On numbers n relatively prime to the nth term of a linear recurrence

ON NUMBERS n RELATIVELY PRIME TO THE nTH TERM OF A LINEAR RECURRENCE

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Abstract. Let $(u_n)_{n\geq 0}$ be a nondegenerate linear recurrence of integers, and let $\mathcal{A}$ be the set of positive integers n such that u_n and n are relatively prime. We prove that $\mathcal{A}$ has an asymptotic density, and that this density is positive unless $(u_n/n)_{n\geq 1}$ is a linear recurrence.

1. Introduction

Let $(u_n)_{n\geq 0}$ be a linear recurrence over the integers, that is, $(u_n)_{n\geq 0}$ is a sequence of integers satisfying

$$u_n = a_1 u_{n-1} + a_2 u_{n-2} + \cdots + a_k u_{n-k},$$

for all integers $n \geq k$, where $a_1, \ldots, a_k \in \mathbf{Z}$ and $a_k \neq 0$. To avoid trivialities, we assume that $(u_n)_{n\geq 0}$ is not identically zero. We refer the reader to [4, Ch. 1-8] for the general terminology and theory of linear recurrences.

The set

$$\mathcal{B}_u := \{n \in \mathbf{N} : n \mid u_n\}$$

has been studied by several researchers. Assuming that $(u_n)_{n\geq 0}$ is nondegenerate and that its characteristic polynomial has only simple roots, Alba González, Luca, Pomerance, and Shparlinski [1, Theorem 1.1] proved that

$$\#\mathcal{B}_u(x) \ll_k \frac{x}{\log x},$$

for all sufficiently large $x > 1$. André-Jeannin [2] and Somer [10] studied the arithmetic properties of the elements of $\mathcal{B}_u$ when $(u_n)_{n\geq 0}$ is a Lucas sequence, that is, $(u_0, u_1, k) = (0, 1, 2)$. In such a case, generalizing a previous result of Luca and Tron [6], Sanna [8] proved the upper bound

$$\#\mathcal{B}_u(x) \leq x^{1-\left(\frac{1}{2}+o(1)\right)\log\log\log x/\log\log x},$$

as $x \to +\infty$, where the $o(1)$ depends on a_1 and a_2. Furthermore, Corvaja and Zannier [3] studied the more general set

$$\mathcal{B}_{u,v} := \{n \in \mathbf{N} : v_n \mid u_n\},$$

where $(v_n)_{n\geq 0}$ is another linear recurrence over the integers. Under some mild hypotheses on $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$, they proved that $\mathcal{B}_{u,v}$ has zero asymptotic density [3, Corollary 2], while Sanna [7] gave the bound

$$\#\mathcal{B}_{u,v}(x) \ll_{u,v} x \cdot \left(\frac{\log\log x}{\log x}\right)^{h_{u,v}},$$

for all $x \geq 3$, where $h_{u,v}$ is a positive integer depending only on $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$.

If $(F_n)_{n\geq 0}$ is the sequence of Fibonacci numbers, Leonetti and Sanna [5] showed that the set

$$\mathcal{G} := \{\gcd(n, F_n) : n \in \mathbf{N}\}$$

has zero asymptotic density, and that

$$\#\mathcal{G}(x) \gg \frac{x}{\log x},$$

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for all $x \geq 2$. Moreover, Sanna and Tron [9] proved that for each positive integer m the set

$$\{n \in \mathbf{N} : \gcd(n, F_n) = m\}$$

has an asymptotic density. They also gave a criterion to establish when this density is positive, and a formula for the density in terms of an infinite series involving the Möbius function and the rank of appearance.

On the other hand, the set

$$\mathcal{A}_u = \{n \in \mathbf{N} : \gcd(n, u_n) = 1\}$$

does not seem to have attracted so much attention. We prove the following result:

Theorem 1.1. *For any nondegenerate linear recurrence of integers $(u_n)_{n\geq 0}$, the asymptotic density $\mathbf{d}(\mathcal{A}_u)$ of $\mathcal{A}_u$ exists. Moreover, if $(u_n/n)_{n\geq 1}$ is not a linear recurrence (of rational numbers) then $\mathbf{d}(\mathcal{A}_u) > 0$. Otherwise, $\mathcal{A}_u$ is finite and, a fortiori, $\mathbf{d}(\mathcal{A}_u) = 0$.*

We remark that given the initial conditions and the coefficients of a linear recurrence $(u_n)_{n\geq 0}$, it is easy to test effectively if $(u_n/n)_{n\geq 1}$ is a linear recurrence or not (see Lemma 2.1, in §2).

Notation. Throughout, the letter p always denotes a prime number. For a set of positive integers $\mathcal{S}$, we put $\mathcal{S}(x) := \mathcal{S} \cap [1, x]$ for all $x \geq 1$, and we recall that the asymptotic density $\mathbf{d}(\mathcal{S})$ of $\mathcal{S}$ is defined as the limit of the ratio $\#\mathcal{S}(x)/x$ as $x \to +\infty$, whenever this exists. We employ the Landau–Bachmann "Big Oh" and "little oh" notations O and o, as well as the associated Vinogradov symbols $\ll$ and $\gg$, with their usual meanings. Any dependence of the implied constants is explicitly stated or indicated with subscripts.

2. Preliminaries

In this section we give some definitions and collect some preliminary results needed in the later proofs. Let f_u be the characteristic polynomial of $(u_n)_{n\geq 0}$, i.e.,

$$f_u(X) = X^k - a_1 X^{k-1} - a_2 X^{k-2} - \cdots - a_k.$$

Moreover, let $\mathbf{K}$ be the splitting field of f_u over $\mathbf{Q}$, and let $\alpha_1, \ldots, \alpha_r \in \mathbf{K}$ be all the distinct roots of f_u. It is well known that there exist $g_1, \ldots, g_r \in \mathbf{K}[X]$ such that

$$u_n = \sum_{i=1}^{r} g_i(n)\, \alpha_i^n, \tag{1}$$

for all integers $n \geq 0$. We define B_u as the smallest positive integer such that all the coefficients of the polynomials $B_u g_1, \ldots, B_u g_r$ are algebraic integers.

We have the following easy lemma.

Lemma 2.1. *$(u_n/n)_{n\geq 1}$ is a linear recurrence (of rational numbers) if and only if*

$$g_1(0) = \cdots = g_r(0) = 0. \tag{2}$$

In such a case, $\mathcal{A}_u$ is finite.

Proof. The first part of the lemma follows immediately from the fact that any linear recurrence can be written as a generalized power sum like (1) in a unique way (assuming the roots $\alpha_1, \ldots, \alpha_r$ are distinct, and up to the order of the addends). For the second part, if (2) holds then for all positive integer n we have that

$$\frac{B_u u_n}{n} = \sum_{i=1}^{r} \frac{B_u g_i(n)}{n}\, \alpha_i^n$$

is both a rational number and an algebraic integer, hence it is an integer. Therefore, $n \mid B_u u_n$, and so $\gcd(n, u_n) = 1$ only if $n \mid B_u$, which in turn implies that $\mathcal{A}_u$ is finite. $\square$

For the rest of this section, we assume that $(u_n)_{n\geq 0}$ is nondegenerate and that f_u has only simple roots, hence, in particular, $r = k$. We write Δ_u for the discriminant of the polynomial f_u, and we recall that Δ_u is a nonzero integer. If $k \geq 2$, then for all integers $x_1, \dots, x_k$ we set

$$D_u(x_1, \dots, x_k) := \det(\alpha_i^{x_j})_{1\leq i,j\leq k},$$

and for any prime number p not dividing a_k we define $T_u(p)$ as the greatest integer $T \geq 0$ such that p does not divide

$$\prod_{1\leq x_2,\dots,x_k\leq T} \max\{1, |N_{\mathbf{K}}(D_u(0, x_2, \dots, x_k))|\},$$

where $N_{\mathbf{K}}(\alpha)$ denotes the norm of $\alpha \in \mathbf{K}$ over $\mathbf{Q}$, and the empty product is equal to 1. It is known that such T exists [4, p. 88]. If $k = 1$, then we set $T_u(p) := +\infty$ for all prime numbers p not dividing a_1. Note that $T_u(p) = 0$ if and only if $k = 2$ and p divides Δ_u.

Finally, for all $\gamma \in \,]0, 1[$, we define

$$\mathcal{P}_{u,\gamma} := \{p : p \nmid a_k, \; T_u(p) < p^\gamma\}.$$

We are ready to state two important lemmas regarding $T_u(p)$ [1, Lemma 2.1, Lemma 2.2].

Lemma 2.2. *For all $\gamma \in \,]0, 1[$ and $x \geq 2^{1/\gamma}$ we have*

$$\#\mathcal{P}_{u,\gamma}(x) \ll_u \frac{x^{k\gamma}}{\gamma \log x}.$$

Lemma 2.3. *Assume that p is a prime number not dividing $a_k B_u \Delta_u$ and relatively prime with at least one term of $(u_n)_{n\geq 0}$. Then, for all $x \geq 1$, the number of positive integers $m \leq x$ such that $u_{pm} \equiv 0 \pmod p$ is*

$$O_k\left(\frac{x}{T_u(p)} + 1\right).$$

Actually, in [1] both Lemma 2.2 and Lemma 2.3 were proved only for $k \geq 2$. However, one can easily check that they are true also for $k = 1$.

3. Proof of Theorem 1.1

For all integers $n \geq 0$, define

$$v_n := B_u \sum_{i=1}^{r} \frac{g_i(n) - g_i(0)}{n} \, \alpha_i^n \quad \text{and} \quad w_n := B_u \sum_{i=1}^{r} g_i(0) \, \alpha_i^n.$$

Note that both $(v_n)_{n\geq 0}$ and $(w_n)_{n\geq 0}$ are linear recurrences of algebraic integers, and that the characteristic polynomial of $(w_n)_{n\geq 0}$ has only simple roots.

Let $\mathcal{G}$ be the Galois group of $\mathbf{K}$ over $\mathbf{Q}$. Since u_n is an integer, for any $\sigma \in \mathcal{G}$ we have that

$$nv_n + w_n = B_u u_n = \sigma(B_u u_n) = \sigma(nv_n + w_n) = n\sigma(v_n) + \sigma(w_n), \tag{3}$$

for all integers $n \geq 0$. In (3) note that both $n\sigma(v_n)$ and $\sigma(w_n)$ are linear recurrences, and the first is a multiple of n, while the characteristic polynomial of the second has only simple roots. Since the expression of a linear recurrence as a generalized power sum is unique, from (3) we get that $w_n = \sigma(w_n)$ for any $\sigma \in \mathcal{G}$, hence w_n is an integer.

Thanks to Lemma 2.1, we know that $(w_n)_{n\geq 0}$ is identically zero if and only if $(u_n/n)_{n\geq 1}$ is a linear recurrence, and in such a case $\mathcal{A}_u$ is finite, so that the claim of Theorem 1.1 is obvious. Hence, we assume that $(w_n)_{n\geq 0}$ is not identically zero.

For the sake of convenience, put $\mathcal{C}_u := \mathbf{N} \setminus \mathcal{A}_u$. Thus we have to prove that the asymptotic density of $\mathcal{C}_u$ exists and is less than 1. For each $y > 0$, we split $\mathcal{C}_u$ into two subsets:

$$\mathcal{C}_{u,y}^- := \{n \in \mathcal{C}_u : p \mid \gcd(n, u_n) \text{ for some } p \leq y\},$$
$$\mathcal{C}_{u,y}^+ := \mathcal{C}_u \setminus \mathcal{C}_{u,y}^-.$$

It is well known that $(u_n)_{n\geq 0}$ is definitively periodic modulo p, for any prime number p. Therefore, it is easy to see that $\mathcal{C}_{u,y}^-$ is an union of finitely many arithmetic progressions and a finite subset of $\mathbf{N}$. In particular, $\mathcal{C}_{u,y}^-$ has an asymptotic density. If we put $\delta_y := \mathbf{d}(\mathcal{C}_{u,y}^-)$, then it is clear that δ_y is a bounded nondecreasing function of y, hence the limit

$$\delta := \lim_{y \to +\infty} \delta_y \tag{4}$$

exists finite. We shall prove that $\mathcal{C}_u$ has asymptotic density δ. Hereafter, all the implied constants may depend on $(u_n)_{n \geq 0}$ and k. If $n \in \mathcal{C}_{u,y}^+(x)$ then there exists a prime $p > y$ such that $p \mid n$ and $p \mid u_n$. Furthermore, $B_u u_n = n v_n + w_n$ implies that $p \mid w_n$. Hence, we can write $n = pm$ for some positive integer $m \leq x/p$ such that $w_{pm} \equiv 0 \pmod p$. For sufficiently large y, we have that p does not divide $f_w(0) B_w \Delta_w$ (actually, $B_w = 1$) and is coprime with at least one term of $(w_s)_{s \geq 0}$, since $(w_s)_{s \geq 0}$ is not identically zero.

Therefore, by applying Lemma 2.3 to $(w_s)_{s \geq 0}$, we get that the number of possible values of m is at most

$$O\!\left(\frac{x}{pT_w(p)} + 1\right).$$

As a consequence,

$$\#\mathcal{C}_{u,y}^+(x) \ll \sum_{y < p \leq x} \left(\frac{x}{pT_w(p)} + 1\right) \ll x \cdot \left(\sum_{p > y} \frac{1}{pT_w(p)} + \frac{1}{\log x}\right), \tag{5}$$

where we also used the Chebyshev's bound for the number of primes not exceeding x. Setting $\gamma := 1/(k+1)$, by partial summation and Lemma 2.2, we have

$$\sum_{\substack{p > y \\ p \in \mathcal{P}_{w,\gamma}}} \frac{1}{pT_w(p)} \leq \sum_{\substack{p > y \\ p \in \mathcal{P}_{w,\gamma}}} \frac{1}{p} = \left[\frac{\#\mathcal{P}_{w,\gamma}(t)}{t}\right]_{t=y}^{+\infty} + \int_y^{+\infty} \frac{\#\mathcal{P}_{w,\gamma}(t)}{t^2} \mathrm{d}t \ll \frac{1}{y^{1-k\gamma}} = \frac{1}{y^\gamma}. \tag{6}$$

On the other hand,

$$\sum_{\substack{p > y \\ p \notin \mathcal{P}_{w,\gamma}}} \frac{1}{pT_w(p)} \leq \sum_{\substack{p > y \\ p \notin \mathcal{P}_{w,\gamma}}} \frac{1}{p^{1+\gamma}} \ll \int_y^{+\infty} \frac{\mathrm{d}t}{t^{1+\gamma}} \ll \frac{1}{y^\gamma} \tag{7}$$

Thus, putting together (5), (6), and (7), we obtain

$$\frac{\#\mathcal{C}_{u,y}^+(x)}{x} \ll \frac{1}{y^\gamma} + \frac{1}{\log x},$$

so that

$$\limsup_{x \to +\infty} \left|\frac{\#\mathcal{C}_u(x)}{x} - \delta_y\right| = \limsup_{x \to +\infty} \left|\frac{\#\mathcal{C}_u(x)}{x} - \frac{\#\mathcal{C}_{u,y}^-(x)}{x}\right| = \limsup_{x \to +\infty} \frac{\#\mathcal{C}_{u,y}^+(x)}{x} \ll \frac{1}{y^\gamma}, \tag{8}$$

hence, by letting $y \to +\infty$ in (8) and by using (4), we get that $\mathcal{C}_u$ has asymptotic density δ.

It remains only to prove that $\delta < 1$. Clearly,

$$\mathcal{C}_{u,y}^- \subseteq \{n \in \mathbf{N} : p \mid n \text{ for some } p \leq y\},$$

so that, by Eratosthenes' sieve and Mertens' third theorem [11, Ch. I.1, Theorem 11], we have

$$\limsup_{x \to +\infty} \frac{\#\mathcal{C}_{u,y}^-(x)}{x} \leq 1 - \prod_{p \leq y} \left(1 - \frac{1}{p}\right) \leq 1 - \frac{c_1}{\log y}, \tag{9}$$

for all $y \geq 2$, where $c_1 > 0$ is an absolute constant. Furthermore, the last part of (8) says that

$$\limsup_{x \to +\infty} \frac{\#\mathcal{C}_{u,y}^+(x)}{x} \leq \frac{c_2}{y^\gamma}, \tag{10}$$

for all sufficiently large y, where $c_2 > 0$ is an absolute constant.

Therefore, putting together (9) and (10), we get

$$\delta = \lim_{x \to +\infty} \frac{\#\mathcal{C}_u(x)}{x} \leq \limsup_{x \to +\infty} \frac{\#\mathcal{C}_{u,y}^-(x)}{x} + \limsup_{x \to +\infty} \frac{\#\mathcal{C}_{u,y}^+(x)}{x} \leq 1 - \left(\frac{c_1}{\log y} - \frac{c_2}{y^\gamma}\right), \tag{11}$$

for all sufficiently large y.

Finally, picking a sufficiently large y, depending on c_1 and c_2, the bound (11) yields $\delta < 1$. The proof of Theorem 1.1 is complete.

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