

Least common multiple of polynomial sequences

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**D. Bazzanella and C. Sanna**

**LEAST COMMON MULTIPLE OF  
POLYNOMIAL SEQUENCES**

**Abstract.** We collect some results and problems about the quantity

$$L_f(n) := \text{lcm}(f(1), f(2), \dots, f(n)),$$

where  $f$  is a polynomial with integer coefficients and  $\text{lcm}$  denotes the least common multiple.

**1. Introduction**

For each positive integer  $n$ , let us define

$$L(n) := \text{lcm}(1, 2, \dots, n),$$

that is, the lowest common multiple of the first  $n$  positive integers. It is not difficult to show that

$$\log L(n) = \psi(n) := \sum_{p \leq n} \log p,$$

where  $\psi$  denotes the first Chebyshev function, and  $p$  runs over all primes numbers not exceeding  $n$ . Hence, bounds for  $L(n)$  are directly related to bounds for  $\psi(n)$  and, consequently, to estimates for the prime counting function  $\pi(n)$ . In particular, since the Prime Number Theorem is equivalent to  $\psi(n) \sim n$  as  $n \rightarrow +\infty$ , we have

$$\log L(n) \sim n.$$

In 1936 Gelfond and Shnirelman, proposed a new elementary and clever method for deriving a lower bound for the prime counting function  $\pi(x)$  (see Gelfond's editorial remarks in the 1944 edition of Chebyshev's Collected Works [15, pag. 287–288]). In 1982 the Gelfond-Shnirelman method was rediscovered and developed by Nair [16, 17]. Their method was actually based on estimating  $L(n)$ , and in its simplest form [16] it gives

$$n \log 2 \leq \log L(n) \leq n \log 4,$$

for every  $n \geq 9$ , which in turn implies

$$(\log 2 + o(1)) \frac{n}{\log n} \leq \pi(n) \leq (\log 4 + o(1)) \frac{n}{\log n},$$

after some manipulations. Later, it was proved [18] that the Gelfond-Shnirelman-Nair method can give lower bound in the form

$$\pi(n) \geq C \frac{n}{\log n},$$

only for constants  $C$  less than 0.87, which is quite far from what is expected by the Prime Number Theorem. (A possible way around this problem has been considered in [13, 14, 19].)

Moving from this initial connection with estimates for  $\pi(n)$  and the Prime Number Theorem, several authors have considered bounds and asymptotic for the following generalization of  $L(n)$  to polynomials. For every polynomial  $f \in \mathbb{Z}[x]$ , let us define

$$L_f(n) := \text{lcm}(f(1), f(2), \dots, f(n)).$$

In the next section we collect some results on  $L_f(n)$ .

## 2. Products of linear polynomials

Stenger [12] used the Prime Number Theorem for arithmetic progressions to show the following asymptotic estimate for linear polynomials:

**THEOREM 1.** *For any linear polynomial  $f(x) = ax + b \in \mathbb{Z}[x]$ , we have*

$$\log L_f(n) \sim n \frac{q}{\varphi(q)} \sum_{\substack{1 \leq r \leq q \\ (q,r)=1}} \frac{1}{r},$$

as  $n \rightarrow +\infty$ , where  $q = a/(a, b)$  and  $\varphi$  denotes the Euler's totient function.

Hong, Qian, and Tan [6] extended this result to polynomials  $f$  which are the product of linear polynomials, showing that an asymptotic of the form  $\log L_f(n) \sim A_f n$  holds as  $n \rightarrow +\infty$ , where  $A_f > 0$  is a constant depending only on  $f$ .

Moreover, effective lower bounds for  $L_f(n)$  when  $f$  is a linear polynomial have been proved by Hong and Feng [3], Hong and Kominers [4], Hong, Tan and Wu [7], Hong and Yang [8], and Oon [9],

## 3. Quadratic polynomials

Cilleruelo [2, Theorem 1] considered irreducible quadratic polynomials and proved the following result:

**THEOREM 2.** *For any irreducible quadratic polynomial with integer coefficients  $f(x) = ax^2 + bx + c$ , we have*

$$\log L_f(n) = n \log n + B_f n + o(n),$$

where

$$B_f := \gamma - 1 - 2 \log 2 - \sum_p \frac{(d/p) \log p}{p-1} + \frac{1}{\varphi(q)} \sum_{\substack{1 \leq r \leq q \\ (r,q)=1}} \log \left( 1 + \frac{r}{q} \right) \\ + \log a + \sum_{p|2aD} \log p \left( \frac{1 + (d/p)}{p-1} - \sum_{k \geq 1} \frac{s(f, p^k)}{p^k} \right),$$

and  $\gamma$  is the Euler–Mascheroni constant,  $D = b^2 - 4ac = d\ell^2$ , where  $d$  is a fundamental discriminant,  $(d/p)$  is the Kronecker symbol,  $q = a/(a, b)$  and  $s(f, p^k)$  is the number of solutions of  $f(x) \equiv 0 \pmod{p^k}$ .

Rué, Šarka, and Zumalacárregui [11, Theorem 1.1] provided a more precise error term for the particular polynomial  $f(x) = x^2 + 1$ ,

**THEOREM 3.** *Let  $f(x) = x^2 + 1$ . For any  $\theta < 4/9$  we have*

$$\log L_f(n) = n \log n + B_f n + O_\theta \left( \frac{n}{(\log n)^\theta} \right).$$

#### 4. Higher degree polynomials

Regarding general irreducible polynomials, Cilleruelo [2] formulated the following conjecture.

**CONJECTURE 1.** *If  $f(x) \in \mathbb{Z}[x]$  is an irreducible polynomial of degree  $d \geq 2$ , then*

$$\log L_f(n) \sim (d-1)n \log n,$$

as  $n \rightarrow +\infty$ .

Except for the result of Theorem 2, no other case of Conjecture 1 is known to date. It can be proved (see [10, p. 2]) that for any irreducible  $f$  of degree  $d \geq 3$ , we have

$$n \log n \ll \log L_f(n) \leq (1 + o(1))(d-1)n \log n.$$

Also, Rudnick and Zehavi [10, Theorem 1.2] proved the following result, which established Conjecture 1 for almost all shifts of a fixed polynomial, in a range of  $n$  depending on the range of shifts.

**THEOREM 4.** *Let  $f(x) \in \mathbb{Z}[x]$  be a monic polynomial of degree  $d \geq 3$ . Then, as  $T \rightarrow +\infty$ , we have that for all  $a \in \mathbb{Z}$  with  $|a| \leq T$ , but a set of cardinality  $o(T)$ , it holds*

$$\log L_{f(x)-a}(n) \sim (d-1)n \log n$$

uniformly for  $T^{1/(d-1)} < n < T/\log T$ .

Regarding lower bounds for  $L_f(n)$ , Hong and Qian [5, Lemma 3.1] proved the following:

**THEOREM 5.** *Let  $f(x) \in \mathbb{Z}[x]$  be a polynomial of degree  $d \geq 1$  and with leading coefficient  $a_d$ . Then for all integers  $1 \leq m \leq n$ , we have*

$$\text{lcm}(f(m), f(m+1), \dots, f(n)) \geq \frac{1}{(n-m)!} \prod_{k=m}^n \left| \frac{f(k)}{a_d} \right|^{1/d}.$$

Shparlinski [1] suggested to study a bivariate version of  $L_f(n)$ , posing the following problem:

**PROBLEM 1.** *Given a polynomial  $f \in \mathbb{Z}[x, y]$ , obtain an asymptotic formula for*

$$\log \text{lcm}\{f(m, n) : 1 \leq m, n \leq N\}$$

*with a power saving in the error term.*

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Danilo BAZZANELLA,  
Department of Mathematical Sciences, Politecnico di Torino  
Corso Duca degli Abruzzi 24, 10129 Torino, Italy  
e-mail: [danilo.bazzanella@polito.it](mailto:danilo.bazzanella@polito.it)

Carlo SANNA,  
Department of Mathematics, Università di Genova  
Via Dodecaneso 35, 16146 Genova, Italy  
e-mail: [carlo.sanna.dev@gmail.com](mailto:carlo.sanna.dev@gmail.com)

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