A finite-time local observer in the original coordinates for nonlinear control systems

A finite-time local observer in the original coordinates for nonlinear control systems

Corrado Possieri, Simone Vidano and Carlo Novara

***Abstract*—In this note, by coupling sliding mode differentiators with a tool capable of inverting in finite time the suspension of the observability map, a local finite-time observer for nonlinear control systems is proposed. Differently from other approaches, this observer does not rely on a change of coordinates and provides the state estimates in the original coordinates.**

***Index Terms*—Nonlinear control systems, observers, sliding mode, observability.**

I. Introduction

In several practical applications, the state of a system has to be determined from the available measurements, either to design a controller or to simply obtain real-time information about the state of a plant [1]. For linear systems, the problem of designing observers can be addressed by using classical techniques, such as Luenberger observers and the Kalman filter [2]. On the other hand, when dealing with nonlinear plants, the problem is significantly more challenging [3]. Several tools have been proposed in the literature to design observers in the nonlinear case such as linearization via output injection [4], the extended Kalman filter (EKF) [5], Luenberger-like approaches [6], high-gain observers [7], [8], and particle filters [9].

A classical method [3] to design state observers for nonlinear system consists in determining an injective change of coordinates that recasts the nonlinear system in the so-called *canonical observability form*, designing an observer for the transformed system (e.g., by using high-gain observers [10], sliding mode differentiators [11], or super twisting algorithms [12]), and using an inverse of the change of coordinates to estimate the state of the system in the original coordinates.

The main objective of this note consists in proposing a local observer for nonlinear systems that converges in finite time to the current state of the plant. Differently from the design strategy reviewed above, such an observer does not require the knowledge of an inverse of the change of coordinates that recasts the system in canonical observability form.

The proposed observer has been designed coupling the sliding mode exact differentiator given in [13] with a novel tool that is able to dynamically invert, in finite-time, the suspension of the observability map. In particular, the latter system has been designed by using a tool similar to that given in [14], [15], which is a modified version of the Newton algorithm.

It is worth noticing that several other works dealt with the problem of inverting a diffeomorphism. For instance, in [16], [17], [18], high-gain approaches have been proposed to invert the observability map of a nonlinear system through its Jacobian matrix. The main difference between these approaches and the one given in this note is that the latter allows us to dynamically invert time-varying mappings in finite time.

Note that other methods have been proposed in the literature to design observers for nonlinear systems in the original coordinates [8], [19], [20]. Differently from [8], [20], the observer proposed in this note converges in finite time and is capable of estimating the state of nonlinear control systems, whereas, differently from [19], we do not need any convexity assumption to ensure local convergence of the given observer.

In order to show the effectiveness of the proposed observer, its sensitivity to measurement noise is characterized and it is shown that its accuracy is asymptotically the best possible among the methods relying on the time derivatives of the output to estimate the state of the system.

II. Observability for nonlinear control systems

Let $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Z}_{\geqslant 0}$ and $\mathbb{R}_{\geqslant 0}$, $\mathbb{R}_{>0}$ denote the set of integer, real, natural, nonnegative real, and positive real numbers, respectively. The symbol $\mathbb{1}(t)$ denotes the step function centered at $t=0$. Letting $x \in \mathbb{R}^n$, $\|x\|_j$ denotes the jth norm of x. The symbol $\times$ denotes the Cartesian product.

Consider the nonlinear dynamical system

$$\dot{x} = f(x,u), \qquad\qquad y = h(x,u), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}$ is the input, $y(t) \in \mathbb{R}$ is the output, $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $h : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ are functions of class $\mathcal{C}^k$ for some sufficiently large $k \in \mathbb{Z}_{\geqslant 0}$. Letting $u : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ be a class $\mathcal{C}^k$ function for some sufficiently large $k \in \mathbb{Z}_{\geqslant 0}$, let $\phi(t, x_0, u)$ be the solution to system (1) from the initial state $x_0 \in \mathbb{R}^n$, which satisfies

$$\phi(0, x_0, u) = x_0, \quad \frac{\mathrm{d}\phi(t,x_0,u)}{\mathrm{d}t} = f(\phi(t,x_0,u), u(t)).$$

For simplicity, assume that the control input $u(t)$ is such that $\phi(t, x_0, u)$ is well defined over $\mathbb{R}_{\geqslant 0}$ for each $x_0 \in \mathbb{R}^n$. Thus, let $u_e^{(i,j)} \doteq [\ u^{(i)} \ \ \cdots \ \ u^{(j)}\ ]^\top$ for each $i, j \in \mathbb{Z}_{\geqslant 0}$, $j \geqslant i$, and define for all $k \in \mathbb{Z}_{\geqslant 0}$

$$D_f^0 h(x, u_e^{(0,0)}) \doteq h(x, u_0), \tag{2a}$$

$$D_f^{k+1} h(x, u_e^{(0,k+1)}) \doteq \frac{\partial D_f^k h(x, u_e^{(0,k)})}{\partial x} f(x, u_0) + \frac{\partial D_f^k h(x, u_e^{(0,k)})}{\partial u_e^{(0,k)}} u_e^{(1,k+1)}. \tag{2b}$$

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Thus, define the *observability matrix* of order $\kappa \in \mathbb{Z}_{\geqslant 0}$,

$$O_{\kappa-1}(x, u_e^{(0,\kappa-1)}) \doteq \begin{bmatrix} D_f^0 h(x, u_e^{(0,0)}) \\ \vdots \\ D_f^{\kappa-1} h(x, u_e^{(0,\kappa-1)}) \end{bmatrix}. \qquad (3)$$

Letting $y^{(i)}(t) = \frac{\mathrm{d}^i y(t)}{\mathrm{d}t^i}$, $u^{(i)}(t) = \frac{\mathrm{d}^i u(t)}{\mathrm{d}t^i}$, $i \in \mathbb{Z}_{\geqslant 0}$, and defining the vector $y_e^{(0,\kappa-1)} \doteq [\ y^{(0)}(t) \ \ \cdots \ \ y^{(\kappa-1)}(t)\ ]^\top$, $\kappa \in \mathbb{Z}_{\geqslant 0}$, $\kappa \geqslant 1$, one has that, for all $t \in \mathbb{R}_{\geqslant 0}$,

$$y_e^{(0,\kappa-1)}(t) = O_{\kappa-1}(\phi(t, x_0, u), u_e^{(0,\kappa-1)}(t)),$$

i.e., $O_{\kappa-1}$ relates the current value of the solution of system (1) and the derivatives of the input $u(t)$ (up to order $\kappa-1$) with the current value of the time derivatives of the output $y(t)$ (up to order $\kappa-1$), $\kappa \in \mathbb{Z}_{\geqslant 0}$, $\kappa \geqslant 1$. Hence, let $\mathcal{U}_e^\kappa \subset \mathbb{R}^n$ denote the set of all the admissible $u_e^{(0,\kappa-1)}(t)$ (i.e., the set of all the admissible control inputs together with their time derivatives up to order $\kappa-1$) and consider the *suspension* $\Phi : \mathbb{R}^n \times \mathcal{U}_e^\kappa \to \mathbb{R}^\kappa \times \mathcal{U}_e^\kappa$ *of the matrix* $O_{\kappa-1}$ [21],

$$\Phi(x, u_e^{(0,\kappa-1)}) \doteq (O_{\kappa-1}(x, u_e^{(0,\kappa-1)}), u_e^{(0,\kappa-1)}).$$

Such a map relates the current state of system (1) and the time derivatives of its input with the time derivatives of the output and of the input. Given $\mathcal{X} \subset \mathbb{R}^n$, let

$$\mathcal{Y} \doteq \bigcup_{(x, u_e^{0,\kappa-1}) \in \mathcal{X} \times \mathcal{U}_e^\kappa} O_{\kappa-1}(x, u_e^{(0,\kappa-1)}) \subset \mathbb{R}^\kappa.$$

System (1) is *strongly κ-differentially observable in* $\mathcal{X} \times \mathcal{U}_e^\kappa \subset \mathbb{R}^n \times \mathbb{R}^\kappa$ if the *restriction* of $\Phi(\cdot,\cdot)$ to $\mathcal{X} \times \mathcal{U}_e^\kappa$,

$$\Phi_{\mathcal{X}, \mathcal{U}_e^\kappa} : \mathcal{X} \times \mathcal{U}_e^\kappa \to \mathcal{Y} \times \mathcal{U}_e^\kappa, \qquad (4)$$

$\Phi_{\mathcal{X}, \mathcal{U}_e^\kappa}(x, u_e^{(0,\kappa-1)}) = \Phi(x, u_e^{(0,\kappa-1)})$ for all $(x, u_e^{(0,\kappa-1)}) \in \mathcal{X} \times \mathcal{U}_e^\kappa$, is an injective immersion [21]. On the other hand, system (1) is *strongly observable* if the suspension $\overline{\Phi}(\cdot,\cdot)$, obtained from $(O_{\kappa-1}(x, u_e^{(0,\kappa-1)}), u_e^{(0,\kappa-1)})$ by taking $\kappa \to +\infty$, is an injective immersion. Clearly, κ-differential observability implies strong observability while the converse need not hold [22]. In addition, differently from linear systems, nonlinear ones may have *singular inputs* that make them unobservable (these inputs are ruled out here by the subsequent assumption that the sets $\mathcal{X}$ and $\mathcal{U}_e^n$ are such that $\Phi_{\mathcal{X}, \mathcal{U}_e^n}$ is a diffeomorphism). However, by [23], κ-differential observability is a generic property for nonlinear systems in the form (1), i.e., almost all the systems that can be written in the form (1) are κ-differentially observable, for some $\kappa \in \mathbb{Z}_{\geqslant 0}$, although, generically, $\kappa > n$.

In order to design a finite-time observer for system (1), all throughout this preliminary paper, we assume that system (1) is n-differentially observable in $\mathcal{X} \times \mathcal{U}_e^n \subset \mathbb{R}^n \times \mathbb{R}^n$ and we show that if the trajectories of system (1) are bounded, then it is possible to design a local state observer that converges to the state $x(t)$ of system (1) in finite time. Toward this objective, in the following section, we propose a technique to dynamically invert, in finite time, a suspension.

III. Finite-time dynamical inverse of a suspension

In this section, by suitably adapting the technique given in [24], a method to dynamically invert a suspension is proposed.

Consider the suspension $L : \mathcal{A} \times \mathcal{V} \to \mathcal{B} \times \mathcal{V}$,

$$L(x, v) = (\ell(x, v), v), \qquad (5)$$

with $\mathcal{A}$ and $\mathcal{B}$ beings suitable subsets of $\mathbb{R}^n$, and assume that $(x, v) \mapsto L(x, v)$ is a diffeormophism in $\mathcal{A} \times \mathcal{V}$.

For any $\varepsilon \in \mathbb{R}_{>0}$, define the sets

$$\begin{aligned} \Omega_\varepsilon &\doteq \{\xi \in \mathbb{R}^n : \|\xi\|_2 < \varepsilon\}, & (6a) \\ \mathcal{A}_\varepsilon &\doteq \{\xi \in \mathcal{A} : \hat{\xi} := \xi - \tilde{\xi} \in \mathcal{A},\ \forall \tilde{\xi} \in \Omega_\varepsilon\}, & (6b) \end{aligned}$$

and assume that ε is sufficiently small so that $\mathcal{A}_\varepsilon \neq \emptyset$.

The main objective of this section is to find an algorithm able to invert $(x, u) \mapsto L(x, u)$ in finite time, i.e., given $\varepsilon \in \mathbb{R}_{>0}$, $x : \mathbb{R}_{\geqslant 0} \to \mathcal{A}_\varepsilon$ and $v : \mathbb{R}_{\geqslant 0} \to \mathcal{V}$, letting

$$(z(t), v(t)) \doteq L(x(t), v(t)), \quad \forall t \in \mathbb{R}_{\geqslant 0},$$

the goal of this section is to construct a dynamical system that, on the basis of measurements of $z(t)$ and $v(t)$ for $t \in [0, T^\star]$, is able to reconstruct the value of $x(t)$ for $t = T^\star$. Such a goal is pursued by using a state observer that implements a modified version of the Newton algorithm. Toward this end, note that $x \mapsto \ell(x, v)$ is a diffeormophism for each $v \in \mathcal{V}$ since $(x, v) \mapsto L(x, v)$ is a diffeomorphism. Hence, the matrix

$$G(x, v) \doteq \frac{\partial \ell(x, v)}{\partial x}$$

has full rank for all $(x, v) \in \mathcal{A} \times \mathcal{V}$ [25]. Thus, consider the following nonlinear system

$$\dot{\hat{x}} = G^{-1}(\hat{x}, v) \left(\mu \, \|\tilde{z}\|_2^\alpha \operatorname{sign}(\tilde{z}) + \dot{z} - \frac{\partial \ell(\hat{x}, v)}{\partial v} \, \dot{v} \right), \qquad (7)$$

where $\mu \in \mathbb{R}_{>0}$, $\alpha \in (0, 1)$, $\tilde{z} \doteq z - \ell(\hat{x}, v) = \ell(x, v) - \ell(\hat{x}, v)$, and $\operatorname{sign}(\cdot)$ denotes the entry-wise $\operatorname{sign}(\cdot)$ operator,

$$\operatorname{sign}(x) = \begin{cases} -1, & \text{if } x < 0, \\ [-1, 1], & \text{if } x = 0, \\ 1, & \text{if } x > 0. \end{cases}$$

To begin with, consider the following lemma, whose proof is wholly similar to [26, Lem. 3.2] and is therefore omitted.

Lemma 1. *Let $\mathcal{V}$ be a compact set. If the map $x \mapsto \ell(x, v)$ is a diffeomorphism on a compact set $\mathcal{A}$ for all $v \in \mathcal{V}$, then there exist constant $M, N \in \mathbb{R}_{>0}$ such that*

$$\begin{aligned} \|x - \hat{x}\|_2 &\leqslant M \, \|\ell(x, v) - \ell(\hat{x}, v)\|_2, \\ \|\ell(x, v) - \ell(\hat{x}, v)\|_2 &\leqslant N \, \|x - \hat{x}\|_2, \end{aligned}$$

for all $x, \hat{x} \in \mathcal{A}$ and all $v \in \mathcal{V}$.

The following theorem guarantees that, under some mild assumptions about the sets $\mathcal{A}$, $\mathcal{V}$, and $\mathcal{B}$, system (7) is able to locally invert the suspension $L(x, u)$.

Theorem 1. *Let $\varepsilon \in \mathbb{R}_{>0}$, the suspension $L : \mathcal{A} \times \mathcal{V} \to \mathcal{B} \times \mathcal{V}$, and $(z(t), v(t)) = L(x(t), v(t)) \in \mathcal{B} \times \mathcal{V}$, with $x(t) \in \mathcal{A}_\varepsilon$ for all $t \in \mathbb{R}_{\geqslant 0}$, be given. Assume that the sets $\mathcal{A}$ and $\mathcal{V}$ are compact. Thus, if $t \mapsto x(t)$ is continuous, then there exist*

$\varkappa \in \mathbb{R}_{>0}$ and $T^\star \in \mathbb{R}_{>0}$ such that, letting $\hat{x}(t)$ be the solution of system (7) *from $\hat{x}(0)$, if $\|x(0)-\hat{x}(0)\|_2 < \varkappa$, then*

$$\hat{x}(t) = x(t), \quad \forall t \geqslant T^\star.$$

Proof. Let $\tilde{x} \doteq x - \hat{x}$ and consider the function

$$V = \frac{1}{2}\tilde{z}^\top \tilde{z} = \frac{1}{2}\|\tilde{z}\|_2^2. \tag{8}$$

Thus, let[1]

$$T \doteq \inf\{t \in \mathbb{R}_{\geqslant 0} : \tilde{x}(t) \notin \Omega_\varepsilon\},$$

that is in $\mathbb{R}_{>0}$ since $\tilde{x}(0) \in \Omega_\varepsilon$ and by the continuity of $t \mapsto x(t)$ and $t \mapsto \hat{x}(t)$. By computing the time derivatives of V along the trajectories of system (7), one obtains that,

$$\begin{aligned}\dot{V} &= \tilde{z}^\top \dot{\tilde{z}} = \tilde{z}^\top \left(\dot{z} - G(\hat{x},v)\,\dot{\hat{x}} - \frac{\partial \ell(\hat{x},v)}{\partial v}\,\dot{v}\right)\\ &= -\mu\|\tilde{z}\|_2^\alpha\,\|\tilde{z}\|_1 \leqslant -\mu\sqrt{n}\,\|\tilde{z}\|_2^{\alpha+1} \leqslant -\mu\sqrt{n}\,V^{\frac{\alpha+1}{2}}, \end{aligned} \tag{9}$$

for all $t \in [0,T)$. Furthermore, since $L : \mathcal{A}\times\mathcal{V} \to \mathcal{B}\times\mathcal{V}$ is a diffeomorphism, there exists a smooth function $\ell^{-1} : \mathcal{B}\times\mathcal{V} \to \mathcal{A}$ such that $\ell^{-1}(\ell(x,v),v) = x$ for all $(x,v) \in \mathcal{A}\times\mathcal{V}$.

This implies that the map $x \mapsto \ell(x,v)$ is a diffeomorphism on $\mathcal{A}$ for all $v \in \mathcal{V}$, and hence, by Lemma 1, there exist $M, N \in \mathbb{R}_{>0}$ such that $\|x-\hat{x}\|_2 \leqslant M\|\ell(x,v)-\ell(\hat{x},v)\|_2$ and $\|\ell(x,v)-\ell(\hat{x},v)\|_2 \leqslant N\|x-\hat{x}\|_2$ for all $x, \hat{x} \in \mathcal{A}$. Let

$$\delta \doteq \frac{\varepsilon}{2\,M}, \qquad \varkappa \doteq \min\left\{\frac{\delta}{N}, \frac{\varepsilon}{2}\right\}.$$

Since $x(t) \in \mathcal{A}_\varepsilon$ and $\tilde{x}(t) \in \Omega_\varepsilon$ for all $t \in [0,T)$, one has $\|\tilde{x}(t)\|_2 \leqslant M\|\tilde{z}(t)\|_2$ and $\|\tilde{z}(t)\|_2 \leqslant N\|\tilde{x}(t)\|_2$, $\forall t \in [0,T)$.

Assume now, by contradiction, that $T < +\infty$, i.e., that there exists $T \in \mathbb{R}_{>0}$ such that $\|\tilde{x}(T)\|_2 = \varepsilon$. This implies that there exists $\tilde{T} \in \mathbb{R}_{>0}$, $\tilde{T} < T$, such that $\|\tilde{z}(\tilde{T})\|_2 \geqslant \frac{\varepsilon}{2\,M}$. This is in contradiction with $\|\tilde{x}(0)\|_2 < \varkappa$ (which implies $\|\tilde{z}(0)\|_2 \leqslant N\|\tilde{x}(0)\|_2 < \delta$) and with the monotonically decreasing behavior of $\|\tilde{z}(t)\|_2$ induced by the condition given in (9) for all $t \in [0,T)$. Therefore, we have that $T = +\infty$. Hence, by classical results about finite-time convergence [27], the inequality given in (9) implies that

$$\tilde{z}(t) = 0, \quad \forall t \geqslant \frac{2^{\frac{\alpha+1}{2}}\,\delta^{1-\alpha}}{(1-\alpha)\,\mu\sqrt{n}} \doteq T^\star.$$

The proof is concluded by the fact that, since $x \mapsto \ell(x,v)$ is a diffeormophism for each $v \in \mathcal{V}$, one has that $\ell(x,v) = \ell(\hat{x},v)$ if and only if $x = \hat{x}$, for each $v \in \mathcal{V}$. □

Note that the proof of Theorem 1 establishes also robustness with respect to inflations and perturbations of the inversion dynamics given in (7). In fact, by Lemma 1, we have that $\|\tilde{z}\|_2 \leqslant N\|\tilde{x}\|_2$, and hence, if $\tilde{x} \in \Omega_\varepsilon$, $z \in \mathcal{B}$, and $z - \tilde{z} \in \mathcal{B}$, the function V given in (8) satisfies

$$\frac{1}{2\,M^2}\|\tilde{x}\|_2^2 \leqslant V \leqslant \frac{N^2}{2}\|\tilde{x}\|_2^2, \tag{10}$$

i.e., V is a local Lyapunov function for the error dynamics. By Chapter 7 of [28], the existence of such a function establishes asymptotic stability of the error to the set $\{0\}$ with respect to (small) measurement errors and to perturbations of the vector field governing the dynamics (7).

Remark 1. Note that the assumptions about the sets $\mathcal{A}$, $\mathcal{V}$, and $\mathcal{B}$ can be weakened by requiring that there exist $\underline{\lambda}, \overline{\lambda} \in \mathbb{R}_{>0}$ such that $\underline{\lambda} I \leqslant G^\top(x,v)G(x,v) \leqslant \overline{\lambda} I$ for all $(x,v) \in \mathcal{A}\times\mathcal{V}$ (thus guaranteeing that both $G(\hat{x},v)$ and $G^{-1}(\hat{x},v)$ are bounded along the trajectories of system (7)) and that the map $(z,v) \mapsto \ell^{-1}(z,v)$ satisfies $\|\ell^{-1}(z,v) - \ell^{-1}(\hat{z},v)\|_2 \leqslant M\|z-\hat{z}\|_2$ for all $z, \hat{z} \in \mathcal{B}$ and $v \in \mathcal{V}$ (thus guaranteeing that $\|\tilde{x}(t)\|_2 \leqslant M\|\tilde{z}(t)\|_2$ for all $t \in [0,T)$). However, the conditions given in the statement of Theorem 1 can be easily verified in practice and hence have been preferred.

IV. Finite-time observers for nonlinear systems

In this section, the methods detailed in the previous section are coupled with the exact differentiator given in Section 6.7 of [13] in order to design a finite-time local observer for a nonlinear control system. Toward this end, it is assumed that the input and its time derivatives are measured for all times, i.e., the vector $u_e^{(0,n)}(t)$ is available for all times $t \in \mathbb{R}_{\geqslant 0}$. Such an assumption is removed in the subsequent Remark 5.

Given $\varepsilon \in \mathbb{R}_{>0}$ and letting $\mathcal{Y} \subset \mathbb{R}^n$, define the sets

$$\begin{aligned}\Omega_\varepsilon &\doteq \{\vartheta \in \mathbb{R}^n : \|\vartheta\|_2 < \varepsilon\},\\ \mathcal{Y}_\varepsilon &\doteq \{\vartheta \in \mathcal{Y} : \hat{\vartheta} := \vartheta - \tilde{\vartheta} \in \mathcal{Y},\ \forall \tilde{\vartheta} \in \Omega_\varepsilon\},\end{aligned}$$

and assume that ε is sufficiently small in order to guarantee that $\mathcal{Y}_\varepsilon$ is nonempty. Thus, let

$$J(x, u_e^{(0,n-1)}) \doteq \frac{\partial O_{n-1}(x, u_e^{(0,n-1)})}{\partial x},$$

and consider the following system

$$\dot{\eta}_0 = \zeta_0, \tag{11a}$$
$$\zeta_0 \doteq \eta_1 - \lambda_n\,L^{\frac{1}{n+1}}\,|\eta_0 - y|^{\frac{n}{n+1}}\,\mathrm{sign}(\eta_0 - y), \tag{11b}$$
$$\dot{\eta}_1 = \zeta_1, \tag{11c}$$
$$\zeta_1 \doteq \eta_2 - \lambda_{n-1}\,L^{\frac{1}{n}}\,|\eta_1 - \zeta_0|^{\frac{n-1}{n}}\,\mathrm{sign}(\eta_1 - \zeta_0), \tag{11d}$$
$$\vdots$$
$$\dot{\eta}_n = \zeta_n, \tag{11e}$$
$$\zeta_n \doteq -\lambda_0\,L\,\mathrm{sign}(\eta_n - \zeta_{n-1}), \tag{11f}$$
$$\begin{aligned}\dot{\hat{x}} = J^{-1}(\hat{x}, u_e^{(0,n-1)})\Big(&\mu\,\|\tilde{\eta}_e^{(0,n-1)}\|_2^\alpha\,\mathrm{sign}(\tilde{\eta}_e^{(0,n-1)})\\ &+ \eta_e^{(1,n)} - \frac{\partial O_{n-1}(\hat{x}, u_e^{(0,n-1)})}{\partial u_e^{(0,n-1)}}\,u_e^{(1,n)}\Big),\end{aligned} \tag{11g}$$

where $\eta_e^{(1,n)} \doteq [\ \eta_1\ \ \cdots\ \ \eta_n\ ]^\top$,

$$\tilde{\eta}_e^{(0,n-1)} \doteq [\ \eta_0\ \ \cdots\ \ \eta_{n-1}\ ]^\top - O_{n-1}(\hat{x}, u_e^{(0,n-1)}), \tag{11h}$$

$\lambda_0, \lambda_1, \ldots, \lambda_n \in \mathbb{R}_{>0}$ are properly chosen constants (see [13, Rem. 6.1] and [29, Prop. 1]), $\alpha \in (0,1)$, $L \in \mathbb{R}_{>0}$ and μ are sufficiently large parameters.

The following theorem guarantees that system (11) is a local observer for system (1).

Theorem 2. *Let $\mathcal{X}$, $\mathcal{U}_e^n$ and $\mathcal{Y}$ be suitable subsets of $\mathbb{R}^n$ such that the suspension $\Phi_{\mathcal{X},\mathcal{U}_e^n}$ given in* (4) *is a diffeomorphism.*

[1] In the following, we use the convention $\inf\emptyset = +\infty$.

Assume that the sets $\mathcal{X}$ and $\mathcal{U}_e^n$ are compact and that the nth time derivative of the output

$$y^{(n)}(t) = D_f^n h(\phi(t, x_0, u), u_e^{(0,n)}(t)),$$

is L-Lipschitz as a function of $t \in \mathbb{R}_{\geqslant 0}$. Thus, there are $\gamma, \mu^\star, \nu, \varrho \in \mathbb{R}_{>0}$ such that if $O_{n-1}(\phi(t, x_0, u), u_e^{(0,n-1)}) \in \mathcal{Y}_\gamma$, $\mu \geqslant \mu^\star$, $\|\tilde{\eta}_e^{(0,n-1)}(0)\|_2 \leqslant \nu$, and $|y_e^{(i)}(0) - \eta_i(0)| \leqslant \varrho$, $i = 0, \ldots, n$, then there exists a finite $T^\star \in \mathbb{R}_{\geqslant 0}$ such that

$$\hat{x}(t) = x(t), \quad \forall t \geqslant T^\star.$$

Proof. By Section 6.7 of [13], there exists $T_1^\star \in \mathbb{R}_{>0}$ such that $\tilde{y}_e^{(0,n)}(t) \doteq y_e^{(0,n)}(t) - [\ \eta_0(t) \ \ \cdots \ \ \eta_n(t)\ ]^\top$ vanishes identically for all times $t \geqslant T_1^\star$. Therefore, letting $\tilde{x} \doteq x - \hat{x}$ and $\eta_e^{(0,n-1)} \doteq [\ \eta_0 \ \ \cdots \ \ \eta_{n-1}\ ]^\top$, if $\|\tilde{x}(T_1^\star)\|_2 \leqslant \varepsilon$ and

$$\begin{aligned}\|\ell(\hat{x}(T_1^\star), v(T_1^\star)) - \ell(x(T_1^\star), v(T_1^\star))\|_2 \\ = \|\ell(\hat{x}(T_1^\star), v(T_1^\star)) - y_e^{(0,n-1)}(T_1^\star)\|_2 \leqslant \delta,\end{aligned}$$

for some sufficiently small $\delta \in \mathbb{R}_{>0}$, then, by Theorem 1, there exists $T_2^\star \in \mathbb{R}_{>0}$ such that $\tilde{x}(t) = 0$ for all $t \geqslant T_2^\star$. Therefore, in order to establish the statement, it suffices to prove that the latter two conditions hold.

By Theorem 6.4 of [13], the dynamics of the estimation error $\tilde{y}_e^{(0,n)}$ are Lyapunov stable, i.e., for each $\gamma \in \mathbb{R}_{\geqslant 0}$ there exists $\varrho \in \mathbb{R}_{\geqslant 0}$ such that if $\|\tilde{y}_e^{(0,n)}(0)\|_2 \leqslant \varrho$ then $\|\tilde{y}_e^{(0,n)}(t)\|_2 \leqslant \frac{\gamma}{2}$ for all $t \in \mathbb{R}_{\geqslant 0}$. Therefore, since, by assumption, $y_e^{(0,n-1)}$ is in $\mathcal{Y}_\gamma$, if $\|\tilde{y}_e^{(0,n)}(0)\|_2 \leqslant \varrho$, then $\eta_e^{(0,n-1)}(t) \in \mathcal{Y}_{\frac{\gamma}{2}}$ for all $t \in \mathbb{R}_{\geqslant 0}$. Due to the injectivity of $\Phi : \mathcal{X} \times \mathcal{U}_e^n \to \mathcal{Y} \times \mathcal{U}_e^n$, for each $(x, u_e^{(0,n-1)}) \in \mathcal{X} \times \mathcal{U}_e^n$, there exists $O^{-1} : \mathcal{Y} \times \mathcal{U}_e^n \to \mathcal{X}$ such that

$$O^{-1}(O_{n-1}(x, u_e^{(0,n-1)}), u_e^{(0,n-1)}) = x,$$

for all $(x, u_e^{0,n-1}) \in \mathcal{X} \times \mathcal{U}_e^n$. Thus, let $\chi \doteq O^{-1}(\eta_e^{(0,n-1)}, u_e^{(0,n-1)})$, which is well defined for all $t \in \mathbb{R}_{\geqslant 0}$ since $\eta_e^{(0,n-1)}(t) \in \mathcal{Y}_{\frac{\gamma}{2}}$ for all $t \in \mathbb{R}_{\geqslant 0}$. Hence, let $\tilde{\chi} \doteq \chi - \hat{x}$ and consider the function

$$V = \frac{1}{2}(\tilde{\eta}_e^{(0,n-1)})^\top \tilde{\eta}_e^{(0,n-1)}. \tag{12}$$

Letting

$$T_3 \doteq \inf\{t \in \mathbb{R}_{\geqslant 0} : \|\tilde{\chi}(t)\|_2 \geqslant \varepsilon\}$$

and by computing the time derivative of V along the trajectories of (11), one has

$$\begin{aligned}\dot{V} = (\tilde{\eta}_e^{(0,n-1)})^\top \Big(\dot{\eta}_e^{(0,n-1)} - \eta_e^{(1,n)} \\ - \mu \|\tilde{\eta}_e^{(0,n-1)}\|_2^\alpha \operatorname{sign}(\tilde{\eta}_e^{(0,n-1)}) \Big),\end{aligned}$$

for all $t \in [0, T_3)$. Hence, let $\sigma_i = \dot{\eta}_i - \eta_{i+1}$, $i = 0, \ldots, n-1$, so that $\dot{\eta}_e^{(0,n-1)} - \eta_e^{(1,n)} = [\ \sigma_0 \ \ \cdots \ \ \sigma_{n-1}\ ]^\top \doteq \sigma$. By (11), it can be easily derived that

$$\sigma_i = -\lambda_{n-i} L^{\frac{1}{n+1-i}} |\eta_i - \zeta_{i-1}|^{\frac{n-i}{n+1-i}} \operatorname{sign}(\eta_i - \zeta_{i-1}), \tag{13}$$

where $\zeta_{-1} = y$, for $i = 0, \ldots, n-1$ and for all $t \in \mathbb{R}_{\geqslant 0}$. Therefore, there exists $S \in \mathbb{R}_{\geqslant 0}$ such that $\|\sigma(t)\| \leqslant S$ for all $t \in \mathbb{R}_{\geqslant 0}$. This implies that

$$\dot{V} \leqslant \|\tilde{\eta}_e^{(0,n-1)}\|_2 (S - \mu \sqrt{n} \|\tilde{\eta}_e^{(0,n-1)}\|_2^\alpha), \tag{14}$$

for all $t \in [0, T_3)$. Thus, letting

$$\mu^\star \doteq \frac{2S}{\sqrt{n}\,\gamma},$$

if $\mu \geqslant \mu^\star$, then $\dot{V} \leqslant 0$ for all $\tilde{\eta}_e^{(0,n-1)}$ such that $\|\tilde{\eta}_e^{(0,n-1)}\|_2 \geqslant \frac{\gamma}{2}$ and for all $t \in [0, T_3)$. Therefore, if $\|\tilde{\eta}_e^{(0,n-1)}(0)\|_2 \leqslant \frac{\gamma}{2}$ and $\mu \geqslant \mu^\star$, we have that $\|\tilde{\eta}_e^{(0,n-1)}(t)\|_2 \leqslant \frac{\gamma}{2}$ for all $t \in [0, T_3)$. Furthermore, since $\eta_e^{(0,n-1)}(t) \in \mathcal{Y}_{\frac{\gamma}{2}}$ for all $t \in \mathbb{R}_{\geqslant 0}$, if $\|\tilde{\eta}_e^{(0,n-1)}(0)\|_2 \leqslant \frac{\gamma}{2}$, then $O_{n-1}(\hat{x}(t), u_e^{0,n-1}(t)) \in \mathcal{Y}$ for all $t \in [0, T_3)$. Since the set $\mathcal{U}_e^n$ is compact and the map $x \mapsto O_{n-1}(x, u_e^{(0,n-1)})$ is a diffeomorphism on $\mathcal{X}$ for all $u_e^{(0,n-1)} \in \mathcal{U}_e^n$, by Lemma 1, there exist $\tilde{M}, \tilde{N} \in \mathbb{R}$ such that $\|\chi - \hat{x}\|_2 \leqslant \tilde{M}\|O_{n-1}(\chi, u_e^{(0,n-1)}) - O_{n-1}(\hat{x}, u_e^{(0,n-1)})\|_2$, and $\|O_{n-1}(\chi, u_e^{(0,n-1)}) - O_{n-1}(\hat{x}, u_e^{(0,n-1)})\|_2 \leqslant N\|\chi - \hat{x}\|_2$ for all $\chi, \hat{x} \in \mathcal{X}$ and all $u_e^{(0,n-1)} \in \mathcal{U}_e^n$. Thus, since $\chi(t), \hat{x}(t) \in \mathcal{X}$ for all $t \in [0, T_3)$, one has that $\|\tilde{\chi}(t)\|_2 \leqslant \tilde{M}\, \|\tilde{\eta}_e^{(0,n-1)}(t)\|_2$ for all $t \in [0, T_3)$. Thus, the same reasoning used in the proof of Theorem 1 can be used to prove that, if $\gamma \leqslant \frac{\varepsilon}{\tilde{M}}$, then $T_3 = +\infty$. Since $\chi(T_1^\star) = x(T_1^\star)$, this implies that if $\|\tilde{\eta}_e^{(0,n-1)}(0)\|_2 \leqslant \min\{\frac{\gamma}{2}, \delta\} \doteq \nu$ and $\|\tilde{y}_e^{(0,n)}(0)\|_2 \leqslant \varrho$, then $\|\tilde{x}(T_1^\star)\|_2 \leqslant \varepsilon$ and $\|\ell(\hat{x}(T_1^\star), v(T_1^\star)) - y_e^{(0,n-1)}(T_1^\star)\|_2 \leqslant \delta$, thus concluding the proof by the reasoning given at the beginning of the proof. □

Theorem 2 guarantees that system (11) is a finite-time local state observer for system (1), i.e., if the initial estimation error is sufficiently small, then such an observer is able to reconstruct the state of system (1) in finite time. The following remark shows that, by allowing a time delay in the dynamics (11), it is possible to relax the conditions about the initial errors $|y_e^{(i)}(0) - \eta_i(0)| \leqslant \varrho$, $i = 0, \ldots, n$.

Remark 2. Let $\tilde{y}_e^{(0,n)} \doteq y_e^{(0,n)} - [\ \eta_0 \ \ \cdots \ \ \eta_n\ ]^\top$. By [30], for each $R \in \mathbb{R}_{\geqslant 0}$, if $\|\tilde{y}_e^{(0,n)}(0)\|_2 \leqslant R$ and $|y^{(n+1)} - \zeta^n| \leqslant R$, then the time $T_1^\star \in \mathbb{R}_{>0}$ such that $\tilde{y}_e^{(0,n)}(t) = 0$ for all $t \geqslant T_1^\star$ can be made arbitrarily small by suitably selecting the parameters $\lambda_0, \lambda_1, \ldots, \lambda_n \in \mathbb{R}_{>0}$ and $L \in \mathbb{R}_{>0}$. Hence, consider the observer given in (11) with (11g) substituted by

$$\begin{aligned}\dot{\hat{x}} = \mathbb{1}(t - T_1^\star) J^{-1}(\hat{x}, u_e^{(0,n-1)}) \Bigg(-\frac{\partial O_{n-1}(\hat{x}, u_e^{(0,n-1)})}{\partial u_e^{(0,n-1)}}\, u_e^{(1,n)} \\ + \eta_e^{(1,n)} + \mu \|\tilde{\eta}_e^{(0,n-1)}\|_2^\alpha \operatorname{sign}(\tilde{\eta}_e^{(0,n-1)}) \Bigg).\end{aligned} \tag{15}$$

Note that, by the continuity of $t \mapsto \phi(t, x_0, u)$, if $\|x(0) - \hat{x}(0)\|_2 < \varepsilon$ then $\|x(T_1) - \hat{x}(T_1)\|_2 = \|x(T_1) - \hat{x}(0)\|_2 \leqslant \varepsilon$, provided that $T_1^\star$ is sufficiently small. Therefore, by the assumption that the set $\mathcal{X}$ is compact, this implies that $\|\ell(\hat{x}(T_1^\star), v(T_1^\star)) - y_e^{(0,n-1)}(T_1^\star)\|_2 \leqslant \delta$, provided that ε is sufficiently small. Therefore, by using (15) rather than (11g) to dynamically invert $\Phi_{\mathcal{X}, \mathcal{U}_e^n}$, in order to guarantee finite time convergence of the proposed observer, it suffices to require that $\|\tilde{y}_e^{(0,n)}(0)\|_2 \leqslant R$, $|y^{(n+1)} - \zeta^n| \leqslant R$, and that $T_1^\star$ is sufficiently small instead of $|y_e^{(i)}(0) - \eta_i(0)| \leqslant \varrho$, $i = 0, \ldots, n$.

Although the tool given in Remark 2 relaxes the assumptions about $\eta_i(0)$, the observer obtained by substituting (11g) with (15) is still local due to the fact that we still have to guarantee that $\|x(0) - \hat{x}(0)\|_2 < \varepsilon$ so that $\|\tilde{x}(T_1^\star)\|_2 \leqslant \varepsilon$. The following corollary extends the result given in Theorem 2 by providing some conditions that guarantee finite-time semi-global convergence [21] of the observer (11).

Corollary 1. *Assume that $\Phi : \mathbb{R}^n \times \mathcal{U}_e^n \to \mathbb{R}^n \times \mathcal{U}_e^n$ is a diffeomorphism, that $\Phi(\mathbb{R}^n \times u_e^{(0,n)}) = \mathbb{R}^n \times u_e^{(0,n)}$ for all $u_e^{(0,n)} \in \mathcal{U}_e^n$, that there exist $\underline{\lambda}, \overline{\lambda} \in \mathbb{R}_{>0}$ such that*

$$\underline{\lambda} I \leqslant J^\top(x, u_e^{(0,n-1)}) J(x, u_e^{(0,n-1)}) \leqslant \overline{\lambda} I \tag{16}$$

for all $(x, u_e^{(0,n-1)}) \in \mathbb{R}^n \times \mathcal{U}_e^n$, and that the nth time derivative of the output $y^{(n)}(t) = D_f^n h(\Phi(t, x_0, u), u_e^{(0,n)}(t))$ is L-Lipschitz as a function of $t \in \mathbb{R}_{\geqslant 0}$. Thus, for any $R \in \mathbb{R}_{\geqslant 0}$ such that $\|y_e^{(0,n)}(0) - [\ \eta_0(0) \ \ \cdots \ \ \eta_n(0)\ ]^\top\|_2 \leqslant R$ and $\|x(0) - \hat{x}(0)\|_2 \leqslant R$, there exist $\mu^\star \in \mathbb{R}_{\geqslant 0}$ and $T^\star \in \mathbb{R}_{>0}$ such that, if $\mu \geqslant \mu^\star$, then

$$\hat{x}(t) = x(t), \quad \forall t \geqslant T^\star.$$

Proof. In this proof, we use the same notation used in the proof of Theorem 2. Since the suspension $\Phi : \mathbb{R}^n \times \mathcal{U}_e^n \to \mathbb{R}^n \times \mathcal{U}_e^n$ is a diffeomorphism, we have that the matrix $J^{-1}(x, u_e^{(0,n-1)})$ is well defined for all $(x, u_e^{(0,n-1)}) \in \mathbb{R}^n \times \mathcal{U}_e^n$ [25]. Furthermore, since $\underline{\lambda} I \leqslant J^\top(x, u_e^{(0,n-1)}) J(x, u_e^{(0,n-1)}) \leqslant \overline{\lambda} I$ for all $(x, u_e^{(0,n-1)}) \in \mathbb{R}^n \times \mathcal{U}_e^n$, we have that both $J(x, u_e^{(0,n-1)})$ and $J^{-1}(x, u_e^{(0,n-1)})$ are bounded for all $(x, u_e^{(0,n-1)}) \in \mathbb{R}^n \times \mathcal{U}_e^n$. Hence, if $\|x(0) - \hat{x}(0)\|_2 \leqslant R$, since $\Phi(\mathbb{R}^n \times \mathcal{U}_e^n) = \mathbb{R}^n \times \mathcal{U}_e^n$, by the same reasoning used in the proofs of Theorems 1 and 2, we have that there exists $P \in \mathbb{R}_{>0}$ such that $\|y_e^{(0,n-1)}(0) - O_{n-1}(\hat{x}(t), u_e^{(0,n-1)}(0))\|_2 \leqslant P$. This, together with the fact that $\|\tilde{y}_e^{(0,n)}(0)\| \leqslant R$, implies that there exists $W_1 \in \mathbb{R}_{\geqslant 0}$ such that $\|\tilde{\eta}_e^{(0,n-1)}(0)\|_2 \leqslant W_1$. Thus, consider the function V defined in (12), whose time derivative satisfies (14) for all $t \in \mathbb{R}_{\geqslant 0}$ by the same reasoning used in the proof of Theorem 2. Hence, if $\mu \geqslant \frac{S}{\sqrt{n}\, W_1} \doteq \mu^\star$ then $\|\tilde{\eta}_e^{(0,n-1)}(t)\|_2 \leqslant W_1$ for all $t \in \mathbb{R}_{\geqslant 0}$. Thus, since, by Theorem 6.1 of [13], $\|\tilde{y}_e^{(0,n)}(0)\| \leqslant R$ implies that there exists W_2 such that $\|\tilde{y}_e^{(0,n)}(t)\| \leqslant W_2$, we have that there exists $W \doteq W_1 + W_2$ such that $\|y_e^{(0,n-1)}(t) - O_{n-1}(\hat{x}(t), u_e^{(0,n-1)}(t))\|_2 \leqslant W$ for all $t \in \mathbb{R}_{\geqslant 0}$, i.e., the output error is uniformly bounded.

By Section 6.7 of [13], there exists $T_1^\star \in \mathbb{R}_{>0}$ such that $\tilde{y}_e^{(0,n-1)}(t) = 0$ and $\sigma(t) = 0$ for all $t \geqslant T_1^\star$. Therefore, for all $t \geqslant T_1^\star$, it results that

$$\dot{V} \leqslant -\mu \sqrt{n}\, V^{\frac{\alpha+1}{2}}, \tag{17}$$

thus implying that, for all $t \geqslant T_1^\star + \frac{2^{\frac{\alpha+1}{2}}\, W_1^{1-\alpha}}{(1-\alpha)\, \mu \sqrt{n}} \doteq T^\star$,

$$y_e^{(0,n-1)}(t) = \eta^{(0,n-1)}(t) = O_{n-1}(\hat{x}(t), u_e^{(0,n-1)}(t)).$$

The proof is concluded by the fact that $O_{n-1}(\hat{x}, u_e^{(0,n-1)}) = O_{n-1}(x, u_e^{(0,n-1)})$ if and only if $x = \hat{x}$. $\square$

The following two remarks provide further insights on the properties of the considered sliding mode observer.

Remark 3. By Corollary 6.1 of [13], since the function V given in (12) satisfies (10) and (17), the stability of the observer (11) is robust with respect to small perturbations. Nonetheless, while the exact differentiator is independent of the vector field f and of the function h, the inversion algorithm relies on the knowledge of the map O_{n-1}, which, in turns, depends on f and h. Thus, although the sliding mode differentiator (11a)–(11f) is insensitive to model uncertainties, the fact that we employ (11g) to reconstruct the state x of system (1) makes the proposed observer sensitive to parameter uncertainties.

Remark 4. The assumptions of Corollary 1 are met by observable linear systems, provided that the state is bounded and that the control input has bounded time derivatives.

In the remainder of this section, we characterize the sensitivity to additive noise of the proposed observer. Hence, let $d : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ be any Lebesgue-measurable function such that $|d(t)| \leqslant D$ for all $t \in \mathbb{R}_{\geqslant 0}$ and let

$$\psi(t) = h(\phi(t, x_0, u), u(t)) + d(t), \quad \forall t \in \mathbb{R}_{\geqslant 0}.$$

The following proposition characterizes the sensitivity of the proposed observer with respect to the noise d.

Theorem 3. *Let the assumptions of Corollary 1 hold and assume that, in* (11a)*, $y(t)$ is substituted by $\psi(t)$ for all $t \in \mathbb{R}_{\geqslant 0}$ (i.e., assume that the measures of y are corrupted by an additive Lebesgue-measurable disturbance with magnitude not exceeding $D \in \mathbb{R}_{>0}$). Thus, the inequality*

$$\|\hat{x} - x\|_2 \leqslant C \sqrt{\frac{(D^2 - 1)\, D^{\frac{2}{n+1}}}{D^{\frac{2}{n+1}} - 1}} + \frac{S}{\sqrt{n}\, \mu} \sqrt{\frac{(D^2 - 1)}{D^{\frac{2}{n+1}} - 1}} \tag{18}$$

is established in finite time for each $\mu \in \mathbb{R}_{>0}$, where C and S are positive constants independent of the disturbance.

Proof. In this proof, the same symbols used in the proofs of Theorem 2 and Corollary 1 are used. By [30], [13] and [29], there exist $C_1 \in \mathbb{R}_{>0}$ and $T_1^\star \in \mathbb{R}_{>0}$ such that $|y^{(i)}(t) - \eta_i(t)| \leqslant C_1 D^{\frac{n+1-i}{n+1}}$ for all times $t \geqslant T_1^\star$, $i = 0, \dots, n$ (see also [31, Thm. 6]). Hence, we have

$$\|\tilde{y}_e^{(0,n)}(t)\|_2 \leqslant C_1 \sqrt{\frac{(D^2-1) D^{\frac{2}{n+1}}}{D^{\frac{2}{n+1}} - 1}}, \quad \forall t \geqslant T_1^\star.$$

Furthermore, in view of (13), we have that there exist $Q_0, \dots, Q_{n-1} \in \mathbb{R}_{>0}$ such that $|\sigma_i(t)| \leqslant Q_i\, |\eta_i(t) - \zeta_{i-1}(t)|^{\frac{n-i}{n+1-i}}$ for all $t \in \mathbb{R}_{\geqslant 0}$, $i = 0, \dots, n-1$. Note that, by the dynamics given in (11), one has that $|\eta_i(t) - \zeta_{i-1}(t)| \leqslant Q_i |\eta_{i-1}(t) - \zeta_{i-2}(t)|^{\frac{n-i+1}{n+2-i}}$ for all $t \in \mathbb{R}_{\geqslant 0}$, $i = 0, \dots, n-1$. By iterating this procedure, one obtains that there exists $S_1 \in \mathbb{R}_{\geqslant 0}$ such that, for all $t \geqslant T_1^\star$,

$$|\sigma_i(t)| \leqslant S_1\, |\eta_{i-j}(t) - \zeta_{i-j-1}(t)|^{\frac{n-i}{n-i+j+1}},$$

for $i = 0, \dots, n-1$ and $j = 0, \dots, i$. Thus, letting $j = i$, we have $|\sigma_i(t)| \leqslant D^{\frac{n-i}{n+1}}$ for all $t \geqslant T_1^\star$, $i = 0, \dots, n-1$. Hence,

$$\|\sigma(t)\|_2 \leqslant S_1 \sqrt{\frac{(D^2 - 1)}{D^{\frac{2}{n+1}} - 1}}, \quad \forall t \geqslant T_1^\star.$$

Thus, by (14), there exists $T_2^\star \geqslant T_1^\star$ such that, for all $t \geqslant T_2^\star$,

$$\begin{aligned} &\|y_e^{(0,n-1)}(t) - O_{n-1}(\hat{x}(t), u_e^{(0,n-1)}(t))\|_2 \\ &\qquad \leqslant C_1 \sqrt{\frac{(D^2 - 1)\, D^{\frac{2}{n+1}}}{D^{\frac{2}{n+1}} - 1}} + \frac{S_1}{\sqrt{n}\, \mu} \sqrt{\frac{(D^2 - 1)}{D^{\frac{2}{n+1}} - 1}}. \end{aligned}$$

Therefore, since $J^{-1}(x, u_e^{(0,n-1)})$ is upper bounded for all $(x, u_e^{(0,n-1)}) \in \mathbb{R}^n \times \mathcal{U}_e^n$, by Lemma 1, there exist $C \in \mathbb{R}_{>0}$ and $S \in \mathbb{R}_{>0}$ such that (18) holds. $\square$

Note that, under the assumptions of Corollary 1, if, additionally, the initial error $|\zeta_n(0) - y^{(n+1)}(0)|$ is bounded, then the transient time may be arbitrarily shortened by the same reasoning given in [30]. Furthermore, in view of Theorem 3 and [13], letting $\mu \to \infty$, the accuracy of the proposed observer is asymptotically the best possible among the methods using the time derivatives of the output to estimate the state. Indeed, by [13, Sec. 6.7, p. 220], the accuracy $\bar{C}\sqrt{\frac{(D^2-1)D^{\frac{2}{n+1}}}{D^{\frac{2}{n+1}}-1}}$ is asymptotically the best possible when estimating the time derivatives of a noisy signal; see [11, Prop. 2, Thm. 3]. Thus, since $O_{n-1}^{-1}(\cdot,\cdot)$ is assumed to be $\mathcal{C}^k$ and hence absolutely continuous, the sensitivity given in (18) is asymptotically the best possible among the methods using $y_e^{(0,n-1)}$. Furthermore, note that, differently form classical high-gain observers [32], such a sensitivity is independent of the frequency of the disturbance affecting the output.

In the following remark, we discuss the hypothesis about measurability of the time derivatives of the input $u(t)$.

Remark 5. If the time derivatives of the input $u(t)$ cannot be measured, a sliding mode observer wholly similar to (11a)–(11f) can be used to estimate them. In fact, assuming that $u^{(n)}(t)$ is Z-Lipschitz, consider the following system:

$$\dot{\upsilon}_0 = \varpi_0, \tag{19a}$$
$$\varpi_0 \doteq \upsilon_1 - \lambda_n Z^{\frac{1}{n+1}} |\upsilon_0 - y|^{\frac{n}{n+1}} \operatorname{sign}(\upsilon_0 - u), \tag{19b}$$
$$\vdots$$
$$\dot{\upsilon}_n = \varpi_n, \tag{19c}$$
$$\varpi_n \doteq -\lambda_0 Z \operatorname{sign}(\upsilon_n - \varpi_{n-1}), \tag{19d}$$

where $\lambda_0, \ldots, \lambda_n \in \mathbb{R}_{\geqslant 0}$ are the same one as in (11). Hence, letting $\tilde{u}_e^{(0,n)} \doteq u_e^{(0,n)}(0) - [\ \upsilon_0(0) \ \cdots \ \upsilon_n(0)\ ]^\top$, if there exists $H \in \mathbb{R}_{\geqslant 0}$ such that $\|\tilde{u}_e^{(0,n)}(0)\|_2 \leqslant H$ and $|\varpi_n(0) - u^{(n+1)}(0)| \leqslant H$, then there exists $T_1^\star \in \mathbb{R}_{>0}$ such that $\tilde{u}_e^{(0,n)}(t) = 0$ for all $t \geqslant T_1^\star$. Furthermore, the time $T_1^\star$ can be made arbitrarily small by suitably selecting the parameters of (19). Hence, by the same reasoning given in Remark 2, if, in the dynamics of the observer (11), (11g) is substituted by

$$\dot{\hat{x}} = \mathbb{1}(t - T_1^\star) J^{-1}(\hat{x}, \upsilon_e^{(0,n-1)}) \Bigg(-\frac{\partial O_{n-1}(\hat{x}, \upsilon_e^{(0,n-1)})}{\partial u_e^{(0,n-1)}} \upsilon_e^{(1,n)} + \eta_e^{(1,n)} + \mu \|\tilde{\eta}_e^{(0,n-1)}\|_2^\alpha \operatorname{sign}(\tilde{\eta}_e^{(0,n-1)}) \Bigg), \tag{20}$$

where $\upsilon_e^{(i,j)} = [\ \upsilon_i \ \cdots \ \upsilon_j\ ]^\top$, $i, j \in \mathbb{Z}_{\geqslant 0}$, $j \geqslant i$, then $\hat{x}(t)$ still converges in finite time to the state $x(t)$ of system (1), provided that the time $T_1^\star$ is made sufficiently small.

Remark 6. In the literature, several techniques have been proposed to design observer in the original coordinates for nonlinear systems; see, e.g., [33], [34]. For instance, by using Lemma 1 and arguments wholly similar to the ones employed in the proof of Theorem 2, it can be easily proved that, letting

$$F(y, y_e^{(0,n-1)}) := \begin{bmatrix} y^{(1)} - \tilde{\lambda}_{n-1} L^{\frac{1}{n}} |y^{(0)} - y|^{\frac{n-1}{n}} \operatorname{sign}(y_0 - y) \\ y^{(2)} - \tilde{\lambda}_{n-2} L^{\frac{2}{n}} |y^{(0)} - y|^{\frac{n-2}{n}} \operatorname{sign}(y_0 - y) \\ \vdots \\ y^{(n-1)} - \tilde{\lambda}_1 L^{\frac{n-1}{n}} |y^{(0)} - y|^{\frac{1}{n}} \operatorname{sign}(y_0 - y) \\ \tilde{\lambda}_0 L \operatorname{sign}(y_0 - y) \end{bmatrix},$$

where $\tilde{\lambda}_{n-1} = \lambda_{n-1}$, $\tilde{\lambda}_i = \lambda_i \tilde{\lambda}_{i+1}^{\frac{i}{i+1}}$, $i = n-2, \ldots, 0$, under the assumptions of Theorem 2, the n-dimensional system

$$\dot{\hat{x}} = J^{-1}(\hat{x}, u_e^{(0,n-1)}) \Bigg(F(y, O(\hat{x}, u_e^{(0,n-1)})) - \frac{\partial O_{n-1}(\hat{x}, u_e^{(0,n-1)})}{\partial u_e^{(0,n-1)}} u_e^{(1,n)} \Bigg), \tag{21}$$

is a local state observer for (1) that converges in finite time. The key advantage of (11) over (21) is that the estimation of the time derivatives of the output y, which is performed via the sliding mode differentiator (11a)–(11f), is independent of the inversion of the suspension $\Phi(x, u_e^{(0,n-1)})$, which is carried out via the dynamical inversion algorithm (11g). Therefore, if the output of system (1) is affected by measurement noise, it is possible to mitigate its effects on the estimate $\hat{x}$ of the state x by substituting in (11g) and (11h) the estimates $\eta_e^{(0,n-1)}$ and $\eta_e^{(1,n)}$ with the vectors $\breve{\eta}_e^{(0,n-1)}$ and $\breve{\eta}_e^{(1,n)}$, obtained by suitably filtering $\eta_e^{(0,n-1)}$ and $\eta_e^{(1,n)}$. In particular, if the output signal $y(t)$ is affected by an additive disturbance of the form $d(t) = R\sin(\omega t + \varphi)$, then this approach is generically more effective than filtering directly the estimate $\hat{x}$. In fact, if R and ω are such that $y^{(n)}(t) + \frac{\mathrm{d}^n d(t)}{\mathrm{d}t^n}$ is an L-Lipschitz function of t, then there exists a finite time T such that $\eta_i(t) = y_i(t) + R\omega^i \sin((-1)^{i+1}(i\frac{\pi}{2} - \omega t - \varphi))$, $i = 0, \ldots, n$, for all $t \geqslant T$. Thus, in such a case, the effects of the disturbance can be easily mitigated via linear filters, provided that the bandwidth of the signal $d(t)$ is known; see the example reported in the subsequent Section V and the numerical examples given in [30]. On the other hand, filtering directly $\hat{x} = O_{n-1}^{-1}(\eta_e^{(0,n-1)}, u_e^{(0,n-1)})$ may lead to large errors due to the fact that the map $O_{n-1}^{-1}(\cdot,\cdot)$ is generically nonlinear. Furthermore, by using the numerical trick given in Remark 2, it is possible to avoid inverting the observability map O_{n-1} during the transient behavior of the exact sliding mode differentiator, thus possibly allowing for larger initial errors in the estimate $\hat{x}(0)$ of $x(0)$.

V. Numerical examples

Let $n = 3$ and consider the following nonlinear system:

$$\dot{x}_1 = \tfrac{u-5}{x_2^2+1} + u\,x_1 + u + x_1 + x_2 + \tfrac{6-10x_1x_2}{(x_2^2+1)^2} - \tfrac{10x_2}{(x_2^2+1)^3}, \tag{22a}$$
$$\dot{x}_2 = -5x_1 - 3x_2 - \frac{5}{x_2^2+1}, \tag{22b}$$
$$y = x_1 + \frac{1}{x_2^2+1}. \tag{22c}$$

By computing the observability map of system (22) as in (3),

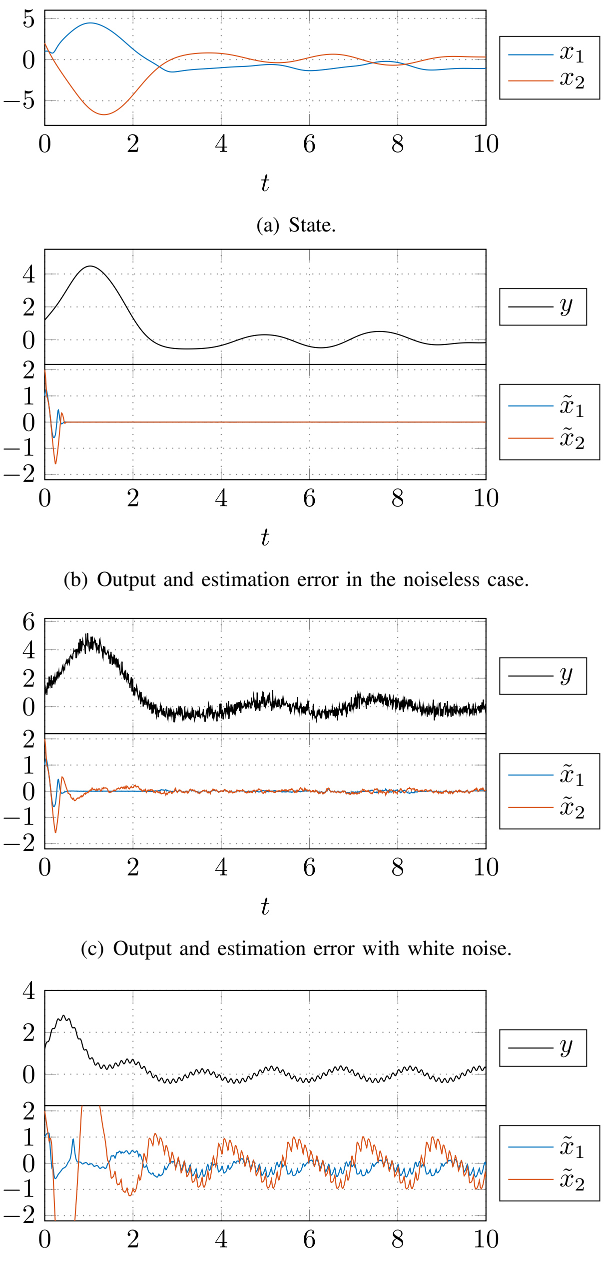
$$O_2(x, u_e^{(0,1)}) = \begin{bmatrix} x_1 + \frac{1}{x_2^2+1} \\ u^{(0)}x_1 + \frac{u^{(0)}+1}{x_2^2+1} + u^{(0)} + x_1 + x_2 \end{bmatrix}.$$

By letting $J(x) := \frac{\partial O_2(x, u_e^{(0,1)})}{\partial x}$, $\mathcal{U}_e^2 = \{u_e^{(0,1)} : |u_e^{(0)}| \leqslant 10\}$, and using the tools given in [35], one obtains that (16) holds with $\underline{\lambda} = 0.00529$ and $\overline{\lambda} = 188.753$. Additionally, by using the algebraic geometry tools given in [36], it can be derived that $O_2 : \mathbb{R}^2 \times \mathcal{U}_e^2 \to \mathbb{R}^2$ is a diffeomorphism. Thus, the hypotheses of Theorem 2 are met, provided that $y^{(2)}(t)$ is an L-Lipschitz function of time and $\hat{x}(0)$ is sufficiently close to $x(0)$. Nonetheless, is rather difficult determine an expression in closed-form for an inverse $O_2^{-1}(x, u_e^{(0,1)})$ of the observability map $O_2(x, u_e^{(0,1)})$ since computing such an inverse requires to find the solution to a set of rational equations in the unknowns x_1, x_2.

Numerical simulations have been carried out to test the observer (11), (19). In all the simulations reported hereafter, the parameters of the observer (11), (19) have been chosen as $\lambda = [\ 1.1 \quad 1.5 \quad 3\ ]$, $L = 10^2$, $Z = 10^3$, $\alpha = 0.1$, $\mu = 10$, the input has been chosen as $u(t) = 0.4\sin(3t) + 0.5\sin(2t) + 0.2\sin(5t)$, the initial condition of system (22) has been assumed to be $x_0 = [\ 1 \quad 2\ ]^T$, and the observer has been initialized at $\hat{x}_0 = 0$, $\eta_e^{(0,n)}(0) = 0$, and $v_e^{(0,n)}(0) = 0$.

In the first simulation, whose results are depicted in Figure 1(b), it has been assumed that the output of system (22) is noiseless, whereas, in the second simulation, whose results are depicted in Figure 1(c), it has been assumed that the output is affected by band-limited white noise with power 10^{-4} and sampling time 10^{-3}. As shown by Figure 1, the observer (11) is capable of reconstructing the state of system (22) also if the output is affected by noise. Moreover, in the noiseless setting, it converges in finite time to the state of system (22). Finally, another simulation has been carried out assuming that the output is affected by the sinusoidal disturbance $d(t) = 0.01\sin(48t)$. Following the construction proposed in Remark 6, we substituted, in (11g) and (11h), the estimates $\eta_e^{(0,n-1)}$ and $\eta_e^{(1,n)}$ with the vectors $\breve{\eta}_e^{(0,n-1)}$ and $\breve{\eta}_e^{(1,n)}$, which have been obtained by filtering each entry of $\eta_e^{(0,n)}$ with a filter having transfer function $Q(s) = \frac{s^2+50^2}{s^2+5s+50^2}$. The results of this simulation are depicted in Figure 1(d). In order to corroborate the effectiveness of this approach, Figure 2 depicts the estimation error obtained by using the n-dimensional observer (21), with the same parameters.

As shown by such figures, despite the proposed observer is more complex than (21) due to the fact that it has $2n$ rather than n states, it allows one to reduce the effects of sinusoidal measurement noise on the estimated state $\hat{x}$ by suitably filtering the estimates $\eta^{(0)}, \dots \eta^{(n)}$.



(a) State.

(b) Output and estimation error in the noiseless case.

(c) Output and estimation error with white noise.

(d) Output and estimation error with sinusoidal noise.

Fig. 1. Results of the numerical simulations.

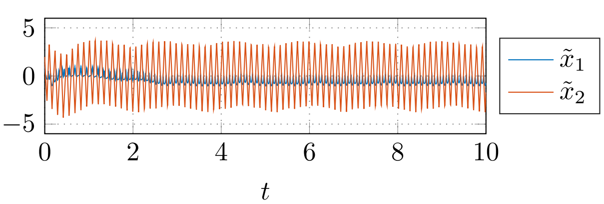


Fig. 2. Estimation error obtained using system (21).

VI. Conclusions

In this note, a local finite time observer for nonlinear control systems has been proposed. In particular, it has been shown that it is possible to design a state observer that, in the absence of measurement noise, reconstruct the state of a controlled nonlinear system in finite time. The observer has been designed by coupling exact differentiators with a tool able to invert, in finite-time, a suspension.

The dimension of the proposed observer is two times the dimension of the system (while other observation schemes, such as the one given in [20], have dimension equal to the one of the system). Nonetheless, as shown in Remark 6 and in Section V, by suitably filtering the additional states $\eta^{(0)}, \dots \eta^{(n)}$ that are estimated through the considered observer, it is possible reduce the effects of sinusoidal measurement noise.

The main differences between the state observer given here and the ones given in [15] and [8] are that it is capable of estimating in finite time both the time derivatives of the output and the state of the system and that it can be used also for nonlinear control systems. However, while the observers given in [15] and [8] allow one to estimate the state of the system if it is κ-differentially observable for some $\kappa \geqslant n$, the observer given in this note is guaranteed to converge only in the n-differentially observable case. The extension to the κ-differentially observable case, with $\kappa > n$, (which may be possibly dealt with by using a coordinate augmentation approach similar [20]) will be the objective of our future work.

Furthermore, note that the proposed observer requires the knowledge of the input u and of its time derivatives. Another objective of our future research is to extend the results given in this note to the case of unmeasurable input.

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