Algebraic approaches for the design of simultaneous observers for linear systems

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# Algebraic approaches for the design of simultaneous observers for linear systems

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**Abstract:** In this paper, algebraic techniques are proposed to design observers capable of estimating the state of multiple linear continuous-time systems. In order to pursue this objective, firstly an algebraic technique is given to compute the set of all the linear inverses of the observability map of a single plant. Such a result is then used to characterize, through algebraic geometry tools, the simultaneous observability of multiple linear systems both in the forced and in the autonomous case. Such a characterization is finally employed to design a single observer that is capable of estimating the state of multiple linear systems.

## 1 Introduction

In several control and identification applications, the state of a dynamical system cannot be fully measured, thus leading to the need of tools capable of estimating unmeasured variables from available measurements. Several solutions to such a problem have been proposed for both linear [1] and nonlinear [2–6] systems.

Although the design of a state observer for a single continuous-time system can be carried out by using classical techniques, such as the Luenberger observer [7] and the Kalman filter [8], the design of a single observer capable of estimating the state of multiple systems is much more challenging. This problem was firstly introduced in [9] and can be summarized as follows: given a set of plants, design a single observer that can estimate the state of each of these plants. This problem is particularly interesting when one aims at obtaining a reliable estimate of the state of a plant affected by known perturbations that arise from sensor or actuator faults [10].

Several techniques have been proposed in the literature to deal with such a problem. For instance, it has been addressed in [9], by using coprime factorization technique, in [11], by using evolutionary strategies, in [12], by using a parametrization in terms of a stable inverse and a stable null space, and in [13], by using a state-space characterization. Furthermore, in [14, 15], common functional observers have been proposed for two linear systems with unknown inputs. Such results have been extended in [16] to deal with three systems with unknown inputs. On the other hand, switching observers for switched systems have been given in [17–20].

The main objective of this paper is to provide algebraic tools to design a simultaneous observer for multiple linear systems. In order to pursue this objective, in Section 3, the set of all the linear embeddings of a continuous-time system is characterized and is used to determine a parametrization of the set of all the linear inverses of its observability map. Such a parametrization is then used in Section 4 to provide necessary and sufficient conditions for the existence of a simultaneous inverse, which holds for almost all inputs, of the observability maps of a set of linear systems. Such a result is specialized in Section 5 to the case of autonomous systems, for which stronger results are obtained. In Section 6, the theoretical results established in Sections 4 and 5 to guarantee simultaneous observability of a set of systems are used to design a state observer capable of estimating the state of multiple plants. Several examples are given all throughout the paper to illustrate the theoretical results.

The main difference between the tools given in this paper and the ones given in [9, 11–13] is that the former, by employing techniques borrowed from algebraic geometry, provides an exact certificate for the simultaneous observability (or lack thereof) of a set of linear continuous-time systems. Namely, given a set of systems, the proposed method allows one to determine a closed-form expression for the simultaneous inverse of the observability map of such systems, if any, which can be directly used to design a simultaneous observer.

## 2 Notation and preliminaries

In this section, some tools of algebraic geometry are reviewed following the exposition in [21, 22].

Let $\mathbb{Z}$, $\mathbb{Z}_{\geqslant 0}$, $\mathbb{R}$, and $\mathbb{R}_{\geqslant 0}$ denote the sets of integer, natural, real, and non-negative real numbers, respectively. For any integer $z \in \mathbb{Z}$, let $\mathbb{Z}_{\geqslant z} := \{a \in \mathbb{Z} : a \geqslant z\}$. Symbols $\boldsymbol{I}_m$ and $\boldsymbol{0}_{m_1,m_2}$ denote the $m$-dimensional identity matrix and the zero matrix of dimensions $m_1 \times m_2$, respectively; symbol $\otimes$ denotes the Kronecker product. Let $\boldsymbol{e}_i^m$ be the $i$-th column of $\boldsymbol{I}_m$, $i \in \{1, \dots, m\}$. For any matrix $\boldsymbol{A} \in \mathbb{R}^{m_1 \times m_2}$, symbol $\mathcal{E}_r(\boldsymbol{A})$ denotes the *reduced row echelon form of* $A$, which can be computed by using the Gauss-Jordan algorithm [23].

A *multi-index* is a vector $\mathfrak{a} = [\ \mathfrak{a}_1 \ \ \dots \ \ \mathfrak{a}_n\ ]^\top \in \mathbb{Z}^n_{\geqslant 0}$, with $n \in \mathbb{Z}_{\geqslant 1}$. Letting $\boldsymbol{x} = [\ x_1 \ \ \dots \ \ x_n\ ]^\top$, symbol $\boldsymbol{x}^{\mathfrak{a}}$ denotes the *monomial* $x_1^{\mathfrak{a}_1} \dots x_n^{\mathfrak{a}_n}$. A *polynomial* is a finite, linear combination of monomials; a *rational function* is a ratio a polynomials, with the denominator being different from the zero polynomial. The *ring* of all polynomials in $\boldsymbol{x}$ with coefficients in the *field* $\mathbb{K}$ is denoted $\mathbb{K}[x_1, \dots, x_n]$ (briefly, $\mathbb{K}[\boldsymbol{x}]$), whereas the *field* of all rational functions in $\boldsymbol{x}$ is denoted $\mathbb{K}(x_1, \dots, x_n)$ (briefly, $\mathbb{K}(\boldsymbol{x})$). On the other hand, the set of all $m$-dimensional vectors (respectively, of all matrices of dimensions $m_1 \times m_2$) whose entries are polynomials in $\mathbb{K}[\boldsymbol{x}]$ and rational functions in $\mathbb{K}(\boldsymbol{x})$ are denoted $\mathbb{K}^m[\boldsymbol{x}]$ and $\mathbb{K}^m(\boldsymbol{x})$ (respectively, $\mathbb{K}^{m_1 \times m_2}[\boldsymbol{x}]$ and $\mathbb{K}^{m_1 \times m_2}(\boldsymbol{x})$), respectively.

A subset $\mathcal{I}$ of $\mathbb{K}[\boldsymbol{x}]$ is an *ideal* of $\mathbb{K}[\boldsymbol{x}]$ if:

- $0 \in \mathcal{I}$, where 0 is the zero polynomial;
- if $f, g \in \mathcal{I}$, then $f + g \in \mathcal{I}$;
- if $f \in \mathcal{I}$ and $h \in \mathbb{K}[\boldsymbol{x}]$, then $h\, f \in \mathcal{I}$.

Given polynomials $p_1, \dots, p_\ell \in \mathbb{K}[\boldsymbol{x}]$, the set

$$\mathbb{V}(p_1, \dots, p_\ell) := \{\boldsymbol{x} \in \mathbb{K}^n : p_i(\boldsymbol{x}) = 0,\ i = 1, \dots, \ell\}$$

is the *variety* generated by $p_1, \dots, p_\ell$, and the set

$$\langle p_1, \dots, p_\ell \rangle := \{\textstyle\sum_{i=1}^{\ell} q_i\, p_i,\ q_i \in \mathbb{K}[\boldsymbol{x}]\}$$

is an ideal (which is referred to as the *ideal generated by* $p_1,\dots,p_\ell$, and the set $\{p_1,\dots,p_\ell\}$ is referred to as a *basis* of such an ideal). The sets $\{0\}$ and $\mathbb{K}[\boldsymbol{x}]$ are ideals of $\mathbb{K}[\boldsymbol{x}]$ and are denoted $\langle\emptyset\rangle$ and $\langle 1\rangle$, respectively. By the Hilbert basis theorem, each ideal $\mathcal{I}$ in $\mathbb{K}[\boldsymbol{x}]$, $\mathcal{I}\neq\langle\emptyset\rangle$, is finitely generated, *i.e.*, there is $\{p_1,\dots,p_\ell\}\subset\mathbb{K}[\boldsymbol{x}]$ such that $\mathcal{I}=\langle p_1,\dots,p_\ell\rangle$. The concepts of ideal and variety are strongly related. Indeed, for any ideal $\mathcal{I}$ in $\mathbb{K}[\boldsymbol{x}]$, the *variety of* $\mathcal{I}$ is

$$\mathbb{V}(\mathcal{I}) := \{\boldsymbol{x}\in\mathbb{K}^n : p(\boldsymbol{x})=0,\ \forall p\in\mathcal{I}\},$$

and the identity

$$\mathbb{V}(\mathcal{I}) = \mathbb{V}(p_1,\dots,p_\ell)$$

holds for any basis $\{p_1,\dots,p_\ell\}$ of $\mathcal{I}$. Similarly, for any subset $\mathcal{R}\subset\mathbb{K}^n$, the set

$$\mathbb{I}(\mathcal{R}) := \{p\in\mathbb{K}[\boldsymbol{x}] : p(\boldsymbol{x})=0,\ \forall\boldsymbol{x}\in\mathcal{R}\}$$

is an ideal of $\mathbb{K}[\boldsymbol{x}]$ even if $\mathcal{R}$ is not a variety (such an ideal is referred to as the *ideal of* $\mathcal{R}$). For any ideal $\mathcal{I}$ of $\mathbb{K}[\boldsymbol{x}]$, one has that $\mathcal{I}\subset\mathbb{I}(\mathbb{V}(\mathcal{I}))$, but the converse inclusion need not hold, unless $\mathbb{K}$ is an algebraically closed field. Similarly, for any $\mathcal{R}\subset\mathbb{K}^n$, one has that $\mathcal{R}\subset\mathbb{V}(\mathbb{I}(\mathcal{R}))$. In particular, the set $\mathcal{Z}(\mathcal{R}) := \mathbb{V}(\mathbb{I}(\mathcal{R}))$ is the smallest variety in $\mathbb{K}^n$ that contains $\mathcal{R}$, and is called the *Zariski closure of* $\mathcal{R}$. A property holds for "*almost all*" $\boldsymbol{x}\in\mathbb{K}^n$ (or, equivalently, it is "*generic*" in $\mathbb{K}^n$) if the Zariski closure of the set where such a property "does not hold" does not coincide with $\mathbb{K}^n$.

A *monomial order* $\succ$ on $\mathbb{K}[\boldsymbol{x}]$ is a total, well ordering relation on the set of monomials $\boldsymbol{x}^{\mathfrak{a}}\in\mathbb{K}[\boldsymbol{x}]$. The *lexicographic order* (briefly, the Lex order), denoted $\succ_{\mathrm{L}}$, is a monomial order and is defined as follows: $\boldsymbol{x}^{\mathfrak{a}}\succ_{\mathrm{L}}\boldsymbol{x}^{\mathfrak{b}}$ if in the vector difference $\mathfrak{a}-\mathfrak{b}$ the first nonzero entry is positive. Hence, let any monomial order $\succ$ be fixed. For any $p\in\mathbb{K}[\boldsymbol{x}]$, the *leading term of* $p$, denoted $\mathrm{LT}(p)$, is the greatest term $c\,\boldsymbol{x}^{\mathfrak{a}}$ appearing in $p$. A polynomial $r\in\mathbb{K}[\boldsymbol{x}]$ is *reduced* with respect to $\{p_1,\dots,p_\ell\}$ if either $r=0$ or no monomial of $r$ is divisible by any $\mathrm{LT}(p_i)$, $i=1,\dots,\ell$. A finite set $\{g_1,\dots,g_\tau\}$ of $\mathbb{K}[\boldsymbol{x}]$ is a *Gröbner basis* of an ideal $\mathcal{I}$ of $\mathbb{K}[\boldsymbol{x}]$ if

$$\langle \mathrm{LT}(g_1),\dots,\mathrm{LT}(g_\tau)\rangle = \langle\{c\,\boldsymbol{x}^{\mathfrak{a}} : \exists f\in\mathcal{I} : \mathrm{LT}(f)=c\,\boldsymbol{x}^{\mathfrak{a}}\}\rangle.$$

A Gröbner basis $\{g_1,\dots,g_\tau\}$ is *reduced* if $g_i$ is reduced with respect to $\{g_1,\dots,g_{i-1},g_{i+1},\dots,g_\tau\}$ and $\mathrm{LT}(g_i)=\boldsymbol{x}^{\mathfrak{a}}$, for some $\mathfrak{a}\in\mathbb{Z}^n_{\geqslant 0}$, $i=1,\dots,\tau$. Each ideal $\mathcal{I}$ of $\mathbb{K}[\boldsymbol{x}]$ has a unique reduced Gröbner basis.

Given two ideals $\mathcal{I}$ and $\mathcal{J}$ of $\mathbb{K}[\boldsymbol{x}]$, the *intersection of* $\mathcal{I}$ *and* $\mathcal{J}$, denoted $\mathcal{I}\cap\mathcal{J}$, is the set of all polynomials belonging to both $\mathcal{I}$ and $\mathcal{J}$; in particular, $\mathcal{I}\cap\mathcal{J}$ is an ideal of $\mathbb{K}[\boldsymbol{x}]$ satisfying

$$\mathcal{I}\cap\mathcal{J} = (t\,\mathcal{I}+(1-t)\mathcal{J})\cap\mathbb{K}[\boldsymbol{x}],$$

where $t$ is a single auxiliary variable.

The notion of ideal can be generalized to deal with polynomial vectors in $\mathbb{K}^m[\boldsymbol{x}]$. Namely, given polynomial vectors $\boldsymbol{p}_1,\dots,\boldsymbol{p}_\ell\in\mathbb{K}^m[\boldsymbol{x}]$, the *sub-module of* $\mathbb{K}^m[\boldsymbol{x}]$ *generated by* $\boldsymbol{p}_1,\dots,\boldsymbol{p}_\ell$ is

$$\langle\boldsymbol{p}_1,\dots,\boldsymbol{p}_\ell\rangle = \{\textstyle\sum_{i=1}^{\ell} q_i\,\boldsymbol{p}_i,\ q_i\in\mathbb{K}[\boldsymbol{x}]\}.$$

Similarly, given a polynomial matrix $\boldsymbol{R}=[\ \boldsymbol{r}_1\ \ \dots\ \ \boldsymbol{r}_{m_2}\ ]\in\mathbb{K}^{m_1\times m_2}[\boldsymbol{x}]$, define the *image of* $\boldsymbol{R}$ as a sub-module of $\mathbb{K}^{m_1}[\boldsymbol{x}]$,

$$\mathrm{Img}(\boldsymbol{R}) := \langle\boldsymbol{r}_1,\dots,\boldsymbol{r}_{m_2}\rangle.$$

On the other hand, for any polynomial matrix $\boldsymbol{R}\in\mathbb{K}^{m_1\times m_2}[\boldsymbol{x}]$, the *syzygy of* $\boldsymbol{R}$ is the sub-module of $\mathbb{K}^{m_2}[\boldsymbol{x}]$ consisting of all polynomial vectors $\boldsymbol{p}\in\mathbb{K}^{m_2}[\boldsymbol{x}]$ such that $\boldsymbol{R}\,\boldsymbol{p}=\boldsymbol{0}_{m_1,1}$, where $\boldsymbol{0}_{m_1,1}$ is the zero polynomial vector.

Also the concept of monomial order can be extended to the deal with polynomial vectors in $\mathbb{K}^m[\boldsymbol{x}]$. Indeed, a *monomial in* $\mathbb{K}^m[\boldsymbol{x}]$ is a product of the form $\boldsymbol{x}^{\mathfrak{a}}\,\boldsymbol{e}_i^m$, so that each $\boldsymbol{p}$ in $\mathbb{K}^m[\boldsymbol{x}]$ can be written as a finite linear combination, with coefficients in $\mathbb{K}$, of monomials in $\mathbb{K}^m[\boldsymbol{x}]$. Hence, a *monomial order* $\succ$ *on* $\mathbb{K}^m[\boldsymbol{x}]$ is a total, well ordering relation on the set of monomials in $\mathbb{K}^m[\boldsymbol{x}]$. For instance, the POT *extension of the* Lex *order* is a monomial order on $\mathbb{K}^m[\boldsymbol{x}]$ and is defined as follows: $\boldsymbol{x}^{\mathfrak{a}}\,\boldsymbol{e}_i^m\succ_{\mathrm{L}}\boldsymbol{x}^{\mathfrak{b}}\,\boldsymbol{e}_j^m$ if $i<j$ or $i=j$ and $\boldsymbol{x}^{\mathfrak{a}}\succ_{\mathrm{L}}\boldsymbol{x}^{\mathfrak{b}}$. Hence, let any monomial order on $\mathbb{K}^m[\boldsymbol{x}]$ be fixed. The *leading term of* $\boldsymbol{p}\in\mathbb{K}^m[\boldsymbol{x}]$, denoted $\mathrm{LT}(\boldsymbol{p})$, is the greatest term $c\,\boldsymbol{x}^{\mathfrak{a}}\,\boldsymbol{e}_i^m$ appearing in $\boldsymbol{p}$. Given a sub-module $\mathcal{M}$ of $\mathbb{K}^m[\boldsymbol{x}]$, let $\mathrm{LT}(\mathcal{M})$ be the sub-module generated by the leading terms of all $\boldsymbol{p}\in\mathcal{M}$ according to $\succ$. Hence, for any sub-module $\mathcal{M}$ of $\mathbb{K}^m[\boldsymbol{x}]$, a finite set $\{\boldsymbol{g}_1,\dots,\boldsymbol{g}_\tau\}$ of $\mathbb{K}^m[\boldsymbol{x}]$ is a *Gröbner basis* of $\mathcal{M}$ if

$$\langle\mathrm{LT}(\mathcal{M})\rangle = \langle\mathrm{LT}(\boldsymbol{g}_1),\dots,\mathrm{LT}(\boldsymbol{g}_\tau)\rangle.$$

Reduced Gröbner bases of sub-modules can be defined as for ideals, and it can be shown that there is a unique reduced Gröbner basis for each sub-module in $\mathbb{K}^m[\boldsymbol{x}]$, once a monomial order has been fixed.

## 3 Input-output embeddings of SISO continuous-time linear systems

Consider the following single-input single-output (briefly, SISO) continuous-time linear system:

$$\dot{\boldsymbol{\xi}}(t) = \boldsymbol{E}\,\boldsymbol{\xi}(t)+\boldsymbol{F}\,\nu(t), \qquad \psi(t)=\boldsymbol{G}\,\boldsymbol{\xi}(t)+H\,\nu(t), \qquad (1)$$

where $\boldsymbol{\xi}(t)\in\mathbb{R}^n$ is the state vector, $\nu(t)\in\mathbb{R}$ is the scalar input, and $\psi(t)\in\mathbb{R}$ is the scalar output at time $t\in\mathbb{R}_{\geqslant 0}$; $\nu(t)$ is assumed to be differentiable a sufficiently high number of times. Let $\nu^{(i)}(t) := \frac{\mathrm{d}^i\nu(t)}{\mathrm{d}t}$ and $\psi^{(i)}(t) := \frac{\mathrm{d}^i\psi(t)}{\mathrm{d}t}$ denote the $i$-th time-derivative of the input and of the output, respectively, $i\in\mathbb{Z}_{\geqslant 0}$. For any $N\in\mathbb{Z}_{\geqslant 0}$, let $\boldsymbol{\nu}_{e,N}(t)=[\ \nu^{(0)}(t)\ \ \dots\ \ \nu^{(N)}(t)\ ]^\top$ be a vector having as entries the input and its time-derivatives and let $\boldsymbol{\psi}_{e,N}(t)=[\ \psi^{(0)}(t)\ \ \dots\ \ \psi^{(N)}(t)\ ]^\top$ be a vector having as entries the corresponding output response and its time-derivatives. Hence, consider the following definition (see [24, 25]).

**Definition 1.** A polynomial $p\in\mathbb{R}[\boldsymbol{\psi}_{e,N},\boldsymbol{\nu}_{e,N}]$ is an *embedding* of system (1) if

$$p(\boldsymbol{\psi}_{e,N}(t),\boldsymbol{\nu}_{e,N}(t))=0, \quad \forall t\in\mathbb{R}_{\geqslant 0}.$$

An embedding is *linear* if $\deg(p)=1$.

In order to characterize the set of all embeddings of system (1), which is an ideal of $\mathbb{R}[\boldsymbol{\psi}_{e,N},\boldsymbol{\nu}_{e,N}]$ by of [24, 26], define, for all $N\in\mathbb{Z}_{\geqslant 0}$, the matrices

$$\boldsymbol{M}_N := \begin{bmatrix} H & 0 & \dots & 0\\ \boldsymbol{G}\,\boldsymbol{F} & H & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \boldsymbol{G}\,\boldsymbol{E}^{N-1}\,\boldsymbol{F} & \boldsymbol{G}\,\boldsymbol{E}^{N-2}\,\boldsymbol{F} & \dots & H\end{bmatrix}, \qquad (2a)$$

$$\boldsymbol{O}_N := \begin{bmatrix}\boldsymbol{G}\\ \boldsymbol{G}\,\boldsymbol{E}\\ \vdots\\ \boldsymbol{G}\,\boldsymbol{E}^N\end{bmatrix}. \qquad (2b)$$

Thus, consider the following lemma.

**Lemma 1.** *For any* $N\in\mathbb{Z}_{\geqslant 0}$, *letting* $\boldsymbol{\xi}(t)$ *be the state response of system* (1), *one has*

$$\boldsymbol{\psi}_{e,N}(t)=\boldsymbol{O}_N\,\boldsymbol{\xi}(t)+\boldsymbol{M}_N\,\boldsymbol{\nu}_{e,N}(t), \quad \forall t\in\mathbb{R}_{\geqslant 0}. \qquad (3)$$

*Proof:* The proof follows directly from the analysis carried out in Section 5.1.2 of [27] and from Proposition 1 of [28]. □

Given $N \in \mathbb{Z}_{\geqslant 0}$, $N \geqslant n-1$, define the *observability map of order N of system* (7),

$$\boldsymbol{\Psi}_N(\boldsymbol{\xi}, \boldsymbol{\nu}_{e,N}) := \boldsymbol{O}_N\, \boldsymbol{\xi} + \boldsymbol{M}_N\, \boldsymbol{\nu}_{e,N},$$

which, by Lemma 1, relates the current state of system (7) and the time-derivatives of its input $\nu$ up to order $N$, with the time-derivatives of the output $\psi$ up to order $N$. Hence, in order to determine the set of all embeddings of system (7), define the matrix

$$\boldsymbol{Q}_N := \left[\begin{array}{ccc} \boldsymbol{O}_N & -\boldsymbol{I}_{N+1} & \boldsymbol{M}_N \end{array}\right] \in \mathbb{R}^{(N+1)\times(2\,N+n+2)},$$

which, by (3), is such that

$$\boldsymbol{Q}_N \left[\begin{array}{c} \boldsymbol{\xi}(t) \\ \boldsymbol{\psi}_{e,N}(t) \\ \boldsymbol{\nu}_{e,N}(t) \end{array}\right] = \boldsymbol{0}_{N+1,1}, \quad \forall t \in \mathbb{R}_{\geqslant 0}. \qquad (4)$$

Thus, let $\boldsymbol{Z}_N := \mathcal{E}_r(\boldsymbol{Q}_N)$ be the reduced row echelon form of $\boldsymbol{Q}_N$ and partition such a matrix as follows:

$$\boldsymbol{Z}_N = \begin{array}{c} \quad n \text{ columns} \quad 2\,N+2 \text{ columns} \\ \left[\begin{array}{c|c} \boldsymbol{Z}_{N,1,1} & \boldsymbol{Z}_{N,1,2} \\ \boldsymbol{0}_{\nu,n} & \boldsymbol{Z}_{N,2,2} \end{array}\right] \end{array},$$

*i.e.*, let $\boldsymbol{Z}_{N,2,2} \in \mathbb{R}^{\nu\times(2\,N+2)}$ be the matrix obtained by retaining only the last $2\,N+2$ columns of $\boldsymbol{Z}_N$ and the rows that contain a pivot. Thus, consider the following theorem.

**Theorem 1.** *Let $\boldsymbol{Z}_N$ be partitioned as above. The set of all linear embeddings of system* (1) *in $\mathbb{R}[\boldsymbol{\psi}_{e,N}, \boldsymbol{\nu}_{e,N}]$ is*

$$\mathcal{L} = \{\textstyle\sum_{j=1}^{\mu} c_j\, q_j,\, \forall c_j \in \mathbb{R}\},$$

*where $q_1, \ldots, q_\mu$ are given by*

$$\left[\begin{array}{ccc} q_1 & \ldots & q_\mu \end{array}\right]^\top = \boldsymbol{Z}_{N,2,2} \left[\begin{array}{c} \boldsymbol{\psi}_{e,N} \\ \boldsymbol{\nu}_{e,N} \end{array}\right].$$

*Furthermore, $\mathcal{J}_N := \langle q_1, \ldots, q_\mu\rangle$ is the ideal of all embeddings in $\mathbb{R}[\boldsymbol{\psi}_{e,N}, \boldsymbol{\nu}_{e,N}]$ of system* (1)*.*

*Proof:* By [23], there exists a matrix $\boldsymbol{\Lambda}_N$ such that $\boldsymbol{Z}_N = \boldsymbol{\Lambda}_N\, \boldsymbol{Q}_N$, and, therefore, it results that,

$$\boldsymbol{Z}_N \left[\begin{array}{c} \boldsymbol{\xi}(t) \\ \boldsymbol{\psi}_{e,N}(t) \\ \boldsymbol{\nu}_{e,N}(t) \end{array}\right] = \boldsymbol{0}_{\nu,1}, \quad \forall t \in \mathbb{R}_{\geqslant 0}.$$

In particular, this implies that the polynomials $q_1, \ldots, q_\mu$ are linear embeddings of system (1). Note that there is no embedding of system (1) in $\mathbb{R}[\boldsymbol{\psi}_{e,N}, \boldsymbol{\nu}_{e,N}]$ that is not in $\mathcal{J}_N$, because the steps carried out to compute $\boldsymbol{Z}_N = \mathcal{E}_r(\boldsymbol{Q}_N)$ are exactly those to compute the reduced Gröbner basis of the ideal generated by the relations given in (3) according to the Lex order with $\xi_1 \succ_{\mathrm{L}} \ldots \xi_n \succ_{\mathrm{L}} \psi^{(0)} \succ_{\mathrm{L}} \ldots \succ_{\mathrm{L}} \psi^{(N)} \succ_{\mathrm{L}} \nu^{(0)} \succ_{\mathrm{L}} \ldots \succ_{\mathrm{L}} \nu^{(N)}$ [21] via the Buchberger's algorithm [29, 30]. Hence, the statement follows directly from Theorem 2 of [24]. $\square$

*Remark* 1. If pair $(\boldsymbol{E}, \boldsymbol{G})$ is observable, then matrix $\boldsymbol{Z}_{N,2,2}$ has exactly $N-n+1$ rows due to the fact that the matrix $\boldsymbol{O}_N$ has rank $n$, and hence $\boldsymbol{Z}_{N,1,1} = \boldsymbol{I}_n$.

By combining the results established in Lemma 1 and in Theorem 1, the set of all linear inverses of the observability map of system (1) is characterized by the following corollary.

**Corollary 1.** *Assume that pair $(\boldsymbol{E}, \boldsymbol{G})$ is observable. For any $N \in \mathbb{Z}_{\geqslant 0}$, $N \geqslant n$, the set of all linear inverses of the observability map $\boldsymbol{\Psi}_N(\cdot,\cdot)$ of system* (7)*,* i.e.*, the set of all linear functions $\Xi_N(\cdot,\cdot)$ such that*

$$\Xi_N(\boldsymbol{\Psi}_N(\boldsymbol{\xi}, \boldsymbol{\nu}_{e,N}), \boldsymbol{\nu}_{e,N}) = \boldsymbol{\xi}, \quad \forall \boldsymbol{\xi} \in \mathbb{R}^n, \forall \boldsymbol{\nu}_{e,N} \in \mathbb{R}^{N+1},$$

*is parametrized by*

$$\Xi_N(\boldsymbol{\psi}_{e,N}, \boldsymbol{\nu}_{e,N}) = \boldsymbol{O}_{n-1}^{-1}\, (\boldsymbol{\psi}_{e,n-1} - \boldsymbol{M}_{n-1}\, \boldsymbol{\nu}_{e,n-1}) + \sum_{j=1}^{N-n+1} \boldsymbol{c}_j\, q_j(\boldsymbol{\psi}_{e,N}, \boldsymbol{\nu}_{e,N}), \qquad (5)$$

*where the polynomials $q_1, \ldots, q_{N-n+1}$ are defined as in Theorem 1 and $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_{N-n+1}$ are arbitrary vectors in $\mathbb{R}^n$.*

*Proof:* By Remark 1, if pair $(\boldsymbol{E}, \boldsymbol{G})$ is observable, then $\nu = N - n + 1$. In view of Lemma 1, the set of all linear formulas relating $\boldsymbol{\xi}(t)$ with $\boldsymbol{\psi}_{e,N}(t)$ and $\boldsymbol{\nu}_{e,N}(t)$ corresponds to the set of all solutions in $\boldsymbol{\xi}$ to the system of equations given in (3). Since pair $(\boldsymbol{E}, \boldsymbol{G})$ is observable, one of the formulas that relates $\boldsymbol{\xi}(t)$ with $\boldsymbol{\psi}_{e,N}(t)$ and $\boldsymbol{\nu}_{e,N}(t)$ is $\boldsymbol{\xi}(t) = \boldsymbol{O}_{n-1}^{-1}\, (\boldsymbol{\psi}_{e,n-1}(t) - \boldsymbol{M}_{n-1}\, \boldsymbol{\nu}_{e,n-1}(t))$, which corresponds to (5) with $\boldsymbol{c}_1 = \ldots = \boldsymbol{c}_\mu = \boldsymbol{0}$. Thus, since (4) is a system of linear equations that hold for all times $t \in \mathbb{R}_{\geqslant 0}$, the formula given in (5) follows by the facts that, by Theorem 1, $\mathcal{L}$ is the set of all the linear embeddings of system (1) and that the set of all the solutions to (4) is given by the sum of a particular solution to (4) and the set of all the solutions of the corresponding homogeneous equation [23]. $\square$

The expression in (5) can be used also to find non-linear formulas relating the current state $\boldsymbol{\xi}(t)$ of system (1) with $\boldsymbol{\psi}_{e,N}(t)$ and $\boldsymbol{\nu}_{e,N}(t)$. Indeed, by a trivial extension of Corollary 1, non-linear formulas relating the state and the time-derivative of the input and of the output are given by

$$\Xi_N(\boldsymbol{\psi}_{e,N}, \boldsymbol{\nu}_{e,N}) = \boldsymbol{O}_{n-1}^{-1}\, (\boldsymbol{\psi}_{e,n-1} - \boldsymbol{M}_{n-1}\, \boldsymbol{\nu}_{e,n-1}) + \sum_{j=1}^{N-n+1} \boldsymbol{c}_j(\boldsymbol{\psi}_{e,N}, \boldsymbol{\nu}_{e,N})\, q_j(\boldsymbol{\psi}_{e,N}, \boldsymbol{\nu}_{e,N}), \qquad (6)$$

where $\boldsymbol{c}_j(\boldsymbol{\psi}_{e,N}, \boldsymbol{\nu}_{e,N})$ are arbitrary vector functions of $\boldsymbol{\psi}_{e,N}$ and $\boldsymbol{\nu}_{e,N}$. The expression given in (6) is exploited in the following Section 4 to characterize the simultaneous observability of a set of linear systems.

The following example illustrates the application of Corollary 1.

**Example 1.** Consider system (1) with

$$\boldsymbol{E} = \left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right], \quad \boldsymbol{F} = \left[\begin{array}{c} 0 \\ 1 \end{array}\right], \quad \boldsymbol{G} = \left[\begin{array}{cc} 1 & 0 \end{array}\right], \quad D = 1.$$

Letting $N = 3$ and by using the result established in Corollary 1, it turns out that the set of all linear inverses of the observability map $\boldsymbol{\Psi}_3(\cdot,\cdot)$ of system (7) can be parametrized as

$$\Xi_3(\boldsymbol{\psi}_{e,3}, \boldsymbol{\nu}_{e,3}) = \left[\begin{array}{c} \psi^{(0)} - \nu^{(0)} \\ \psi^{(1)} - \nu^{(1)} \end{array}\right] + \boldsymbol{c}_1\, (\psi^{(1)} + \psi^{(3)} - 2\nu^{(1)} - \nu^{(3)}) + \boldsymbol{c}_2\, (\psi^{(0)} + \psi^{(2)} - 2\nu^{(0)} - \nu^{(2)}),$$

where $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ are arbitrary vectors in $\mathbb{R}^2$.

## 4 Simultaneous inverse of the observability map for multiple SISO continuous-time linear systems

Consider the SISO continuous-time linear system:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\,\boldsymbol{x}(t) + \boldsymbol{B}\,u(t), \qquad y(t) = \boldsymbol{C}\,\boldsymbol{x}(t) + D\,u(t), \tag{7}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, and $y(t) \in \mathbb{R}$ denote the state vector, the input, and the output, respectively, at time $t \in \mathbb{R}_{\geqslant 0}$; $u(t)$ is assumed to be differentiable a sufficiently high number of times. Assume that $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $D$ are not known, but that it is known that the tuple $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, D)$ belongs to the following finite set:

$$\mathcal{S} := \{(\boldsymbol{A}_1, \boldsymbol{B}_1, \boldsymbol{C}_1, D_1), \ldots, (\boldsymbol{A}_s, \boldsymbol{B}_s, \boldsymbol{C}_s, D_s)\},$$

with $s \in \mathbb{Z}_{\geqslant 2}$.

The main objective of this section is, given the set $\mathcal{S}$, to compute a function, if any, that relates the current state $\boldsymbol{x}(t)$ of system (7) with the time-derivatives of $u(t)$ and $y(t)$ up to order $N$ ($\boldsymbol{u}_{e,N} = [\ u^{(0)} \ \ \ldots \ \ u^{(N)}\ ]^\top$, $u^{(i)}(t) = \frac{\mathrm{d}^i u(t)}{\mathrm{d}t^i}$, $i = 0, \ldots, N$ and $\boldsymbol{y}_{e,N} = [\ y^{(0)} \ \ \ldots \ \ y^{(N)}\ ]^\top$, $y^{(i)}(t) = \frac{\mathrm{d}^i y(t)}{\mathrm{d}t^i}$, $i = 0, \ldots, N$, respectively), for some $N \in \mathbb{Z}_{\geqslant 0}$.

By Lemma 1, letting $\boldsymbol{M}_{k,N}$ and $\boldsymbol{O}_{k,N}$ be the matrices obtained by substituting, in the expressions given in (2) for $\boldsymbol{M}_N$ and $\boldsymbol{O}_N$, $\boldsymbol{E}$ with $\boldsymbol{A}_k$, $\boldsymbol{F}$ with $\boldsymbol{B}_k$, $\boldsymbol{G}$ with $\boldsymbol{C}_k$, and $H$ with $D_k$, $k = 1, \ldots, s$, it results that there exists $k \in \{1, \ldots, s\}$ such that

$$\boldsymbol{y}_{e,N}(t) = \boldsymbol{O}_{k,N}\,\boldsymbol{x}(t) + \boldsymbol{M}_{k,N}\,\boldsymbol{u}_{e,N}(t).$$

Thus, define the observability map of order $N$ of the $k$-th system,

$$\boldsymbol{\Psi}_{k,N}(\boldsymbol{x}, \boldsymbol{u}_{e,N}) := \boldsymbol{O}_{k,N}\,\boldsymbol{x} + \boldsymbol{M}_{k,N}\,\boldsymbol{u}_{e,N}, \tag{8}$$

which relates the current state of system (1) and the time-derivatives of $u(t)$ and $y(t)$ up to order $N$, provided that $\boldsymbol{A} = \boldsymbol{A}_k$, $\boldsymbol{B} = \boldsymbol{B}_k$, $\boldsymbol{C} = \boldsymbol{C}_k$, and $D = D_k$. Therefore, the main objective of this section is to compute, if any, a function $\boldsymbol{\Phi}_N(\cdot,\cdot)$ such that

$$\boldsymbol{\Phi}_N(\boldsymbol{\Psi}_{k,N}(\boldsymbol{x}, \boldsymbol{u}_{e,N}), \boldsymbol{u}_{e,N}) = \boldsymbol{x},$$

for "almost all" $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{u}_{e,N} \in \mathbb{R}^{N+1}$, and for all $k \in \{1, \ldots, s\}$. If such a function exists, then it is called a *simultaneous inverse of the observability maps* $\boldsymbol{\Psi}_{k,N}(\cdot,\cdot)$, $k = 1, \ldots, s$.

**Assumption 1.** Pair $(\boldsymbol{A}_k, \boldsymbol{C}_k)$ is observable, $k = 1, \ldots, s$.

Clearly, if Assumption 1 does not hold, then there does not exist a simultaneous inverse of the observability maps $\boldsymbol{\Psi}_{k,N}(\cdot,\cdot)$, $k = 1, \ldots, s$, due to the fact that the relation given in (8) is not invertible for some $k \in \{1, \ldots, s\}$. In addition, as well known, "almost all" pairs $(\boldsymbol{A}_k, \boldsymbol{C}_k)$ are observable.

Under Assumption 1, let $N \in \mathbb{Z}_{\geqslant 0}$, $N \geqslant n$, and let

$$\mathcal{G}_{k,N} := \{g_{k,N,1}, \ldots, g_{k,N,\ell}\}, \tag{9}$$

be the reduced Gröbner basis according to the Lex order with $y^{(0)} \succ_{\mathrm{L}} \ldots \succ_{\mathrm{L}} y^{(N)} \succ_{\mathrm{L}} u^{(0)} \succ_{\mathrm{L}} \ldots \succ_{\mathrm{L}} u^{(N)}$ of the set all embeddings of

$$\begin{aligned} \dot{\boldsymbol{\chi}}_k(t) &= \boldsymbol{A}_k\,\boldsymbol{\chi}_k(t) + \boldsymbol{B}_k\,u(t), \\ y(t) &= \boldsymbol{C}_k\,\boldsymbol{\chi}_k(t) + D_k\,u(t), \end{aligned} \tag{10}$$

for $k = 1, \ldots, s$. Note that, by Remark 1, the set $\mathcal{G}_{k,N}$ is composed by exactly $\ell := N - n + 1$ linear polynomials and can be easily computed by using the method illustrated just above Theorem 1. Hence, for $k = 1, \ldots, s$, let

$$\boldsymbol{J}_{k,N} := \boldsymbol{I}_n \otimes [\ g_{k,N,1} \ \ \cdots \ \ g_{k,N,\ell}\ ] = \begin{bmatrix} g_{k,N,1} & \cdots & g_{k,N,\ell} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & g_{k,N,1} & \cdots & g_{k,N,\ell} \end{bmatrix},$$

and define the following vector (being linear in $\boldsymbol{y}_{e,n-1}$ and $\boldsymbol{u}_{e,n-1}$)

$$\boldsymbol{\theta}_k := \boldsymbol{O}_{k,n-1}^{-1}(\boldsymbol{y}_{e,n-1} - \boldsymbol{M}_{k,n-1}\,\boldsymbol{u}_{e,n-1}). \tag{11}$$

Define the matrices

$$\boldsymbol{W}_N := \begin{bmatrix} \boldsymbol{J}_{1,N} & -\boldsymbol{J}_{2,N} & \cdots & \boldsymbol{0}_{n,n\,\ell} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{J}_{1,N} & \boldsymbol{0}_{n,n\,\ell} & \cdots & -\boldsymbol{J}_{s,N} \end{bmatrix}, \tag{12a}$$

$$\boldsymbol{\Theta} := \begin{bmatrix} \boldsymbol{\theta}_1 - \boldsymbol{\theta}_2 \\ \vdots \\ \boldsymbol{\theta}_1 - \boldsymbol{\theta}_s \end{bmatrix}, \tag{12b}$$

and consider the following theorem.

**Theorem 2.** *Let $\mathcal{S}$ be given and $N \in \mathbb{Z}_{\geqslant n}$; under Assumption 1, consider the following sub-module of $\mathbb{R}^{n\,(s-1)}[\boldsymbol{y}_{e,N}, \boldsymbol{u}_{e,N}]$:*

$$\mathcal{P} := \langle [\ \boldsymbol{W}_N \quad \boldsymbol{\Theta}\ ] \rangle.$$

*Let $\boldsymbol{S}_N \in \mathbb{R}^{(n\,\ell\,s+1)\times\varrho}[\boldsymbol{y}_{e,N}, \boldsymbol{u}_{e,N}]$ be the reduced Gröbner basis of the syzygy of $[\ \boldsymbol{W}_N \quad \boldsymbol{\Theta}\ ]$ with respect to the* POT *extension of the* Lex *order with $y^{(0)} \succ_{\mathrm{L}} \ldots \succ_{\mathrm{L}} y^{(N)} \succ_{\mathrm{L}} u^{(0)} \succ_{\mathrm{L}} \ldots \succ_{\mathrm{L}} u^{(N)}$. Thus, there exists a simultaneous inverse $\boldsymbol{\Phi}_N(\cdot,\cdot)$ of the observability maps $\boldsymbol{\Psi}_{k,N}(\cdot,\cdot)$, $k = 1, \ldots, s$, that holds for "almost all" $\boldsymbol{u}_{e,N}$ if and only if there is $\omega \in \mathbb{R}[\boldsymbol{u}_{e,N}]$, $\omega \neq 0$, such that the polynomial matrix $\boldsymbol{S}_N$ can be partitioned as*

$$\boldsymbol{S}_N = \left[\begin{array}{c|c|c} \boldsymbol{S}_{N,1,1} & \boldsymbol{S}_{N,1,2} & \boldsymbol{S}_{N,1,3} \\ \boldsymbol{0}_{1,\rho} & \omega & \boldsymbol{S}_{N,2,3} \end{array}\right] \begin{array}{l} n\,\ell\,s \text{ rows} \\ 1 \text{ row} \end{array}.$$

*In such a case, by letting $\boldsymbol{S}_{N,1,2}$ be partitioned as*

$$\boldsymbol{S}_{N,1,2} = \begin{bmatrix} \boldsymbol{S}_{N,1,2,1} \\ \vdots \\ \boldsymbol{S}_{N,1,2,s} \end{bmatrix} \begin{array}{l} n\,\ell \text{ rows} \\ \\ n\,\ell \text{ rows} \end{array},$$

*a simultaneous inverse of the observability maps $\boldsymbol{\Psi}_{k,N}(\cdot,\cdot)$, $k = 1, \ldots, s$, holding for "almost all" $\boldsymbol{u}_{e,N}$, is*

$$\boldsymbol{\Phi}_N = \boldsymbol{\theta}_1 + \frac{1}{\omega}(\boldsymbol{J}_{1,N}\,\boldsymbol{S}_{N,1,2,1}). \tag{13}$$

*Proof:* Since $\boldsymbol{S}_N$ is a syzygy matrix of the polynomial sub-module $\mathcal{P}$, whose basis is $[\ \boldsymbol{W}_N \quad \boldsymbol{\Theta}\ ]$, it holds that

$$\boldsymbol{W}_N\,\boldsymbol{S}_{N,1,2} + \boldsymbol{\Theta}\,\omega = \boldsymbol{0}_{n\,(s-1),1}.$$

In particular, by the definition of the polynomial matrices $\boldsymbol{W}_N$ and $\boldsymbol{\Theta}$, the following polynomial relations hold:

$$\frac{1}{\omega}\boldsymbol{J}_{1,N}\,\boldsymbol{S}_{N,1,2,1} + \boldsymbol{\theta}_1 = \frac{1}{\omega}\boldsymbol{J}_{2,N}\,\boldsymbol{S}_{N,1,2,2} + \boldsymbol{\theta}_2,$$
$$\vdots$$
$$\frac{1}{\omega}\boldsymbol{J}_{1,N}\,\boldsymbol{S}_{N,1,2,1} + \boldsymbol{\theta}_1 = \frac{1}{\omega}\boldsymbol{J}_{s,N}\,\boldsymbol{S}_{N,1,2,s} + \boldsymbol{\theta}_s.$$

Hence, letting $\boldsymbol{\Phi}_N(\boldsymbol{y}_{e,N}, \boldsymbol{u}_{e,N})$ be defined as in (13), by Theorem 1, $\boldsymbol{\Phi}_N(\cdot,\cdot)$ is a simultaneous inverse of the observability maps

$\boldsymbol{\Psi}_{k,N}(\cdot,\cdot)$, $k = 1,\ldots,s$. Furthermore, it holds for "almost all" $\boldsymbol{u}_{e,N}$ because the set where it cannot be applied is the variety $\mathbb{V}(\omega) \subset \mathbb{R}^{N+1}$, which does not coincide with $\mathbb{R}^{N+1}$ since $\omega \neq 0$.

On the other hand, assume that there exists a simultaneous inverse $\boldsymbol{\Phi}_N(\cdot,\cdot)$ of the observability maps $\boldsymbol{\Psi}_{k,N}(\cdot,\cdot)$, $k = 1,\ldots,s$, that holds for "almost all" $\boldsymbol{u}_{e,N}$. By Theorem 1, since $\boldsymbol{\Phi}_N(\cdot,\cdot)$ can be used for "almost all" $\boldsymbol{u}_{e,N}$, there exist $\boldsymbol{c}_{k,j} \in \mathbb{R}^n(\boldsymbol{y}_{e,N},\boldsymbol{u}_{e,N})$ whose entries are rational functions with numerator in $\mathbb{R}[\boldsymbol{y}_{e,N},\boldsymbol{u}_{e,N}]$ and denominator in $\mathbb{R}[\boldsymbol{u}_{e,N}]$ such that

$$\Phi_N = \boldsymbol{O}_{k,n-1}^{-1}\left(\boldsymbol{y}_{e,n-1} - \boldsymbol{M}_{k,n-1}\,\boldsymbol{u}_{e,n-1}\right) + \sum_{j=1}^{N-n+1} \boldsymbol{c}_{k,j}\, g_{k,N,j},$$

for $k = 1,\ldots,s$. By taking pairwise differences of the expression above for $k \in \{1,\ldots,s\}$, one obtains that

$$\begin{gathered}\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2 + \sum_{j_1=1}^{N-n+1} \boldsymbol{c}_{1,j_1}\, g_{1,j_1} - \sum_{j_2=1}^{N-n+1} \boldsymbol{c}_{2,j_2}\, g_{2,j_2} = 0,\\ \vdots \\ \boldsymbol{\theta}_1 - \boldsymbol{\theta}_s + \sum_{j_1=1}^{N-n+1} \boldsymbol{c}_{1,j_1}\, g_{1,j_1} - \sum_{j_s=1}^{N-n+1} \boldsymbol{c}_{s,j_s}\, g_{s,j_s} = 0,\end{gathered}$$

which implies that there exists a vector in the reduced Gröbner basis of the syzygy of $\mathcal{P}$ with respect to the POT extension of the Lex order with $y^{(0)} \succ_{\mathrm{L}} \ldots \succ_{\mathrm{L}} y^{(N)} \succ_{\mathrm{L}} u^{(0)} \succ_{\mathrm{L}} \ldots \succ_{\mathrm{L}} u^{(N)}$ whose last entry is a polynomial in $\mathbb{R}[\boldsymbol{u}_{e,N}]$. □

Theorem 2 provides necessary and sufficient conditions for the existence of a simultaneous inverse $\boldsymbol{\Phi}_N(\cdot,\cdot)$ of the observability maps $\boldsymbol{\Psi}_{k,N}(\cdot,\cdot)$, $k = 1,\ldots,s$, together with computational tools to compute a closed-form expression of such an inverse.

The next example illustrates the application of such a technique.

**Example 2.** Let $s = 2$ and

$$\boldsymbol{A}_1 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \quad \boldsymbol{B}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{C}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad D_1 = 1,$$
$$\boldsymbol{A}_2 = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}, \quad \boldsymbol{B}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{C}_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D_2 = 1.$$

By letting $N = 4$, and computing the sets $\mathcal{G}_{k,4}$, $k = 1, 2$, as in (9), one obtains the following polynomials of $\mathbb{R}[\boldsymbol{y}_{e,4},\boldsymbol{u}_{e,4}]$:

$$\mathcal{G}_{1,4}\begin{cases} g_{1,1} = y^{(4)} + 2u^{(2)} - u^{(3)} - u^{(4)},\\ g_{1,2} = y^{(3)} + 2u^{(1)} - u^{(2)} - u^{(3)},\\ g_{1,3} = y^{(2)} + 2u^{(0)} - u^{(1)} - u^{(2)},\end{cases}$$
$$\mathcal{G}_{2,4}\begin{cases} g_{2,1} = 3y^{(2)} - y^{(4)} - 2u^{(2)} + u^{(3)} + u^{(4)},\\ g_{2,2} = 3y^{(1)} - y^{(3)} - 2u^{(1)} + u^{(2)} + u^{(3)},\\ g_{2,3} = 9y^{(0)} - y^{(4)} - 6u^{(0)} + 3u^{(1)} + u^{(2)} + u^{(3)} + u^{(4)}.\end{cases}$$

On the other hand, by computing $\boldsymbol{\Theta}$ as in (12b), one obtains

$$\boldsymbol{\Theta} = \begin{bmatrix} -\frac{1}{2}y^{(0)} + \frac{1}{4}y^{(1)} + \frac{1}{4}u^{(0)} - \frac{1}{4}u^{(1)} \\ y^{(0)} - \frac{3}{4}y^{(1)} - \frac{1}{4}u^{(0)} + \frac{3}{4}u^{(1)} \end{bmatrix}.$$

Finally, by computing the reduced Gröbner basis of the syzygy of $[\ \boldsymbol{W}_4 \quad \boldsymbol{\Theta}\ ]$, one of its elements is:

$$\begin{bmatrix} -2y^{(0)} + y^{(1)} + u^{(0)} - u^{(1)} \\ 0 \\ 6y^{(0)} - 3y^{(1)} - 3u^{(0)} + 3u^{(1)} \\ 4y^{(0)} - 3y^{(1)} - u^{(0)} + 3u^{(1)} \\ 0 \\ -12y^{(0)} + 9y^{(1)} + 3u^{(0)} - 9u^{(1)} \\ 2y^{(0)} - y^{(1)} - u^{(0)} + u^{(1)} \\ 0 \\ 0 \\ -4y^{(0)} + 3y^{(1)} + u^{(0)} - 3u^{(1)} \\ 0 \\ 0 \\ 24u^{(0)} - 12u^{(1)} - 12u^{(2)} \end{bmatrix}.$$

Therefore, since the last entry of such a vector is in $\mathbb{R}[\boldsymbol{u}_{e,4}]$, by Theorem 2, there exists a simultaneous inverse $\boldsymbol{\Phi}_4(\cdot,\cdot)$ of the observability maps $\boldsymbol{\Psi}_{k,4}(\cdot,\cdot)$, $k = 1, 2$, that holds for "almost all" $\boldsymbol{u}_{e,4}$. In particular, such an inverse can be obtained by using (13),

$$x_1 = -\frac{(u^{(0)}-u^{(1)}-2y^{(0)}+y^{(1)})(2u^{(2)}-u^{(3)}-u^{(4)}-3y^{(2)}+y^{(4)})}{12(-2u^{(0)}+u^{(1)}+u^{(2)})},$$
$$x_2 = \frac{(u^{(0)}-3u^{(1)}-4y^{(0)}+3y^{(1)})(2u^{(2)}-u^{(3)}-u^{(4)}-3y^{(2)}+y^{(4)})}{12(-2u^{(0)}+u^{(1)}+u^{(2)})},$$

and can be used for all times $t \in \mathbb{R}_{\geqslant 0}$ such that

$$u_{e,4}(t) \notin \mathbb{V}(\omega), \tag{14}$$

where $\omega = 2u^{(0)} - u^{(1)} - u^{(2)}$. It is worth noticing that if $u_{e,4}(t) \in \mathbb{V}(\omega)$ for all times $t \in \mathbb{R}_{\geqslant 0}$, then system (7) may be unobservable without knowing which of the tuples $(\boldsymbol{A}_1,\boldsymbol{B}_1,\boldsymbol{C}_1,D_1)$ and $(\boldsymbol{A}_2,\boldsymbol{B}_2,\boldsymbol{C}_2,D_2)$ is governing the dynamics of the system. Indeed, letting

$$u(t) = \left(\frac{c_0}{3} - \frac{c_1}{3}\right)\exp(-2t) + \left(\frac{2c_0}{3} + \frac{c_1}{3}\right)\exp(t), \tag{15}$$

one has that $u_{e,4}(t) \in \mathbb{V}(\omega)$ for all $t \in \mathbb{R}$ and for all $c_1, c_2 \in \mathbb{R}$. Hence, letting $u$ be as in (15), if the dynamics of system (7) are governed by $(\boldsymbol{A}_1,\boldsymbol{B}_1,\boldsymbol{C}_1,D_1)$ and the initial condition is

$$\boldsymbol{x}_{1,0} = \begin{bmatrix} -\frac{3c_0}{4} - \frac{c_1}{4} \\ \frac{c_1}{4} - \frac{c_0}{4} \end{bmatrix},$$

then the state and output response of system (7) are given by

$$\boldsymbol{x}_1(t) = \begin{bmatrix} -\frac{2c_0+c_1}{3}\exp(t) + \frac{c_1-c_0}{12}\exp(-2t) \\ \frac{c_1-c_0}{4}\exp(-2t) \end{bmatrix},$$
$$y_1(t) = 0,$$

respectively. On the other hand, letting $u$ be the input given in (15), if the dynamics of system (7) are governed by $(\boldsymbol{A}_2,\boldsymbol{B}_2,\boldsymbol{C}_2,D_2)$ and the initial condition of the system is

$$\boldsymbol{x}_{2,0} = \begin{bmatrix} -c_0 \\ -\frac{c_1}{2} \end{bmatrix},$$

then the state and output response of system (7) are given by

$$\boldsymbol{x}_2(t) = \begin{bmatrix} -\frac{2c_0+c_1}{3}\exp(t) + \frac{c_1-c_0}{3}\exp(-2t) \\ -\frac{2c_0+c_1}{6}\exp(t) + \frac{c_0-c_1}{3}\exp(-2t) \end{bmatrix},$$
$$y_2(t) = 0,$$

respectively. Hence, since $y_1(t) = y_2(t)$ for all $t \in \mathbb{R}_{\geqslant 0}$, but $\boldsymbol{x}_1(t)$ need not be equal to $\boldsymbol{x}_2(t)$ for all $t \in \mathbb{R}_{\geqslant 0}$, it is not possible to reconstruct the state of system (7) from measurements of $u$ and $y$ without knowing which one among the tuples $(\boldsymbol{A}_1, \boldsymbol{B}_1, \boldsymbol{C}_1, D_1)$ and $(\boldsymbol{A}_2, \boldsymbol{B}_2, \boldsymbol{C}_2, D_2)$ is governing its dynamics.

Finally, consider the following definition.

**Definition 2.** A polynomial $p \in \mathbb{R}[\boldsymbol{y}_{e,N}, \boldsymbol{u}_{e,N}]$ is an *embedding* of system (7) if, $\forall \boldsymbol{x} \in \mathbb{R}^n$, $\forall \boldsymbol{u}_{e,N} \in \mathbb{R}^{N+1}$, $\forall k \in \{1, \ldots, s\}$,

$$p(\boldsymbol{\Psi}_{k,N}(\boldsymbol{x}, \boldsymbol{u}_{e,N}), \boldsymbol{u}_{e,N}) = 0.$$

Note that Definition 2 generalizes Definition 1 to deal with systems whose dynamical matrices are not known. Indeed, by the construction given above, it results that if $p \in \mathbb{R}[\boldsymbol{y}_{e,N}, \boldsymbol{u}_{e,N}]$ is an embedding of system (7), then

$$p(\boldsymbol{y}_{e,N}(t), \boldsymbol{u}_{e,N}(t)) = 0, \quad \forall t \in \mathbb{R}_{\geqslant 0},$$

independently of which $(\boldsymbol{A}_k, \boldsymbol{B}_k, \boldsymbol{C}_k, D_k) \in \mathcal{S}$ is governing the dynamics of the system. The following proposition is a straightforward corollary of Theorem 1.

**Proposition 1.** *Let the polynomials $g_{k,N,1}, \ldots, g_{k,N,\ell}$ be defined as in (9), $k = 1, \ldots, s$. The set of all embeddings of system (7) is*

$$\mathcal{T}_N = \bigcap_{k=1}^{s} \langle g_{k,N,1}, \ldots, g_{k,N,\ell} \rangle. \tag{16}$$

The interest in the set $\mathcal{T}_N$ defined in (16) relies on the fact that it allows to construct several simultaneous inverses of the observability maps $\boldsymbol{\Psi}_{k,N}(\cdot,\cdot)$, $k = 1, \ldots, s$. Indeed, by the same reasoning used to prove Corollary 1, letting the assumptions of Theorem 2 hold, letting $\boldsymbol{\Phi}_N(\cdot,\cdot)$ be defined as in (13), and letting $\{\lambda_1, \ldots, \lambda_\varkappa\} \subset \mathbb{R}[\boldsymbol{y}_{e,N}, \boldsymbol{u}_{e,N}]$ be the reduced Gröbner basis of the ideal $\mathcal{T}_N$ given in (16), the function $(\boldsymbol{y}_{e,N}, \boldsymbol{u}_{e,N}) \mapsto \boldsymbol{\Phi}_N(\boldsymbol{y}_{e,N}, \boldsymbol{u}_{e,N}) + \sum_{j=1}^{\varkappa} \boldsymbol{c}_j(\boldsymbol{y}_{e,N}, \boldsymbol{u}_{e,N})\, \lambda_j(\boldsymbol{y}_{e,N}, \boldsymbol{u}_{e,N})$, where $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_\varkappa$ are arbitrary vectors in $\mathbb{R}^n[\boldsymbol{y}_{e,N}, \boldsymbol{u}_{e,N}]$, is a simultaneous inverse of the maps $\boldsymbol{\Psi}_{k,N}(\cdot,\cdot)$, $k = 1, \ldots, s$, that holds for "almost all" $\boldsymbol{u}_{e,N}$.

## 5 The case of autonomous systems

In this section, the analysis carried out in Section 4 is specialized to the case of the following autonomous systems:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\,\boldsymbol{x}(t), \quad y(t) = \boldsymbol{C}\,\boldsymbol{x}(t). \tag{17}$$

Assume that the matrices $\boldsymbol{A}$ and $\boldsymbol{C}$ are not known, but that it is known that pair $(\boldsymbol{A}, \boldsymbol{C})$ belongs to the set

$$\hat{\mathcal{S}} = \{(\boldsymbol{A}_1, \boldsymbol{C}_1), \ldots, (\boldsymbol{A}_s, \boldsymbol{C}_s)\},$$

where $s \in \mathbb{Z}_{\geqslant 2}$.The results stated in Theorem 2 still hold for system (17), and the main objective of this section is to show such a statement can be simplified when dealing with autonomous systems. Toward this end, consider the following example in which the results of Theorem 2 are applied to a systems being autonomous.

**Example 3.** Let $s = 3$ and consider the matrices

$$A_1 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \qquad C_1 = [\ 1 \quad 0\ ], \tag{18a}$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}, \qquad C_2 = [\ 0 \quad 1\ ] \tag{18b}$$

$$A_3 = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}, \qquad C_3 = [\ 1 \quad 1\ ]. \tag{18c}$$

By letting $N = 5$, and computing the sets $\mathcal{G}_{k,5}$, $k = 1, 2, 3$, as in (9), one obtains the following polynomials of $\mathbb{R}[\boldsymbol{y}_{e,5}]$:

$$\mathcal{G}_{1,5} \quad \begin{cases} g_{1,1} = y^{(3)} - 2y^{(4)} + y^{(5)}, \\ g_{1,2} = y^{(2)} - 3y^{(4)} + 2y^{(5)}, \\ g_{1,3} = y^{(1)} - 4y^{(4)} + 3y^{(5)}, \\ g_{1,4} = y^{(0)} - 5y^{(4)} + 4y^{(5)}, \end{cases}$$

$$\mathcal{G}_{2,5} \quad \begin{cases} g_{2,1} = 2y^{(3)} + y^{(4)} - y^{(5)}, \\ g_{2,2} = 4y^{(2)} - 3y^{(4)} + y^{(5)}, \\ g_{2,3} = 8y^{(1)} + 5y^{(4)} - 3y^{(5)}, \\ g_{2,4} = 16y^{(0)} - 11y^{(4)} + 5y^{(5)}, \end{cases}$$

$$\mathcal{G}_{3,5} \quad \begin{cases} g_{3,1} = 3y^{(3)} - 3y^{(4)} + y^{(5)}, \\ g_{3,2} = 3y^{(2)} - 2y^{(4)} + y^{(5)}, \\ g_{3,3} = 9y^{(1)} - 3y^{(4)} + 2y^{(5)}, \\ g_{3,4} = 9y^{(0)} - y^{(4)} + y^{(5)}. \end{cases}$$

Hence, by considering that

$$\begin{bmatrix} g_{1,1} \\ g_{1,2} \\ g_{2,1} \\ g_{2,2} \\ g_{2,3} \\ g_{2,4} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 & -3 & 2 \\ 0 & 0 & 0 & 2 & 1 & -1 \\ 0 & 0 & 4 & 0 & -3 & 1 \\ 0 & 8 & 0 & 0 & 5 & -3 \\ 16 & 0 & 0 & 0 & -11 & 5 \end{bmatrix} \begin{bmatrix} y^{(0)} \\ y^{(1)} \\ y^{(2)} \\ y^{(3)} \\ y^{(4)} \\ y^{(5)} \end{bmatrix},$$

$$\begin{bmatrix} g_{1,3} \\ g_{1,4} \\ g_{3,1} \\ g_{3,2} \\ g_{3,3} \\ g_{3,4} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & -4 & 3 \\ 1 & 0 & 0 & 0 & -5 & 4 \\ 0 & 0 & 0 & 3 & -3 & 1 \\ 0 & 0 & 3 & 0 & -2 & 1 \\ 0 & 9 & 0 & 0 & -3 & 2 \\ 9 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} y^{(0)} \\ y^{(1)} \\ y^{(2)} \\ y^{(3)} \\ y^{(4)} \\ y^{(5)} \end{bmatrix},$$

and that the matrices on the right-hand side of the expressions above have full rank, by the same reasoning used in the proof of Theorem 1, one has that

$$\langle g_{1,1}, g_{1,2}, g_{2,1}, g_{2,2}, g_{2,3}, g_{2,4} \rangle = \langle y^{(0)}, y^{(1)}, \ldots, y^{(5)} \rangle,$$
$$\langle g_{1,3}, g_{1,4}, g_{3,1}, g_{3,2}, g_{3,3}, g_{3,4} \rangle = \langle y^{(0)}, y^{(1)}, \ldots, y^{(5)} \rangle.$$

Therefore, it results that, letting $\boldsymbol{W}_5$ be defined as in (12a),

$$\mathrm{Img}(\boldsymbol{W}_5) = \langle y^{(0)}\boldsymbol{e}_1^4, \ldots, y^{(5)}\boldsymbol{e}_1^4, y^{(0)}\boldsymbol{e}_2^4, \ldots, y^{(5)}\boldsymbol{e}_4^4 \rangle,$$

*i.e.*, there exists a constant matrix $\boldsymbol{K} \in \mathbb{R}^{24\times 24}$ such that

$$\boldsymbol{W}_5\,\boldsymbol{K} = \boldsymbol{I}_4 \otimes \boldsymbol{y}_{e,5}^{\top}.$$

Hence, since the polynomial vector $\boldsymbol{\Theta}$ has linear entries in $\boldsymbol{y}_{e,1}$, *i.e.*, there exists a matrix $\boldsymbol{\Upsilon} \in \mathbb{R}^{4\times 2}$ such that

$$\boldsymbol{\Theta} = \boldsymbol{\Upsilon}\,\boldsymbol{y}_{e,1},$$

by Lemma 2.2 of [31] and Lemma 4 of [32], there exists a constant vector in the syzygy of $[\ \boldsymbol{W}_5 \quad \boldsymbol{\Theta}\ ]$ (the explicit expression of such a vector is omitted for brevity). Thus, by Theorem 2, there exists a simultaneous inverse $\boldsymbol{\Phi}_5(\cdot)$ of the observability maps $\boldsymbol{\Psi}_{k,5}(\cdot,\cdot)$, $k = 1, 2, 3$, that, by using (13), is given by

$$\boldsymbol{\Phi}_5(\boldsymbol{y}_{e,5}) = \begin{bmatrix} -\frac{11}{14} & \frac{7}{2} & -\frac{5}{7} & -\frac{31}{14} & \frac{3}{2} & \frac{2}{7} \\ \frac{73}{14} & \frac{35}{4} & \frac{1}{28} & \frac{51}{7} & \frac{19}{4} & \frac{27}{28} \end{bmatrix} \boldsymbol{y}_{e,5}.$$

In view of the construction illustrated in Example 3, the following theorem specializes the results given in Theorem 2 to the case of autonomous systems.

**Theorem 3.** *Let $s \in \mathbb{Z}_{\geqslant 2}$. If $N \geqslant n\,s - 1$, then, for "almost all" $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_s \in \mathbb{R}^{n\times n}$ and $\boldsymbol{C}_1, \ldots, \boldsymbol{C}_s \in \mathbb{R}^{1\times n}$, there exists a linear simultaneous inverse of the observability maps $\boldsymbol{x} \mapsto \boldsymbol{O}_{k,N}\,\boldsymbol{x}$, $k = 1, \ldots, s$,* i.e.*, there exists $\boldsymbol{L}_N \in \mathbb{R}^{n\times(N+1)}$ such that*

$$\boldsymbol{L}_N\,\boldsymbol{O}_{k,N}\,\boldsymbol{x} = \boldsymbol{x}, \quad \forall \boldsymbol{x} \in \mathbb{R}^n,\ k = 1, \ldots, s. \tag{19}$$

*Proof:* Firstly, note that, for "almost all" $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_s \in \mathbb{R}^{n\times n}$ and $\boldsymbol{C}_1, \ldots, \boldsymbol{C}_s \in \mathbb{R}^{1\times n}$, pairs $(\boldsymbol{A}_k, \boldsymbol{C}_k)$ are observable [27]. Let $\mathcal{G}_{k,N}$ be defined as in (9), $k = 1, \ldots, s$: in view of Remark 1, such sets are composed by exactly $N - n + 1$ linear polynomials for "almost all" matrices $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_s \in \mathbb{R}^{n\times n}$ and $\boldsymbol{C}_1, \ldots, \boldsymbol{C}_s \in \mathbb{R}^{1\times n}$. By using the construction made in Example 3, there exists a matrix $\boldsymbol{H}_k \in \mathbb{R}^{(N+1)\times(N+1)}$ such that

$$\begin{bmatrix} g_{1,(k-2)n+1} & \cdots & g_{1,(k-1)\,n} & \cdots & g_{k,N,N-n+1} \end{bmatrix}^\top = \boldsymbol{H}_k\,\boldsymbol{y}_{e,N},$$

for $k = 2, \ldots, s$. Furthermore, such a matrix has full rank for "almost all" $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_s \in \mathbb{R}^{n\times n}$ and $\boldsymbol{C}_1, \ldots, \boldsymbol{C}_s \in \mathbb{R}^{1\times n}$. Thus, by the same reasoning used in Example 3, it results that

$$\mathrm{Img}(\boldsymbol{W}_N) = \langle y^{(0)}\boldsymbol{e}_1^{n(s-1)}, \ldots, y^{(N)}\boldsymbol{e}_1^{n(s-1)}, y^{(0)}\boldsymbol{e}_{n(s-1)}^{n(s-1)}, \ldots, y^{(N)}\boldsymbol{e}_{n(s-1)}^{n(s-1)}\rangle.$$

Therefore, by Lemma 2.2 of [31] and [32, 33], there exists a constant vector in the syzygy of $[\ \boldsymbol{W}_N \quad \boldsymbol{\Theta}\ ]$; therefore, by (13), there exists a linear inverse of the observability maps $\boldsymbol{x} \mapsto \boldsymbol{O}_{k,N}\,\boldsymbol{x}$, $k = 1, \ldots, s$, for "almost all" matrices $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_s \in \mathbb{R}^{n\times n}$ and $\boldsymbol{C}_1, \ldots, \boldsymbol{C}_s \in \mathbb{R}^{1\times n}$. ☐

Theorem 3 provides a "generic" result on the number of time-derivatives of the output that have to be taken into account in order to allow one to jointly invert the observability maps $\boldsymbol{x} \mapsto \boldsymbol{O}_{k,N}\,\boldsymbol{x}$, $k = 1, \ldots, s$. Finally, the following proposition characterizes the number of time-derivatives of the output to be taken into account to guarantee the existence of a linear embedding for system (17).

**Proposition 2.** *Let the set $\hat{\mathcal{S}}$ be given and let the ideal $\mathcal{T}_N$ of $\mathbb{R}[\boldsymbol{y}_{e,N}]$ be defined as in* (16)*, with $\boldsymbol{B}_k = \boldsymbol{0}_{n,1}$ and $D_k = 0$, $k = 1, \ldots, s$. If $N \geqslant n\,s$, then there exists a linear polynomial $p \in \mathcal{T}_N$.*

*Proof:* Define $\boldsymbol{A}_e := \mathrm{diag}(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_s) \in \mathbb{R}^{(n\,s)\times(n\,s)}$, $\boldsymbol{C}_e := [\ \boldsymbol{C}_1 \quad \ldots \quad \boldsymbol{C}_s\ ] \in \mathbb{R}^{1\times(n\,s)}$, and consider the extended system

$$\boldsymbol{\zeta} = \boldsymbol{A}_e\,\boldsymbol{\zeta}, \qquad \varpi = \boldsymbol{C}_e\,\boldsymbol{\zeta}. \tag{20}$$

It can be easily derived that if $p$ is an embedding of system (20), then it is also an embedding of system (17). Thus, the result follows directly from Theorem 1. ☐

The following example illustrates the application of Proposition 1.

**Example 4.** Consider again the matrices given in (18). By letting $N = 6$, computing the sets $\mathcal{G}_{k,6}$, $k = 1, 2, 3$, as in (9), and computing the ideal $\mathcal{T}_N$ as in (16), one obtains that the polynomial

$$p = 6y^{(0)} - 15y^{(1)} + 8y^{(2)} + 9y^{(3)} - 13y^{(4)} + 6y^{(5)} - y^{(6)}$$

is in the ideal $\mathcal{T}_N$ and hence it is a (linear) embedding of system (17).

## 6 State observer design

In this section, it is shown how the tools given in Sections 4 and 5 can be used to design a state observer for systems (7) and (17). In particular, in Subsection 6.1, it is shown how the methods proposed in Section 5 can be used to design a state observer for system (17) without knowing which of the pairs $(\boldsymbol{A}_1, \boldsymbol{C}_1), \ldots, (\boldsymbol{A}_s, \boldsymbol{C}_s)$ is governing its dynamics, whereas in Subsection 6.2, it is shown how the techniques given in Section 4 can be used to design a state observer for system (7), without knowing which of the tuples $(\boldsymbol{A}_1, \boldsymbol{B}_1, \boldsymbol{C}_1, D_1), \ldots, (\boldsymbol{A}_s, \boldsymbol{B}_s, \boldsymbol{C}_s, D_s)$ is governing its dynamics. Assumption 1 is supposed to hold throughout this section.

### *6.1 Design of state observers for multiple autonomous systems*

Assume that $N \geqslant n\,s - 1$, let $\boldsymbol{L}$ be such that (19) holds, and let

$$p = \alpha_0\,y_0 + \ldots + \alpha_N\,y_N + y_{N+1} = \boldsymbol{\alpha}^\top\,\boldsymbol{y}_{e,N} + y_{N+1},$$

be a linear embedding of system (17) in $\mathcal{T}_{N+1}$ (note that, in view of the results established in Section 5, such $\boldsymbol{L}$ and $\boldsymbol{\alpha}$ exist for "almost all" $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_s \in \mathbb{R}^{n\times n}$ and $\boldsymbol{C}_1, \ldots, \boldsymbol{C}_s \in \mathbb{R}^{1\times n}$). Since $p$ is a linear embedding of system (17), independently of which of the pairs $(\boldsymbol{A}_1, \boldsymbol{C}_1), \ldots, (\boldsymbol{A}_s, \boldsymbol{C}_s)$ is governing its dynamics, the time-derivatives of the output $y$ satisfy

$$\dot{\boldsymbol{y}}_{e,N}(t) = \boldsymbol{V}\,\boldsymbol{y}_{e,N}(t), \tag{21}$$

where

$$\boldsymbol{V} = \begin{bmatrix} \boldsymbol{0}_{N,1} & \boldsymbol{I}_N \\ & -\boldsymbol{\alpha}^\top \end{bmatrix}. \tag{22}$$

Thus, a state observer for system (17) that is independent of which of the pairs $(\boldsymbol{A}_1, \boldsymbol{C}_1), \ldots, (\boldsymbol{A}_s, \boldsymbol{C}_s)$ is governing its dynamics can be "generically" given by

$$\dot{\hat{\boldsymbol{y}}}_{e,N}(t) = (\boldsymbol{V} - \boldsymbol{T}\,\boldsymbol{Y})\,\hat{\boldsymbol{y}}_{e,N}(t) + \boldsymbol{T}\,y(t), \tag{23a}$$

$$\hat{\boldsymbol{x}}(t) = \boldsymbol{L}\,\hat{\boldsymbol{y}}_{e,N}(t), \tag{23b}$$

where $\boldsymbol{Y} = [\ 1 \quad \boldsymbol{0}_{1,N}\ ]$, $\boldsymbol{T}$ is such that $\boldsymbol{V} - \boldsymbol{T}\,\boldsymbol{Y}$ has eigenvalues with negative real part, $\hat{\boldsymbol{y}}_{e,N}(t)$ is an estimate of $\boldsymbol{y}_{e,N}(t)$ and $\hat{\boldsymbol{x}}(t)$ is an estimate of $\boldsymbol{x}(t)$. As a matter of fact, system (23) is a classical Luenberger observer [1] for system (21) and hence the estimation error $\hat{\boldsymbol{y}}_{e,N}(t) - \boldsymbol{y}_{e,N}(t)$ converges exponentially to 0. Therefore, since $\boldsymbol{x}(t) = \boldsymbol{L}\,\boldsymbol{y}_{e,N}(t)$, it results that also the estimation error $\hat{\boldsymbol{x}}(t) - \boldsymbol{x}(t)$ converges exponentially to 0, *i.e.*, system (23) is an exponential state observer for system (17). System (23) is referred to as a *simultaneous state observer for multiple autonomous systems*. The next example illustrates the application of this observer.

**Example 5.** Consider the mechanical system depicted in Figure 1, which is constituted by two bodies having mass m and two springs having stiffness k. Let the output $y$ be the position of the first body.

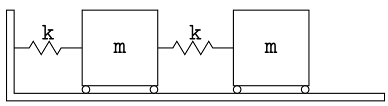


**Fig. 1**: A mechanical system with two masses and two springs.

Letting the masses be unitary and assuming that it is not knwon whether the stiffness is k = 1 or k = 2, the dynamics of such a system are given by system (17) with $s = 2$ and

$$\boldsymbol{A}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}, \quad \boldsymbol{A}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & 0 \end{bmatrix},$$

$$\boldsymbol{C}_1 = [\ 1 \quad 0 \quad 0 \quad 0\ ], \qquad \boldsymbol{C}_2 = [\ 1 \quad 0 \quad 0 \quad 0\ ].$$

Using the results established in Section 5 (see Examples 3 and 4 for the explicit steps that have to be carried out in order to compute the matrix $\boldsymbol{L}$ and the vector $\boldsymbol{\alpha}$), one obtains

$$\boldsymbol{L} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{4}{3} & 0 & -\frac{7}{6} & 0 & -\frac{7}{6} & 0 & -\frac{1}{6} & 0 \\ 0 & \frac{4}{3} & 0 & -\frac{7}{6} & 0 & -\frac{7}{6} & 0 & -\frac{1}{6} \end{bmatrix},$$

$$\boldsymbol{\alpha}^{\top} = [\ -4 \quad 0 \quad -18 \quad 0 \quad -23 \quad 0 \quad -9 \quad 0\ ].$$

Hence, letting $\boldsymbol{V}$ be defined as in (22), one has that the matrix

$$\boldsymbol{T} = \left[16.64\ 121.3\ 470.1\ 830.3\ -508.1\ -4601\ -1830\ 21922\right]^{\top},$$

is such that $\boldsymbol{V} - \boldsymbol{T}\boldsymbol{Y}$ has eigenvalues with negative real part. Thus, the simultaneous state observer (23) provides an exponentially converging estimate of the state of the mechanical system depicted in Figure 1 independently of whether $\mathtt{k} = 1$ or $\mathtt{k} = 2$.
Numerical simulations have been carried out to test such an observer. Figure 2 depicts the behavior of the mechanical system, the time history of the state of the observer (23) and the estimation error obtained when $\mathtt{k} = 1$, $\boldsymbol{x}(0) = [\ 1 \quad 0 \quad -1 \quad 0\ ]^{\top}$, $\hat{\boldsymbol{y}}_{e,7}(0) = \boldsymbol{0}_{7,1}$.

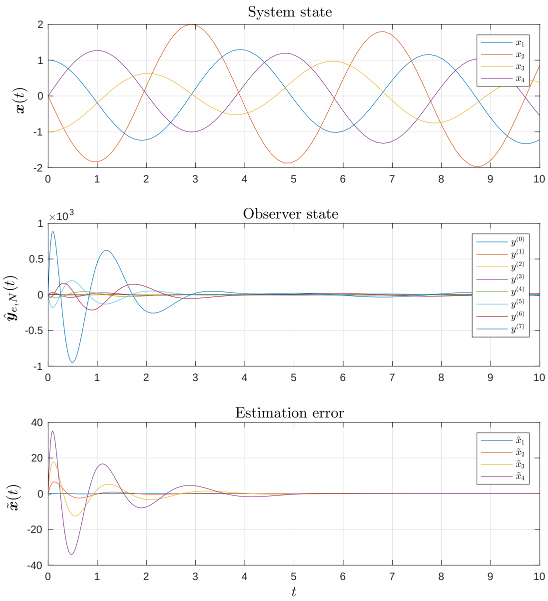


**Fig. 2**: Time behavior of the state of the mechanical system, time history of the state of the state observer (23), and estimation error $\tilde{\boldsymbol{x}} = \boldsymbol{x} - \hat{\boldsymbol{x}}$ obtained when $\mathtt{k} = 1$.

On the other hand, Figure 3 depicts the time behavior of the mechanical system, the time history of the state of the observer (23) and the estimation error obtained when $\mathtt{k} = 2$, $\boldsymbol{x}(0) = [\ 1 \quad 0 \quad -1 \quad 0\ ]^{\top}$, $\hat{\boldsymbol{y}}_{e,7}(0) = \boldsymbol{0}_{7,1}$.
In order to compare the proposed observer with the ones existing in the literature, the simultaneous observation scheme given in Section III of [13] has been implemented for the mechanical system depicted in Figure 1 by designing the observer gains as for the observer (23). Figures 4 and 5 depict the results of numerical simulations of such an observer in which the system has been initialized at the same initial

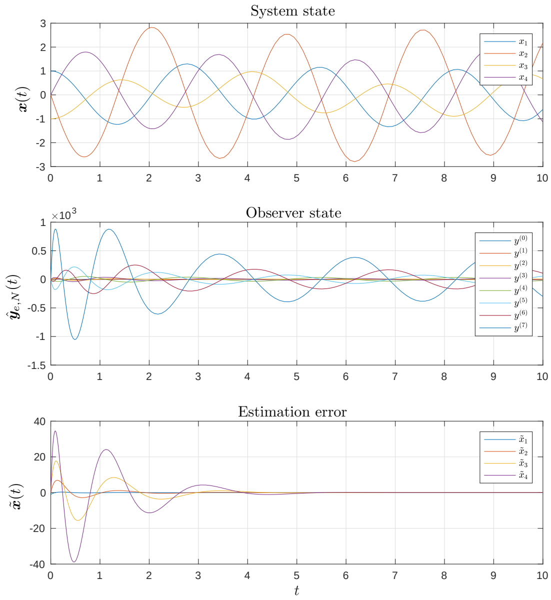


**Fig. 3**: Time behavior of the state of the mechanical system, time history of the state of the observer (23), and estimation error $\tilde{\boldsymbol{x}} = \boldsymbol{x} - \hat{\boldsymbol{x}}$ obtained when $\mathtt{k} = 2$.

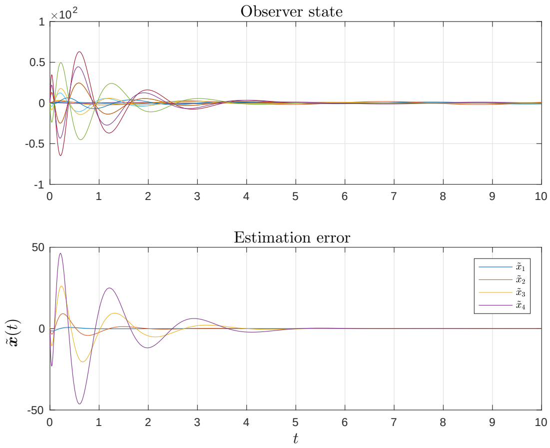


**Fig. 4**: Time history of the state of the observer given in Section III of [13] and estimation error $\tilde{\boldsymbol{x}} = \boldsymbol{x} - \hat{\boldsymbol{x}}$ with $\mathtt{k} = 1$.

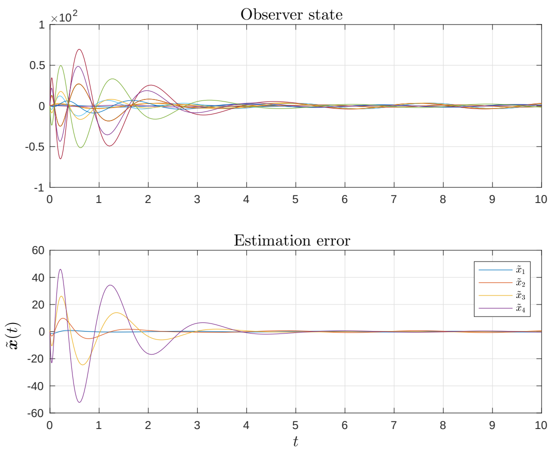


**Fig. 5**: Time history of the state of the observer given in Section III of [13] and estimation error $\tilde{\boldsymbol{x}} = \boldsymbol{x} - \hat{\boldsymbol{x}}$ with $\mathtt{k} = 2$.

condition of the ones reported in Figures 2 and 3 and the observer has been initialized at $\boldsymbol{0}_{12,1}$.

As shown by such figures, although the transient behavior of the observer given in Section III of [13] is similar to the one of the observer (23) and both are capable of estimating the state of the mechanical system without knowing whether the stiffness of the springs is $\mathtt{k} = 1$ or $\mathtt{k} = 2$, the observer given in [13] has a higher state dimension; as a matter of fact, it has 12 states whereas the observer (23) has 8 states. Nonetheless, the computations to be performed off-line to design the proposed observer are slightly more complex than the ones required to design the observer given in Section III of [13]. In fact, while the latter can be designed by determining two stabilizing gains for two linear plants, the former requires first the computation of the matrix $\boldsymbol{L}$ and of the vector $\boldsymbol{\alpha}$ (see Examples 3 and 4 for the explicit steps that have to be carried out) and, secondly, the computation of the stabilizing gain $\boldsymbol{T}$.

### 6.2 Design of state observers for multiple systems with inputs

When dealing with systems with inputs, the design strategy proposed in Subsection 6.1 cannot be directly applied due to the fact that there need not exist a linear embedding for system (7) that holds independently of which of the tuples $(\boldsymbol{A}_1, \boldsymbol{B}_1, \boldsymbol{C}_1, D_1), \dots, (\boldsymbol{A}_s, \boldsymbol{B}_s, \boldsymbol{C}_s, D_s)$ is governing its dynamics. However, it is still possible to design a state observer for such a system by using the results given in [34].

Let $N \in \mathbb{Z}_{\geqslant 0}$ be fixed so that there exists a simultaneous inverse $\boldsymbol{\Phi}_N(\cdot,\cdot)$ of the observability maps $\boldsymbol{\Psi}_{k,N}(\cdot,\cdot)$, $k = 1, \dots, s$, that holds for "almost all" $\boldsymbol{u}_{e,N}$ (see Theorem 2 for the conditions ensuring that such an assumption hold). Hence, assuming that, for all times $t \in \mathbb{R}_{\geqslant 0}$, the input $u$ is such that $|\frac{\mathrm{d}^{N+1}}{\mathrm{d}t^{N+1}} u(t)| \leqslant \Delta$ for some $\Delta \in \mathbb{R}_{\geqslant 0}$ and that the corresponding state response of system (7) is such that $|\boldsymbol{x}(t)| \leqslant \Lambda$ for some $\Lambda \in \mathbb{R}_{\geqslant 0}$, an observer for system (7) that is independent of which of the tuples $(\boldsymbol{A}_1, \boldsymbol{B}_1, \boldsymbol{C}_1, D_1), \dots, (\boldsymbol{A}_s, \boldsymbol{B}_s, \boldsymbol{C}_s, D_s)$ is governing its dynamics is "generically" given by

$$\dot{\hat{y}}^{(0)}(t) = \hat{y}^{(1)}(t) + \frac{\kappa_1}{\varepsilon}(y(t) - \hat{y}^{(0)}(t)), \tag{24a}$$

$$\dot{\hat{y}}^{(1)}(t) = \hat{y}^{(2)}(t) + \frac{\kappa_2}{\varepsilon^2}(y(t) - \hat{y}^{(0)}(t)), \tag{24b}$$

$$\vdots$$

$$\dot{\hat{y}}^{(N)}(t) = \frac{\kappa_{N+1}}{\varepsilon^{N+1}}(y(t) - \hat{y}^{(0)}(t)), \tag{24c}$$

$$\dot{\hat{u}}^{(0)}(t) = \hat{u}^{(1)}(t) + \frac{\kappa_1}{\varepsilon}(u(t) - \hat{u}^{(0)}(t)), \tag{24d}$$

$$\dot{\hat{u}}^{(1)}(t) = \hat{u}^{(2)}(t) + \frac{\kappa_2}{\varepsilon^2}(u(t) - \hat{u}^{(0)}(t)), \tag{24e}$$

$$\vdots$$

$$\dot{\hat{u}}^{(N)}(t) = \frac{\kappa_{N+1}}{\varepsilon^{N+1}}(u - \hat{u}^{(0)}), \tag{24f}$$

$$\hat{\boldsymbol{x}}(t) = \boldsymbol{\Phi}_N(\hat{\boldsymbol{y}}_{e,N}(t), \hat{\boldsymbol{u}}_{e,N}(t)) \tag{24g}$$

where $\hat{\boldsymbol{y}}_{e,N}(t) = [\ \hat{y}^{(0)}(t)\ \ \dots\ \ \hat{y}^{(N)}(t)\ ]^\top$ is the estimate of $\boldsymbol{y}_{e,N}(t)$, $\hat{\boldsymbol{u}}_{e,N}(t) = [\ \hat{u}^{(0)}(t)\ \ \dots\ \ \hat{u}^{(N)}(t)\ ]^\top$ is the estimate of $\boldsymbol{u}_{e,N}(t)$, the coefficients $\kappa_1, \dots, \kappa_{N+1}$ are such that the polynomial $\varsigma^{N+1} + \kappa_1 \varsigma^N + \dots + \kappa_N \sigma + \kappa_{N+1}$ is Hurwitz, $\varepsilon$ is a sufficiently small positive real parameter, and $\hat{\boldsymbol{x}}(t)$ is an estimate of $\boldsymbol{x}(t)$. As a matter of fact, by [34], if $|\frac{\mathrm{d}^{N+1}}{\mathrm{d}t^{N+1}} u(t)| \leqslant \Delta$ and $|\boldsymbol{x}(t)| \leqslant \Lambda$ (thus implying by Lemma 1 that there exists $\bar{\Lambda}$ such that $|\frac{\mathrm{d}^{N+1}}{\mathrm{d}t^{N+1}} u(t)| \leqslant \bar{\Lambda}$), then the estimation errors $|\boldsymbol{y}_{e,N}(t) - \hat{\boldsymbol{y}}_{e,N}(t)|$ and $|\boldsymbol{u}_{e,N}(t) - \hat{\boldsymbol{u}}_{e,N}(t)|$ can be made arbitrarily small in an arbitrarily small amount of time by letting the parameter $\varepsilon$ be sufficiently small. Therefore, in view of the absolute continuity of the function $\boldsymbol{\Phi}_N(\cdot,\cdot)$ in its domain, system (24) is able of "practically" estimate (*i.e.*, with arbitrarily small estimation error and convergence time) the state of system (7), without requiring the knowledge of which of the tuples $(\boldsymbol{A}_1, \boldsymbol{B}_1, \boldsymbol{C}_1, D_1), \dots, (\boldsymbol{A}_s, \boldsymbol{B}_s, \boldsymbol{C}_s, D_s)$ is governing its dynamics. Such a system is referred to as a *simultaneous state observer for multiple systems with inputs*.

The next example illustrates the application of such an observer.

**Example 6.** Consider the electric circuit depicted in Figure 6, which is composed by a capacitor with capacitance $\mathtt{c}$, an inductor with inductance $\mathtt{l}$ two resistors with resistance $\mathtt{r}$, and two switches. Assume that the position of the two switches is not known and let the output $y$ be the voltage across the resistor and let the input $u$ be the current delivered by the current source.

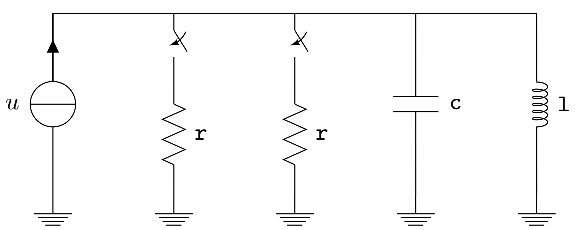


**Fig. 6**: An electric circuit with two switches.

Letting the values of the parameters be unitary, the dynamics of the circuit are modeled by system (7) with $s = 3$,

$$\boldsymbol{A}_1 = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & 0 \end{bmatrix}, \quad \boldsymbol{B}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{C}_1 = [\ 0 \quad 1\ ],$$

$$\boldsymbol{A}_2 = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & -1 \end{bmatrix}, \quad \boldsymbol{B}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{C}_2 = [\ 0 \quad 1\ ],$$

$$\boldsymbol{A}_3 = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}, \quad \boldsymbol{B}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{C}_3 = [\ 0 \quad 1\ ],$$

and $D_1 = D_2 = D_3 = 0$, corresponding to the possible configurations of the switches. Letting $N = 6$ and using the results established in Section 4 (see Example 2 for the explicit steps that have to be carried out), one obtains a simultaneous inverse $\Phi_6(\cdot,\cdot)$ of the observability maps $\boldsymbol{\Psi}_{k,6}(\cdot,\cdot)$, $k = 1, 2, 3$, (whose explicit expression is omitted for compactness), which can be used for all input functions $u(t)$ such that $u^{(2)}(t)$ does not vanish (or vanishes at isolated time instants). Hence, such a function can be used to design the simultaneous state observer (24).
Numerical simulations have been carried out to test such an observer. Figure 7 depicts the time behavior of the state of the electrical circuit and the estimation error obtained by using the state observer (24) with $\varepsilon = 10^{-3}$, $\hat{\boldsymbol{y}}_{e,6}(0) = \boldsymbol{0}_{7,1}$, and $\kappa_1 = 7$, $\kappa_2 = 21$, $\kappa_3 = 35$, $\kappa_4 = 35$, $\kappa_5 = 21$, $\kappa_6 = 7$, $\kappa_7 = 1$, when no switch is closed, $\boldsymbol{x}(0) = [\ 1 \quad 0\ ]^\top$, $u(t) = 0.01\, t^2$.

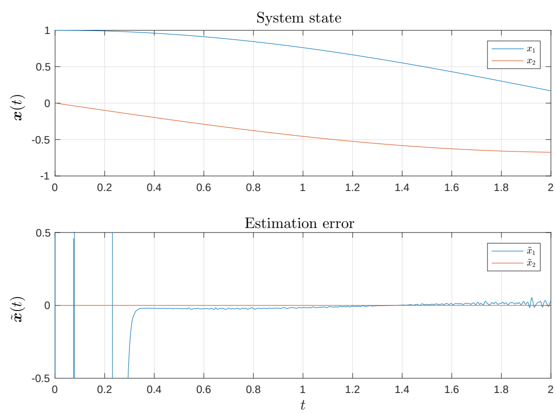


**Fig. 7**: Time behavior of the state of the electrical circuit and estimation error $\tilde{\boldsymbol{x}} = \boldsymbol{x} - \hat{\boldsymbol{x}}$ obtained when no switch is closed.

Figure 8 depicts the behavior of the electrical circuit system and the estimation error obtained by using the state observer (24) with the same parameters as above, when one switch is closed, $\boldsymbol{x}(0) = [\ 1 \quad 0\ ]^\top$ and $u(t) = 0.01\, t^2$.

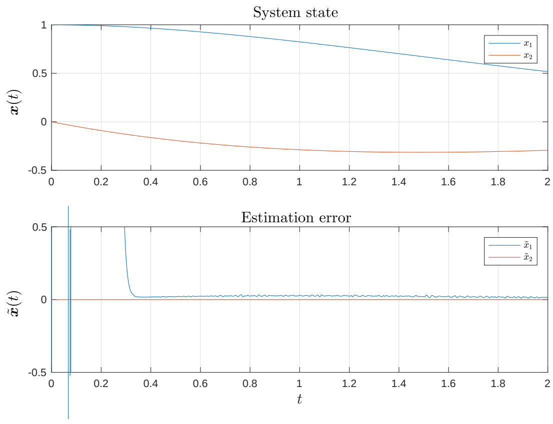


**Fig. 8**: Time behavior of the state of the electrical circuit and estimation error $\tilde{\boldsymbol{x}} = \boldsymbol{x} - \hat{\boldsymbol{x}}$ obtained when one switch is closed.

Finally, Figure 9 depicts the behavior of the electrical circuit system and the estimation error obtained by using the observer (24) with the same parameters as above, when two switches are closed, $\boldsymbol{x}(0) = [\ 1 \quad 0\ ]^\top$ and $u(t) = 0.01\, t^2$.

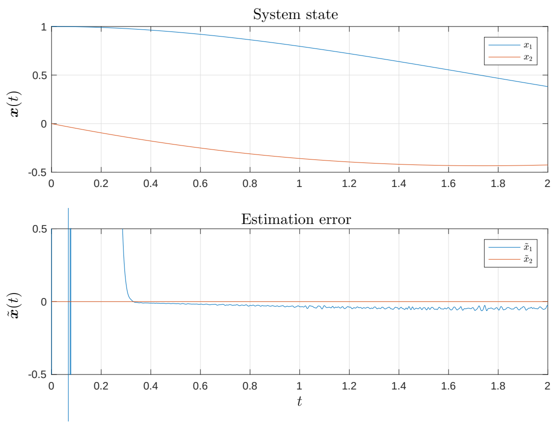


**Fig. 9**: Time behavior of the state of the electrical circuit and estimation error $\tilde{\boldsymbol{x}} = \boldsymbol{x} - \hat{\boldsymbol{x}}$ obtained when two switches are closed.

As shown by Figures 7, 8, and 9, the simultaneous state observer (24) is capable of practically reconstructing the state of the electrical system without requiring the knowledge of the position of the switches.
Note that the results given in [13] cannot be directly applied to design an observer for the considered electrical circuit since [13] does not deal with the case of three systems with inputs. Furthermore, also the approach given in [16] cannot be directly used since Assumption A2 of [16] does not hold for the considered system.

## 7 Conclusions

In this paper, algebraic geometry tools have been proposed to characterize the simultaneous observability of a set of linear systems and to design a simultaneous state observer. In order to pursue this objective, firstly an algebraic technique has been proposed to compute the set of all the embeddings of a single linear system and it has been used to find a parametrization of all the linear inverses of its observability map. Such a parametrization has been used to provide necessary and sufficient conditions for the existence of a simultaneous inverse, holding for almost all inputs, of the observability maps of multiple linear systems and to provide an algebraic geometry technique capable of computing such an inverse. The results given for single-input single-output linear system have been then specialized to the case of autonomous systems, for which much stronger results hold. In particular, a "generic" result has been given on the number of time-derivatives of the output that have to be taken into account in order to allow one to jointly invert $k$ observability maps. Finally, it has been shown how such techniques can be directly used to design simultaneous observers for multiple linear systems by coupling them with "practical" high-gain observers. The theoretical results have been corroborated and illustrated by several examples reported all throughout the paper.

The main advantage of the tools given in this paper with respect to others given in the literature [9, 11–13] is that they provide an exact certificate for the simultaneous observability of a set of linear system or for the lack thereof. Furthermore, they allow one to compute directly a closed-form expression for the simultaneous inverse of the observability maps of the systems in the set, which can be readily used to design simultaneous observers by interfacing it with high-gain observers.

It is worth pointing out that although, for simplicity, the results given in Section 6.2 have been illustrated assuming that the input is $C^k$ for some sufficiently large $k \in \mathbb{Z}_{\geqslant 0}$, the proposed technique can be employed also if the input is discontinuous, provided that there is a minimum dwell time between two consecutive discontinuities (see [24] for further details).

As shown in Example 5, the proposed simultaneous state observer for multiple autonomous systems has state dimension smaller than others available in the literature, although the computations that have to be carried out off-line to design the observer may be slightly more complex. Furthermore, as shown in Example 6, the given simultaneous state observer for multiple systems with inputs can be used in some cases in which other design procedures cannot be used, although it requires more strict hypotheses on the input, which has to be piecewise $C^k$ for some sufficiently large $k \in \mathbb{Z}_{\geqslant 0}$, with a minimum dwell time between two consecutive discontinuities.

It is worth mentioning that the proposed algebraic technique for a single system can be extended so to deal with unknown inputs. In fact, by [35, 36], the state $\boldsymbol{\xi}$ of system (1) is observable with unknown inputs if and only if system (1) is differentially flat. In such a case, by letting $\mathcal{I}_N := \langle \boldsymbol{\psi}_{e,N} - \boldsymbol{O}_N\, \boldsymbol{\xi} - \boldsymbol{M}_N\, \boldsymbol{\nu}_{e,N} \rangle$ be the ideal generated by the relations given in (3), an inverse of the observability map $\boldsymbol{\Psi}_N(\boldsymbol{\xi}, \boldsymbol{\nu}_{e,N})$ that is independent of the input $\nu$ and its time derivatives can be determined by computing the Gröbner basis of $\mathcal{K}_N := \mathcal{I}_N \cap \mathbb{R}[\boldsymbol{\xi}, \boldsymbol{\psi}_{e,N}]$ according to the Lex order with $\psi^{(0)} \succ_{\mathrm{L}} \ldots \succ_{\mathrm{L}} \psi^{(N)} \succ_{\mathrm{L}} \xi_1 \succ_{\mathrm{L}} \ldots \xi_n$. In particular, there exists a rational inverse of the observability map $\boldsymbol{\Psi}_N(\boldsymbol{\xi}, \boldsymbol{\nu}_{e,N})$ that is independent of the input $\nu$ and its time derivatives if and only if there are polynomials $g_1, \ldots, g_n \in \mathcal{K}_N$ such that $\mathrm{LT}(g_i) = \varphi(\boldsymbol{\psi}_{e,N}) x_i$, $i = 1, \ldots, n$ (see [24] for further details). Designing a common unknown input observer is therefore easy with the method proposed here if such polynomials exist, as shown in the following example.

**Example 7.** Consider system (7) with $s = 2$,

$$\boldsymbol{A}_1 = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ -\frac{5}{3} & \frac{1}{3} \end{bmatrix}, \quad \boldsymbol{B}_1 = \begin{bmatrix} -3 \\ 3 \end{bmatrix}, \quad \boldsymbol{C}_1 = [\ \tfrac{1}{3} \quad \tfrac{1}{3}\ ],$$

$$\boldsymbol{A}_2 = \begin{bmatrix} \frac{7}{3} & \frac{10}{3} \\ -\frac{13}{3} & -\frac{7}{3} \end{bmatrix}, \quad \boldsymbol{B}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \boldsymbol{C}_2 = [\ \tfrac{1}{3} \quad \tfrac{1}{3}\ ].$$

By defining the ideal $\mathcal{I}_{1,1} := \langle \boldsymbol{y}_{e,1} - \boldsymbol{O}_{1,1}\, \boldsymbol{x} - \boldsymbol{M}_{1,1}\, \boldsymbol{u}_{e,1} \rangle$ and computing the reduced Gröbner basis of $\mathcal{K}_{1,1} := \mathcal{I}_{1,1} \cap \mathbb{R}[\boldsymbol{x}, \boldsymbol{y}_{e,1}]$ according to the Lex order with $y^{(0)} \succ_{\mathrm{L}} y^{(1)} \succ_{\mathrm{L}} x_1 \succ_{\mathrm{L}} x_2$, one obtains that $\mathcal{K}_{1,1} = \langle g_1, g_2 \rangle$ with

$$g_1 = x_2 - 2y_0 - y_1,$$
$$g_2 = x_1 - y_0 + y_1,$$

which implies that

$$\boldsymbol{\theta}_1 = \begin{bmatrix} y_0 - y_1 \\ 2y_0 + y_1 \end{bmatrix}$$

is an inverse of the observability map $\boldsymbol{\Psi}_{1,1}(\boldsymbol{x}, \boldsymbol{u}_{e,1})$ that is independent of the input $u$ and its time derivative. Furthermore, by computing the reduced Gröbner basis of the syzygy of $[\ \boldsymbol{W}_3 \quad \boldsymbol{\Theta}\ ]$, one of its elements is $[\ 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1\ ]^\top$, and hence $\boldsymbol{\theta}_1$ is a common inverse of the observability maps $\boldsymbol{\Psi}_{1,3}(\boldsymbol{x}, \boldsymbol{u}_{e,3})$ and $\boldsymbol{\Psi}_{2,3}(\boldsymbol{x}, \boldsymbol{u}_{e,3})$ that is independent of the input $u$ and its time derivatives. Therefore, a common unknown input observer for the considered system can be designed by coupling the observer (24a)–(24c) with such an inverse.

On the other hand, if there do not exist polynomials $g_{k,1}, \ldots, g_{k,n} \in \langle \boldsymbol{y}_{e,N} - \boldsymbol{O}_{k,N}\, \boldsymbol{x} - \boldsymbol{M}_{k,N}\, \boldsymbol{u}_{e,N} \rangle \cap \mathbb{R}[\boldsymbol{y}_{e,N}, \boldsymbol{x}]$ such that $\mathrm{LT}(g_{k,i}) = \varphi(\boldsymbol{\psi}_{e,N})x_i$, $i = 1, \ldots, n$, $k = 1, \ldots, s$, it is not easy to extend the techniques proposed in this paper; this would correspond to design unknown input observers for multiple systems in the case of unknown input detectability, as in [14–16].

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