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Dichotomy and Stability of Disturbed Systems with Periodic Nonlinearities

Vera B. Smirnova[1], Anton V. Proskurnikov[2], Natalia V. Utina[3] and Roman V. Titov[4]

***Abstract*— Systems that can be decomposed as feedback interconnections of stable linear blocks and periodic nonlinearities arise in many physical and engineering applications. The relevant models e.g. describe oscillations of a viscously damped pendulum, synchronization circuits (phase, frequency and delay locked loops) and networks of coupled power generators. A system with periodic nonlinearities usually has multiple equilibria (some of them being locally unstable). Many tools of classical stability and control theories fail to cope with such systems. One of the efficient methods, elaborated to deal with periodic nonlinearities, stems from the celebrated Popov method of "integral indices", or integral quadratic constraints; this method leads, in particular, to frequency-domain criteria of the solutions' convergence, or, equivalently, global stability of the equilibria set. In this paper, we further develop Popov's method, addressing the problem of *robustness* of the convergence property against external disturbances that do not oscillate at infinity (allowing the system to have equilibria points). Will the forced solutions also converge to one of the equilibria points of the disturbed system? In this paper, a criterion for this type of robustness is offered.**

I. Introduction

Many applications deal with a special class of nonlinear systems that can be decomposed as a feedback interconnection of a linear time invariant system and a periodic nonlinearity (henceforth referred to as the systems with periodic nonlinearities). The simplest example of such a system is a viscously damped pendulum; other examples include, but are not limited to, vibrational units, electric motors, power generators and various synchronization circuits such as e.g. phase and frequency locked loops (PLL/FLL) [1]–[5].

Systems with periodic nonlinearities are featured by multiple stable and unstable equilibria points, which lead to many effects (such as e.g. existence of periodic trajectories and other "hidden" attractors [6]–[10]) that cannot be examined by tools of classical stability theory. An important problem, concerned with dynamics of PLLs and other synchronization circuits, is the convergence of all solutions to equilibria points (or, equivalently, global stability of the equilibria set). This counterpart of global asymptotic stability in systems with unique equilibria (sometimes called *gradient-like* behavior [11], [12]) excludes, in particular, "hidden" attractors.

Finite-dimensional systems with periodic nonlinearities have been thoroughly studied [6], [9], [11]–[15]; one of the methods for their investigation [11], [14] exploits special non-quadratic Lyapunov functions, whose existence is proved via the Kalman-Yakubovich-Popov lemma. Much less studied are their *infinite-dimensional* counterparts, describing e.g. synchronization systems with delays [16], [17] and non-rational low-pass filters [18]. The central method proposed to study them [19]–[24] stems from the seminal V.M. Popov's technique [25], [26], which is referred to as the method of "a priory integral indices" and has given rise to the method of *integral quadratic constraints* (IQC) [27]–[29]. Using this approach, conditions for the solutions' convergence, as well as estimates for their transient behavior, have been obtained [19]–[24]. These criteria may be called "frequency-algebraic" and reduce to frequency-domain conditions, involving the transfer function of the linear part and some parameters that are constrained by algebraic inequalities.

In this paper, we are interested in the convergence of systems with periodic nonlinearities in presence of uncertain disturbances. Obviously, if such a disturbance persistently excites the solution (being e.g. harmonic or other periodic oscillatory signal), the solution no longer converges to an equilibrium point but oscillates around it. In synchronization systems, such disturbances are typically modeled as combinations of stationary random signals and polyharmonic signals [30]–[32] to be attenuated. In this paper, we deal with other type of disturbances that have limits at infinity and thus enable the disturbed system to have equilibria. Such disturbances are often considered in the works on *cycle slipping* [33], [34], whose main concern is the solution's "switching" between the basins of attraction of different equilibria points (in synchronization systems, this effect is considered as undesirable since it leads to demodulation errors). We address the following question: given a convergent system and disturbance that has a limit at infinity, does each of the *forced* solutions converge to some equilibrium of the disturbed system? In this paper, we establish novel "frequency-algebraic" criteria that ensure the convergence of forced solutions under the disturbances of the aforementioned type and "relaxed" versions of these criteria, ensuring the property of *dichotomy* (convergence of each bounded solution). This paper extends our previous

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[1] Vera B. Smirnova is with Department of Mathematics, St.Petersburg State University of Architecture and Civil Engineering. She is also with the Department of Mathematics and Mechanics, St.Petersburg State University, St.Petersburg, Russia; root@al2189.spb.edu

[2] Anton V. Proskurnikov is with Delft Center for Systems and Control, Delft University of Technology, Delft, The Netherlands. He is also with the Institute for Problems of Mechanical Engineering of the Russian Academy of Sciences (IPME RAS), St.Petersburg, Russia anton.p.1982@ieee.org

[3] Natalia V. Utina is with Department of Mathematics, St.Petersburg State University of Architecture and Civil Engineering unv74@mail.ru

[4] Roman V. Titov is with Department of Mathematics, St.Petersburg State University of Architecture and Civil Engineering spb_titov_roman@mail.ru

work [35], confined the case of static scalar nonlinearity.

II. Problem setup.

Consider a control system described by integro–differential equations

$$\dot\sigma(t) = b(t) + R(\psi(\sigma(t-h)) + f(t-h)) - \\ - \int_0^t \gamma(t-\tau)(\psi(\sigma(\tau)) + f(\tau))\,d\tau \quad (t>0). \tag{1}$$

Here $\sigma(t) = (\sigma_1(t), \dots, \sigma_l(t))^\top$, $\psi : \mathbb{R}^l \to \mathbb{R}^l$ and $\psi(\sigma) = (\psi_1(\sigma_1), \dots, \psi_l(\sigma_l))^\top$, $f : [-h, +\infty) \to \mathbb{R}^l$, $b : [0, +\infty) \to \mathbb{R}^l$, $\gamma : [0, +\infty) \to \mathbb{R}^{l\times l}$, $R \in \mathbb{R}^{l\times l}$, $h \geq 0$. The solution of (1) is defined by initial condition

$$\sigma(t)|_{t\in[-h,0]} = \sigma^0(t). \tag{2}$$

In all theorems demonstrated throughout the paper, the following conditions are adopted.

Assumption 1: The system (1) and its initial condition (2) satisfy the following restrictions:

1) the function $b(\cdot)$ is continuous and $b(t) \to 0$ as $t \to \infty$; the function $\gamma(\cdot)$ is piece–wise continuous;
2) $$|b(t)| \in L_1[0, +\infty) \tag{3}$$
$$|\gamma(t)|e^{rt} \in L_2[0, +\infty) \quad (r > 0); \tag{4}$$
3) the function $\sigma^0(\cdot)$ is continuous and $\sigma(0+0) = \sigma^0(0)$;
4) the function $f(t)$ is continuous and
$$\lim_{t\to+\infty} f(t) = L, \tag{5}$$
where $L = (L_1, \dots, L_l)^\top$, $L_j \in \mathbb{R}$;
5) each map ψ_j is Δ_j-periodic ($\psi_j(\sigma_j + \Delta_j) = \psi_j(\sigma_j)$); it is C^1-smooth with
$$\alpha_{1j} := \inf_{\zeta\in[0,\Delta_j)} \psi_j'(\zeta); \alpha_{2j} := \sup_{\zeta\in[0,\Delta_j)} \psi_j'(\zeta) \tag{6}$$
(it is clear that $\alpha_{1j}\alpha_{2j} < 0$);
6) the functions
$$\varphi_j(\zeta) \triangleq \psi_j(\zeta) + L_j \tag{7}$$
have simple isolated roots.

By (5) and (7) the system (1) can be rewritten in the form

$$\begin{cases} \dot\sigma(t) = b(t) + R\xi(t-h) - \int_0^t \gamma(t-\tau)\xi(\tau)\,d\tau \ (t>0), \\ \xi(t) = \varphi(\sigma(t)) + g(t), \end{cases} \tag{8}$$

where $\varphi(\sigma) = (\varphi_1(\sigma_1), \dots, \varphi_l(\sigma_l))^\top$, $g(t) = f(t) - L$. It is clear that

$$g(t) \to 0 \text{ as } t \to +\infty \tag{9}$$

In this paper we are going to investigate the asymptotic behavior of the system (1) (or of the equivalent system (8)). The goal of the paper is to establish the properties of $g(t)$ which guarantee certain types of asymptotic behavior of the system. Thus the paper inherits [22], [20], devoted to the system (1) without external disturbances ($f(t) \equiv 0$), and [36] considering the case of $g(t) \in L_2[0, +\infty)$. We shall study here the following aspects of asymptotic behavior: the dichotomy which means that any bounded solution converges and the gradient-like behavior when any solution converges.

Our study is based on Popov's method [25], [26]. Its main idea is to determine a functional of $\dot\sigma$ and ξ in the form of a positive inner product in Hilbert space and to rewrite it in the frequency domain, using the Plancherel theorem. If the real part of the frequency response (with certain additions) does not change its sign, one can obtain the convergence of quadratic functionals, which in turn implies the convergence of solutions. For this reason, all the results are formulated in terms of the transfer matrix of the linear part of (1)

$$K(p) = -Re^{-ph} + \int_0^\infty \gamma(t)e^{-pt}\,dt \quad (p \in \mathbb{C}) \tag{10}$$

and involve a certain frequency–domain inequality. The parameters of the latter inequality vary in some set, described by nonlinear algebraic constraints.

Let

$$m_{1j} \leq \alpha_{1j}, \quad m_{2j} \geq \alpha_{2j}. \tag{11}$$

Notice that m_{ij} $(i = 1, 2; j = 1, \dots, l)$ may be either a certain number or ∞. In the latter case we put $m_{ij}^{-1} = 0$. Let $M_i = diag\left\{m_{i1}^{-1}, \dots, m_{il}^{-1}\right\}$ $(i = 1, 2)$. Introduce diagonal matrices $\varkappa = diag\left\{\varkappa_1, \dots, \varkappa_l\right\}$, $\varepsilon = diag\left\{\varepsilon_1, \dots, \varepsilon_l\right\}$, $\tau = diag\left\{\tau_1, \dots, \tau_l\right\}$ and $\delta = diag\left\{\delta_1, \dots, \delta_l\right\}$ and determine the frequency–domain inequality

$$\Pi(\omega) \triangleq \Re e\{\varkappa K(\imath\omega) - (K(\imath\omega) + \imath\omega M_1)^*\tau(K(\imath\omega) + \\ + \imath\omega M_2) - K(\imath\omega)^*\varepsilon K(\imath\omega)\} - \delta \geq 0. \tag{12}$$

Here $\imath^2 = -1$, the symbol $(*)$ means Hermitian conjugation and

$$\Re eH \triangleq \frac{1}{2}(H + H^*), \quad H \in \mathbb{R}^{l\times l}. \tag{13}$$

III. Frequency-domain conditions of dichotomy

.

We shall establish frequency-domain conditions of the dichotomy for several various classes of $g(t)$.

Class 1. Let us assume that in addition to (9) $g(t)$ is differentiable and

$$\dot g(t) \in L_1(0, +\infty) \bigcap L_2(0, +\infty). \tag{14}$$

Theorem 1. *Let the inclusions* (14) *be true. Suppose for a certain set of parameters* $\varkappa$, $\varepsilon > 0$, $\delta > 0$, $\tau > 0$, $M_1\, M_2$ *the frequency-domain inequality* (12) *holds for all* $\omega \geq 0$. *Then any bounded solution of* (1) *converges in the sense*

$$\sigma_j(t) \xrightarrow[t\to\infty]{} \sigma_{jeq} \quad (j = 1, 2, \dots, l), \tag{15}$$
$$\dot\sigma(t) \xrightarrow[t\to\infty]{} 0, \tag{16}$$

where σ_{jeq} *is a root of the equation*

$$\psi_j(\sigma_{jeq}) = -L_j.$$

Proof: We follow here the standard scheme of Popov's method in case of differentiable nonlinearity.

Let $\eta(t)=\varphi(\sigma(t))$, then $\xi(t)=\eta(t)+g(t)$. Introduce the auxiliary function

$$\mu(t)\triangleq\begin{cases}0, & t<0,\\ t, & 0\le t\le 1,\\ 1. & t>1.\end{cases}\tag{17}$$

For $T\ge 1$ and a given solution of (1) consider the functions

$$\xi_T(t)\triangleq\begin{cases}\mu(t)\xi(t), & t<T,\\ \xi(T)e^{c(T-t)} & t\ge T\quad (c>0);\end{cases}\tag{18}$$

$$\sigma_0(t)=b(t)+R\xi(t-h)(1-\mu(t-h))-\\ -\int\limits_0^t(1-\mu(\tau))\gamma(t-\tau)\xi(\tau)\,d\tau.\tag{19}$$

It is easy to see that

$$|\sigma_0(t)|\in L_1[0,+\infty)\bigcap L_2[0,+\infty).\tag{20}$$

Introduce the function

$$\sigma_T(t)=R\xi_T(t-h)-\int\limits_0^t\gamma(t-\tau)\xi_T(\tau)\,d\tau.\tag{21}$$

It is clear that

$$\dot\sigma(t)=\sigma_0(t)+\sigma_T(t)\quad\text{for } t\in[0,T].\tag{22}$$

Consider a set of functionals ($T\ge 1$)

$$J_T\triangleq\int\limits_0^\infty\{\sigma_T^*\varkappa\xi_T+\xi_T^*\delta\xi_T+\\ +\sigma_T^*\varepsilon\sigma_T+(\sigma_T-M_1\dot\xi_T)^*\tau(\sigma_T-M_2\dot\xi_T)\}\,dt.\tag{23}$$

Due to Plancherel theorem we have

$$J_T=\tfrac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\{\mathfrak{F}(\sigma_T)^*\varkappa\mathfrak{F}(\xi_T)+\mathfrak{F}(\xi_T)^*\delta\mathfrak{F}(\xi_T)+\\ +\mathfrak{F}(\sigma_T)^*\varepsilon\mathfrak{F}(\sigma_T)+(\mathfrak{F}(\sigma_T)-M_1\mathfrak{F}(\dot\xi_T))^*\tau(\mathfrak{F}(\sigma_T)-\\ -M_2\mathfrak{F}(\dot\xi_T))\}\,d\omega,\tag{24}$$

where $\mathfrak{F}$ stands for the Fourier transform. Since

$$\mathfrak{F}(\sigma_T)(\imath\omega)=-K(\imath\omega)\mathfrak{F}(\xi_T)(\imath\omega)\tag{25}$$

and

$$\mathfrak{F}(\dot\xi_T)(\imath\omega)=\imath\omega\mathfrak{F}(\xi_T)(\imath\omega).\tag{26}$$

We have

$$J_T=-\tfrac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\mathfrak{F}^*(\xi_T)(\imath\omega)\Pi(\omega)\mathfrak{F}(\xi_T)(\imath\omega)\,d\omega\tag{27}$$

and by virtue of (12)

$$J_T\le 0.\tag{28}$$

On the other hand the following inequality is true

$$J_T>\rho_T+J_0+J_{1T}+J_{2T},\tag{29}$$

where

$$\rho_T\triangleq\int\limits_0^T\{\dot\sigma^*\varkappa\xi+\xi^*\delta\xi+\dot\sigma^*\varepsilon\dot\sigma+\\ +(\dot\sigma-M_1\dot\xi)^*\tau(\dot\sigma-M_2\dot\xi)\}\,dt,\tag{30}$$

$$J_{1T}\triangleq-\int\limits_0^T\{\sigma_0^*\varkappa\xi_T+2\sigma_0^*\varepsilon\dot\sigma+\\ +\sigma_0^*\tau(\dot\sigma-M_2\dot\xi_T)+(\dot\sigma-M_1\dot\xi_T)^*\tau\sigma_0\}\,dt,\tag{31}$$

$$J_{2T}\triangleq\int\limits_T^\infty\{\sigma_T^*\varkappa\xi(T)e^{c(T-t)}+\xi(T)^*\delta\xi(T)e^{2c(T-t)}+\\ +c^2\xi(T)^*M_1\tau M_2\xi(T)e^{2c(T-t)}-\\ -ce^{c(T-t)}\xi(T)^*(M_1+M_2)\tau\sigma_T\}\,dt,\tag{32}$$

and J_0 is an integral of a continuous function from 0 to 1.

We shall demonstrate that certain functionals are bounded by constants independent of T, denoting the latter by C with subscripts.

From the assumptions 1)–5) it follows that $\eta(t)$, $\dot\sigma(t)$, $\dot\eta(t)$ are bounded on $\mathbb{R}_+$. Hence and from (20) and (14) it follows that

$$|J_{1T}|\le C_2.\tag{33}$$

It is easy to see that

$$|J_{2T}|<C_3.\tag{34}$$

Then inequalities (28), (33), (34) together with (29) imply that

$$\rho_T<C_0,\quad\forall T>1.\tag{35}$$

Let us consider some summands of ρ_T:

i)

$$\int\limits_0^T\dot\sigma^*\varkappa\xi\,dt=\sum_{j=1}^l\int\limits_{\sigma_j(0)}^{\sigma_j(T)}\varkappa_j\varphi_j(\zeta)\,d\zeta+\\ +\sum_{j=1}^l\varkappa_j\int\limits_0^T\dot\sigma_j g_j(t)\,dt.\tag{36}$$

The first summand in right-hand side of (36) is bounded since σ and φ are bounded on $[0,+\infty)$. Besides

$$\int\limits_0^T\dot\sigma_j g_j(t)\,dt=\sigma_j(T)g_j(T)-\sigma_j(0)g_j(0)-\int\limits_0^T\sigma_j\dot g_j(t)\,dt.\tag{37}$$

Then it follows from (14) that the second summand in right part (36) is bounded. So

$$\Big|\int\limits_0^T\dot\sigma^*\varkappa\xi\,dt\Big|<C_1.\tag{38}$$

ii)

$$\rho_{1T} \triangleq \int_0^T \{(\dot\sigma - M_1\dot\xi)^*\tau(\dot\sigma - M_2\dot\xi)\}\,dt =$$
$$= \int_0^T \{(\dot\sigma - M_1\dot\eta)^*\tau(\dot\sigma - M_2\dot\eta)\}\,dt -$$
$$- \int_0^T \{(\dot g^* M_1^*\tau(\dot\sigma - M_2\dot\eta)\}\,dt -$$
$$- \int_0^T \{(\dot\sigma - M_1\dot\eta)^*\tau M_2\dot g\}\,dt + \int_0^T \{\dot g^* M_1^*\tau(M_2\dot g)\}\,dt. \tag{39}$$

The first summand in right-hand side of (39) is nonnegative in virtue of (11). The last term is bounded in virtue of (14). On the other hand for any $\varepsilon_0 > 0$

$$|\dot g^* M_1^*\tau(\dot\sigma - M_2\dot\eta)| + |(\dot\sigma - M_1\dot\eta)^*\tau M_2\dot g| \le$$
$$\le \varepsilon_0(|\dot\sigma - M_2\dot\eta|^2 + |\dot\sigma - M_1\dot\eta|^2) + \tag{40}$$
$$+\tfrac{1}{4\varepsilon_0}(\max_{j=1,\dots,l}(\tau_j m_{1j}^{-1})^2 + \max_{j=1,\dots,l}(\tau_j m_{2j}^{-1})^2)|\dot g|^2$$

and

$$|\dot\sigma - M_i\dot\eta|^2 \le C_4|\dot\sigma|^2. \tag{41}$$

So in virtue of (14)

$$\Big|\int_0^T (\dot g^* M_1^*\tau(\dot\sigma - M_2\dot\eta) + \tag{42}$$
$$+(\dot\sigma - M_1\dot\eta)^*\tau M_2\dot g\,dt\Big| \le 2\varepsilon_0 C_4 \int_0^T |\dot\sigma|^2\,dt + C_5.$$

As a result

$$\rho_{1T} \ge -2\varepsilon_0 C_4 \int_0^T |\dot\sigma|^2\,dt - C_6 \ (C_6 > 0). \tag{43}$$

Let ε_0 be small enough and $\bar\varepsilon \triangleq \varepsilon - 2\varepsilon_0 C_4 E_l > 0$, where E_l is a unit matrix. Then it follows from (30), (35), (38), (43) that

$$\int_0^T (\dot\sigma^*\bar\varepsilon\dot\sigma + \xi^*\delta\xi)\,dt \le C_7 \ (C_7 > 0), \tag{44}$$

which implies that

$$\dot\sigma_j(t) \in L_2[0,+\infty),$$
$$\varphi_j(\sigma_j(t)) + g_j(t) \in L_2[0,+\infty) \quad (j = 1,\dots,l). \tag{45}$$

Any $\varphi_j(\sigma_j(t))$ and $g_j(t)$ is uniformly continuous on $[0,+\infty)$. So according to Barbalat lemma [25] $\varphi_j(\sigma_j(t))$ tends to zero as $t \to +\infty$. Then $\sigma_j(t)$ tends to a zero of $\varphi_j(\zeta)$ as $t \to +\infty$. Since the functions $\dot\sigma_j(t)$ are uniformly continuous on $[0,+\infty)$, they tend to zero as $t \to +\infty$. Theorem 1 is proved. ■

Class 2. Let us strengthen the restrictions on $g(t)$: $g(t)$ satisfies (9) and

$$g(t) \in L_2[0,+\infty). \tag{46}$$

It turns out then that the restrictions on $\dot g(t)$ can be weakened:

$$\dot g(t) \in L_2[0,+\infty). \tag{47}$$

Theorem 2. *Theorem 1 remains valid replacing* (14) *by* (46) *and* (47)

Proof: If (14) is replaced by (47) all the arguments of Theorem 1 are valid except the assertion that $\int_0^T \dot\sigma_j g_j\,dt$ is bounded independently of T. That is why instead of formula (37) we shall use the inequality

$$\int_0^T \dot\sigma_j g_j\,dt \le \varepsilon_0 \int_0^T \dot\sigma_j^2\,dt + \frac{1}{4\varepsilon_0}\int_0^T g_j^2\,dt. \tag{48}$$

If ε_0 is small enough we can conclude from (30), (35), (36), (43), (46) and (48) that

$$\int_0^T (\dot\sigma^*\hat\varepsilon\dot\sigma + \xi^*\delta\xi)\,dt \le C_8 \tag{49}$$

where $\hat\varepsilon$ is positive definite. The estimate (49) is just alike the estimate (44). Thus Theorem 2 is proved. ■

Class 3. Let us assume that $g(t)$ satisfies (9) and

$$|g(t)| \in L_1[0,+\infty). \tag{50}$$

Theorem 3. *Theorem 1 remains valid replacing* (14) *by* (50).

Proof: Let us rewrite the system (8) as

$$\dot\sigma(t) = \beta(t) + R\varphi(\sigma(t-h)) - \int_0^t \gamma(t-\tau)\varphi(\sigma(\tau))\,d\tau, \tag{51}$$

where

$$\beta(t) = b(t) + Rg(t-h) - \int_0^t \gamma(t-\tau)g(\tau)\,d\tau. \tag{52}$$

Consider

$$v(t) \triangleq \Big|\int_0^t \gamma(t-\tau)g(\tau)\,d\tau\Big|. \tag{53}$$

We have

$$\int_0^\infty v(t)\,dt \le \int_0^\infty \int_0^t |\gamma(t-\tau)||g(\tau)|\,d\tau\,dt =$$
$$= \int_0^\infty |g(\tau)| \int_\tau^\infty |\gamma(t-\tau)|\,dt\,d\tau = \tag{54}$$
$$= \int_0^\infty |g(\tau)| \int_0^\infty |\gamma(\lambda)|\,d\lambda\,d\tau,$$

which implies that

$$v(t) \in L_1[0, +\infty). \tag{55}$$

On the other hand since $|\gamma(t)| \in L_1[0, +\infty)$ and $g(t) \to 0$ as $t \to +\infty$ it follows from [37] that

$$v(t) \to 0 \text{ as } t \to +\infty. \tag{56}$$

Thus the functions $\beta(t)$ and $b(t)$ have the same properties on $[0, +\infty)$. Then Theorem 3 arises from Theorem 1. ■

Notice that Theorem 1, Theorem 2 and Theorem 3 give conditions for convergence of *bounded* solutions, being thus criteria of *dichotomy*: any solution is either convergent or unbounded. This result may serve to prove convergence of *any* solution in the case where all solutions are bounded (Lagrange stability of (1)).

One sufficient condition for this has been found in [38]. In [38] the case of $l = 1$ is considered and the following conditions are adopted.

Assumption 2: The restrictions 1), 2), 3) of Assumption 1 are preserved with (3) replaced by

$$|b(t)|e^{rt} \in L_2[0, +\infty). \tag{57}$$

The disturbance $f(t)$ is continuous and bounded on $[0, +\infty)$. The function $\psi(\sigma)$ is continuous and Δ–periodic.

Proposition 1: [38] *Let $l = 1$ and Assumption 2 hold. Suppose there exist numbers $\varepsilon > 0$, $\lambda \in (0, \frac{r}{2})$ and γ_1, γ_2 such that the following conditions are fulfilled:*
1) the functions $\psi(\sigma) + \gamma_i$ $(i = 1, 2)$ have two simple zeros on $[0, \Delta)$;
2) $\Re e\{K(\imath\omega - \lambda)\} - \varepsilon|K(\imath\omega - \lambda)|^2 \geq 0$, $\forall \omega \geq 0$;
3) the equations

$$\ddot{\sigma} + 2\sqrt{\lambda\varepsilon}\dot{\sigma} + \psi(\sigma) + \gamma_i = 0 \quad (i = 1, 2) \tag{58}$$

are Lagrange stable;
4) $\gamma_2 \leq f(t) + \frac{1}{2\lambda}\dot{f}(t) \leq \gamma_1$ $\forall t \in \mathbb{R}_+$, $f(0) \in [\gamma_1, \gamma_2]$.
Then the equation (1) *is Lagrange stable.*

The Proposition 1 is proved in [38] by nonlocal reduction method [20], which presupposes the injecting the trajectories of (58) into Popov functionals for (1).

Two other conditions for convergence of solutions, being nonlinear algebraic constraints on the parameters, are given in next section.

IV. From dichotomy to gradient-like behavior.

In this section we consider two classes of the external disturbances.

Class 2. We assume that $|g(t)|$ and $|\dot{g}(t)|$ are from $L_2[0, +\infty)$. This case has been studied in [36]. The conditions of gradient-like behavior are given in two criteria established there. To formulate them we introduce the functions

$$\Phi_j(\zeta) \triangleq \sqrt{\left(1 - m_{1j}^{-1}\varphi_j'(\zeta)\right)\left(1 - m_{2j}^{-1}\varphi_j'(\zeta)\right)} \tag{59}$$

and

$$P_j(\zeta; \alpha, \beta) \triangleq \sqrt{1 + \frac{\alpha}{\beta}\Phi_j^2(\zeta)} \tag{60}$$

where $\alpha > 0$ and $\beta > 0$ are parameters.

We shall also need the constants

$$\nu_j := \frac{\int_0^{\Delta_j} \varphi_j(\zeta)d\zeta}{\int_0^{\Delta_j} |\varphi_j(\zeta)|d\zeta}, \quad \nu_{0j} := \frac{\int_0^{\Delta_j} \varphi_j(\zeta)d\zeta}{\int_0^{\Delta_j} \Phi_j(\zeta)|\varphi_j(\zeta)|d\zeta}. \tag{61}$$

$$\nu_{1j}(\alpha, \beta) := \frac{\int_0^{\Delta_j} \varphi_j(\zeta)\, d\zeta}{\int_0^{\Delta_j} |\varphi_j(\zeta)|P_j(\zeta; \alpha, \beta)\, d\zeta}. \tag{62}$$

Theorem 4. [36] *Suppose there exist positive definite matrices δ, τ, ε, matrices $\varkappa$, M_1 and M_2, and numbers $a_j \in [0, 1]$ $(j = 1, \ldots, l)$ such that the following conditions are fulfilled:*
1) for all $\omega \geq 0$ the frequency–domain inequality (12) *is true,*
2) the quadratic forms

$$\begin{aligned} Q_j(x, y, z) := \varepsilon_j x^2 + \delta_j y^2 + \tau_j z^2 + \varkappa_j a_j \nu_j xy + \\ + \varkappa_j(1 - a_j)\nu_{0j} yz \quad (j = 1, \ldots, l) \end{aligned} \tag{63}$$

are positive definite.
Then any solution of (1) *converges, i.e. the limit relations* (15) *and* (16) *are true.*

Theorem 5. [36] *Suppose there exist positive definite matrices δ, τ, ε and matrices $\varkappa$, M_1 and M_2 such that for all $\omega \geq 0$ frequency–domain inequality* (12) *holds and algebraic inequalities*

$$2\sqrt{\varepsilon_j \delta_j} > |\nu_{1j}(\tau_j, \varepsilon_j)|\varkappa_j \quad (j = 1, \ldots, l) \tag{64}$$

are true.
Then the conclusion of Theorem 1 is valid.

Class 3. Let $|g(t)| \in L_1[0, +\infty)$. It follows from the previous section that in this case the given system (1) can be reduced to the system (51) with all the attributes of (1) but with $f(t) \equiv 0$. So Theorems 4 and 5 are valid here.

Example 1. Consider the phase–locked loop (PLL) with sine-shaped characteristic and the integrating filter

$$K(p) = \frac{T}{Tp + 1} \quad (T > 0). \tag{65}$$

The mathematical equation of the corresponding system (1) coincides with that of the disturbed pendulum

$$\ddot{\sigma} + \frac{1}{T}\dot{\sigma} + (\sin\sigma - \beta) + f(t) = 0. \tag{66}$$

Using Theorem 4, it is possible to estimate the set of the coefficients $\{(T, \beta)\}$, for which all solutions of the system (66) converge, provided that either $f, \dot{f} \in L_2[0, \infty)$ or $f \in L_1[0, \infty)$. In Fig. 1, we compare this estimate of the stability domain in the parameter space with the *exact* stability domain, computed in [39] for the undisturbed

system ($f \equiv 0$) by using qualitative-numerical methods. The stability domain obtained by Theorem 4 is of course smaller than the exact one but the approximation is good enough.

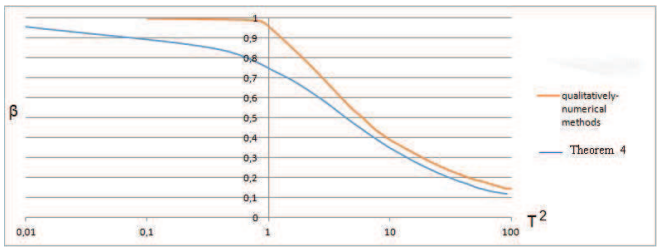


Fig. 1. The exact stability domain for the system (66) vs. its estimate.

V. Conclusion

In this paper we study forced solutions of systems with periodic nonlinearities, affected by uncertain disturbances that do not oscillate at infinity and enable the disturbed system to have equilibria. We offer novel frequency–domain criteria for convergence of any forced solution. These critera are based on techniques, stemming from V.M. Popov's method of integral quadratic constraints. To modify these criteria for Lagrange stability of (1) in case of bounded disturbances, having no limits at infinity, is a non-trivial problem, which is a subject of ongoing research.

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