Synchronization of networked oscillators under nonlinear integral coupling ⋆

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Abstract: In this paper, we consider synchronization of dynamical systems interconnected via nonlinear *integral coupling*. Integral coupling allows one to achieve synchronization with lower interaction levels (coupling gains) than with linear coupling. Previous results on this topic were obtained for synchronization of several systems with all-to-all interconnections. In this paper, we relax the requirement of all-to-all interconnections and provide two results on exponential synchronization under nonlinear integral coupling for networks with topologies different from all-to-all interconnections. In particular, we provide a high-gain result for an arbitrary interconnection topology and a non-high-gain method for analysis of synchronization for specific topologies. The results are illustrated by simulations of Hindmarsh-Rose neuron oscillators.

Keywords: Synchronization, nonlinear systems, Hindmarsh-Rose oscillators, neural dynamics, networked systems.

1. INTRODUCTION

The phenomenon of synchronization in networks of coupled oscillators and chaotic systems receives huge attention in scientific literature. The co-existence of very complex, chaotic or "irregular" dynamics of relatively simple systems, on one hand, and the possibility of "spontaneous order" and synchrony (Strogatz, 2003) in such interconnected systems, on the other hand, forms an intriguing combination for specialists in physics, mathematics, control, neuroscience and biology, thus generating a seemingly endless sequence of various results on this subject. This interest is also explained by a number of applications, already implemented or potential, of synchronization phenomena in various fields of science and technology, see, e.g., (Caroll and Pecora, 1991; Oud and Tyukin, 2004; Levine, 2004; Pogromsky, 1998; Steur, 2011; Belykh and Porfiri, 2016; Abrams et al., 2016).

Often, synchronization is studied for the case of linear diffusive couplings between the systems (Pogromsky and Nijmeijer, 2001). In reality, interactions between physical systems can be nonlinear, whereas linear couplings serve only as their local approximations. Therefore, it is of interest to investigate conditions on general *nonlinear* couplings that would guarantee synchronization of interconnected systems. Several approaches in this direction were taken in (Liu and Chen, 2008; He and Yang, 2008; Proskurnikov, 2014; Proskurnikov and Matveev, 2015). In (Pavlov et al., 2009), a specific form of nonlinear couplings has been introduced: nonlinear integral coupling. It equals a definite integral of a non-negative weighting function with the limits being the outputs of the interconnected systems. This coupling is, in some sense, a generalized "distance" between the outputs of the systems, which, unlike the usual distance, can be zero even if the outputs differ. In (Pavlov et al., 2009), a procedure was offered to find the synchronizing weighting function (we call it nonlinear integral gain) for a class of nonlinear systems such that synchronization is achieved. For the well-known example of Hindmarsh and Rose neuron model (Hindmarsh and Rose, 1984) it was shown that such nonlinear integral coupling can lead to synchronization of systems with average gains that are much lower than the synchronizing gains of the linear coupling functions. This opened a new perspective on analysis and design of synchronizing systems.

That result obtained in Pavlov et al. (2009) was limited to systems with all-to-all interconnections. The question

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of extending those results to arbitrary interconnection topologies has proved to be challenging and remained open for a long time. In this paper, we present two results in this direction. One of those is a high-gain result corresponding to networks with arbitrary undirected connected topologies. The other result is, in fact, a method to choose coupling with minimal interaction gains, being in line with the original motivation of Pavlov et al. (2009) and demonstrated by an example. The presented results open further ways for studying synchronization of nonlinear oscillatory systems. In particular, these will allow further analysis in the direction of finding minimal interactions between systems that would lead to their synchronization.

The paper is organized as follows. The controlled synchronization problem and the concept of integral coupling are presented in Section 2. Section 3 presents some preliminary results on integrally coupled networks. Sections 4 and 5 contain the main results, illustrated with numerical simulations in Section 6. Conclusions are drawn in Section 7.

2. CONTROLLED SYNCHRONIZATION PROBLEM

Consider N identical dynamical systems of the form

$$\dot{x}_i = f(x_i, u_i), \quad y_i = h(x_i), \quad i = 1, \dots N, \tag{1}$$

with $x_i \in \mathbb{R}^n$, $y_i, u_i \in \mathbb{R}$. It is assumed that f and h are C^1-smooth, and for $u_i = 0$ the system has bounded solutions that oscillate (i.e., have non-trivial ω-limit sets).

The problem of controlled synchronization studied in this paper is to find control laws for each u_i that render asymptotic synchronization of the systems' states:

$$|x_i(t) - x_j(t)| \to 0, \quad \text{as} \quad t \to \infty, \quad \forall i, j.$$

For each system i, the input u_i should depend only on the output y_i and on the outputs of the systems interacting to system i. We also require that for identical outputs $y_1 = y_2 = \ldots = y_N$, the controls satisfy $u_1 = u_2 = \ldots = u_N = 0$, such that in exact synchrony the systems exhibit the oscillatory dynamics of the unforced system (1).

The results presented in this paper are based on the notion of integral coupling between systems (Pavlov et al., 2009). For $N = 2$ systems , this coupling takes the following form:

$$u_1 = \int_{y_1}^{y_2} \lambda(s) ds, \qquad u_2 = \int_{y_2}^{y_1} \lambda(s) ds. \tag{2}$$

Here $\lambda(s) \geq 0$ is a continuous function, called nonlinear integral gain. Obviously, for $y_1 = y_2 = y$ one has $u_1 = u_2 = 0$. Notice that for a constant integral gain $\lambda(s) \equiv \lambda$, integral coupling (2) becomes the linear diffusive coupling (Pogromsky and Nijmeijer, 2001) $u_1 = \lambda(y_2 - y_1)$, $u_2 = \lambda(y_1 - y_2)$, which is well studied in literature.

The work Pavlov et al. (2009) has studied synchronization of $N \geq 2$ and more systems with all-to-all interactions and presented constructive conditions for finding $\lambda(s)$ for a certain class of systems. Below, the requirement of all-to-all interconnections is relaxed. We introduce the adjacency matrix $A = (\alpha_{ij}) = A^\top$ corresponding to an undirected topology of mutual coupling among the systems and examine the following control policy

$$u_i = \sum_{j=1}^{n} \alpha_{ij} \int_{y_i}^{y_j} \lambda(s) ds, \quad i = 1, \dots, N. \tag{3}$$

3. PRELIMINARIES

In this section, we recall some key definitions and results.

Definition 1. (Pogromsky (1998)). A system

$$\dot{x} = f(x, u), \quad y = h(x), \quad x \in \mathbb{R}^n, \ y, u \in \mathbb{R},$$

is C^1-semipassive if there exist a C^1-smooth function $V : \mathbb{R}^n \to [0, \infty)$ and a function $H : \mathbb{R}^n \to \mathbb{R}$ such that

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x, u) \leq y^T u - H(x) \quad \forall x \in \mathbb{R}^n$$

and $H(x) \geq 0$ for any x outside some ball ($|x| \geq \rho$).

Being interconnected through an integral coupling of the form (2) with a non-negative weight function $\lambda(s)$, two semipassive systems (1) will have bounded solutions defined up to $t = \infty$ (Pavlov et al., 2009). This result remains valid for $N > 2$ systems interconnected through an undirected network with bidirectional couplings (3).

Lemma 1. Suppose that each system (1) is C^1-semipassive with a radially unbounded storage function $V(x_i)$. Then all solutions of N interconnected systems (1), (3) with nonlinear coupling gain $\lambda(s) \geq 0$ are defined and bounded over the infinite time interval $t \geq 0$.

Lemma 1 is proved similarly to the case of two systems in Pavlov et al. (2009); its proof is omitted here.

Consider two systems

$$\dot{x}_i = f(x_i) + Bu_i, \quad y_i = Cx_i, \tag{4}$$

where $x_i \in \mathbb{R}^n$,$y_i, u_i \in \mathbb{R}$, $i = 1, 2$, B and C are constant matrices of appropriate dimensions and function $f(x)$ is C^1. For these systems one can find sufficient conditions on $\lambda(\cdot)$ that ensure synchronization (Pavlov et al., 2009).

Theorem 1. Let system (4) be C^1-semipassive. Assume that matrices $P = P^\top > 0, R = R^\top > 0$ and a continuous function $\lambda(s) \geq 0$, $s \in \mathbb{R}$, exist that satisfy the conditions

$$\begin{aligned} P\frac{\partial f(x)}{\partial x} + \frac{\partial f^T(x)}{\partial x} P - 2C^T C \lambda(Cx) < -R \, \forall x \in \mathbb{R}^n, \\ PB = C^T. \end{aligned} \tag{5}$$

Then, all solutions of (4), (2) are bounded and satisfy

$$|x_1(t) - x_2(t)| \leq \mu e^{-\nu t} |x_1(0) - x_2(0)|,$$

for some constant $\mu > 0$, $\nu > 0$.

4. HIGH-GAIN SYNCHRONIZATION OF N SYSTEMS

The question of extending results of Theorem 1 to $N > 2$ systems and arbitrary topologies turns to be non-trivial. This section presents the first result in this direction.

We first introduce a class of functions $\lambda(s)$, serving as integral coupling gains.

Assumption 1. The function $\lambda(s)$ is continuous and has finite support. Furthermore, for any $\varepsilon > 0$ the set $\{s : \lambda(s) \geq \varepsilon\}$ is connected, thus being either a closed interval $[a_\varepsilon, b_\varepsilon]$ or the empty set. In addition, $[a_{\varepsilon_2}, b_{\varepsilon_2}] \subseteq [a_{\varepsilon_1}, b_{\varepsilon_1}]$ whenever $\varepsilon_2 > \varepsilon_1$.

This assumption holds, e.g., when $\lambda(\cdot)$ is concave or unimodal, as in the example from Pavlov et al. (2009) (see Fig. 1).

The next theorem extends Theorem 1 to synchronization of N systems, assuming that nonlinear integral coupling between the systems is "strong" enough.

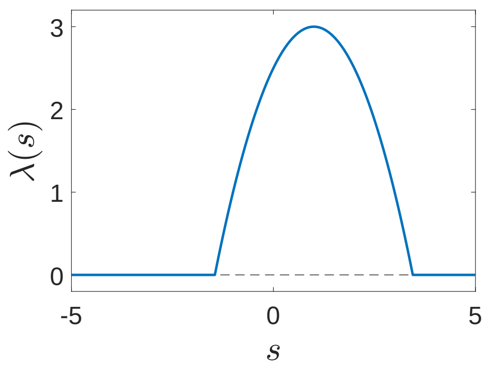

Fig. 1. An example of admissible function $\lambda(s)$.

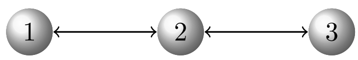

Fig. 2. The chain of three systems.

Theorem 2. Let $N \geq 2$ systems (4) be C^1-semipassive. Suppose that $\lambda(s)$ satisfies Assumption 1 and (5) for some $P = P^\top > 0, R = R^\top > 0$. Let the undirected graph, given by the adjacency matrix $A = A^\top$, be connected. Then, for sufficiently large $k > 0$, the following integral coupling protocol

$$u_i = k \sum_{j=1}^{n} \alpha_{ij} \int_{y_i}^{y_j} \lambda(s) ds \tag{6}$$

exponentially synchronizes the trajectories, that is,

$$\sum_{i,j=1}^{n} |x_i(t) - x_j(t)| \leq \mu e^{-\nu t} \sum_{i,j=1}^{n} |x_i(0) - x_j(0)|,$$

for some constant $\mu, \nu > 0$.

As follows from the proof of the theorem (see Appendix), the lower bound on the gain k under which this theorem guarantees synchronization depends on the function $\lambda(s)$, matrix R and on the algebraic connectivity of the interconnections graph. Although this bound can be found explicitly, the estimate may be rather conservative. Theorem 2 is thus mainly an *existence* result, establishing synchronization under sufficiently high gain $k > 0$. At the same time, the original motivation for the integral coupling (Pavlov et al., 2009) was to demonstrate that by adopting nonlinear couplings between the systems we can demonstrate that synchronization can be achieved with a lower "interaction level" than for linear diffusive coupling. The high-gain argument, to certain extent, is opposite to this motivation. Therefore, we present an alternative strategy in the next section.

5. TOWARDS NON-HIGH-GAIN PROTOCOLS

An alternative approach to the result above can be based on the reasoning illustrated by the following example. Consider $N = 3$ systems (4) interconnected into a chain as shown in Fig. 2. and coupled via the following protocol

$$\begin{aligned} u_1 &= \int_{y_1}^{y_2} \lambda(s) ds, \\ u_2 &= \int_{y_2}^{y_1} \lambda(s) ds + \int_{y_2}^{y_3} \lambda(s) ds, \\ u_3 &= \int_{y_3}^{y_2} \lambda(s) ds. \end{aligned} \tag{7}$$

Systems 1 and 3, in view of (4) and (7) are written as

$$\begin{aligned} \dot{x}_1 &= f(x_1) + B \int_{y_1}^{y_2} \lambda(s) ds = \tilde{f}(x_1) + Bg(x_2(t)), \\ \dot{x}_3 &= f(x_3) + B \int_{y_3}^{y_2} \lambda(s) ds = \tilde{f}(x_3) + Bg(x_2(t)) \end{aligned} \tag{8}$$

where $\tilde{f}(x), g(x)$ are defined as follows:

$$\tilde{f}(x) = f(x) + \int_{Cx}^{0} \lambda(s) ds, \quad g(x) := \int_{0}^{Cx} \lambda(s) ds. \tag{9}$$

Therefore, $x_1(t)$ and $x_3(t)$ are, in fact, solutions to system

$$\dot{x} = \tilde{f}(x) + Bg(x).$$

Due to (5), $\tilde{f}(x)$ satisfies the so-called Demidovich condition (Demidovich, 1967; Pavlov et al., 2004)

$$P \frac{\partial \tilde{f}}{\partial x}(x) + \frac{\partial \tilde{f}^T}{\partial x}(x) P < -R,$$

entailing (Pavlov et al., 2004) that $\forall x_A, x_B$ it holds that

$$\begin{aligned} &2(x_A - x_B)^T P(\tilde{f}(x_A) - \tilde{f}(x_B)) \leq \\ &\qquad \leq -(x_A - x_B)^T R(x_A - x_B). \end{aligned} \tag{10}$$

Therefore, if we calculate the derivative of $V_{13} = 1/2(x_1 - x_3)^T P(x_1 - x_3)$ along the solution of (8), we obtain

$$\dot{V}_{13} \leq -W(x_1 - x_3) \leq -c_1 |x_1 - x_3|^2 < 0,$$

where $W(x) = 1/2x^T Rx$ and $c_1 > 0$ is the smallest eigenvalue of $R/2$, i.e., x_1, x_3 exponentially synchronize.

In a similar way, we can treat the difference between $x_1(t)$ and $x_2(t)$. Due to (7), we can write

$$\dot{x}_1 - \dot{x}_2 = \tilde{f}(x_1) - \tilde{f}(x_2) + 2B \int_{y_1}^{y_2} \lambda(s) ds + B \int_{y_3}^{y_1} \lambda(s) ds.$$

Therefore, for $V_{12} = 1/2(x_1 - x_2)^T P(x_1 - x_2)$, taking into account (10), one obtains

$$\begin{aligned} \dot{V}_{12} \leq -c_1 |x_1 - x_2|^2 + 2(x_1 - x_2) PB \int_{y_1}^{y_2} \lambda(s) ds + \\ + (x_1 - x_2) PB \int_{y_3}^{y_1} \lambda(s) ds. \end{aligned}$$

Since $PB = C^T$ due to (5) and $\lambda(\cdot)$ is a nonnegative bounded function, one notices that

$$(x_1 - x_2) PB \int_{y_1}^{y_2} \lambda(s) ds = (y_1 - y_2) \int_{y_1}^{y_2} \lambda(s) ds \leq 0,$$

and the third term can be estimated as

$$(x_1 - x_2) PB \int_{y_3}^{y_1} \lambda(s) ds \leq c_2 |x_1 - x_2||x_1 - x_3|$$

for some constant $c_2 > 0$. By choosing $V = \beta V_{13} + V_{12}$ with $\beta > 0$, we obtain

$$\dot{V} \leq -\beta c_1 |x_1 - x_3|^2 - c_1 |x_1 - x_2|^2 + c_2 |x_1 - x_2||x_1 - x_3|.$$

Choosing $\beta > 0$ in such a way that $c_2^2 < \beta c_1^2$, the right-hand side of the last inequality will be a negative-definite quadratic form of the arguments $x_1 - x_2$ and $x_1 - x_3$. This implies complete exponential synchronization of x_1, x_2, x_3.

This example illustrates an alternative approach to establishing synchronization in networks without all-to-all synchronization and the requirement of high coupling gain. Extension of this method to general networks and analysis of its applicability to more general cases is out of the scope of this paper and will appear in its journal version.

6. SIMULATIONS

In this section, we illustrate our results by a numerical simulation of $N = 3$ identical Hindmarsh-Rose (HR) oscillators

$$\begin{aligned} \dot{y}_i &= -ay_i^3 + by_i^2 + z_{i,1} - z_{i,2} + I + u_i, \\ \dot{z}_{i,1} &= c - dy_i^2 - z_{i,1} \\ \dot{z}_{i,2} &= \varepsilon(m(y_i + y_0) - z_{i,2}), \end{aligned} \tag{11}$$

coupled in the chain shown in Fig. 2. The parameters are $a = c = 1, b = 3, d = 5, m = 4, I = 3.25, y_0 = 1.618, \varepsilon = 0.005$. As shown in Pavlov et al. (2009), the inequality (5) is satisfied by the function

$$\lambda(s) = \max\left\{0, \varepsilon - 3as^2 + 2bs + \tfrac{(1-\gamma ds)^2}{2(\gamma-\varepsilon)}, \right\},$$

where γ depends on the parameters. For the above listed parameters, we choose $\gamma = 0.2$. This function $\lambda(\cdot)$ is shown in Fig. 1.

Fig. 3 shows solutions of the HR oscillators with coupling (3). These solutions are obtained by numerical integration of the ODEs with Matlab using the `ode45` solver, where the initial condition of each state is chosen uniformly at random from the interval $[-5,\ 5]$. Fig. 4 shows the functions

$$g_{ij}(t) = \frac{\int_{y_i(t)}^{y_j(t)} \lambda(s)\, ds}{y_i(t) - y_j(t)},$$

which are the "variable gains" (Pavlov et al., 2009) of the nonlinear integral couplings. Both g_{12} and g_{23} vary between 0 and 3 and have a mean value of 1.22. (These mean values are computed over the time-interval $[0,\ 10000]$.)

The best estimate of the linear diffusive coupling gain that we are aware of is 3, which can be computed using the results of Belykh et al. (2005). Note that $\max_{s\in\mathbb{R}} \lambda(s) = 3$ and thus we obtain the same estimate of the linear coupling gain that guarantees global synchronization. Nevertheless, the average gain effectively implemented through this nonlinear coupling gain strategy is significantly lower than needed for the linear strategy.

7. CONCLUSIONS

In this paper we presented results on exponential synchronization of dynamical systems interconnected through nonlinear integral coupling. These results relax the condition of all-to-all interaction obtained in the previous work on this subject. Similar to the result in the original work on nonlinear integral coupling, the presented results can be utilized to demonstrate that synchronization can be achieved by nonlinear couplings with average gains much lower than the gains of linear diffusive couplings.

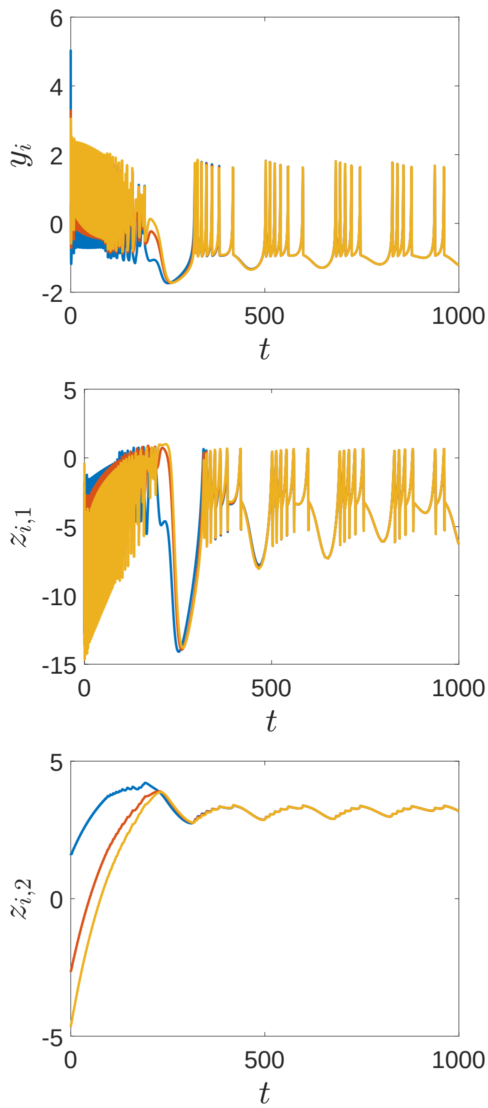

Fig. 3. Synchronization of the HR oscillators (11).

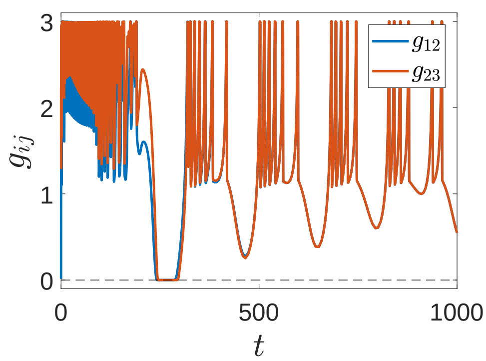

Fig. 4. Variable gains of the integral coupling functions.

APPENDIX

In this appendix we will prove Theorem 2. Re-scaling k in (6), one can assume without loss of generality that

$$\int_{-\infty}^{\infty} \lambda(s) = 1. \tag{12}$$

For the proof, we introduce the maximal value $\lambda^* = \max_{s\in\mathbb{R}} \lambda(s)$ of the function $\lambda(\cdot)$. Recall that the set $\{s :$

$\lambda(s) \geq \varepsilon\}$, unless it is empty, is supposed to be an interval $[a_\varepsilon, b_\varepsilon]$, and $[a_{2\varepsilon}, b_{2\varepsilon}] \subseteq [a_\varepsilon, b_\varepsilon]$ for any $\varepsilon > 0$.

By G we denote the connected undirected graph corresponding to the adjacency matrix A in the formulation of the theorem. Consider the *algebraic connectivity* μ_2 of the graph G, which is defined as follows:

$$\begin{aligned} \mu_2 =& N \min_{z\in\Upsilon} \frac{\sum_{i,j=1}^N \alpha_{ij}(z_j-z_i)^2}{\sum_{i,j=1}^N (z_j-z_i)^2}, \\ \Upsilon :=& \big\{z \in \mathbb{R}^n : z_k \neq z_j \text{ for some } j,k\big\}. \end{aligned} \tag{13}$$

The following key lemma will be used.

Lemma 2. For any $\varepsilon > 0$ there exists a number $k_* = k_*(\varepsilon)$ such that for any $k > k_*$ and $y \in \mathbb{R}^n$ one has

$$\sum_{i,j=1}^N \left[k\alpha_{ij}(y_j-y_i)\int_{y_i}^{y_j}\lambda(s)\,ds + 2\varepsilon|y_i-y_j|^2\right] \geq \\ \geq \sum_{i,j=1}^N (y_j-y_i)\int_{y_i}^{y_j}\lambda(s)\,ds. \tag{14}$$

Proof. Note first that (14) automatically holds (with any $k \geq 0$) if $M = \max_{i,j}|y_i-y_j| \geq M_* := N/(2\varepsilon)$. Indeed, let $M \geq M_*$ and $M = |y_p-y_q|$ for some p,q. Then

$$\sum_{i,j=1}^N (y_j-y_i)\int_{y_i}^{y_j}\lambda(s)\,ds \overset{(12)}{\leq} \sum_{i,j=1}^N |y_j-y_i| \leq \\ \leq NM < 2\varepsilon M^2 = 2\varepsilon|y_p-y_q|^2 \leq 2\varepsilon\sum_{i,j=1}^N|y_i-y_j|^2, \tag{15}$$

entailing (14) (since $\lambda(s) \geq 0$, the first sum in (14) is always non-negative). Similarly, (14) automatically holds with any $k \geq 0$ for any vector $y \in \mathbb{R}^n$ such that $y_i \leq a_{2\varepsilon}\forall i$ or $y_i \geq b_{2\varepsilon}\forall i$. For any such vector one has $\lambda(s) \leq 2\varepsilon$ for any $s \in [\min_i y_i, \max_i y_i]$, which yields in

$$\left|(y_j-y_i)\int_{y_i}^{y_j}\underbrace{\lambda(s)}_{\leq 2\varepsilon}\,ds\right| \leq 2\varepsilon\sum_{i,j=1}^N|y_i-y_j|^2.$$

Hence, we can confine ourselves to vectors $y \in \mathbb{R}^N$ such that $\min_i y_i < b_{2\varepsilon}$, $\max_i y_i > a_{2\varepsilon}$ and $0 \leq \max_i y_i - \min_i y_i \leq M_*$; in view of (15) this implies that

$$\sum_{i,j=1}^n (y_j-y_i)\int_{y_i}^{y_j}\lambda(s)\,ds \leq NM_*. \tag{16}$$

We consider three situations.

Case 1. For any i, $y_i \in [a_\varepsilon, b_\varepsilon]$. In this situation, the inequality (14) holds whenever $k > k_1 := N\lambda^*/(\varepsilon\mu_2)$. Indeed, thanks to (13) for any $k > k_1$ one has

$$k\sum_{i,j=1}^N \alpha_{ij}(y_j-y_i)\int_{y_i}^{y_j}\underbrace{\lambda(s)}_{\geq\varepsilon}\,ds \geq \\ \geq k\varepsilon\sum_{i,j=1}^N\alpha_{ij}(y_j-y_i)^2 \overset{(13)}{\geq} \frac{k\varepsilon\mu_2}{N}\sum_{i,j=1}^N(y_j-y_i)^2 \geq \\ \geq \lambda^*\sum_{i,j=1}^N(y_j-y_i)^2 \geq \sum_{i,j=1}^N(y_j-y_i)\int_{y_i}^{y_j}\underbrace{\lambda(s)}_{\leq\lambda^*}\,ds.$$

Case 2. None of the elements y_i belongs to $[a_{2\varepsilon}, b_{2\varepsilon}]$. We claim that (14) holds for $k > k_2 := nM_*/[2\varepsilon(b_{2\varepsilon}-a_{2\varepsilon})^2]$. Let $I = \{i : y_i < a_{2\varepsilon}\}$ and $J = \{j : y_j > b_{2\varepsilon}\}$; by assumption, $I \cup J = \{1,\ldots,N\}$. By assumption, $\min_i y_i < b_{2\varepsilon}$, and hence $\min_i y_i < a_{2\varepsilon}$; similarly, $\max_i y_i > b_{2\varepsilon}$. Hence the sets of indices I, J are non-empty. Since the graph G is connected, an arc from I to J exists. In other words, there exist indices v, w such that $\alpha_{vw} = 1$ and $y_v < a_{2\varepsilon} < b_{2\varepsilon} < y_w$. Therefore,

$$k\sum_{i,j=1}^n\alpha_{ij}(y_j-y_i)\int_{y_i}^{y_j}\lambda(s)\,ds \geq \\ gek(y_w-y_v)\underbrace{\int_{y_v}^{y_w}\lambda(s)ds}_{\geq\int_{a_{2\varepsilon}}^{b_{2\varepsilon}}\lambda(s)ds} \geq 2\varepsilon k(b_{2\varepsilon}-a_{2\varepsilon})^2 > nM_*,$$

which implies (14) in view of (16).

Case 3. The vector y has two elements $y_q \notin [a_\varepsilon, b_\varepsilon]$ and $y_p \in [a_{2\varepsilon}, b_{2\varepsilon}]$. Denote $\delta := \min(a_{2\varepsilon}-a_\varepsilon, b_\varepsilon-b_{2\varepsilon})/2$. We are going to show that (14) holds for

$$k > \frac{nM_*(n-1)^2}{\delta^2\varepsilon}.$$

To prove this, consider a path $i_0 = p \mapsto i_1 \mapsto \ldots \mapsto i_{d-1} \mapsto i_d = q$ in the graph connecting nodes p and q. Let i_{r+1} be the first node on the path such that $y_{i_{r+1}} \notin [a_\varepsilon, b_\varepsilon]$, whereas $y_p = y_{i_0}, \ldots, y_{i_r} \in [a_\varepsilon, b_\varepsilon]$. For definiteness, we assume that $y_{i_{r+1}} > b_\varepsilon$, the case $y_{i_{r+1}} < a_\varepsilon$ can be considered similarly. Consider now two subcases.

Case 3a. $y_{i_r} \leq b_{2\varepsilon} + \delta$. Let $v = i_r$, $w = i_{r+1}$. Then, $y_w - y_v > b_\varepsilon - (b_{2\varepsilon}+\delta) \geq \delta$, whence

$$k\sum_{i,j=1}^n\alpha_{ij}(y_j-y_i)\int_{y_i}^{y_j}\lambda(s)\,ds \geq k(y_w-y_v)\int_{y_v}^{y_w}\lambda(s)ds \geq \\ \geq k\delta\int_{b_{2\varepsilon}+\delta}^{b_\varepsilon}\underbrace{\lambda(s)}_{\geq\varepsilon}\,ds \geq k\delta^2\varepsilon \geq nM_*,$$

implying (14) in view of (16).

Case 3b. $y_{i_r} > b_{2\varepsilon}+\delta \geq y_p+\delta$. In this case, we have a path of length $r \leq n-1$ from $p = i_0$ to i_r, such that all nodes i on this path correspond to the elements $y_i \in [a_\varepsilon, b_\varepsilon]$. There are two adjacent nodes $v = i_l, w = i_{l+1}$, $0 \leq l < r$ with $y_w - y_v \geq \delta/(n-1)$. Hence,

$$k\sum_{i,j=1}^n\alpha_{ij}(y_j-y_i)\int_{y_i}^{y_j}\lambda(s)\,ds \geq \\ \geq k(y_w-y_v)\int_{y_v}^{y_w}\underbrace{\lambda(s)}_{\geq\varepsilon}\,ds \geq \\ \geq k\varepsilon(y_w-y_v)^2 \geq k\varepsilon\delta^2/(n-1)^2 \geq nM_*,$$

which again entails (14) due to (16). Notice that all thresholds k_1, k_2, k_3 depend only on ε, $\lambda(\cdot)$ and the algebraic connectivity μ_2. Hence, choosing $k_* = \max(k_1, k_2, k_3)$, the inequality (14) holds for any $k > k_*$ and $y \in \mathbb{R}^n$.

Remark 1. In fact, we have not used the graph undirectedness in the proof. Lemma 1 remains valid for any strongly connected directed graph.

Below follows another technical lemma needed for the proof of Theorem 2.

Lemma 3. Consider two sequences $y_1, \dots, y_n \in \mathbb{R}$ and $u_1, \dots, u_n \in \mathbb{R}$ such that

$$u_i = \sum_{j=1}^{n} \alpha_{ij} \int_{y_i}^{y_j} \lambda(s) ds \quad \forall i = 1, \dots, n.$$

Then, the following equality holds

$$\sum_{i=1}^{n} (u_i - u_j)(y_i - y_j) = \sum_{i,j=1}^{n} \alpha_{ij}(y_j - y_i) \int_{y_i}^{y_j} \lambda(s) ds. \quad (17)$$

Proof. Denote $\lambda_{ij} = \int_{y_i}^{y_j} \lambda(s) ds$. Since $\lambda_{ij} = -\lambda_{ji}$ and $\alpha_{ij}\lambda_{ij} = -\alpha_{ji}\lambda_{ji}$ due to the graph's symmetry, one has

$$S := \sum_{i,j=1}^{n} \alpha_{ij}\lambda_{ij}(y_j - y_i) = \sum_{i,j=1}^{n} \underbrace{\alpha_{ij}\lambda_{ij}}_{-\alpha_{ji}\lambda_{ji}} y_j -$$

$$- \sum_{i,j=1}^{n} \alpha_{ij}\lambda_{ij} y_i = -2 \sum_{i=1}^{n} \underbrace{\sum_{j=1}^{n} \alpha_{ij}\lambda_{ij}}_{=u_i} y_i = -2 \sum_{i=1}^{n} u_i y_i$$

We now note that $\sum_i u_i = \sum_{i,j} \alpha_{ij}\lambda_{ij} = 0$, thus

$$\sum_{i,j=1}^{n} (u_j - u_i)(y_j - y_i) = 2n \sum_{i=1}^{n} u_i y_i = -nS.$$

Now we are ready to prove Theorem 2.

Proof. The condition (5) entails that

$$P\tilde{f}'(x) + \tilde{f}'(x)^\top P < -R,$$

where $\tilde{f}(x) = f(x) + \int_{Cx}^{0} \lambda(s) ds$. Let X stand for the joint state vector of the system, obtained by stacking x_i one on top of each other. Let $V_{ij}(X) = (x_j - x_i)^\top P(x_j - x_i)$ and $V(X) = \sum_{i,j} V_{i,j}(X)$. Choosing $\varepsilon > 0$ so small that $R_\varepsilon = R - 2\varepsilon C^\top C > 0$ and denote $W_{ij}(X) = (x_j - x_i)^\top R_\varepsilon (x_j - x_i)$, $W(X) = \sum_{i,j} W_{i,j}(X)$, one obtains

$$2(x_j - x_i)^\top P(\tilde{f}(x_j) - \tilde{f}(x_i)) \le$$
$$\le -(x_i - x_j)^\top R(x_i - x_j) = -W_{ij}(X) - 2\varepsilon |y_i - y_j|^2,$$

which results, using (4) and (5), in

$$\dot{V}_{ij}(X) \le -W_{ij}(X) - 2\varepsilon |y_i - y_j|^2 +$$
$$+ (y_j - y_i) \int_{y_i}^{y_j} \lambda(s)\, ds + (u_j - u_i)(y_j - y_i).$$

The summation of the latter inequalities yields in

$$\dot{V}(X) \le -W(X) + \sum_{i,j} (y_j - y_i) \int_{y_i}^{y_j} \lambda(s)\, ds +$$
$$+ \sum_{i,j} (u_j - u_i)(y_j - y_i) - 2\varepsilon \sum_{i,j} |y_i - y_j|^2 \overset{(6),(17)}{=}$$
$$= -W(X) + \sum_{i,j} (y_j - y_i) \int_{y_i}^{y_j} \lambda(s)\, ds -$$
$$- k \sum_{i,j} \alpha_{ij} (y_j - y_i) \int_{y_i}^{y_j} \lambda(s)\, ds.$$

Choosing $k \ge k_*$, where k_* is the margin from Lemma 1, one shows that $\dot{V}(X) \le -W(X)$, which implies the exponential synchronization.

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