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COVERING AN ARITHMETIC PROGRESSION WITH GEOMETRIC PROGRESSIONS AND VICE VERSA

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We show that there exists a positive constant C such that the following holds: Given an infinite arithmetic progression $\mathcal{A}$ of real numbers and a sufficiently large integer n (depending on $\mathcal{A}$), there needs at least Cn geometric progressions to cover the first n terms of $\mathcal{A}$. A similar result is presented, with the role of arithmetic and geometric progressions reversed.

Keywords: Arithmetic progressions; geometric progressions; covering problems; square-free numbers.

Mathematics Subject Classification 2010: 11B25, 11A99

1. Introduction

Arithmetic and geometric progressions are always an active research topic in Number Theory. In particular, problems concerning arithmetic progressions and covering, mostly over the integers, are well studied (for example, see [3]). For $v \geq 0$ and $d > 0$, let

$$\mathcal{A}(v, d) := \{v, v + d, v + 2d, v + 3d, \ldots\}$$

be the arithmetic progression with first term v and common difference d. Also, for $u > 0$ and $q > 1$, let

$$\mathcal{G}(u, q) := \{u, uq, uq^2, uq^3, \ldots\}$$

be the geometric progression with first term u and ratio q. Furthermore, for a positive integer n, let $\mathcal{A}^{(n)}$, respectively $\mathcal{G}^{(n)}$, be the set of the first n terms of the arithmetic progression $\mathcal{A}$, respectively the geometric progression $\mathcal{G}$. Now, for a finite set $\mathcal{S}$ of nonnegative real numbers, denote by $g(\mathcal{S})$ the least positive integer h such that there exist h geometric progressions $\mathcal{G}_1, \ldots, \mathcal{G}_h$ covering $\mathcal{S}$, i.e., $\mathcal{S} \subseteq \bigcup_{i=1}^{h} \mathcal{G}_i$. Similarly, denote by $a(\mathcal{S})$ the least positive integer h such that there exist h arithmetic progressions covering $\mathcal{S}$. Since given any two distinct nonnegative real numbers there is an arithmetic progression, respectively a geometric progression, containing them; it follows easily that $a(\mathcal{S}), g(\mathcal{S}) \leq (|\mathcal{S}|+1)/2$. On the other hand, obviously, $a(\mathcal{A}^{(n)}) = g(\mathcal{G}^{(n)}) = 1$ for each arithmetic progression $\mathcal{A}$ and each geometric progression $\mathcal{G}$. We are interested in lower bounds for $g(\mathcal{A}^{(n)})$ and $a(\mathcal{G}^{(n)})$. Our first result is the following theorem.

Theorem 1.1. *There exists a positive constant C_1 such that for each arithmetic progression $\mathcal{A} = \mathcal{A}(v,d)$ we have $g(\mathcal{A}^{(n)}) \geq C_1 n$ for n sufficiently large (how large depending only on v/d). In particular, we can take $C_1 = 1/\pi^2 - \varepsilon$ with $\varepsilon > 0$.*

Regarding a lower bound for $a(\mathcal{G}^{(n)})$, with $\mathcal{G} = \mathcal{G}(u,q)$, the situation is a little bit different. In fact, we need to distinguish according to whether q is a root of a rational number > 1 or not.

Theorem 1.2. *Let $q = r^{1/m}$ with $r > 1$ a rational number and let m be a positive integer such that $q^{m'}$ is irrational for any positive integer $m' < m$. Then $a(\mathcal{G}^{(n)}) \leq m$ for each geometric progression $\mathcal{G} = \mathcal{G}(u,q)$ and each integer $n \geq 1$, with equality if $n \geq 2m$.*

Theorem 1.3. *There exists a positive constant C_2 such that if $q \neq r^{1/m}$ for all rationals $r > 1$ and all positive integers m, then $a(\mathcal{G}^{(n)}) \geq C_2 n$ for each geometric progression $\mathcal{G}$ and each integer $n \geq 1$. In particular, we can take $C_2 = 1/6$.*

A natural question, open to us, is the evaluation of the best constants C_1 and C_2 in Theorem 1.1 and Theorem 1.3, i.e., to find

$$\inf_{\mathcal{A}} \liminf_{n\to\infty} \frac{g(\mathcal{A}^{(n)})}{n} \quad \text{and} \quad \inf_{\mathcal{G}} \liminf_{n\to\infty} \frac{a(\mathcal{G}^{(n)})}{n},$$

where $\mathcal{A}$ runs over all the arithmetic progressions and $\mathcal{G}$ runs over all the geometric progressions $\mathcal{G} = \mathcal{G}(u,q)$, with ratio q not a root of a rational number > 1. The results above give $1/\pi^2 \leq C_1 \leq 1/2$ and $1/6 \leq C_2 \leq 1/2$.

Notation

Hereafter, $\mathbf{N}$ denotes the set of positive integers and $\mathbf{N}_0 := \mathbf{N} \cup \{0\}$. The letter p is reserved for prime numbers and $\upsilon_p(\cdot)$ denotes the p-adic valuation over the field of rational numbers $\mathbf{Q}$.

2. Preliminaries

The fundamental tool for our results is the following theorem of A. Dubickas and J. Jankauskas, regarding the intersection of arithmetic and geometric progressions [1, Theorem 3 and 4].

Theorem 2.1. *Suppose that the ratio $q > 1$ is not or the form $r^{1/m}$, with $r > 1$ a rational number and $m \in \mathbf{N}$, then $|\mathcal{A} \cap \mathcal{G}| \leq 6$ for each arithmetic progression $\mathcal{A}$ and each $\mathcal{G} = \mathcal{G}(u,q)$.*

If the ratio q of the geometric progression $\mathcal{G}$ is a root of a rational number > 1, then, without further assumptions, $|\mathcal{A} \cap \mathcal{G}|$ can take any nonnegative integer value, or even be infinite [1, Theorem 1 and 2]. However, we have the following:

Lemma 2.2. *Suppose that $q = r^{1/m}$, with $r > 1$ rational and $m \in \mathbf{N}$ such that $q^{m'}$ is irrational for any positive integer $m' < m$. If $|\mathcal{A}(v,d) \cap \mathcal{G}(u,q)| \geq 3$ then v/d is rational and $u/d = sq^{-\ell}$ for some $s \in \mathbf{Q}$ and some $\ell \in \{0, 1, \ldots, m-1\}$. Moreover, for each $uq^k \in \mathcal{A}(v,d) \cap \mathcal{G}(u,q)$ it results $k \equiv \ell$ (mod m).*

Proof. Since $|\mathcal{A}(v,d) \cap \mathcal{G}(u,q)| \geq 3$, there exist $k_1, k_2, k_3 \in \mathbf{N}_0$ pairwise distinct and such that $uq^{k_i} = v + dh_i$, with $h_i \in \mathbf{N}_0$ $(i = 1, 2, 3)$. Set $t := v/d$ and $\xi := u/d$, so that $\xi q^{k_i} = t + h_i$ for each i. Then, $q^{k_1} \neq q^{k_3}$ and

$$\frac{q^{k_1} - q^{k_2}}{q^{k_1} - q^{k_3}} = \frac{h_1 - h_2}{h_1 - h_3} \in \mathbf{Q},$$

so that $q^{k_1}, q^{k_2}, q^{k_3}$ are linearly dependent over $\mathbf{Q}$. Since $x^m - r$ is the minimal polynomial of q over the rationals (as it follows at once from our assumptions), we have that $q^0, q^1, \ldots, q^{m-1}$ are linearly independent over $\mathbf{Q}$. It follows that at least two of k_1, k_2, k_3 lie in the same class modulo m. Without loss of generality, we can assume $k_1 \equiv k_2 \pmod{m}$, so that $q^{k_1 - k_2}$ is rational. Now

$$t + h_1 = \xi q^{k_1} = q^{k_1 - k_2} \xi q^{k_2} = q^{k_1 - k_2}(t + h_2),$$

thus, on the one hand,

$$t = \frac{h_1 - q^{k_1 - k_2} h_2}{q^{k_1 - k_2} - 1} \in \mathbf{Q},$$

and on the other hand $\xi = (t + h_1)q^{-k_1} = sq^{-\ell}$ for some $s \in \mathbf{Q}$ and some $\ell \in \{0, 1, \ldots, m-1\}$ such that $\ell \equiv k_1 \pmod{m}$. In conclusion, for each $uq^k \in \mathcal{A}(v,d) \cap \mathcal{G}(u,q)$ we have $\xi q^k = t + h$ for some $h \in \mathbf{N}_0$, so $q^{k-\ell} = (t+h)/s \in \mathbf{Q}$ and necessarily $k \equiv \ell \pmod{m}$. $\square$

Finally, we need to state the following lemma about the asymptotic density of squarefree integers in an arithmetic progression [2].

Lemma 2.3. *Let a, b be integers with $b \geq 1$ and $\gcd(a,b) = 1$. Then*

$$|\{k \in \mathbf{N}_0 : k \leq x \text{ and } a + bk \text{ is squarefree}\}| \sim \frac{6}{\pi^2} \prod_{p \mid b} \left(1 - \frac{1}{p^2}\right)^{-1} x$$

as $x \to \infty$, where for $b = 1$ the empty product is interpreted as 1.

3. Proof of Theorem 1.1

Let $\mathcal{A} = \mathcal{A}(v,d)$ and $n \in \mathbf{N}$. For the sake of brevity, set $g := g(\mathcal{A}^{(n)})$ and let $\mathcal{G}_1, \ldots, \mathcal{G}_g$ be geometric progressions such that $\mathcal{A}^{(n)} \subseteq \bigcup_{i=1}^{g} \mathcal{G}_i$. Suppose that

$|\mathcal{A} \cap \mathcal{G}_i| \leq 6$ for $i = 1, 2, \dots, g$. Then

$$n = |\mathcal{A}^{(n)}| \leq \sum_{i=1}^{g} |\mathcal{A}^{(n)} \cap \mathcal{G}_i| \leq \sum_{i=1}^{g} |\mathcal{A} \cap \mathcal{G}_i| \leq 6g,$$

which implies $g \geq n/6 > n/\pi^2$.

Suppose now that there exists $i_0 \in \{1, 2, \dots, g\}$ such that $|\mathcal{A} \cap \mathcal{G}_{i_0}| > 6$. For a moment, let $\mathcal{G}_{i_0} = \mathcal{G}(u, q)$. It follows from Theorem 2.1 that $q = r^{1/m}$ for some rational number $r > 1$ and $m \in \mathbf{N}$. In particular, we can assume that $q^{m'}$ is irrational for any positive integer $m' < m$. Therefore, Lemma 2.2 implies that $t := v/d$ is rational, $u/d = sq^{-\ell}$ with $s \in \mathbf{Q}$, $\ell \in \{0, 1, \dots, m-1\}$, and that for each $uq^k \in \mathcal{A} \cap \mathcal{G}_{i_0}$ we have $k \equiv \ell \pmod{m}$. Since $t \geq 0$ is rational, we can write $t = a/b$, where $a \geq 0$ and $b \geq 1$ are relatively prime integers. On the other hand, if $uq^k \in \mathcal{A} \cap \mathcal{G}_{i_0}$ then $k = mj + \ell$ and $uq^k = v + dh$ for some $j, h \in \mathbf{N}_0$. As a consequence, $sr^j = t + h$ and $bsr^j = a + bh$. Now we claim that there exist at most two $j \in \mathbf{N}_0$ such that bsr^j is a squarefree integer. In fact, since $r > 1$, there exists a prime p such that $\upsilon_p(r) \neq 0$. So $\upsilon_p(bsr^j) = \upsilon_p(bs) + j\upsilon_p(r)$ is a strictly monotone function of j and can take the values 0 or 1, which is a necessary condition for bsr^j to be a squarefree integer, for at most two $j \in \mathbf{N}_0$. Consequently, if we define

$$\mathcal{B}^{(n)} := \{v + dh \in \mathcal{A}^{(n)} : a + bh \text{ is squarefree}\},$$

then $|\mathcal{B}^{(n)} \cap \mathcal{G}_{i_0}| \leq 2$. Note that the definition of $\mathcal{B}^{(n)}$ depends only on v, d and n, so we can conclude that $|\mathcal{B}^{(n)} \cap \mathcal{G}_i| \leq 6$ for all $i = 1, 2, \dots, g$. In fact, on the one hand, if $|\mathcal{A}^{(n)} \cap \mathcal{G}_i| \leq 6$ then it is straightforward that $|\mathcal{B}^{(n)} \cap \mathcal{G}_i| \leq 6$, since $\mathcal{B}^{(n)} \subseteq \mathcal{A}^{(n)}$. On the other hand, if $|\mathcal{A}^{(n)} \cap \mathcal{G}_i| > 6$ then we have proved that $|\mathcal{B}^{(n)} \cap \mathcal{G}_i| \leq 2$. Now, Lemma 2.3 yields

$$|\mathcal{B}^{(n)}| \sim \frac{6}{\pi^2} \prod_{p \mid b} \left(1 - \frac{1}{p^2}\right)^{-1} n,$$

as $n \to \infty$, so that $|\mathcal{B}^{(n)}| \geq 6(1/\pi^2 - \varepsilon)n$ for n sufficiently large, depending only on a, b, i.e., t.

In conclusion,

$$6\left(\frac{1}{\pi^2} - \varepsilon\right)n \leq |\mathcal{B}^{(n)}| \leq \sum_{i=1}^{g} |\mathcal{B}^{(n)} \cap \mathcal{G}_i| \leq 6g,$$

hence $g \geq n(1/\pi^2 - \varepsilon)$, for sufficiently large n. This completes the proof.

4. Proofs of Theorem 1.2 and 1.3

Let $\mathcal{G} = \mathcal{G}(u, q)$ and $n \in \mathbf{N}$. For the sake of brevity, set $a := a(\mathcal{G}^{(n)})$ and let $\mathcal{A}_1, \dots, \mathcal{A}_a$ be arithmetic progressions such that $\mathcal{G}^{(n)} \subseteq \bigcup_{i=1}^{a} \mathcal{A}_i$. Suppose $q = r^{1/m}$, for a rational number $r > 1$ and $m \in \mathbf{N}$ such that $q^{m'}$ is irrational for all positive integers $m' < m$. Since $r > 1$ is rational, we can write $r = r_1/r_2$, where r_1 and r_2 are coprime positive integers. Then, for $k = 0, 1, \dots, n-1$, we have

$$uq^k = uq^{(k \bmod m)} r^{\lfloor k/m \rfloor} = 0 + \frac{uq^{(k \bmod m)}}{r_2^n} \cdot r_1^{\lfloor k/m \rfloor} r_2^{n - \lfloor k/m \rfloor} \in \mathcal{A}(0, uq^{(k \bmod m)}/r_2^n),$$

so that $\mathcal{G}^{(n)} \subseteq \bigcup_{i=0}^{m-1} \mathcal{A}(0, uq^i/r_2^n)$ and $a \le m$.

Now we shall prove that $a(G^{(n)}) = m$ holds for each $n \ge 2m$. We define the sets $J := \{1, 2, \dots, a\}$,

$$J_1 := \{i \in J : \exists\, uq^{k_1}, uq^{k_2} \in \mathcal{A}_i \cap \mathcal{G}^{(n)} \text{ such that } k_1 \ne k_2,\ k_1 \equiv k_2 \pmod m\},$$

and $J_2 := J \setminus J_1$. Clearly, $\{J_1, J_2\}$ is a partition of J. For $i \in J$, suppose that there exist $uq^{k_1}, uq^{k_2} \in \mathcal{A}_i \cap \mathcal{G}^{(n)}$ such that $k_1 < k_2$. This implies that if $\mathcal{A}_i = \mathcal{A}(v, d)$ then

$$d = \tfrac{1}{s}(uq^{k_2} - uq^{k_1}),$$

for some positive integer s. Furthermore, if $uq^k \in \mathcal{A}_i \cap \mathcal{G}^{(n)}$ then

$$uq^k = uq^{k_1} + dh = uq^{k_1} + \tfrac{h}{s}(uq^{k_2} - uq^{k_1}),$$

for some integer h, hence

$$q^k = (1 - \tfrac{h}{s})q^{k_1} + \tfrac{h}{s} q^{k_2}. \tag{4.1}$$

On the one hand, if $i \in J_1$ then we can assume $k_1 \equiv k_2 \pmod m$, thus q^{k_1} and q^{k_2} are linearly dependent over $\mathbf{Q}$. From (4.1) it follows that also q^k and q^{k_1} are linearly dependent over $\mathbf{Q}$, i.e., $k \equiv k_1 \pmod m$. On the other hand, if $i \in J_2$ then we can assume $k_1 \not\equiv k_2 \pmod m$, thus q^{k_1} and q^{k_2} are linearly independent over $\mathbf{Q}$. From (4.1) it follows that necessarily $h = 0$ and $k = k_1$, or $h = s$ and $k = k_2$. To summarize, we have found that for a fixed $i \in J_1$ it results that all the $uq^k \in \mathcal{A}_i \cap \mathcal{G}^{(n)}$ have k in the same class modulo m, while for $i \in J_2$ we have $|\mathcal{A}_i \cap \mathcal{G}^{(n)}| \le 2$. As a consequence, if

$$R := \Big\{ k_1 \in \{0, 1, \dots, m-1\} : uq^k \in \bigcup_{i \in J_1} (\mathcal{A}_i \cap \mathcal{G}^{(n)}) \text{ for some } k \equiv k_1 \pmod m \Big\},$$

then $|J_1| \ge |R|$. Also, if $k_1 \in \{0, 1, \dots, m-1\} \setminus R$ and $uq^k \in \mathcal{G}^{(n)}$, with $k \equiv k_1 \pmod m$, then $uq^k \notin \bigcup_{i \in J_1} (\mathcal{A}_i \cap \mathcal{G}^{(n)})$. But $uq^k \in \bigcup_{i \in J} (\mathcal{A}_i \cap \mathcal{G}^{(n)})$ and so $uq^k \in \bigcup_{i \in J_2} (\mathcal{A}_i \cap \mathcal{G}^{(n)})$. Thus, it follows that

$$\bigcup_{k_1 \in \{0,1,\dots,m-1\} \setminus R} \{uq^k \in \mathcal{G}^{(n)} : k \equiv k_1 \pmod m\} \subseteq \bigcup_{i \in J_2} (\mathcal{A}_i \cap \mathcal{G}^{(n)}). \tag{4.2}$$

The set on the left hand side of (4.2) is an union of $(m - |R|)$ pairwise disjoint sets, each of them has at least $\lfloor n/m \rfloor$ elements, so it has at least

$$\lfloor n/m \rfloor (m - |R|) \ge 2(m - |R|)$$

elements. This and (4.2) yield

$$2(m - |R|) \le \Big| \bigcup_{i \in J_2} (\mathcal{A}_i \cap \mathcal{G}^{(n)}) \Big| \le \sum_{i \in J_2} |\mathcal{A}_i \cap \mathcal{G}^{(n)}| \le 2|J_2|,$$

so that $|J_2| \geq m - |R|$. In conclusion,

$$a = |J| = |J_1| + |J_2| \geq |R| + (m - |R|) = m,$$

hence $a = m$. This completes the proof of Theorem 1.2.

Suppose now that q is not of the form $r^{1/m}$, with $r > 1$ a rational number and $m \in \mathbf{N}$. From Theorem 2.1 it follows that $|\mathcal{A}_i \cap \mathcal{G}| \leq 6$ for all $i \in J$. Then, for all $n \in \mathbf{N}$,

$$n = |\mathcal{G}^{(n)}| \leq \sum_{i=1}^{a} |\mathcal{A}_i \cap \mathcal{G}^{(n)}| \leq \sum_{i=1}^{a} |\mathcal{A}_i \cap \mathcal{G}| \leq 6a,$$

and so $a \geq n/6$. This completes the proof of Theorem 1.3.

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References

[1] A. Dubickas and J. Jankauskas, *On the intersection of infinite geometric and arithmetic progressions*, Bull. Braz. Math. Soc. (N.S.) **41** (2010), 551–566.

[2] K. Prachar, *Über die kleinste quadratfreie zahl einer arithmetischen reihe*, Monatsh. Math. **62** (1958), 173–176.

[3] Zhi-Wei Sun, *Covering the integers by arithmetic sequences*, Acta Arith. **72** (1995), 109–129.