic stiffness coefficients $\hat{\mathcal{C}}_{11}$ (transverse) and $\hat{\mathcal{C}}_{33}$ (axial) with the experimental and theoretical data reported in [59] (and references therein) corresponding to mineralized turkey tendon leg, human femur and mice bone.

280 For completeness in the analysis we show in Fig. 6 the shear effective elastic coefficients $\hat{\mathcal{C}}_{44}$, $\hat{\mathcal{C}}_{55}$ and
 281 $\hat{\mathcal{C}}_{66}$ with respect to the mineral volume fraction. As shown in Fig. 6, the shear coefficients $\hat{\mathcal{C}}_{44}$, $\hat{\mathcal{C}}_{55}$ and

282 $\hat{\mathcal{C}}_{66}$ increase with increasing tissue's mineralization. Furthermore, the coefficients $\hat{\mathcal{C}}_{44}$ and $\hat{\mathcal{C}}_{55}$ coincide. We
 283 remark that the homogenized elasticity tensor has tetragonal symmetry (6 independent elastic coefficients),
 284 i.e. the matrix representation of $\hat{\mathcal{C}}$ (in Voigt notation) is

$$[\hat{\mathcal{C}}] = \begin{pmatrix} \hat{\mathcal{C}}_{11} & \hat{\mathcal{C}}_{12} & \hat{\mathcal{C}}_{13} & 0 & 0 & 0 \\ \hat{\mathcal{C}}_{12} & \hat{\mathcal{C}}_{11} & \hat{\mathcal{C}}_{13} & 0 & 0 & 0 \\ \hat{\mathcal{C}}_{13} & \hat{\mathcal{C}}_{13} & \hat{\mathcal{C}}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \hat{\mathcal{C}}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \hat{\mathcal{C}}_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \hat{\mathcal{C}}_{66} \end{pmatrix}. \quad (30)$$

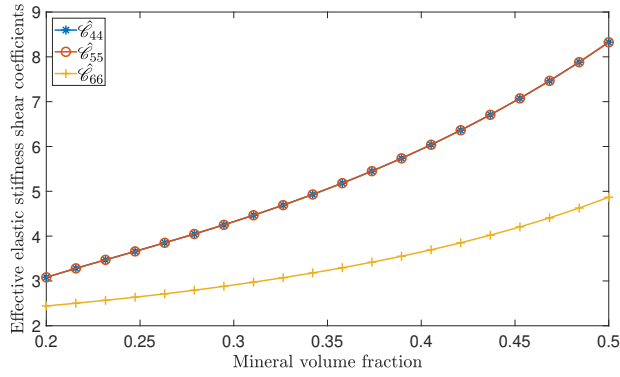


Figure 6: Shear effective elastic stiffness coefficients plotted with respect to the mineral volume fraction.

285 We now turn the attention to the computation of the effective Young's modulus (\hat{E}), shear modulus
 286 ($\hat{\nu}$) and Poisson's ratio ($\hat{\nu}$) of the hierarchical composite tissue. In particular, the effective shear modulus
 287 for hierarchical fiber-reinforced composites has been recently studied in the previous work [47]. Here, we
 288 adapt the computational scheme developed therein to the present framework. In the present study, via the
 289 homogenization process, the resulting homogenized mineralized tissue shows characteristics of a tetragonal
 290 material. Therefore, using Voigt notation, we have that

$$\hat{E}_1 = \frac{\Delta}{(\hat{\mathcal{C}}_{23})^2 - \hat{\mathcal{C}}_{22}\hat{\mathcal{C}}_{33}}, \quad \hat{\nu}_{12} = \hat{\nu}_{21} = \frac{\hat{\mathcal{C}}_{13}\hat{\mathcal{C}}_{23} - \hat{\mathcal{C}}_{12}\hat{\mathcal{C}}_{33}}{(\hat{\mathcal{C}}_{23})^2 - \hat{\mathcal{C}}_{22}\hat{\mathcal{C}}_{33}}, \quad (31a)$$

$$\hat{E}_2 = \frac{\Delta}{(\hat{\mathcal{C}}_{13})^2 - \hat{\mathcal{C}}_{11}\hat{\mathcal{C}}_{33}}, \quad \hat{\nu}_{13} = \hat{\nu}_{31} = \frac{\hat{\mathcal{C}}_{12}\hat{\mathcal{C}}_{23} - \hat{\mathcal{C}}_{13}\hat{\mathcal{C}}_{22}}{(\hat{\mathcal{C}}_{23})^2 - \hat{\mathcal{C}}_{22}\hat{\mathcal{C}}_{33}}, \quad (31b)$$

$$\hat{E}_3 = \frac{\Delta}{(\hat{\mathcal{C}}_{12})^2 - \hat{\mathcal{C}}_{11}\hat{\mathcal{C}}_{22}}, \quad \hat{\nu}_{23} = \hat{\nu}_{32} = \frac{\hat{\mathcal{C}}_{12}\hat{\mathcal{C}}_{13} - \hat{\mathcal{C}}_{11}\hat{\mathcal{C}}_{23}}{(\hat{\mathcal{C}}_{13})^2 - \hat{\mathcal{C}}_{11}\hat{\mathcal{C}}_{33}}, \quad (31c)$$

291 where

$$\Delta = (\hat{\mathcal{C}}_{13})^2\hat{\mathcal{C}}_{22} - 2\hat{\mathcal{C}}_{12}\hat{\mathcal{C}}_{13}\hat{\mathcal{C}}_{23} + \hat{\mathcal{C}}_{11}(\hat{\mathcal{C}}_{23})^2 + (\hat{\mathcal{C}}_{12})^2\hat{\mathcal{C}}_{33} - \hat{\mathcal{C}}_{11}\hat{\mathcal{C}}_{22}\hat{\mathcal{C}}_{33}. \quad (32)$$

292 Figure 7 shows the predicted effective Young's moduli (top left), shear moduli (top right) and Poisson's
 293 ratio (bottom). We remark that it has been difficult to find experimental data measuring the anisotropic
 294 properties of MMTs and validating the computations reported in Fig. 7. Additionally, as details regarding
 295 the mineral content in the tissue are often not available in experimental studies, we cannot establish a logical

296 correspondence with the numerical results shown in Fig. 7, as we did previously in Fig. 5. However, in
 297 what follows, we make a qualitative comparison with the data available in the scientific literature. In this
 298 respect, bone has been an extensively discussed hierarchical tissue, and several experimental techniques,
 299 such as micromechanical tests or nanoindentation [63], have been used in the measurement of its mechanical
 300 properties. For instance, the experimental studies conducted in [32] for bone tissues show that the magnitude
 301 of Young's and shear moduli increase with the degree of mineralization. This trend is captured by our
 302 computations as shown in Fig. 7 (top left and top right panels). In addition, Young's moduli and Poisson's
 303 ratio of single trabeculae in three orthogonal material directions were measured in [25] using compression
 304 tests. Therein, it was reported Young's modulus values in the trabeculae longitudinal direction significantly
 305 higher than those on the transverse directions. This experimental findings are in agreement with the predicted
 306 results from the present theoretical approach as shown in Fig. 7 (top left panel). Moreover, the data collected
 307 in the review paper [63] shows Young's modulus of trabecular bone varying between 0 Gpa ando 25 Gpa (see
 308 Fig. 5 in [63]), which is in the range of the results obtained for low mineral concentrations. Finally, we
 309 observe that $\hat{\nu}_{12}$ decreases, and that $\hat{\nu}_{13} = \hat{\nu}_{23}$ increases, with the augment of tissue's mineralization.

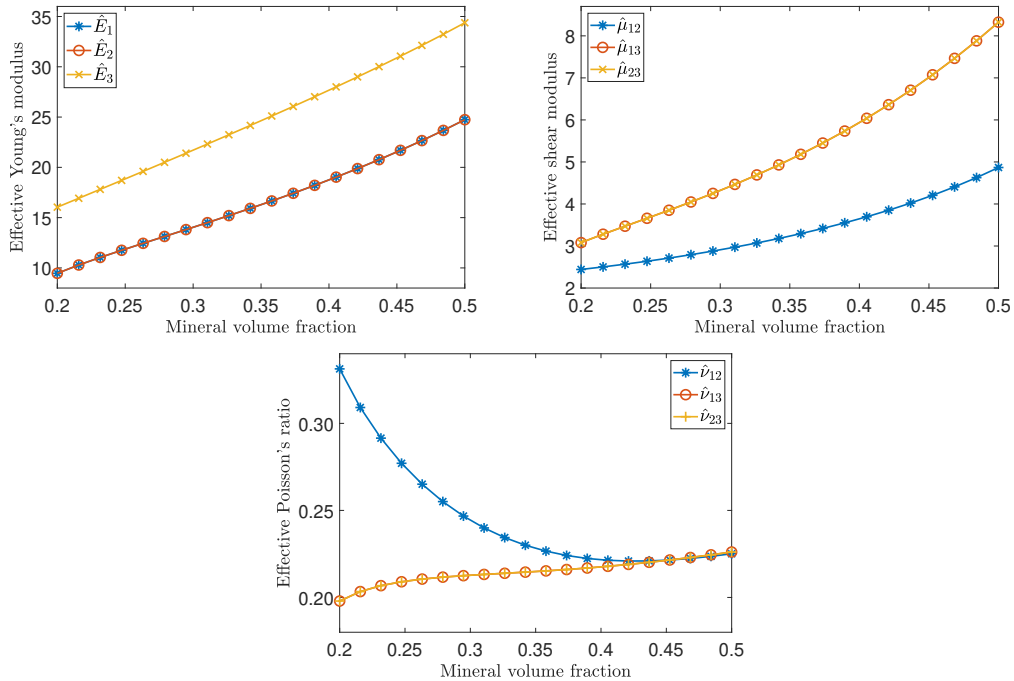


Figure 7: Comparison of the predicted effective Young's modulus, shear modulus and Poisson's ratio of the musculoskeletal mineralized tissue with respect to the mineral volume fraction. (Top) \hat{E}_1 , \hat{E}_2 and \hat{E}_3 , (middle) $\hat{\mu}_{12}$, $\hat{\mu}_{13}$ and $\hat{\mu}_{23}$, (bottom) $\hat{\nu}_{12}$, $\hat{\nu}_{13}$ and $\hat{\nu}_{23}$.

310 6 Conclusions

311 In the present work we have depicted a three-scale asymptotic homogenization procedure to investigate the
 312 effective properties of multiscale, linear elastic composite materials. Using this approach we compute the
 313 effective properties of a linear elastic, fiber reinforced hierarchical material using an analytical resolution
 314 process, allowing us to reduce the computational cost necessary to calculate the homogenized properties.
 315 Furthermore, the three-scale scheme was employed in a biological scenario of interest, that is, the modeling

316 of the macroscopic properties of MMTs. Specifically, we conducted a parametric study by varying the min-
 317 eralization of the heterogeneous tissue, and we compared the effective axial and transverse elastic stiffnes
 318 constants with theoretical and experimental values. In the study, we take advantage of Reuss’ lower formula
 319 to model the properties of the extrafibrillar space. In this sense, we hypothesize that performing an asymp-
 320 totic homogenization approach to describe the extrafibrillar space will produce more accurate outcomes for
 321 the description of MMTs. Finally, we computed the effective Young’s and shear moduli, and Poisson’s ratio,
 322 and we showed that the predictions are consistent with experimental findings concerning bone tissues.

323 Minerals content can substantially affect the macroscopic tissue behavior [37, 41, 18]. To avoid modeling
 324 the complex interplay between mineral crystals and water, we embrace a simplified approach by modeling
 325 the effective behavior of the extrafibrillar space by means of Reuss’ lower-bound formula. In this direction,
 326 we aim to account for another scale in the homogenization process, and to solve the related local problem by
 327 means of the finite elements method [41]. Further developments of this work include: (i) the generalization
 328 to a nonlinear framework (e.g. considering hyperelasticity) [46, 11, 49] and (ii) the consideration of growth
 329 of the tissue and remodelling of its internal structure [56, 44, 38, 12, 11, 49]. Another issue that could
 330 arise in our formulation is that of a non-macroscopically uniform medium. In other words, a medium in
 331 which the periodic cells are not independent of the macroscale and thus, the geometry can be varying
 332 over the multiple scales, not only the elastic constants. In this particular case, the generalized Reynold’s
 333 transport theorem (see e.g. [24]) has to be enforced as done, for instance in [40] and in [39] in the context
 334 of poro-mechanics. Alternative approaches that are rapidly emerging in the literature also involve a more
 335 explicit definition of the normal vector [9], which has been used to investigate the role of porosity gradients
 336 to optimize filter efficiency [15]. Also, the macroscopic uniformity assumption may also not be suitable
 337 for modelling peculiar situations, such as, for example, localized deformations and damage phenomena
 338 that can violate the periodicity constraint. In this context, hierarchical computational schemes have been
 339 developed for overcoming this issue [65, 17, 19]. In an idealized setting, one may think of reinterpreting the
 340 small parameter ε_2 as e.g. the damage length-scale and perform an analytical three-scale homogenization
 341 approach.

342 Finally, we remark that the technique has the advantage of reducing the intrinsic geometrical complexities
 343 when studying heterogeneous materials, and it ciphers the constituent’s properties at the several scales in
 344 the effective coefficients.

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352 A Solution of the cell problems

353 Following the procedure given in [45, 52, 54, 8], we present an analytical approach to find the solution of
 354 the cell problems \mathcal{P}_α^{qq} ($q = 1, 2, 3$) and \mathcal{P}_α^{12} . In particular, the choice of the microstructure and material
 355 symmetry allow us to focus on only one hierarchical level.

356 A.1 Theoretical background

357 In the present section we list some theoretical results that will be useful in the remainder of the text.

358 **Definition 1** Let w_1 and w_2 two linearly independent complex numbers on \mathbb{R} , i.e., there exists no pair of
 359 real numbers a and b , with $a, b \neq 0$, such that $aw_1 + bw_2 = 0$. We define a lattice, the set of all complex
 360 numbers of the form

$$w = mw_1 + nw_2, \quad m, n \in \mathbb{Z}, \quad (33)$$

361 which is denoted by $L = [w_1, w_2]$.

362 **Proposition 1** The Laurent series expansion of the $(k-1)$ -th ($k = 2, 3, \dots$) derivative of Weierstrass'
 363 function (ζ) and Natanzon's function (Q) in zero are, respectively,

$$\zeta^{(k-1)}(z) = \frac{(k-1)!}{z^k} - (k-1)! \sum_{l=1}^{\infty o} \Delta_{kl} z^l \quad \text{and} \quad Q^{(k-1)}(z) = (k-1)! \sum_{l=1}^{\infty o} \mathring{\Delta}_{kl} z^l, \quad (34a)$$

364 where

$$\Delta_{kl} = -\binom{k+l-1}{l} S_{k+l} \quad \text{and} \quad \mathring{\Delta}_{kl} = k \binom{k+l}{l} T_{k+l}. \quad (35)$$

365 The superscript "o" over the sum operator indicates that the sum is carried out only over odd natural
 366 numbers. The reticulate sums (which contains the geometrical information of the problem) are defined by
 367 $S_{k+l} = \sum_{w \in L^*} \frac{1}{w^{k+l}}$ ($k+l \geq 2$) and $T_{k+l} = \sum_{w \in L^*} \frac{\bar{w}}{w^{k+l+1}}$ ($k+l \geq 3$). The series S_{k+l} vanishes when $k+l$
 368 is not a multiple of 4. Furthermore, the series T_{k+l} vanishes when $k+l$ is not of the form $4t-1$ for $t \in \mathbb{N}$
 369 [20]. Moreover, L^* represents the lattice excluding the number $w = 0$ and \bar{w} denotes the conjugate of the
 370 complex number w .

371 **Proposition 2** Weierstrass' function and Natanzon's function possess the following properties of quasi-
 372 periodicity [36]

$$\zeta(z + w_p) - \zeta(z) = \delta_p, \quad \zeta^{(k)}(z + w_p) - \zeta^{(k)}(z) = 0, \quad \forall k \geq 1 \quad (36a)$$

$$Q(z + w_p) - Q(z) = \bar{w}_p P(z) + \xi_p, \quad Q^{(k)}(z + w_p) - Q^{(k)}(z) = \bar{w}_p P^{(k)}(z), \quad \forall k \geq 1, \quad (36b)$$

373 where $P(z) = -\zeta'(z)$, $\delta_p = 2\zeta(w_p/2)$ and $\xi_p = 2Q(w_p/2) - \bar{w}_p P(w_p/2)$. Moreover, Legendre's relations are
 374 fulfilled, i.e.,

$$\delta_1 w_2 - \delta_2 w_1 = 2\pi i, \quad (37a)$$

$$\delta_1 \bar{w}_2 - \delta_2 \bar{w}_1 = \xi_2 w_1 - \xi_1 w_2. \quad (37b)$$

375 **Remark 1** In the case of a square array of periodic cells, that is, for $w_1 = 1$ and $w_2 = i$, we have that
 376 $\delta_1 = \pi$, $\delta_2 = -i\pi$, $\xi_1 = -\frac{5S_4}{\pi}$ and $\xi_2 = i\frac{5S_4}{\pi}$.

377 A.2 Solution of the in-plane cell problems \mathcal{P}^{qq}

378 The structure of the in-plane cell problems \mathcal{P}^{qq} ($q = 1, 2, 3$) given in (20a) is of plane-strain and therefore,
 379 the theory of harmonic functions and the Kolosov-Muskhelishvili complex potentials [57] are applicable
 380 [45, 52, 54, 8]. The Kolosov-Muskhelishvili complex potentials are related to ω_{1qq} and ω_{2qq} , and to the stress
 381 components by means of the formulae,

$$2\mathcal{E}_{1212}^\gamma (\omega_{1qq}^\gamma + i\omega_{2qq}^\gamma) = \chi^\gamma \varphi^{qq\gamma} - z(\overline{\varphi^{qq\gamma}})' - \bar{\psi}^\gamma, \quad (38a)$$

$$\sigma_{11}^{qq\gamma} + \sigma_{22}^{qq\gamma} = 2((\varphi^{qq\gamma})' + (\overline{\varphi^{qq\gamma, \alpha}})'), \quad (38b)$$

$$\sigma_{22}^{qq\gamma} - \sigma_{11}^{qq\gamma} = 2(\bar{z}(\varphi^{qq\gamma})'' + (\psi^{qq\gamma})'), \quad (38c)$$

382 where $\chi^\gamma = 3 - 4\nu^\gamma$ and $\nu^\gamma = \mathcal{C}_{1122}^\gamma / (\mathcal{C}_{1111}^\gamma + \mathcal{C}_{1122}^\gamma)$. The notation φ' indicates the derivative of φ with
 383 respect to the complex variable z . Following [45, 52, 54, 8], the complex potentials $\varphi^{qq\gamma}$ and $\psi^{qq\gamma}$ can be
 384 written as

$$\varphi^{qqm}(z) = \frac{a_0^{qq}}{R}z + \sum_{k=1}^{\infty o} a_k^{qq} R^k \frac{\zeta^{(k-1)}(z)}{(k-1)!}, \quad \varphi^{qqf}(z) = \sum_{k=1}^{\infty o} \frac{z^k}{R^k} c_k^{qq}, \quad (39a)$$

$$\psi^{qqm}(z) = \frac{b_0^{qq}}{R}z + \sum_{k=1}^{\infty o} b_k^{qq} R^k \frac{\zeta^{(k-1)}(z)}{(k-1)!} + \sum_{k=1}^{\infty o} a_k^{qq} R^k \frac{Q^{(k-1)}(z)}{(k-1)!}, \quad \psi^{qqf}(z) = \sum_{k=1}^{\infty o} \frac{z^k}{R^k} d_k^{qq}, \quad (39b)$$

385 where a_k^{qq} , b_k^{qq} ($k = 0, 1, 3, \dots$), and c_k^{qq} , d_k^{qq} ($k = 1, 3, \dots$) are complex coefficients to be determined. The
 386 radius of the fiber's circular cross section is denoted with R .

387 Using Proposition 1 the complex potentials φ^{qqm} and ψ^{qqm} can be rewritten as follows

$$\varphi^{qqm}(z) = \frac{a_0^{qq}}{R}z + \sum_{l=1}^{\infty o} \left(a_l^{qq} \frac{R^l}{z^l} + A_l^{qq} \frac{z^l}{R^l} \right), \quad (40a)$$

$$\psi^{qqm}(z) = \frac{b_0^{qq}}{R}z + \sum_{l=1}^{\infty o} \left(b_l^{qq} \frac{R^l}{z^l} + B_l^{qq} \frac{z^l}{R^l} + \mathring{A}_l^{qq} \frac{z^l}{R^l} \right), \quad (40b)$$

388 where $A_l^{qq} = \sum_{k=1}^{\infty o} \Lambda_{kl} a_k^{qq}$, $B_l^{qq} = \sum_{k=1}^{\infty o} \Lambda_{kl} b_k^{qq}$ and $\mathring{A}_l^{qq} = \sum_{k=1}^{\infty o} \mathring{\Lambda}_{kl} a_k^{qq}$, with $\Lambda_{kl} = \Delta_{kl} R^{k+l}$ and $\mathring{\Lambda}_{kl} =$
 389 $\mathring{\Delta}_{kl} R^{k+l}$.

390 Then, to find the solution of problem (20a) is equivalent to determine the unknowns a_k^{qq} , b_k^{qq} , c_k^{qq} and
 391 d_k^{qq} . In particular, we show that for computing the effective coefficients, it is sufficient to find a_1^{qq} . In the
 392 following, we outline in three steps, the procedure in [45, 52, 54, 8].

393 **Step 1:** By taking into account the continuity conditions on ω_{1qq} and ω_{2qq} and the two expressions in
 394 (38a) for $\gamma = m$ and $\gamma = f$, we can deduce that

$$\chi^* (\chi^m \varphi^{qqm} - z \overline{(\varphi^{qqm})'}) - \overline{\psi^{qqm}} = \chi^f \varphi^{qqf} - z \overline{(\varphi^{qqf})'} - \overline{\psi^{qqf}}, \quad (41)$$

395 where $\chi^* = \mathcal{C}_{1212}^f / \mathcal{C}_{1212}^m$. Furthermore, the continuity conditions for traction on the interface $\tilde{\Gamma} = Re^{i\theta}$,
 396 $\theta \in [0, 2\pi]$, lead us to the following relation

$$\begin{aligned} & (\sigma_{22}^{qqm} + 2i\sigma_{12}^{qqm} - \sigma_{11}^{qqm})e^{i\theta} - (\sigma_{11}^{qqm} + \sigma_{22}^{qqm})e^{-i\theta} + 2\beta_1^{qq}e^{i\theta} - 2\beta_2^{qq}e^{-i\theta} \\ & = (\sigma_{22}^{qqf} + 2i\sigma_{12}^{qqf} - \sigma_{11}^{qqf})e^{i\theta} - (\sigma_{11}^{qqf} + \sigma_{22}^{qqf})e^{-i\theta}, \end{aligned} \quad (42)$$

397 where

$$\beta_j^{qq} = \begin{cases} \frac{[\mathcal{C}_{1122}] + (-1)^j [\mathcal{C}_{1111}]}{2}, & q = 1, \\ (-1)^j \beta_j^{11}, & q = 2, \\ \frac{1 + (-1)^j}{2} [\mathcal{C}_{1133}], & q = 3, \end{cases} \quad (43)$$

398 with $j = 1, 2$.

399 **Step 2:** Subsequently, let us evaluate (38a) (for $\gamma = m$) in z and $z + w_p$ and subtract the results of
 400 these evaluations. Using the expansions (40a) and (40b), the properties of quasiperiodicity (36a)–(36b), the
 401 periodic properties of the functions involved and Legendre's relations, we obtain that

$$a_0^{qq} + \overline{a_0^{qq}} = [(\tau_2 - \chi^m \tau_1) a_1^{qq} + (\overline{\tau_2} - \chi^m \overline{\tau_1}) \overline{a_1^{qq}} + (\tau_3 + \overline{\tau_3}) b_1^{qq}] \frac{R^2}{\chi^m - 1}, \quad (44a)$$

$$a_0^{qq} - \overline{a_0^{qq}} = [-(\tau_2 + \chi^m \tau_1) a_1^{qq} + (\overline{\tau_2} + \chi^m \overline{\tau_1}) \overline{a_1^{qq}} - (\tau_3 - \overline{\tau_3}) b_1^{qq}] \frac{R^2}{\chi^m - 1}, \quad (44b)$$

$$\overline{b_0^{qq}} = (\tau_4 \chi^m a_1^{qq} + \overline{\tau_5 a_1^{qq}} - \overline{\tau_6 b_1^{qq}}) R^2, \quad (44c)$$

402 where

$$\tau_1 = (\overline{w_1} \delta_2 - \overline{w_2} \delta_1) / W, \quad \tau_4 = -(w_1 \delta_2 - w_2 \delta_1) / W, \quad (45a)$$

$$\overline{\tau_2} = (\overline{w_1} \xi_2 - \overline{w_2} \xi_1) / W, \quad \overline{\tau_5} = (w_1 \overline{\xi_2} - w_2 \overline{\xi_1}) / W, \quad (45b)$$

$$\overline{\tau_3} = (\overline{w_1} \delta_2 - \overline{w_2} \delta_1) / W, \quad \overline{\tau_6} = -(w_1 \overline{\delta_2} - w_2 \overline{\delta_1}) / W, \quad (45c)$$

403 where $W = \overline{w_1} w_2 - w_1 \overline{w_2}$. Furthermore, substituting the Kolosov-Muskhelishvili relationships (38b) and
404 (38c) in equation (42), we obtain

$$z(\overline{\varphi^{qqm}})' + \overline{\psi^{qqm}} + \varphi^{qqm} + \overline{z} \beta_1^{qq} + z \beta_2^{qq} = z(\overline{\varphi^{qqf}})' + \overline{\psi^{qqf}} + \varphi^{qqf}. \quad (46)$$

405 **Step 3:** Now, substituting the Laurent expansions (40a) and (40b) in (41) and in (46), we obtain the
406 following infinite linear system in the unknowns $\tilde{a}_l^{qq} = a_l^{qq} / (R \beta_2^{qq})$ ($q = 1, 2, 3$ and $l = 1, 3, 5, \dots$)

$$\tilde{a}_l^{qq} + \mathcal{H}_l^1 \tilde{a}_1^{qq} + \mathcal{H}_l^2 \overline{\tilde{a}_1^{qq}} + \sum_{k=1}^{\infty} \mathcal{W}_{kl} \tilde{a}_k^{qq} + \sum_{k=1}^{\infty} \mathcal{M}_{kl} \overline{\tilde{a}_k^{qq}} = \mathcal{H}_l^{qq}, \quad (47)$$

407 where

$$\mathcal{H}_l^1 = [2\tau_4 \chi^m (\chi^m - 1) \chi^{m*} R^2 \delta_{1l} + (\overline{\Lambda_{1l}} - \overline{\tau_6} R^2 \delta_{1l}) (\tau_2 - \chi^m \tau_1) \Upsilon R^2] / [2(\chi^m - 1)], \quad (48a)$$

$$\mathcal{H}_l^2 = [2\overline{\tau_5} (\chi^m - 1) \chi^{m*} R^2 \delta_{1l} + (\overline{\Lambda_{1l}} - \overline{\tau_6} R^2 \delta_{1l}) (\overline{\tau_2} - \chi^m \overline{\tau_1}) \Upsilon R^2] / [2(\chi^m - 1)], \quad (48b)$$

$$\mathcal{W}_{kl} = \chi^{mf*} \mathcal{V}_{kl} + \frac{1}{2} (\overline{\Lambda_{1l}} - \overline{\tau_6} R^2 \delta_{1l}) \Upsilon \Lambda_{k1}, \quad (48c)$$

$$\mathcal{M}_{kl} = \chi^{m*} \mathcal{N}_{kl} + \frac{1}{2} (\overline{\Lambda_{1l}} - \overline{\tau_6} R^2 \delta_{1l}) \Upsilon \overline{\Lambda_{k1}}, \quad (48d)$$

$$\mathcal{N}_{kl} = (l+2) \overline{\Lambda_{k(l+2)}} + k \overline{\Lambda_{(k+2)l}} + \overline{\Lambda_{kl}}, \quad (48e)$$

$$\mathcal{V}_{kl} = \sum_{j=1}^{\infty} \Lambda_{k(j+2)} \overline{\Lambda_{(j+2)l}}, \quad (48f)$$

$$\mathcal{H}_l^{qq} = (\theta \beta_1^{qq} / \beta_2^{qq} - \overline{\tau_6} R^2 \Upsilon^*) \delta_{1l} + \overline{\Lambda_{1l}} \Upsilon^*, \quad (48g)$$

$$\alpha_0 = \chi^* [1 - Re(\tau_3) R^2] + (\chi^f - 1) \left[\frac{Re(\tau_3) R^2}{\chi^m - 1} + \frac{1}{2} \right], \quad (48h)$$

$$\theta = -(\chi^* \chi^m + 1)^{-1}, \quad (48i)$$

$$\chi^{m*} = (1 - \chi^*) (\chi^* \chi^m + 1)^{-1}, \quad (48j)$$

$$\chi^{mf*} = (\chi^{m*} (\chi^* \chi^m - \chi^f)) (\chi^* + \chi^f)^{-1}, \quad (48k)$$

$$\Upsilon = (\chi^{m*} (1 + \chi^* \chi^m - \chi^* - \chi^f)) \alpha_0^{-1}, \quad (48l)$$

$$\Upsilon^* = (\chi^{m*} (\chi^f - 1)) (2\alpha_0)^{-1}. \quad (48m)$$

408 In particular, the linear system (47) can be equivalently rewritten as

$$\begin{pmatrix} \tilde{A}_r^{qq} \\ \tilde{A}_i^{qq} \end{pmatrix} = \begin{pmatrix} \mathcal{I} + \check{\mathcal{M}}_r + \check{\mathcal{W}}_r & \check{\mathcal{M}}_i - \check{\mathcal{W}}_i \\ \check{\mathcal{M}}_i + \check{\mathcal{W}}_i & \mathcal{I} + \check{\mathcal{M}}_r - \check{\mathcal{W}}_r \end{pmatrix}^{-1} \begin{pmatrix} \mathcal{H}_r^{qq} \\ \mathcal{H}_i^{qq} \end{pmatrix}, \quad (49)$$

409 where $\tilde{\mathcal{A}}_r^{qq} = (Re(\tilde{a}_1^{qq}), Re(\tilde{a}_3^{qq}), \dots)^T$, $\tilde{\mathcal{A}}_i^{qq} = (Im(\tilde{a}_1^{qq}), Im(\tilde{a}_3^{qq}), \dots)^T$, with \mathbf{a}^T denoting the operation of
410 transposition of the vector \mathbf{a} . Moreover, \mathcal{I} is the infinite identity matrix, $\check{\mathcal{M}}_r = Re(\check{\mathcal{M}})$, $\check{\mathcal{W}}_r = Re(\check{\mathcal{W}})$,
411 $\check{\mathcal{M}}_i = Im(\check{\mathcal{M}})$, $\check{\mathcal{W}}_i = Im(\check{\mathcal{W}})$, $\mathcal{H}_r^{qq} = Re(\mathcal{H}^{qq})$ and $\mathcal{H}_i^{qq} = Im(\mathcal{H}^{qq})$, where $Re(\Phi)$ and $Im(\Phi)$ denote
412 the operators that extract the real and imaginary parts of Φ , respectively. The matrices $\check{\mathcal{M}}$ and $\check{\mathcal{W}}$ are
413 decomposed additively as follows, $\check{\mathcal{M}} = \mathcal{U} + \mathcal{M}$ and $\check{\mathcal{W}} = \mathcal{Q} + \mathcal{W}$, where the components of \mathcal{U} and \mathcal{Q} , are
414 given by the following expressions,

$$U_{kl} = \begin{cases} [2\tau_4\chi^m(\chi^m - 1)\chi^{m*}R^2\delta_{1l} + (\overline{\Lambda_{11}} - \overline{\tau_6}R^2\delta_{1l})(\tau_2 - \chi^m\tau_1)\Upsilon R^2][2(\chi^m - 1)]^{-1}, & k = 1, \\ 0, & k > 1, \end{cases} \quad (50a)$$

$$Q_{kl} = \begin{cases} [2\overline{\tau_5}(\chi^m - 1)\chi^{m*}R^2\delta_{1l} + (\overline{\Lambda_{11}} - \overline{\tau_6}R^2\delta_{1l})(\overline{\tau_2} - \chi^m\overline{\tau_1})\Upsilon R^2][2(\chi^m - 1)]^{-1}, & k = 1, \\ 0, & k > 1. \end{cases} \quad (50b)$$

415 Equation (49) is an infinite linear system with an infinite number of unknowns for which is possible to
416 obtain a solution by truncation through a convergent sequence of solutions [27, 52, 54, 8].

417 A.3 Solution of the problem \mathcal{P}^{12}

418 The solution of the in-plane problem \mathcal{P}^{12} (20b) can be found following a similar procedure to the one
419 outlined above. In such a case, the following infinite linear system in the unknowns \tilde{a}_l^{12} ($l = 1, 3, 5, \dots$) is
420 obtained

$$\begin{pmatrix} \tilde{\mathcal{A}}_r^{12} \\ \tilde{\mathcal{A}}_i^{12} \end{pmatrix} = \begin{pmatrix} \mathcal{I} + \check{\mathcal{M}}_r + \check{\mathcal{W}}_r & \check{\mathcal{M}}_i - \check{\mathcal{W}}_i \\ \check{\mathcal{M}}_i + \check{\mathcal{W}}_i & \mathcal{I} + \check{\mathcal{M}}_r - \check{\mathcal{W}}_r \end{pmatrix}^{-1} \begin{pmatrix} \mathcal{H}_r^{12} \\ \mathcal{H}_i^{12} \end{pmatrix}, \quad (51)$$

421 where $\tilde{\mathcal{A}}_r^{12} = (Re(\tilde{a}_1^{12}), Re(\tilde{a}_3^{12}), \dots)^T$, $\tilde{\mathcal{A}}_i^{12} = (Im(\tilde{a}_1^{12}), Im(\tilde{a}_3^{12}), \dots)^T$, $\mathcal{H}_l^{12} = -i\theta\delta_{1l}$ and $\tilde{a}_l^{12} = a_l^{12}/(R[\mathcal{C}_{1212}])$.

422 B Effective coefficients

423 The fact that $\mathcal{C}^{\varepsilon_2}$ is isotropic, together with the assumption that the cell's cross section corresponds to a
424 square embedding a single circle, induce that the tensor \mathcal{C} has tetragonal symmetric structure. This result
425 together with the isotropy assumption of the constituent $\Omega_m^{\varepsilon_1}$ imply that the effective tensor $\hat{\mathcal{C}}$ is at most
426 monoclinic, that is, $\hat{\mathcal{C}}$ has at most 13 independent effective elastic coefficients. In the following, we will
427 consider two elasticity tensors \mathcal{C}^m and \mathcal{C}^f having tetragonal symmetric structure. In this way, the results
428 will apply to both hierarchical levels.

429 B.1 The in-plane effective coefficients

430 Taking into account the major and minor symmetries of the elasticity tensor, the non-zero effective coefficients
431 corresponding to the in-plane problems \mathcal{P}^{qq} are

$$\hat{\mathcal{C}}_{11qq} = \langle \mathcal{C}_{1111}^{\varepsilon} \frac{\partial \omega_{1qq}}{\partial y_1} + \mathcal{C}_{1122}^{\varepsilon} \frac{\partial \omega_{2qq}}{\partial y_2} + \mathcal{C}_{11qq}^{\varepsilon} \rangle, \quad (52a)$$

$$\hat{\mathcal{C}}_{12qq} = \langle \mathcal{C}_{1221}^{\varepsilon} \frac{\partial \omega_{2qq}}{\partial y_1} + \mathcal{C}_{1212}^{\varepsilon} \frac{\partial \omega_{1qq}}{\partial y_2} \rangle, \quad (52b)$$

$$\hat{\mathcal{C}}_{21qq} = \langle \mathcal{C}_{2121}^{\varepsilon} \frac{\partial \omega_{2qq}}{\partial y_1} + \mathcal{C}_{2112}^{\varepsilon} \frac{\partial \omega_{1qq}}{\partial y_2} \rangle, \quad (52c)$$

$$\hat{\mathcal{C}}_{22qq} = \langle \mathcal{C}_{2211}^{\varepsilon} \frac{\partial \omega_{1qq}}{\partial y_1} + \mathcal{C}_{2222}^{\varepsilon} \frac{\partial \omega_{2qq}}{\partial y_2} + \mathcal{C}_{22qq}^{\varepsilon} \rangle, \quad (52d)$$

$$\hat{\mathcal{C}}_{33qq} = \langle \mathcal{C}_{3311}^\varepsilon \frac{\partial \omega_{1qq}}{\partial y_1} + \mathcal{C}_{3322}^\varepsilon \frac{\partial \omega_{2qq}}{\partial y_2} + \mathcal{C}_{33qq}^\varepsilon \rangle. \quad (52e)$$

432 We observe that the variable y plays the role of η and ς since the procedure to obtain the effective coefficients,
433 for this particular case, is the same.

434 Working with the expressions (52a)–(52e), applying Green's theorem to find the integrals involved, taking
435 into account the periodicity properties of the involved functions, the continuity conditions on the interface
436 $\tilde{\Gamma}$, the Kolosov-Muskhelishvili formula (38a), the Laurent expansions of φ^{qqm} and ψ^{qqm} , the orthogonality
437 property of the system of functions $\{e^{in\theta}\}_{n=-\infty}^{+\infty}$ in the interval $[-\pi, \pi]$, we can write

$$\begin{aligned} \hat{\mathcal{C}}_{11qq} = & \langle \mathcal{C}_{11qq} \rangle - V_f \beta_2^{qq} [\beta_2^{11} [2\chi^* \chi^{m*} (\chi^f + 1) \mathcal{C}_{1212}^m]^{-1} Re(\chi^f \Xi^{qq} - \overline{\Xi^{qq}}) \\ & + Re((\chi^m + 1) \overline{a_1^{qq}} + \beta_1^{qq} (\beta_2^{qq})^{-1})], \end{aligned} \quad (53a)$$

$$\begin{aligned} \hat{\mathcal{C}}_{22qq} = & \langle \mathcal{C}_{22qq} \rangle - V_f \beta_2^{qq} [\beta_2^{11} [2\chi^* \chi^{m*} (\chi^f + 1) \mathcal{C}_{1212}^m]^{-1} Re(\chi^f \Xi^{qq} - \overline{\Xi^{qq}}) \\ & - Re((\chi^m + 1) \overline{a_1^{qq}} + \beta_1^{qq} (\beta_2^{qq})^{-1})], \end{aligned} \quad (53b)$$

$$\hat{\mathcal{C}}_{33qq} = \langle \mathcal{C}_{33qq} \rangle - V_f \beta_2^{33} \beta_2^{qq} [2\chi^* \chi^{m*} (\chi^f + 1) [\mathcal{C}_{1212}^m]^{-1} Re(\chi^f \Xi^{qq} - \overline{\Xi^{qq}})], \quad (53c)$$

$$\hat{\mathcal{C}}_{12qq} = V_f \beta_2^{qq} Im((\chi^m + 1) \overline{a_1^{qq}} + \beta_1^{qq} (\beta_2^{qq})^{-1}), \quad (53d)$$

438 where $V_f = \pi R^2$ represents the volume fraction of the circular inclusion and

$$\begin{aligned} \Xi^{qq} = & \{[(\chi^{m*} \chi_-^m + \Upsilon \beta_0) \tau_2 - (\chi^{m*} \chi_-^* + \Upsilon \beta_0) \tau_1 \chi^m] R^2\} (\chi^m - 1)^{-1} \overline{a_1^{qq}} + \{[(\chi^{m*} \chi_-^* + \Upsilon \beta_0) \overline{\tau_2} \\ & - (\chi^{m*} \chi_-^m + \Upsilon \beta_0) \overline{\tau_1} \chi^m] R^2\} (\chi^m - 1)^{-1} \overline{a_1^{qq}} + (\chi^{m*} \chi_+ + \Upsilon \beta_0) \tilde{A}_1^{qq} + (\chi^{m*} \chi_- + \Upsilon \beta_0) \overline{\tilde{A}_1^{qq}} \\ & + \chi^{m*} (\chi^f + 1) - 2\beta_0 \Upsilon^*, \end{aligned} \quad (54a)$$

$$\beta_0 = (\chi^f + 1) \left[\frac{Re(\tau_3) R^2}{\chi^m - 1} + \frac{1}{2} \right] - i \chi^* Im(\tau_3) R^2, \quad (54b)$$

$$\chi_-^m = \chi^f + 1 - \chi^* \chi^m + \chi^*, \quad (54c)$$

$$\chi_-^* = \chi^f + 1 + \chi^* \chi^m - \chi^*, \quad (54d)$$

$$\chi_+ = \chi^f + 1 + \chi^* \chi^m + \chi^*, \quad (54e)$$

$$\chi_- = \chi^f + 1 - \chi^* \chi^m - \chi^*. \quad (54f)$$

439 In (54a), we denote by $\tilde{A}_l^{qq} = \sum_{k=1}^{\infty} \Lambda_{kl} \tilde{a}_l^{qq}$.

440 Resuming, formulae (53a), (53b), (53c) and (53d) give the effective coefficients $\hat{\mathcal{C}}_{11qq}$, $\hat{\mathcal{C}}_{22qq}$, $\hat{\mathcal{C}}_{33qq}$ and
441 $\hat{\mathcal{C}}_{12qq}$, respectively. As anticipated, the effective coefficients depend solely on the unknowns a_1^{qq} .

442 Finally, proceeding in an analogous way, the only one non-zero effective coefficient corresponding to the
443 in-plane problem \mathcal{P}^{12} is

$$\hat{\mathcal{C}}_{1212} = \mathcal{C}_{1212}^m - [\mathcal{C}_{1212}] V_f Im((\chi^m + 1) \tilde{a}_1^{12}). \quad (55)$$

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