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# PRACTICAL NUMBERS IN LUCAS SEQUENCES

CARLO SANNA

ABSTRACT. A *practical number* is a positive integer  $n$  such that all the positive integers  $m \leq n$  can be written as a sum of distinct divisors of  $n$ . Let  $(u_n)_{n \geq 0}$  be the Lucas sequence satisfying  $u_0 = 0$ ,  $u_1 = 1$ , and  $u_{n+2} = au_{n+1} + bu_n$  for all integers  $n \geq 0$ , where  $a$  and  $b$  are fixed nonzero integers. Assume  $a(b+1)$  even and  $a^2 + 4b > 0$ . Also, let  $\mathcal{A}$  be the set of all positive integers  $n$  such that  $|u_n|$  is a practical number. Melfi proved that  $\mathcal{A}$  is infinite. We improve this result by showing that  $\#\mathcal{A}(x) \gg x/\log x$  for all  $x \geq 2$ , where the implied constant depends on  $a$  and  $b$ . We also pose some open questions regarding  $\mathcal{A}$ .

## 1. INTRODUCTION

A *practical number* is a positive integer  $n$  such that all the positive integers  $m \leq n$  can be written as a sum of distinct divisors of  $n$ . The term “practical” was coined by Srinivasan [7]. Let  $\mathcal{P}$  be the set of practical numbers. Estimates for the counting function  $\#\mathcal{P}(x)$  were given by Hausman and Shapiro [1], Tenenbaum [10], Margenstern [2], Saias [5], and, finally, Weingartner [12], who proved that there exists a constant  $C > 0$  such that

$$\#\mathcal{P}(x) = \frac{Cx}{\log x} \cdot \left(1 + O\left(\frac{\log \log x}{\log x}\right)\right)$$

for all  $x \geq 3$ , settling a conjecture of Margenstern [2].

In analogy with well-known conjectures about prime numbers, Melfi [4] proved that every positive even integer is the sum of two practical numbers, and that there are infinitely many triples  $(n, n+2, n+4)$  of practical numbers. Let  $(u_n)_{n \geq 0}$  be a Lucas sequence, that is, a sequence of integers satisfying  $u_0 = 0$ ,  $u_1 = 1$ , and  $u_{n+2} = au_{n+1} + bu_n$  for all integers  $n \geq 0$ , where  $a$  and  $b$  are two fixed nonzero integers. Also, let  $\mathcal{A}$  be the set of all positive integers  $n$  such that  $|u_n|$  is a practical number. From now on, we assume  $a^2 + 4b > 0$  and  $a(b+1)$  even. We remark that, in the study of  $\mathcal{A}$ , assuming  $a(b+1)$  even is not a loss of generality. Indeed, if  $a(b+1)$  is odd then  $u_n$  is odd for all  $n \geq 1$  and, since 1 is the only odd practical number, it follows that  $\mathcal{A} = \{1\}$ . Melfi [3, Theorem 10] proved the following result.

**Theorem 1.1.** *The set  $\mathcal{A}$  is infinite. Precisely,  $2^j \cdot 3 \in \mathcal{A}$  for all sufficiently large positive integers  $j$ , how large depending on  $a$  and  $b$ , and hence*

$$\#\mathcal{A}(x) \gg \log x,$$

for all sufficiently large  $x > 1$ .

In this paper, we improve Theorem 1.1 to the following:

**Theorem 1.2.** *For all  $x \geq 2$ , we have*

$$\#\mathcal{A}(x) \gg \frac{x}{\log x},$$

where the implied constant depends on  $a$  and  $b$ .

We leave the following open questions to the interested readers:

(Q1) Does  $\mathcal{A}$  have zero natural density?

(Q2) Can a nontrivial upper bound for  $\#\mathcal{A}(x)$  be proved?

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(Q3) Are there infinitely many nonpractical  $n$  such that  $|u_n|$  is practical?

(Q4) Are there infinitely many practical  $n$  such that  $|u_n|$  is nonpractical?

(Q5) What about practical numbers in general integral linear recurrences over the integers?

**Notation.** For any set of positive integers  $\mathcal{S}$ , we put  $\mathcal{S}(x) := \mathcal{S} \cap [1, x]$  for all  $x \geq 1$ , and  $\#\mathcal{S}(x)$  denotes the counting function of  $\mathcal{S}$ . We employ the Landau–Bachmann “Big Oh” notation  $O$ , as well as the associated Vinogradov symbols  $\ll$  and  $\gg$ , with their usual meanings. Any dependence of the implied constants is explicitly stated. As usual, we write  $\mu(n)$ ,  $\varphi(n)$ ,  $\sigma(n)$ , and  $\omega(n)$ , for the Möbius function, the Euler’s totient function, the sum of divisors, and the number of prime factors of a positive integer  $n$ , respectively.

## 2. PRELIMINARIES ON LUCAS SEQUENCES

In this section we collect some basic facts about Lucas sequences. Let  $\alpha$  and  $\beta$  be the two roots of the characteristic polynomial  $X^2 - aX - b$ . Since  $a^2 + 4b > 0$  and  $b \neq 0$ , we have that  $\alpha$  and  $\beta$  are real, nonzero, and distinct. It is well known that the *generalized Binet’s formula*

$$(1) \quad u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

holds for all integers  $n \geq 0$ . Define

$$\Phi_n := \prod_{\substack{1 \leq k \leq n \\ \gcd(n,k)=1}} \left( \alpha - e^{2\pi i k/n} \beta \right),$$

for each positive integer  $n$ . It can be proved that  $\Phi_n \in \mathbb{Z}$  for all integers  $n > 1$  (see, e.g., [9, p. 428]). Furthermore, we have

$$(2) \quad u_n = \prod_{\substack{d|n \\ d>1}} \Phi_d$$

and, by the Möbius inversion formula,

$$(3) \quad \Phi_n = \prod_{d|n} u_{n/d}^{\mu(d)}$$

for all integers  $n > 1$ . Changing the sign of  $a$  changes the signs of  $\alpha, \beta$  and turns  $u_n$  into  $(-1)^{n+1}u_n$ , which is not a problem, since for the study of  $\mathcal{A}$  we are interested only in  $|u_n|$ . Hence, without loss of generality, we can assume  $a > 0$  and  $\alpha > |\beta|$ , which in turn implies that  $u_n, \Phi_n > 0$  for all integers  $n > 0$ . We conclude this section with an easy lemma regarding the growth of  $u_n$  and  $\Phi_n$ .

**Lemma 2.1.** *For all integers  $n > 0$ , we have*

$$(i) \quad u_n \geq u_{n-1};$$

$$(ii) \quad u_n = \alpha^{n+O(1)};$$

$$(iii) \quad \Phi_n = \alpha^{\varphi(n)+O(1)};$$

where the implied constants depend on  $a$  and  $b$ .

*Proof.* If  $b > 0$ , then (i) is clear from the recursion for  $u_n$ . Hence, suppose  $b < 0$ , so that  $\beta > 0$ . After a bit of manipulations, (i) is equivalent to  $\alpha^{n-1}(\alpha - 1) \geq \beta^{n-1}(\beta - 1)$ , which in turn follows easily since  $\alpha > \beta > 0$ . Claim (ii) is a consequence of (1). Setting  $\gamma := \beta/\alpha$ , by (1) and (3), we get

$$\Phi_n = \alpha^{\varphi(n)} \prod_{d|n} \left( \frac{1 - \gamma^{n/d}}{\alpha - \beta} \right)^{\mu(d)} = \alpha^{\varphi(n)} \prod_{d|n} \left( 1 - \gamma^{n/d} \right)^{\mu(d)},$$

for all integers  $n > 1$ , where we used the well-known formulas  $\sum_{d|n} \mu(d) \frac{n}{d} = \varphi(n)$  and  $\sum_{d|n} \mu(d) = 0$ . Therefore, since  $|\gamma| < 1$ , we have

$$|\log(\Phi_n/\alpha^{\varphi(n)})| \leq \sum_{d|n} |\log(1 - \gamma^d)| \ll \sum_{d=1}^{\infty} |\gamma|^d \ll 1,$$

and also (iii) is proved. □

### 3. PRELIMINARIES ON PRACTICAL NUMBERS AND CLOSE RELATIVES

The following lemma on practical numbers will be fundamental later.

**Lemma 3.1.** *If  $n$  is a practical number and  $m \leq 2n$  is a positive integer, then  $mn$  is a practical number.*

*Proof.* See [4, Lemma 1]. □

Close relatives of practical numbers are  $\varphi$ -practical numbers. A  $\varphi$ -practical number is a positive integer  $n$  such that all the positive integers  $m \leq n$  can be written as  $m = \sum_{d \in \mathcal{D}} \varphi(d)$ , where  $\mathcal{D}$  is a subset of the divisors of  $n$ . This notion was introduced by Thompson [11] while studying the degrees of the divisors of the polynomial  $X^n - 1$ . Indeed,  $\varphi$ -practical numbers are exactly the positive integers  $n$  such that  $X^n - 1$  has a divisor of every degree up to  $n$ .

We need a couple of results regarding  $\varphi$ -practical numbers.

**Lemma 3.2.** *Let  $n$  be a  $\varphi$ -practical number and  $p$  be a prime number not dividing  $n$ . Then  $pn$  is  $\varphi$ -practical if and only if  $p \leq n + 2$ . Moreover,  $p^j n$  is  $\varphi$ -practical if and only if  $p \leq n + 1$ , for every integer  $j \geq 2$ .*

*Proof.* See [11, Lemma 4.1]. □

**Lemma 3.3.** *If  $n$  is an even  $\varphi$ -practical number, and if  $d_1, \dots, d_s$  are all the divisors of  $n$  ordered so that  $\varphi(d_1) \leq \dots \leq \varphi(d_s)$ , then*

$$(4) \quad \varphi(d_{j+1}) \leq \sum_{i=1}^j \varphi(d_i),$$

for all positive integers  $j < s$ .

*Proof.* It is not difficult to see that  $n$  is  $\varphi$ -practical if and only if

$$(5) \quad \varphi(d_{j+1}) \leq 1 + \sum_{i=1}^j \varphi(d_i),$$

for all positive integers  $j < s$  (see [11, p. 1041]). Hence, we have only to prove that  $n$  even ensures that in (5) the equality cannot happen. If  $j = 1$  then (4) is obvious since  $\{d_1, d_2\} = \{1, 2\}$ , so we can assume  $1 < j < s$ . At this point  $\varphi(d_{j+1})$  is even, while

$$1 + \sum_{i=1}^j \varphi(d_i)$$

is odd, because  $\varphi(m)$  is even for all integers  $m > 2$ . Thus, in (5) the equality is not satisfied. □

Let  $\theta$  be a real-valued arithmetic function, and define  $\mathcal{B}_\theta$  as the set containing  $n = 1$  and all those  $n = p_1^{a_1} \cdots p_k^{a_k}$ , where  $p_1 < \dots < p_k$  are prime numbers and  $a_1, \dots, a_k$  are positive integers, which satisfy

$$p_j \leq \theta \left( \prod_{i=1}^{j-1} p_i^{a_i} \right),$$

for  $j = 1, \dots, k$ , where the empty product is equal to 1. If  $\theta(n) := \sigma(n) + 1$ , then  $\mathcal{B}_\theta$  is the set of practical numbers. This is a characterization given by Stewart [8] and Sierpiński [6].

Weingartner proved a general and strong estimate for  $\#\mathcal{B}_\theta(x)$ . The following is a simplified version adapted just for our purposes.

**Theorem 3.4.** *Suppose  $\theta(1) \geq 2$  and  $n \leq \theta(n) \leq An$  for all positive integers  $n$ , where  $A \geq 1$  is a constant. Then, we have*

$$\#\mathcal{B}_\theta(x) \sim \frac{c_\theta x}{\log x},$$

as  $x \rightarrow +\infty$ , where  $c_\theta > 0$  is a constant.

*Proof.* See [12, Theorems 1.2 and 5.1]. □

#### 4. PROOF OF THEOREM 1.2

The key tool of the proof is the following technical lemma.

**Lemma 4.1.** *Suppose that  $n$  is a sufficiently large positive integer, how large depending on  $a$  and  $b$ . Let  $p$  be a prime number and write  $n = p^v m$  for some nonnegative integer  $v$  and some positive integer  $m$  not divisible by  $p$ . If  $m$  is an even  $\varphi$ -practical number,  $n \in \mathcal{A}$ , and  $p < m$ , then  $p^k n \in \mathcal{A}$  for all positive integers  $k$ .*

*Proof.* Clearly, it is enough to prove the claim for  $k = 1$ . Let  $d_1 = 1, d_2 = 2, \dots, d_s$  be all the divisors of  $m$ , ordered to that  $\varphi(d_1) \leq \dots \leq \varphi(d_s)$ . Furthermore, define

$$N_j := u_n \prod_{i=1}^j \Phi_{p^{v+1}d_i},$$

for  $j = 1, \dots, s$ . We shall prove that each  $N_j$  is practical. This implies the thesis, since  $N_s = u_{pn}$  by (2).

We proceed by induction on  $j$ . First, by (2) and Lemma 2.1(i), we have

$$\Phi_{p^{v+1}d_1} = \Phi_{p^{v+1}} \leq u_{p^{v+1}} \leq u_{p^v m} = u_n,$$

since  $p < m$ . Hence, applying Lemma 3.1 and the fact that  $u_n$  is practical, we get that  $N_1 = u_n \Phi_{p^{v+1}d_1}$  is practical.

Now assuming that  $N_j$  is practical we shall prove that  $N_{j+1}$  is practical. Again, since  $N_{j+1} = \Phi_{p^{v+1}d_{j+1}} N_j$ , thanks to Lemma 3.1 it is enough to show that the inequality

$$(6) \quad \Phi_{p^{v+1}d_{j+1}} \leq u_n \prod_{i=1}^j \Phi_{p^{v+1}d_i}$$

holds. In turn, by Lemma 2.1(ii) and (iii), we have that (6) is implied by

$$(7) \quad n + \varphi(p^{v+1}) \left[ -\varphi(d_{j+1}) + \sum_{i=1}^j \varphi(d_i) \right] \geq C(j+1),$$

where  $C > 0$  is a constant depending only on  $a$  and  $b$ .

On the one hand, since  $m$  is an even  $\varphi$ -practical number, by Lemma 3.3 we have that the term of (7) in square brackets is nonnegative. On the other hand, for sufficiently large  $n$ , we have

$$n \geq C(\log n / \log 2 + 1) \geq C(\omega(n) + 1) \geq C(j+1).$$

Therefore, (7) holds and the proof is complete. □

We are ready to prove Theorem 1.2. Pick a sufficiently large positive integer  $h$ , depending on  $a$  and  $b$ , such that the claim of Lemma 4.1 holds for all integers  $n \geq 2^h \cdot 3$ . Moreover, by Theorem 1.1, we can assume that  $2^j \cdot 3 \in \mathcal{A}$  for all integers  $j \geq h$ . Put  $\mathcal{B} := \mathcal{B}_\theta \setminus \{1\}$ , where  $\theta(n) := \max\{2, n\}$ . Note that, as a consequence of Lemma 3.2, all the elements of  $\mathcal{B}$  are even  $\varphi$ -practical numbers. We shall prove that for all  $n \in \mathcal{B}$  we have  $2^h \cdot 3n \in \mathcal{A}$ . In this way, thanks to Theorem 3.4, we get

$$\#\mathcal{A}(x) \geq \#\mathcal{B}\left(\frac{x}{2^h \cdot 3}\right) \gg \frac{x}{\log x},$$

for all sufficiently large  $x$ . Hence, since  $1 \in \mathcal{A}$ , Theorem 1.2 follows.

We proceed by induction on the number of prime factors of  $n \in \mathcal{B}$ . If  $n \in \mathcal{B}$  has exactly one prime factor, then it follows easily that  $n = 2^j$  for some positive integer  $j$ . Hence, we have  $2^h \cdot 3n = 2^{h+j} \cdot 3 \in \mathcal{A}$ , as claimed.

Now, assuming that the claim is true for all  $n \in \mathcal{B}$  with exactly  $k \geq 1$  prime factors, we shall prove it for all  $n \in \mathcal{B}$  having  $k+1$  prime factors. Write  $n = p_1^{a_1} \cdots p_{k+1}^{a_{k+1}}$ , where  $p_1 < \cdots < p_{k+1}$  are prime numbers and  $a_1, \dots, a_{k+1}$  are positive integers. Put also  $m := p_1^{a_1} \cdots p_k^{a_k}$ . Since  $n \in \mathcal{B}$ , we have  $m \in \mathcal{B}$  and  $p_{k+1} < m$ . On the one hand, by the induction hypothesis,  $2^h \cdot 3m \in \mathcal{A}$ . On the other hand, it is easy to see that  $m \in \mathcal{B}$  implies  $2^h m \in \mathcal{B}$  and  $2^h \cdot 3m \in \mathcal{B}$ .

First, suppose  $p_{k+1} > 3$ . Since  $2^h \cdot 3m$  is an even  $\varphi$ -practical number,  $2^h \cdot 3m \in \mathcal{A}$ , and  $p_{k+1} < 2^h \cdot 3m$ , by Lemma 4.1 we get that  $2^h \cdot 3n = 2^h \cdot 3mp_{k+1}^{a_{k+1}} \in \mathcal{A}$ , as claimed.

On the other hand, if  $p_{k+1} = 3$  the situation is similar. Since  $2^h m$  is an even  $\varphi$ -practical number,  $2^h \cdot 3m \in \mathcal{A}$ , and  $p_{k+1} < 2^h m$ , by Lemma 4.1 we get that  $2^h \cdot 3n = 2^h \cdot 3mp_{k+1}^{a_{k+1}} \in \mathcal{A}$ , as claimed. The proof is complete.

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UNIVERSITÀ DEGLI STUDI DI TORINO, DEPARTMENT OF MATHEMATICS, TURIN, ITALY  
*E-mail address:* carlo.sanna.dev@gmail.com