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Reliable a posteriori mesh adaptivity in discrete fracture network flow simulations [☆]

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Abstract

A new approach for the flow simulation in very complex Discrete Fracture Networks (DFNs) based on PDE-constrained optimization has been recently proposed in [1, 2] with the aim of improving robustness with respect to geometrical complexities. In [3] a rigorous derivation of “a posteriori” error estimates has been performed, proving an equivalence relation between the discretization error and suitable quantities that are computable from the discrete solution and the problem data.

In this paper, the previous results are applied to several fracture networks in order to investigate mesh adaptivity applied to complex realistic fracture networks.

Keywords: Discrete Fracture Network flow simulations, Simulations in complex geometries, Mesh adaptivity, A posteriori error estimates
2010 MSC: 65N30, 65N50, 68U20, 86-08, 86A05

1. Introduction

2 The simulation of underground phenomena in fractured media is a key aspect in
3 many applications such as aquifers monitoring, nuclear waste disposal or the control
4 of contaminant dispersion in the subsoil. In practical applications, many simulations
5 need to be performed on randomly generated configurations of fractures generated from
6 experimentally deduced probability distributions of the physical properties of the soil
7 [4, 5]. The reliability of these simulations is of crucial importance, for instance, for
8 the subsequent application of uncertainty quantification techniques for the estimation
9 of relevant quantities of interest [6, 7].

10 In those situations where the rock matrix can be considered perfectly impervious,
11 the Discrete Fracture Network model is often used, representing fractures as planar

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12 polygons intersecting randomly in the 3D space [8, 9]. These kind of domains usually
 13 present many geometrical challenges, in particular when one needs to mesh the domain
 14 while respecting some kind of conformity requirements, for example in order to apply
 15 standard finite element approaches. To circumvent these difficulties, many approaches
 16 have been devised, relying on domain decomposition techniques [10, 11, 12], or devis-
 17 ing particular meshing strategies [13, 14, 15, 16, 17], or using extensions of the finite
 18 element method [18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33].

19 In [1, 34, 2], a PDE-constrained optimization approach was devised, enabling for
 20 the use of completely non-conforming meshes, where finite element, extended finite
 21 element [35] or virtual element [36] spaces can be used. A scalable parallel implemen-
 22 tation of such approach is proposed in [37]. In [3], a residual a posteriori error estimate
 23 is devised, taking into account both the discretization error and the error introduced by
 24 the non-conformity of the mesh with respect to intersections. In the present work, the
 25 estimate is extended in order to deal with general boundary conditions and is used to
 26 devise an adaptive meshing algorithm [38, 39, 40, 41]. In sections 2 and 3 we intro-
 27 duce the DFN model and define some useful notations, in section 4 we introduce the
 28 variational problem and discretize it as an optimization problem for a suitably defined
 29 functional, in section 5 we define the error estimators and state the a posteriori error
 30 estimate (Theorem 1), in section 6 we describe the adaptive algorithm and in section
 31 7 we validate its application to some test cases with known solution and analyse some
 32 results on some more realistic DFNs.

33 2. Discrete Fracture Networks

34 A DFN is a model representing fractures inside an impervious rock matrix as planar
 35 polygons. We assume that the flow occurs only through fractures and across fracture-
 36 intersections in the normal direction. Let us denote by Ω the DFN, composed of N
 37 intersecting fractures. Each fracture F_i , $i \in \mathcal{I} = \{1, \dots, N\}$ has boundary ∂F_i and the
 38 boundary of Ω is $\partial\Omega = \bigcup_{i \in \mathcal{I}} \partial F_i$. Ω is supposed to be connected and we call *traces* the
 39 intersections between exactly two fractures, denoted by Γ_m , with $m \in \mathcal{M} = \{1, \dots, M\}$.
 40 For each $m \in \mathcal{M}$, let $\mathcal{I}_m = (i, j) \in \mathbb{N}^2$ be the ordered pair of indices of fractures such
 41 that $\Gamma_m = F_i \cap F_j$. For each fracture F_i , let \mathcal{M}_i be the ordered set of indices of traces
 42 belonging to \bar{F}_i , and denote $M_i = \#\mathcal{M}_i$. $\mathcal{M}_i(k)$, for $k = 1, \dots, M_i$, denotes the k -th
 43 index of a trace in \mathcal{M}_i . For each $i \in \mathcal{I}$ and each $m \in \mathcal{M}_i$, $\mathbf{n}_{m,i}$ denotes a fixed unit vector
 44 normal to the trace Γ_m on F_i .

45 3. Notation

46 In this section we define some useful notations very close to those used in [3].

For any bounded open set ω , we denote by $(\cdot, \cdot)_\omega$ and by $\|\cdot\|_\omega$ the $L^2(\omega)$ scalar
 product and norm, respectively, and by $\|\cdot\|_\omega$ the norm in $H_0^1(\omega)$. Further, $(\cdot, \cdot)_{\alpha,\omega}$ and
 $\|\cdot\|_{\alpha,\omega}$ denote the scalar product and the norm in $H^\alpha(\omega)$, respectively. For any given
 segment $\sigma \subset F_i$, $i \in \mathcal{I}$, $\gamma_\sigma^i : H^1(F_i) \rightarrow H^{\frac{1}{2}}(\sigma)$ is the trace operator and

$$\langle \mu, \beta \rangle_\sigma = {}_{H^{-\frac{1}{2}}(\sigma)} \langle \mu, \beta \rangle_{{H^{\frac{1}{2}}(\sigma)}} \quad \forall \mu \in H^{-\frac{1}{2}}(\sigma), \forall \beta \in H^{\frac{1}{2}}(\sigma)$$

is the duality between $H^{\frac{1}{2}}(\sigma)$ and $H^{-\frac{1}{2}}(\sigma)$. For any given function $v \in H^1(F_i)$, $\gamma_{\mathcal{M}_i}(v) \in \prod_{m \in \mathcal{M}_i} H^{\frac{1}{2}}(\Gamma_m)$ is the tuple of the functional traces on the geological traces of the fracture f_i , $\gamma_{\Gamma_m}^i(v)$, $m \in \mathcal{M}_i$ sorted by increasing trace index m , and we denote

$$\forall \mu \in \prod_{m \in \mathcal{M}_i} H^{-\frac{1}{2}}(\Gamma_m), \forall \beta \in \prod_{m \in \mathcal{M}_i} H^{\frac{1}{2}}(\Gamma_m), \langle \mu, \beta \rangle_{\mathcal{M}_i} := \sum_{m \in \mathcal{M}_i} \langle \mu_m, \beta_m \rangle_{\Gamma_m}.$$

Moreover, we denote by

$$\gamma_{\mathcal{M}}(\mathbf{v}) = (\gamma_{\mathcal{M}_1}(v_1), \dots, \gamma_{\mathcal{M}_N}(v_N)) \quad \forall \mathbf{v} \in V,$$

the tuple of the tuples of traces on each fracture. We introduce the functional space $V := V_1 \times \dots \times V_N$, with $V_i := H^1(F_i)$, $\forall i \in \mathcal{I}$. For any function $\mathbf{g} \in V$, we define, $\forall m \in \mathcal{M}$, and $\mathcal{I}_m = (i, j)$ the jump operator across a trace Γ_m , $\llbracket \mathbf{g} \rrbracket_{\Gamma_m} := \gamma_{\Gamma_m}^i(g_i) - \gamma_{\Gamma_m}^j(g_j)$. $\forall i \in \mathcal{I}$, $\llbracket \mathbf{g} \rrbracket_{\mathcal{M}_i}$ is the vector of jumps of \mathbf{g} across the traces in \mathcal{M}_i , sorted according to trace index:

$$\llbracket \mathbf{g} \rrbracket_{\mathcal{M}_i} := (\llbracket \mathbf{g} \rrbracket_{\Gamma_{\mathcal{M}_i(1)}}, \dots, \llbracket \mathbf{g} \rrbracket_{\Gamma_{\mathcal{M}_i(\mathcal{M}_i)}}) \in \prod_{m \in \mathcal{M}_i} H^{\frac{1}{2}}(\Gamma_m).$$

Similarly, given the diffusion parameter $\mu_i \in L^\infty(F_i)$ uniformly positive and a function $g_i \in V_i$, $\llbracket \mu_i \frac{\partial g_i}{\partial \mathbf{n}_{m,i}} \rrbracket_{\Gamma_m}$ is the jump of the co-normal derivative across Γ_m on F_i , and

$$\llbracket \boldsymbol{\mu} \frac{\partial \mathbf{g}}{\partial \mathbf{n}} \rrbracket_{\mathcal{M}_i} := \left(\llbracket \mu_i \frac{\partial g_i}{\partial \mathbf{n}_{\mathcal{M}_i(1),i}} \rrbracket_{\Gamma_{\mathcal{M}_i(1)}}, \dots, \llbracket \mu_i \frac{\partial g_i}{\partial \mathbf{n}_{\mathcal{M}_i(\mathcal{M}_i),i}} \rrbracket_{\Gamma_{\mathcal{M}_i(\mathcal{M}_i)}} \right) \in \prod_{m \in \mathcal{M}_i} H^{-\frac{1}{2}}(\Gamma_m),$$

47 is the tuple of the jumps of co-normal derivatives on the fracture F_i .

48 4. Problem Formulation

49 In this section, we briefly state the formulation of our optimization approach as
50 proposed in [1, 2]. We are interested in computing the hydraulic head distribution
51 within the DFN, that means we aim to solve local Darcy problems on each fracture,
52 coupled together by suitable matching conditions.

53 We denote the hydraulic head in Ω as $\mathbf{h} = (h_1, \dots, h_N) \in V$, where h_i is the hy-
54 draulic head on each fracture F_i . Let $\Gamma_D \subset \partial\Omega$ be the portion of the boundary of
55 Ω where a Dirichlet condition $\mathbf{g}_D \in \prod_{i \in \mathcal{I}} H^{\frac{1}{2}}(\Gamma_{D,i})$ is set, which we suppose to be
56 not empty, where $\Gamma_{D,i} = \Gamma_D \cap F_i$. Let Γ_N be the boundary with Neumann conditions
57 $\mathbf{g}_N \in \prod_{i \in \mathcal{I}} H^{-\frac{1}{2}}(\Gamma_{N,i})$, where $\Gamma_{N,i} = \Gamma_N \cap F_i$. We assume $\Gamma_N \cap \Gamma_D = \emptyset$ and $\Gamma_N \cup \Gamma_D = \partial\Omega$.
58 Furthermore, we denote by $V^D \subset V$ the sub-space of V of functions whose value on Γ_D
59 is g_D , V^0 the sub-space of functions with zero value on Γ_D .

For each $i \in \mathcal{I}$, the local hydraulic head h_i is the solution of the following varia-
tional problem: find $\mathbf{h} \in V^D$ such that, $\forall i \in \mathcal{I}$

$$(\mu_i \nabla h_i, \nabla v)_{F_i} = (f_i, v)_{F_i} + \left\langle \llbracket \mu_i \frac{\partial h_i}{\partial \mathbf{n}} \rrbracket_{\mathcal{M}_i}, \gamma_{\mathcal{M}_i}(v) \right\rangle_{\mathcal{M}_i} + \langle g_{N,i}, \gamma_{\Gamma_{N,i}}(v) \rangle_{\Gamma_{N,i}} \quad \forall v \in V_i^0, \quad (1)$$

where $f_i \in L^2(F_i)$ and $\mu_i \in L^\infty(F_i)$, $\forall i \in \mathcal{I}$. All the results are applicable to the case of uniformly positive definite tensors μ_i . The local problems are coupled together by some natural matching conditions, enforcing continuity of the hydraulic head and conservation of fluxes: $\forall m \in \mathcal{M}$, $\mathcal{I}_m = (i, j)$

$$\gamma_{\Gamma_m}^i(h_i) - \gamma_{\Gamma_m}^j(h_j) = 0, \quad (2)$$

$$\left[\left[\mu_i \frac{\partial h_i}{\partial \mathbf{n}_{m,i}} \right] \right]_{\Gamma_m} + \left[\left[\mu_j \frac{\partial h_j}{\partial \mathbf{n}_{m,j}} \right] \right]_{\Gamma_m} = 0. \quad (3)$$

60 4.1. Formulation as an optimization problem

61 Following [1, 2], problem (1)-(3) can be written equivalently as a PDE-constrained
62 minimization problem of a suitable functional representing the matching conditions.

Let $\alpha \in \mathbb{R}$ be fixed and let, $\forall m \in \mathcal{M}$, $\Lambda_m: H^{\frac{1}{2}}(\Gamma_m) \rightarrow H^{-\frac{1}{2}}(\Gamma_m)$ be the natural embedding from $H^{\frac{1}{2}}(\Gamma_m)$ to $H^{-\frac{1}{2}}(\Gamma_m)$. We define

$$\begin{aligned} \forall i \in \mathcal{I}, \forall m \in \mathcal{M}_i, u_i^m &:= \alpha \Lambda_m(\gamma_m^i(h_i)) + \left[\left[\mu_i \frac{\partial h_i}{\partial \mathbf{n}_{m,i}} \right] \right]_{\Gamma_m} \in H^{-\frac{1}{2}}(\Gamma_m), \\ \forall i \in \mathcal{I}, \mathbf{u}_i &:= (u_i^{M_i(1)}, \dots, u_i^{M_i(M_i)}) \in U_i := \prod_{m \in \mathcal{M}_i} H^{-\frac{1}{2}}(\Gamma_m), \\ \mathbf{u} &:= (\mathbf{u}_1, \dots, \mathbf{u}_N) \in U := \prod_{i \in \mathcal{I}} U_i. \end{aligned}$$

For any $\mathbf{w} \in U$ we set $\{\{\mathbf{w}\}\}_{\Gamma_m} = w_i^m + w_j^m$, $\forall m \in \mathcal{M}$ with $\mathcal{I}_m = (i, j)$ and denote by $\{\{\mathbf{w}\}\}_{\mathcal{M}_k}$ the vector whose k -th component is $\{\{\mathbf{w}\}\}_{\Gamma_{M_k(k)}}$. Finally, we denote by U_i^* and U^* the dual spaces of U_i and U , respectively and by $\Lambda: U^* \rightarrow U$ the vectorized embedding such that

$$\begin{aligned} \Lambda(\mathbf{v}) &= (\Lambda_{\mathcal{M}_1}(\mathbf{v}_1), \dots, \Lambda_{\mathcal{M}_N}(\mathbf{v}_N)) \\ &\text{where, } \forall i \in \mathcal{I}, \Lambda_{\mathcal{M}_i}(\mathbf{v}_i) = (\Lambda_{M_i(1)}(v_i), \dots, \Lambda_{M_i(M_i)}(v_i)) \in U_i. \end{aligned}$$

To define our functional, we first need to introduce the operator $\mathcal{H}: U \rightarrow V^D$, that associates to each vector $\mathbf{w} \in U$ a vector $\mathcal{H}(\mathbf{w}) = (h_1^w, \dots, h_N^w) \in V^D$ such that

$$\begin{aligned} (\mu_i \nabla h_i^w, \nabla v_i)_{F_i} + \alpha (\gamma_{\mathcal{M}_i}(h_i), \gamma_{\mathcal{M}_i}(v_i))_{\mathcal{M}_i} &= (f_i, v_i)_{F_i} + \langle \mathbf{w}_i, \gamma_{\mathcal{M}_i}(v_i) \rangle_{\mathcal{M}_i} \\ &+ \langle g_{N,i}, \gamma_{\Gamma_{N,i}}(v_i) \rangle_{\Gamma_{N,i}} \quad \forall i \in \mathcal{I}, \forall \mathbf{v} \in V^0. \quad (4) \end{aligned}$$

63 Using the above operator, we define, for each $m \in \mathcal{M}$, the functional $J_m: U \rightarrow \mathbb{R}$

$$J_m(\mathbf{w}) = \left\| \left[\mathcal{H}(\mathbf{w}) \right]_{\Gamma_m} \right\|_{\frac{1}{2}, \Gamma_m}^2 + \left\| \left\{ \left\{ \mathbf{w} - \alpha \Lambda(\gamma_{\mathcal{M}}(\mathcal{H}(\mathbf{w}))) \right\} \right\}_{\Gamma_m} \right\|_{-\frac{1}{2}, \Gamma_m}^2. \quad (5)$$

64 Then, problem (1)-(3) can be seen as a constrained optimization problem:

$$\text{find } \mathbf{u} \in U \text{ such that } \mathbf{u} = \arg \min_{\mathbf{w} \in U} J(\mathbf{w}), \quad \text{where } J(\mathbf{u}) = \sum_{m \in \mathcal{M}} J_m(\mathbf{u}). \quad (6)$$

The well-posedness of (6) and its equivalence with problem (1)-(3) is proved in [2]. Introducing the Lagrange multiplier $\mathbf{p} \in V$, it can be proved that the solution \mathbf{u} to (6), satisfies the following system:

$$\left\langle \left\{ \mathbf{u} - \alpha \Lambda \left(\gamma_{\mathcal{M}}(\mathbf{h}) \right) \right\}_{\mathcal{M}_i}, \mu_i \right\rangle_{\mathcal{M}_i} = - \left(\gamma_{\mathcal{M}_i}(p_i), \mu_i \right)_{\mathcal{M}_i} \quad \forall \mu_i \in U_i^*, \quad (7)$$

$$\begin{aligned} (\mu_i \nabla p_i, \nabla v_i)_{F_i} + \alpha \left(\gamma_{\mathcal{M}_i}(p_i), \gamma_{\mathcal{M}_i}(v_i) \right)_{\mathcal{M}_i} &= \left(\llbracket \mathbf{h} \rrbracket_{\mathcal{M}_i}, \gamma_{\mathcal{M}_i}(v_i) \right)_{\mathcal{M}_i} \\ + \left\langle \alpha \left\{ \alpha \Lambda \left(\gamma_{\mathcal{M}}(\mathbf{h}) \right) - \mathbf{u} \right\}_{\mathcal{M}_i}, \gamma_{\mathcal{M}_i}(v_i) \right\rangle_{\mathcal{M}_i} &\quad \forall v_i \in V_i^0, \end{aligned} \quad (8)$$

$$\begin{aligned} (\mu_i \nabla h_i, \nabla v_i)_{F_i} + \alpha \left(\gamma_{\mathcal{M}_i}(h_i), \gamma_{\mathcal{M}_i}(v_i) \right)_{\mathcal{M}_i} &= (f_i, v_i)_{F_i} + \left\langle \mathbf{u}_i, \gamma_{\mathcal{M}_i}(v_i) \right\rangle_{\mathcal{M}_i} \\ + \left\langle g_{N,i}, \gamma_{\Gamma_{N,i}}(v_i) \right\rangle_{\Gamma_{N,i}} &\quad \forall v_i \in V_i^0. \end{aligned} \quad (9)$$

65 From (2) - (3), we see that $\llbracket \mathbf{h} \rrbracket_{\mathcal{M}}$ is the null vector as well as $\left\{ \mathbf{u} - \alpha \Lambda \left(\gamma_{\mathcal{M}}(\mathbf{h}) \right) \right\}_{\mathcal{M}}$,
66 therefore, the exact solution of (8) corresponds to $p_i = 0 \forall i \in \mathcal{I}$.

67 4.2. Discretization of the optimization problem

68 The optimization problem (6) can be discretized using local finite element spaces
69 on each fracture, defined on local triangulations that are completely independent of
70 trace positions.

71 Let $\mathcal{T}_{\delta i}$ be a regular triangulation of F_i , and $\mathcal{T}_{\delta} = \bigcup_{i \in \mathcal{I}} \mathcal{T}_{\delta i}$, δ being a mesh pa-
72 rameter characterizing the mesh size. Let us define the finite dimensional subspaces
73 $V_{\delta i}^D \subset V_i^D$ and $V_{\delta i}^0 \subset V_i^0$, $\forall i \in \mathcal{I}$. Moreover, given a partition of each Γ_m in subinter-
74 vals, let $U_{\delta i}^m \subset L^2(\Gamma_m) \subset H^{-\frac{1}{2}}(\Gamma_m) = U_i^m$, $\forall i \in \mathcal{I}$, $\forall m \in \mathcal{M}_i$ be given and let us set
75 $U_{\delta i} := \prod_{m \in \mathcal{M}_i} U_{\delta i}^m$, $\forall i \in \mathcal{I}$, $U_{\delta} := \prod_{i \in \mathcal{I}} U_{\delta i}$, $V_{\delta}^D := \prod_{i \in \mathcal{I}} V_{\delta i}^D$, $V_{\delta}^0 := \prod_{i \in \mathcal{I}} V_{\delta i}^0$.

Let $\mathcal{H}_{\delta}: U_{\delta} \rightarrow V_{\delta}^D$ such that the components of $\mathcal{H}_{\delta}(\mathbf{w}) = (h_{\delta 1}^w, \dots, h_{\delta N}^w)$ are the solutions of

$$\begin{aligned} (\mu_i \nabla h_{\delta i}^w, \nabla v_{\delta i})_{F_i} + \alpha \left(\gamma_{\mathcal{M}_i}(h_{\delta i}^w), \gamma_{\mathcal{M}_i}(v_{\delta i}) \right)_{\mathcal{M}_i} &= (f_i, v_{\delta i})_{F_i} + \left(\mathbf{w}_i, \gamma_{\mathcal{M}_i}(v_{\delta i}) \right)_{\mathcal{M}_i} \\ + \left\langle g_{N,i}, \gamma_{\Gamma_{N,i}}(v_{\delta i}) \right\rangle_{\Gamma_{N,i}} &\quad \forall i \in \mathcal{I}, \forall \mathbf{w}_{\delta} \in V_{\delta}^0. \end{aligned} \quad (10)$$

76 Then, we can define the following ‘‘discrete’’ functional $J_{\delta m}: U_{\delta} \rightarrow \mathbb{R}$ that approxi-
77 mates (5):

$$J_{\delta m}(\mathbf{w}_{\delta}) = \left\| \llbracket \mathcal{H}_{\delta}(\mathbf{w}_{\delta}) \rrbracket_{\Gamma_m} \right\|_{\Gamma_m}^2 + \left\| \left\{ \mathbf{w}_{\delta} - \alpha \Lambda \left(\gamma_{\mathcal{M}}(\mathcal{H}_{\delta}(\mathbf{w}_{\delta})) \right) \right\}_{\Gamma_m} \right\|_{\Gamma_m}^2, \quad (11)$$

78 and write the following discrete version of problem (6)

$$\text{find } \mathbf{u}_{\delta} \in U_{\delta} \text{ such that } \mathbf{u}_{\delta} = \arg \min_{\mathbf{w}_{\delta} \in U_{\delta}} J_{\delta}(\mathbf{w}_{\delta}), \quad \text{where } J_{\delta}(\mathbf{u}_{\delta}) = \sum_{m \in \mathcal{M}} J_{\delta m}(\mathbf{u}_{\delta}). \quad (12)$$

The well-posedness of problem (12) is proved in [2], where it is also seen to be equiv-

alent to the following problem: find $\mathbf{h}_\delta \in V_\delta^D$, $\mathbf{p}_\delta \in V_\delta^0$, $\mathbf{u}_\delta \in U_\delta$ such that, $\forall i \in \mathcal{I}$,

$$\left(\left\{ \left\{ \mathbf{u}_\delta - \alpha \gamma_{\mathcal{M}}(\mathbf{h}_\delta) \right\} \right\}_{\mathcal{M}_i}, \mu_{\delta i} \right)_{\mathcal{M}_i} = - \left(\gamma_{\mathcal{M}_i}(p_{\delta i}), \mu_{\delta i} \right)_{\mathcal{M}_i} \quad \forall \mu_{\delta i} \in U_{\delta i}, \quad (13)$$

$$\left(\mu_i \nabla p_{\delta i}, \nabla v_{\delta i} \right)_{F_i} + \alpha \left(\gamma_{\mathcal{M}_i}(p_{\delta i}), \gamma_{\mathcal{M}_i}(v_{\delta i}) \right) = \left(\llbracket \mathbf{h}_\delta \rrbracket \right)_{\mathcal{M}_i} + \alpha \left(\left\{ \left\{ \alpha \mathbf{h}_\delta - \mathbf{u}_\delta \right\} \right\}_{\mathcal{M}_i}, \gamma_{\mathcal{M}_i}(v_{\delta i}) \right)_{\mathcal{M}_i} \quad (14)$$

$$\forall v_{\delta i} \in V_{\delta i}^0,$$

$$\left(\mu_i \nabla h_{\delta i}, \nabla v_{\delta i} \right)_{F_i} + \alpha \left(\gamma_{\mathcal{M}_i}(h_{\delta i}), \gamma_{\mathcal{M}_i}(v_{\delta i}) \right)_{\mathcal{M}_i} = \left(f_i, v_{\delta i} \right)_{F_i} + \left(\mathbf{u}_{\delta i}, \gamma_{\mathcal{M}_i}(v_{\delta i}) \right)_{\mathcal{M}_i} \quad (15)$$

$$+ \left\langle g_{N,i}, \gamma_{\Gamma_{N,i}}(v_{\delta i}) \right\rangle_{\Gamma_{N,i}} \quad \forall v_{\delta i} \in V_{\delta i}^0.$$

79 The above equivalent formulation is then used in our a posteriori error analysis.

80 5. Error estimates

81 Before proceeding to the error estimate, we need to establish some useful notation.
 82 Following [3], we denote by \mathcal{E}_δ the set of edges of the triangles in \mathcal{T}_δ . $\mathcal{E}_{\delta i}$ is the
 83 subset of \mathcal{E}_δ containing the edges defined on fracture F_i . For all $T \in \mathcal{T}_\delta$, let T be the
 84 interior of T , and we denote by \mathcal{M}_T the set of indices of those traces having non empty
 85 intersection with T , and define $\ell_T^m = \Gamma_m \cap T$, $\forall m \in \mathcal{M}_T$. Moreover, $\forall \sigma \in \mathcal{E}_\delta$ we indicate
 86 by \mathcal{M}_σ the set of those $m \in \mathcal{M}$ such that $|\Gamma_m \cap \sigma| \neq \emptyset$. On each trace Γ_m , let $\mathcal{I}_m = (i, j)$
 87 be the couple of the two fracture indices sharing Γ_m , we fix two discretizations $\Lambda_{m,i}$
 88 and $\Lambda_{m,j}$ associated to the two fractures F_i and F_j , respectively. Finally, the symbol $h_\#$
 89 denotes the diameter of an arbitrary geometrical object $\#$.

To introduce our error measure, we first define the quantities

$$e_{\mathbf{h}} = \mathbf{h} - \mathbf{h}_\delta, \quad e_{\mathbf{p}} = \mathbf{p}_\delta, \quad e_{\mathbf{u}} = \mathbf{u} - \mathbf{u}_\delta. \quad (16)$$

90 Then, the error we want to estimate is the quantity

$$err := \left[\sum_{i \in \mathcal{I}} \left(\|\mu_i \nabla e_{\mathbf{h}}\|_{F_i}^2 + \|\mu_i \nabla e_{\mathbf{p}}\|_{F_i}^2 + \|e_{\mathbf{u}}\|_U^2 \right) \right]^{\frac{1}{2}}. \quad (17)$$

91 In [3], upper and lower bounds for this error measure were proved in the case where
 92 each fracture has Dirichlet boundary conditions that allows $\alpha = 0$. The extension to
 93 the more general case presented here is straightforward and leads to the definition of
 94 the following error fracture-based **local** estimators $\forall i \in \mathcal{I}$:

95 **Residual estimator for the Darcy problem:** $\forall T \in \mathcal{T}_{\delta i}$,

$$\eta_{H,T} := h_T \|f_i + \nabla \cdot (\mu_i \nabla h_{\delta i})\|_T, \quad (18)$$

Estimator for the approximation of the flux through edges: $\forall \sigma \in \mathcal{E}_{\delta i}$, if $\sigma \notin \Gamma_{N,i}$,

$$\xi_{F,\sigma} := (h_\sigma)^{\frac{1}{2}} \left\| \left[\left[\mu_i \frac{\partial h_{\delta i}}{\partial \mathbf{n}_\sigma} \right] \right]_\sigma - \tilde{u}_{\delta i,\sigma} \right\|_\sigma$$

$$\text{where } \tilde{u}_{\delta i,\sigma} := \begin{cases} u_{\delta i}^m - \alpha \gamma_{\Gamma_m}^i(h_{\delta i}) & \text{on } \Gamma_m \cap \sigma, \forall m \in \mathcal{M}_\sigma, \\ 0 & \text{elsewhere.} \end{cases} \quad (19)$$

96 If $\sigma \in \Gamma_{N,i}$,

$$\xi_{F,\sigma} := (h_\sigma)^{\frac{1}{2}} \left\| \left[\left[\mu_i \frac{\partial h_{\delta i}}{\partial \mathbf{n}_\sigma} \right] \right]_\sigma - g_{N,i} \right\|_\sigma. \quad (20)$$

97 **Estimator for the nonconformity of the discretization:** $\forall T \in \mathcal{T}_{\delta i}, \forall m \in \mathcal{M}_T$,

$$\xi_{NC,T}^m := (h_{\ell_T^m})^{\frac{1}{2}} \|u_{\delta i}^m - \alpha \gamma_m^j(h_{\delta i})\|_{\ell_T^m}. \quad (21)$$

98 **Estimator for the pressure induced by discontinuity:** $\forall T \in \mathcal{T}_{\delta i}$,

$$\eta_{P,T} := \|\mu_i \nabla p_{\delta i}\|_T. \quad (22)$$

99 **Estimator for the pressure induced by the unbalancing of fluxes:** $\forall m \in \mathcal{M}_i, \forall \lambda \in \Lambda_{m,i}$,

$$\xi_{P,\lambda}^i := h_\lambda^{\frac{1}{2}} \|\gamma_{\Gamma_m}^j(p_{\delta i})\|_\lambda. \quad (23)$$

101 **Estimator of the minimization error:** $\forall m \in \mathcal{M}_i, \lambda \in \Lambda_{m,i}$,

$$J_{\delta,\lambda}(\mathbf{u}_\delta) := h_\lambda \left((1 + \alpha)^2 \|\llbracket \mathbf{u}_\delta - \alpha \mathbf{h}_\delta \rrbracket_{\Gamma_m} \rrbracket_\lambda^2 + \|\llbracket \mathbf{h}_\delta \rrbracket_{\Gamma_m} \rrbracket_\lambda^2 \right). \quad (24)$$

Using the above quantities, we define our error estimate as follows:

$$\begin{aligned} est_\delta := & \left\{ \sum_{i \in \mathcal{I}} \left[\sum_{T \in \mathcal{T}_{\delta i}} \left(\eta_{H,T}^2 + \eta_{P,T}^2 + \sum_{m \in \mathcal{M}_T} (\xi_{NC,T}^m)^2 \right) + \sum_{\sigma \in \mathcal{E}_\delta} (\xi_{F,\sigma}^m)^2 \right. \right. \\ & \left. \left. + \sum_{m \in \mathcal{M}_i} \sum_{\lambda \in \Lambda_{m,i}} \left((\xi_{P,\lambda}^i)^2 + J_{\delta,\lambda}(\mathbf{u}_\delta) \right) \right] \right\}^{\frac{1}{2}}. \quad (25) \end{aligned}$$

102 Then, we can prove the following results, stating the equivalence between the error and
103 the error estimator.

Theorem 1. *Let err be defined by (17) and est_δ by (25). Then, there exists a constant $C^* > 0$ independent of δ such that*

$$err \leq C^* est_\delta.$$

Moreover, there exists a constant $C_* > 0$ independent of δ such that

$$est_\delta \leq C_* \left\{ \|\llbracket e_{\mathbf{p}} \rrbracket\|^2 + C_{NC} \left[\max_{\sigma \in \mathcal{E}_\delta} \{1, h_\sigma\} (\|\llbracket e_{\mathbf{h}} \rrbracket\|^2 + \|e_{\mathbf{u}}\|_U^2) + \max_{\substack{i \in \mathcal{I} \\ T \in \mathcal{T}_{\delta i}}} h_T^2 \|f_i - f_T\|_T^2 \right] \right\}^{\frac{1}{2}},$$

where f_T is the integral mean of f_i on a triangle $T \in \mathcal{T}_{\delta i}$ and

$$C_{NC} := \max_{T \in \mathcal{T}_\delta} \left(\frac{1 + h_{NC,T}}{1 - h_{NC,T}} \right).$$

104 *Proof.* The proof is omitted since it follows the same techniques used in [3] with some
105 straightforward extensions due to presence of the additional terms involved by the more
106 general boundary conditions that require $\alpha \neq 0$. \square

107 **6. Adaptive algorithm**

108 In this section we describe an adaptive algorithm exploiting the numerical estimate
 109 obtained above. The aim of the procedure is to reduce the error estimator under a given
 110 percentage τ_p of the initial estimator.

111 For each $i \in \mathcal{I}$, and for any triangle $T \in \mathcal{T}_{\delta,i}$, we define the local estimator $est_{\delta,T}$ as
 112 follows:

$$est_{\delta,T} := \left[\eta_{H,T}^2 + \eta_{P,T}^2 + \sum_{\sigma \in \mathcal{E}_\delta} (\xi_{F,\sigma})^2 + \sum_{m \in \mathcal{M}_T} \sum_{\lambda \in \Lambda_{m,i}} \left((\xi_{NC,T}^m)^2 + (\xi_{P,\ell_T^m}^i)^2 + J_{\delta,\lambda \cap T}(\mathbf{u}_\delta) \right) \right]^{\frac{1}{2}}. \quad (26)$$

113 The adaptive algorithm we chose is based on the equidistribution [42] of the error on
 114 the triangles and is as follows:

- 115 1. choose an estimate reduction threshold $\tau_p \in (0.0, 1.0]$;
- 116 2. set $k = 0$, choose an initial mesh parameter δ_k and build \mathcal{T}_{δ_k} independently on
 117 each fracture and independently of the traces;
- 118 3. do

- 119 **SOLVE** the problem on \mathcal{T}_{δ_k} ;
- 120 **ESTIMATE** the error by computing est_{δ_k} ;
- 121 **MARK** triangles $T \in \mathcal{T}_{\delta_k}$ such that $est_{\delta_k,T} > \frac{est_{\delta_k}}{\#\mathcal{T}_\delta}$ for refinement;
- 122 **REFINE** marked triangles by a chosen constant factor c , obtaining $\mathcal{T}_{\delta_{k+1}}$;
- 123 Increase k ;
- 124 while $est_{\delta_k}^2 > \tau_p est_{\delta_0}^2$ and $0 \leq k \leq k_{\max}$.

125 The reader may refer to [2, 37] for a detailed description of how the minimization
 126 process can be performed using the conjugate gradient method, also in parallel form.

127 **7. Numerical results**

128 In the following sections we apply the adaptive algorithm previously described to
 129 some DFNs with variable complexity in order to assess the reliability of the numerical
 130 solution and its viability to realistic DFNs. First, we consider two test cases with
 131 known solution and compute the effectivity indices at each refinement step to verify
 132 it is independent of the mesh size, then we investigate the behaviour of the adaptive
 133 method on some more realistic test case.

134 In all the following tests, the partitions $\Lambda_{m,i}$ and $\Lambda_{m,j}$ for the discretization of u_{δ_i}
 135 and u_{δ_j} on trace Γ_m such that $\mathcal{I}_m = (i, j)$ are staggered uniform discretizations with a
 136 number of intervals that is the mean value of the number of intersections between Γ_m
 137 and the triangulations \mathcal{T}_{δ_i} and \mathcal{T}_{δ_j} . The discrete spaces used for \mathbf{h}_δ and \mathbf{p}_δ are linear
 138 finite elements, and a discontinuous linear finite element discretization is used for \mathbf{u}_δ
 139 on the traces. Table 1 reports the parameters used for each of the following simula-
 140 tions: the parameter δ_0 is the minimum number of cells required on each fracture at the
 141 beginning of the refinement process, CG-MaxIt is the maximum number of iterations
 142 allowed for the conjugate gradient and CG-ResTo1 is the tolerance set on the relative
 143 residual.

Table 1: Parameters used for the simulations

	τ_p	δ_0	CG-MaxIt	CG-ResTol	c	k_{max}
Problem 1	0.025	100	80000	10^{-12}	3	20
Problem 2	0.025	100	80000	10^{-12}	3	20
Frac6	0.025	100	80000	10^{-12}	3	20
Frac86	0.025	100	80000	10^{-6}	3	20
Frac15K	0.025	100	80000	10^{-6}	3	30

144 *7.1. Numerical Results for Problems with known solutions*

The first set of tests aims at validating the equivalence relation between the error and the error estimator during the adaptive process. In order to compute the error, we consider two simple DFNs on which we can define a problem whose exact solution is known. We apply to these problems our adaptive algorithm and compare at each refinement iteration the error and the error estimator computing the *effectivity index*

$$\varepsilon = \frac{err}{est_\delta},$$

145 and verify that it is independent of the mesh obtained by adaptive refinements. See [3,
146 Section 7] for the same analysis performed with meshes obtained by uniform refine-
147 ment.

148 *7.1.1. Problem 1*

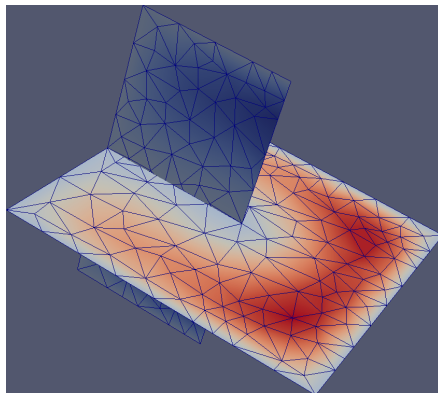
149 The geometry and the parameters of this test problem are described in detail in [34,
150 Section 5.1]. In Figure 1 we report some of the meshes generated by the refinement
151 algorithm, with the parameters reported in Table 1. We remark that in this test problem
152 the presence of a known forcing function on the fractures induces a refinement in the
153 whole domain, this will not be the same for the flow simulations in which the DFNs is
154 considered impervious (Section 7.2).

155 In Figure 2a we report the convergence behaviour of the error and of the error esti-
156 mator and the final rates of convergence α with respect to the total number of degrees of
157 freedom $est_\delta \sim (\#\text{Dofs})^\alpha$ computed on the basis of the last three refinement iterations.
158 We can clearly appreciate a parallel behaviour of error and error estimator. In Figure
159 2b we report the behaviour of the standard deviation of the local error estimators $est_{\delta,T}$,
160 defined by

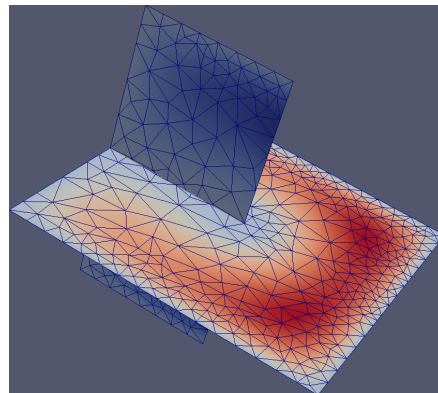
$$\sigma = \sqrt{\frac{\sum_{T \in \mathcal{T}_\delta} (est_{\delta,T} - \overline{est_\delta})^2}{\#\mathcal{T}_\delta}}, \quad \text{being } \overline{est_\delta} = \frac{est_\delta}{\#\mathcal{T}_\delta}. \quad (27)$$

161 From the decreasing behaviour of this plot we can appreciate that the error has been
162 efficiently equidistributed by the refinement algorithm. In Table 2 we report the most
163 significant quantities to describe the refinement process: Ω -Cells is the total number
164 of triangles on the fractures, Γ -Cells is the total number of sub-intervals on the traces
165 for the discretization of the control variable \mathbf{u}_δ , Dofs is the total numbers of degrees
166 of freedom of the problem, ε is the effectivity index, σ is the standard deviation of the
167 local estimator, CG-It is the number of conjugate gradient iterations performed and J

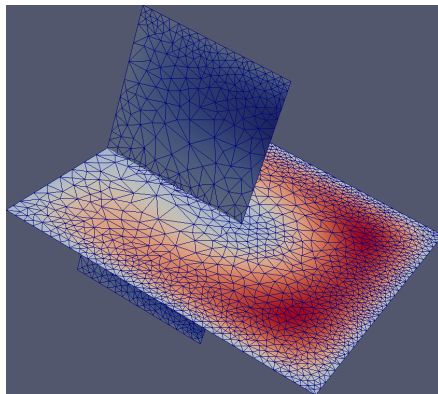
Figure 1: Problem 1. DFN with meshes at some refining steps.



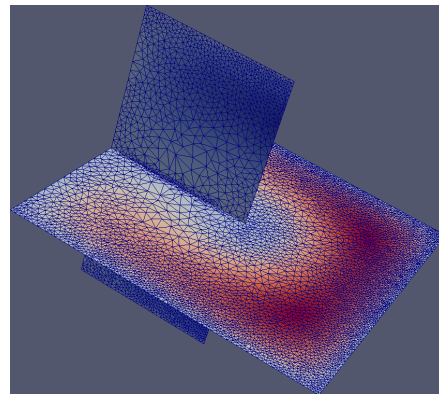
(a) Initial mesh



(b) Step 3



(c) Step 6



(d) Step 9

Figure 2: Problem 1. Plots

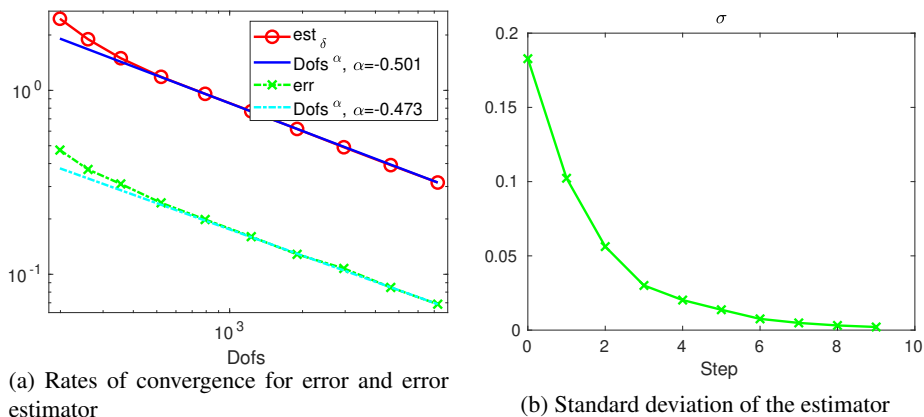


Table 2: Problem 1. Data during the refinement process

Step	Ω -Cells	Γ -Cells	Dofs	ε	σ	CG-It	J
0	305	10	199	0.1933	1.83E-01	27	2.41E-05
1	417	10	260	0.1959	1.02E-01	27	3.13E-05
2	589	10	355	0.2073	5.62E-02	27	4.79E-05
3	891	12	521	0.2065	3.01E-02	33	2.99E-05
4	1393	15	792	0.2073	2.02E-02	43	1.93E-05
5	2215	19	1228	0.2081	1.37E-02	55	5.92E-06
6	3500	26	1899	0.2081	7.56E-03	79	1.61E-06
7	5556	28	2962	0.2193	4.84E-03	85	1.54E-06
8	8790	34	4628	0.2163	3.14E-03	106	7.87E-07
9	13871	52	7234	0.2180	2.09E-03	177	1.94E-07

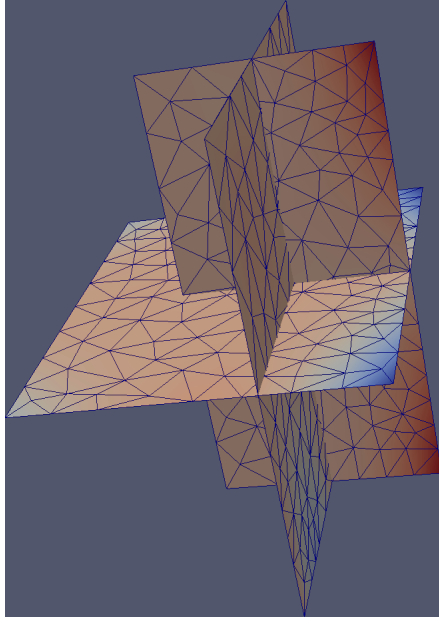
168 if the value of the functional at the last iteration. We highlight the very small variations
 169 of ε with respect to the large variations of number of cells and degrees of freedom.

170 *7.1.2. Problem 2*

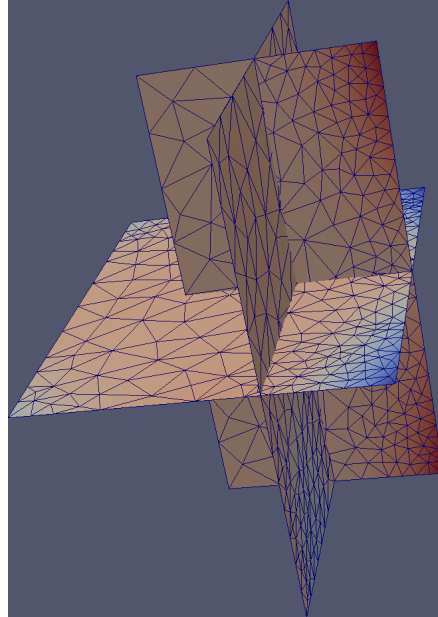
171 The geometry and the parameters of this test problem are described in detail in [18,
 172 Section 6.1]. In Figure 3, we report some of the meshes generated by the refinement
 173 process, with the parameters reported in Table 1. Similarly to the previous test, a
 174 refinement is induced also far from traces due to the presence of a non null forcing
 175 term.

176 In Figure 4a we report the convergence behaviour of the error and of the error
 177 estimator with the final rates of convergence (α), computed on the basis of the last
 178 three refinement iterations: again we have a very good agreement between them. In
 179 Figure 4b we report the behaviour of the standard deviation of the elemental error
 180 estimator, defined by (27). Like for the previous example, also in this case we see that
 181 the error is efficiently equidistributed. In Table 3 we report the same quantities reported

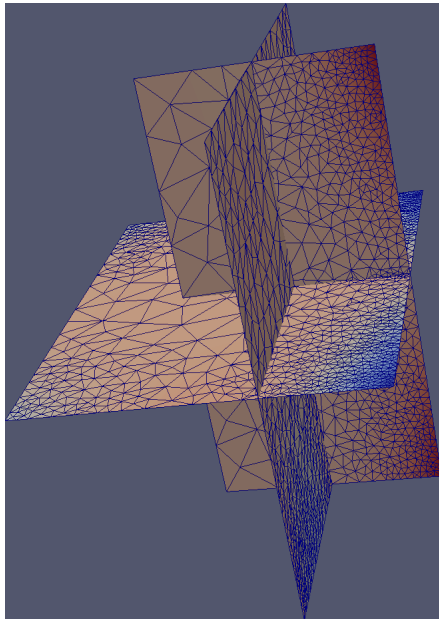
Figure 3: Problem 2. DFN with meshes at some refining steps



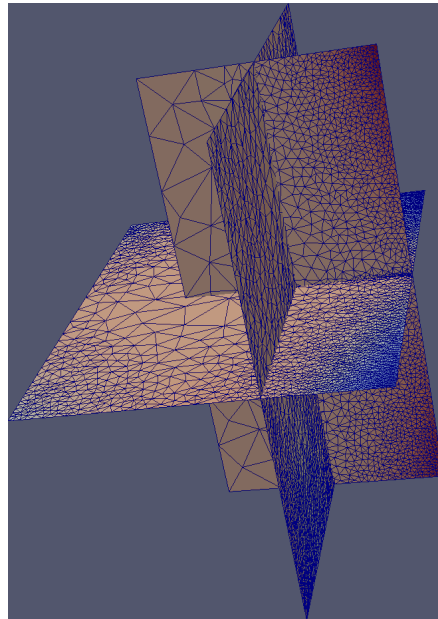
(a) Initial mesh



(b) Step 3



(c) Step 6



(d) Step 8

Figure 4: Problem 2. Plots

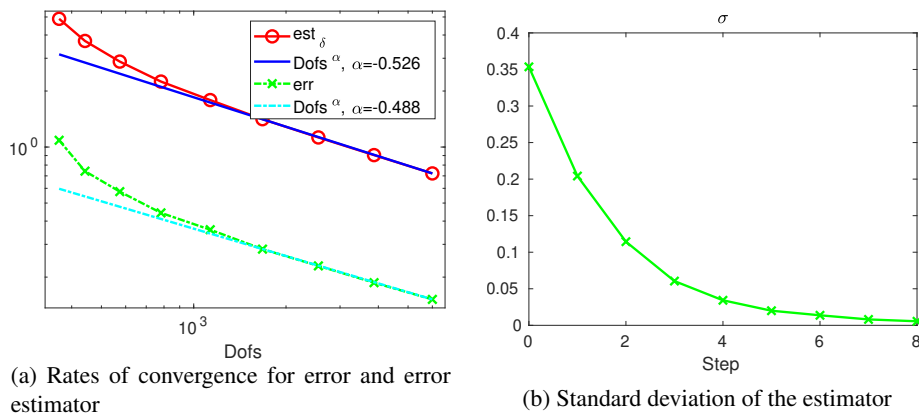


Table 3: Problem 2. Data during the refinement process

Step	Ω -Cells	Γ -Cells	Dofs	ε	σ	CG-It	J
0	458	47	364	0.2225	3.54E-01	151	8.45E-04
1	594	47	442	0.1997	2.04E-01	150	4.50E-04
2	842	47	574	0.2000	1.14E-01	150	6.81E-04
3	1212	49	782	0.1970	6.05E-02	160	3.27E-04
4	1867	51	1132	0.2011	3.42E-02	164	3.19E-04
5	2902	54	1676	0.2003	2.00E-02	171	3.39E-04
6	4566	58	2550	0.2041	1.37E-02	194	3.04E-04
7	7095	70	3873	0.2063	8.07E-03	232	3.02E-04
8	11209	82	6001	0.2104	5.52E-03	283	2.68E-04

182 in Table 2 to describe the refinement process. We observe again that the effectivity
 183 index is independent of the meshsize.

184 *7.2. Numerical Results on DFNs*

185 This second set of tests is used to show the behaviour of the adaptive algorithm
 186 when applied to DFN flow simulations with increasing geometrical complexities. In
 187 Table 4 we summarize the geometrical features of the networks that we consider. In
 188 these cases, we do not impose any forcing term, and the flux is driven by the presence
 189 of two Dirichlet boundary conditions imposed on two sides of two different fractures,
 190 while homogeneous Neumann boundary conditions are imposed on all the other bound-
 191 aries.

192 *7.2.1. Frac6*

193 The first DFN we consider is made up of six fractures that intersect each other
 194 orthogonally, as depicted in Figure 5, where we display the meshes generated by the

Table 4: Number of fractures and traces for each DFN

Problem	#Fractures	#Traces	Min #Tr	Max #Tr	$\frac{\#Traces}{\#Fractures}$
Frac6	6	6	1	4	1
Frac86	86	159	1	11	1.85
Frac15K	15102	27952	1	243	1.85

Table 5: Frac6. Data during the refinement process

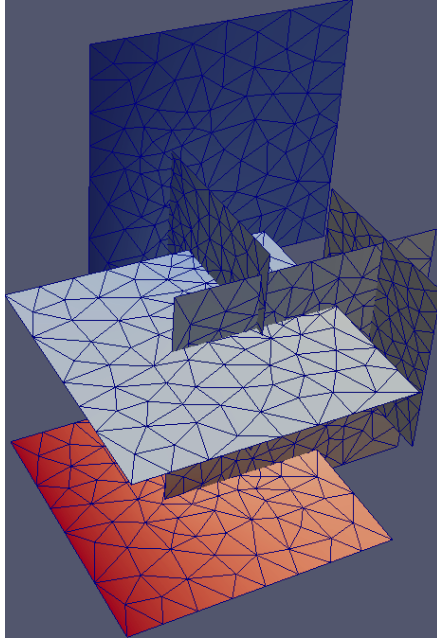
Step	Ω -Cells	Γ -Cells	Dofs	σ	CG-It	J
0	873	68	653	3.91E-01	136	1.04E-01
1	1097	76	783	2.68E-01	156	5.75E-02
2	1461	90	998	2.04E-01	192	3.47E-02
3	1956	112	1297	1.43E-01	225	2.05E-02
4	2689	140	1728	1.00E-01	295	1.38E-02
5	3783	173	2353	7.92E-02	388	1.14E-02
6	5339	237	3271	6.37E-02	548	7.28E-03
7	7836	295	4657	4.16E-02	605	5.37E-03
8	11111	402	6525	2.96E-02	922	4.18E-03
9	16161	512	9295	1.96E-02	1100	2.77E-03
10	23683	689	13437	1.45E-02	1365	1.79E-03
11	34569	977	19506	1.00E-02	1905	1.23E-03
12	50711	1295	28256	7.08E-03	2409	7.78E-04

195 adaptive process, with the parameters reported in Table 1. As expected, since the forc-
 196 ing term is null, the refinement is mainly concentrated in the neighbourhood of traces,
 197 while in the interior of fractures the refinement follows the flux stream: the reader can
 198 observe a non uniform refinement on the highest horizontal fracture, being the flux
 199 essentially flowing between the two intersecting fractures. In Figure 6a we report the
 200 convergence behaviour of the error estimator with the final rate of convergence (α),
 201 computed on the basis of the last three refinement steps. In Figure 6b we report the
 202 behaviour of the standard deviation of the elemental error estimator. In Table 5 we
 203 report the characteristic quantities of the refinement process as in the previous cases.

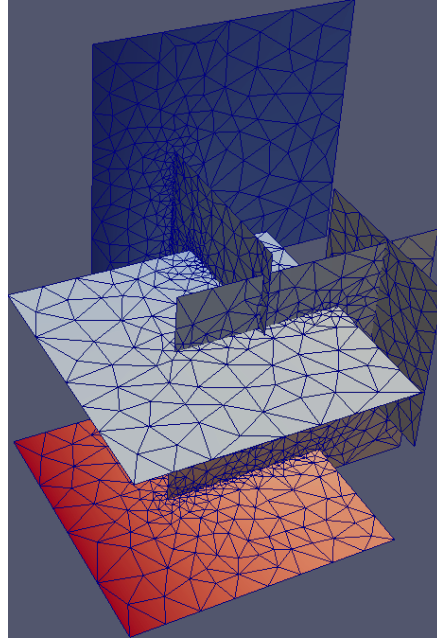
204 7.2.2. Frac86

205 This second network has 86 fracture and is more dense than the previous one, with
 206 an average of 1.85 traces per fracture (see Table 4). As for the previous tests, we
 207 show in Figure 7 the convergence graphs and in Table 6 the parameters describing
 208 the behaviour of the adaptive process. The results are in agreement with the previous
 209 considerations, moreover, from Table 6, we can also appreciate a good effect of the
 210 adaptively refined meshes on the conjugate gradient solver, being the number of iter-
 211 ations required to satisfy the relative tolerance (column CG-It) quite stable after the
 212 first four iterations also with an increasing number of degrees of freedom of the control
 213 variable (column Γ -Cells).

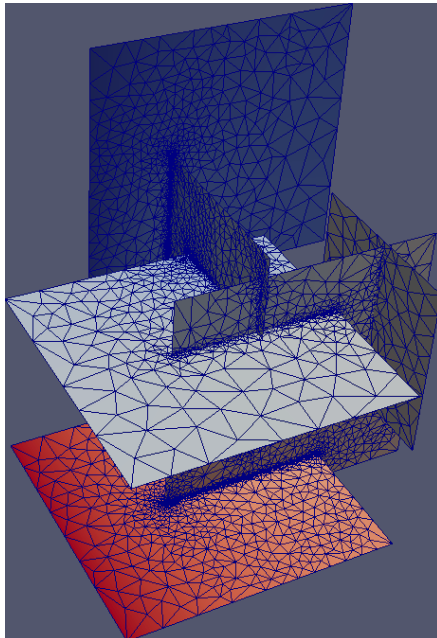
Figure 5: Frac6. DFN with meshes at some refining steps



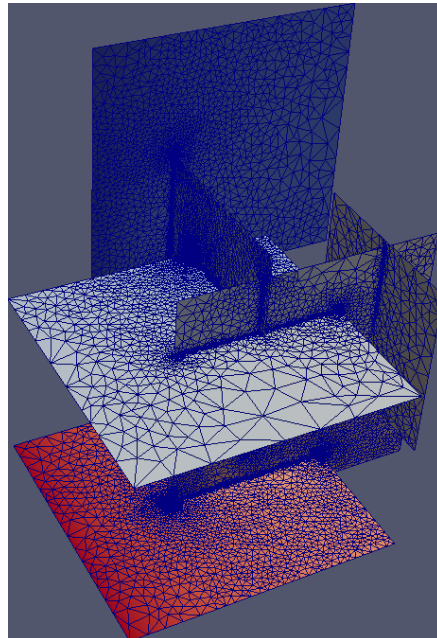
(a) Initial mesh



(b) Step 4



(c) Step 8



(d) Step 12

Figure 6: Frac6. Plots

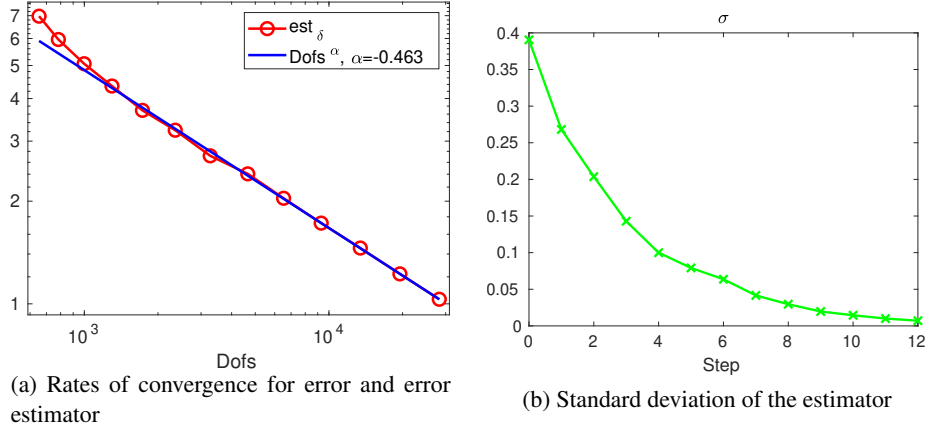


Figure 7: Frac86. Plots

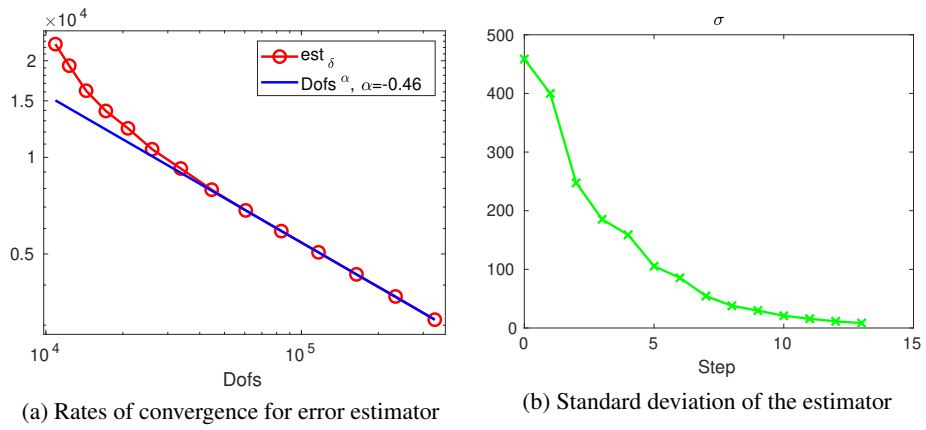


Figure 8: Frac15K. Full DFN

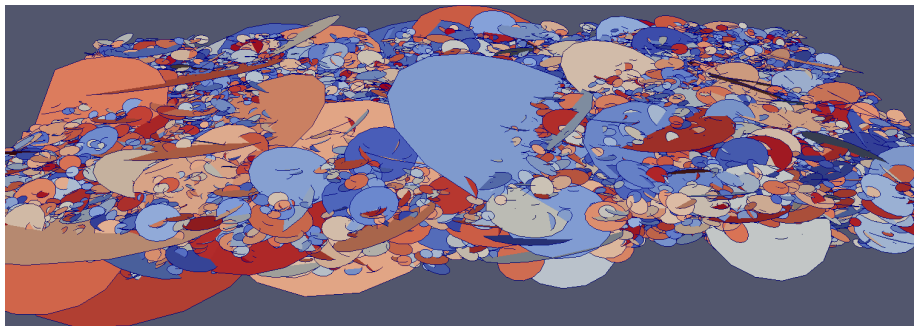


Table 6: Frac86. Data during the refinement process

Step	Ω -Cells	Γ -Cells	Dofs	σ	CG-It	J
0	13765	1354	10890	4.59E+02	9722	7.04E-01
1	16093	1476	12335	4.00E+02	10257	6.58E-01
2	19373	1636	14362	2.47E+02	5508	4.99E-01
3	24050	1826	17156	1.85E+02	2909	3.87E-01
4	30459	2073	20950	1.59E+02	2206	3.04E-01
5	39252	2369	26043	1.05E+02	2157	2.52E-01
6	52592	2788	33705	8.55E+01	1995	1.93E-01
7	71656	3361	44557	5.44E+01	1712	1.59E-01
8	99618	4240	60530	3.79E+01	1619	1.23E-01
9	140479	5345	83449	2.97E+01	1661	9.09E-02
10	199936	6919	116680	2.09E+01	1742	6.84E-02
11	285895	8979	164218	1.58E+01	1719	5.00E-02
12	411647	11972	233611	1.13E+01	1813	3.76E-02
13	593352	15760	332707	8.38E+00	2270	2.56E-02

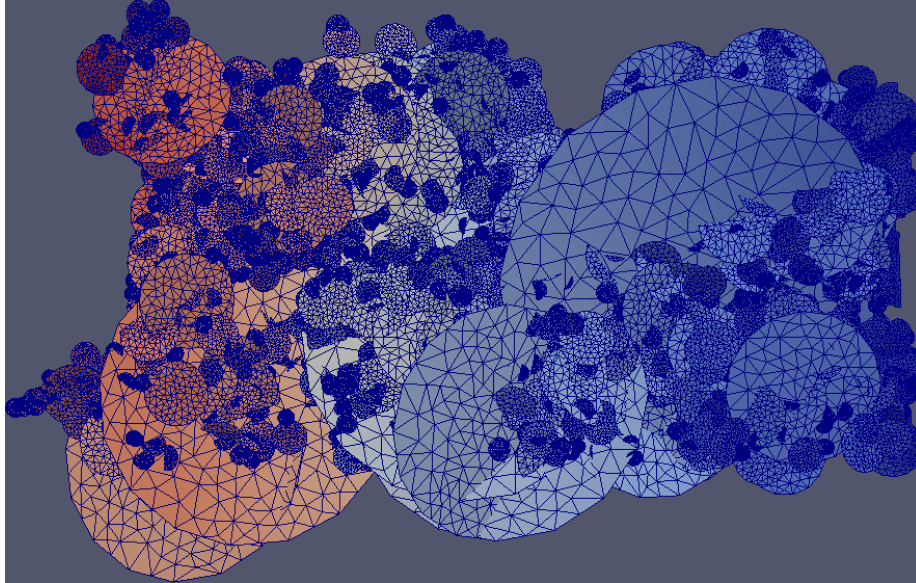
214 7.2.3. *Frac15K*

215 The last network that we analyse is depicted in Figure 8 and it is characterized by
216 many geometrical complexities, such as small fractures intersecting very larger ones,
217 almost parallel traces or traces intersecting with very small angles: for a further discus-
218 sion of these issues we refer to [43]. In Figure 9a we report the initial mesh on a portion
219 of the fractures of the DFN, whereas in Figure 9b the mesh obtained after 8 refinement
220 steps on the same fractures is depicted. We note that in this mesh we can see the re-
221 finement around the traces, this is especially evident around the intersections of the
222 fractures present in the picture with fractures not included in the picture. In Figure 10a
223 a detail of this mesh is reported, and we can notice that the mesh is not refined around
224 all traces in the same way but only around some of the traces that should be that with
225 a more relevant flux. In Figure 10b a closer view of the mesh is reported and this non-
226 uniform refinement around the traces is even more evident. In Table 7 we report the
227 most significant quantities of the problem during the adaptive refinement. We notice
228 that, although the number of the control variables is increasing during refinement, the
229 number of conjugate gradient iterations decreases after step 7, this denotes a positive
230 effect of the refinement on the conditioning of the discrete problem.

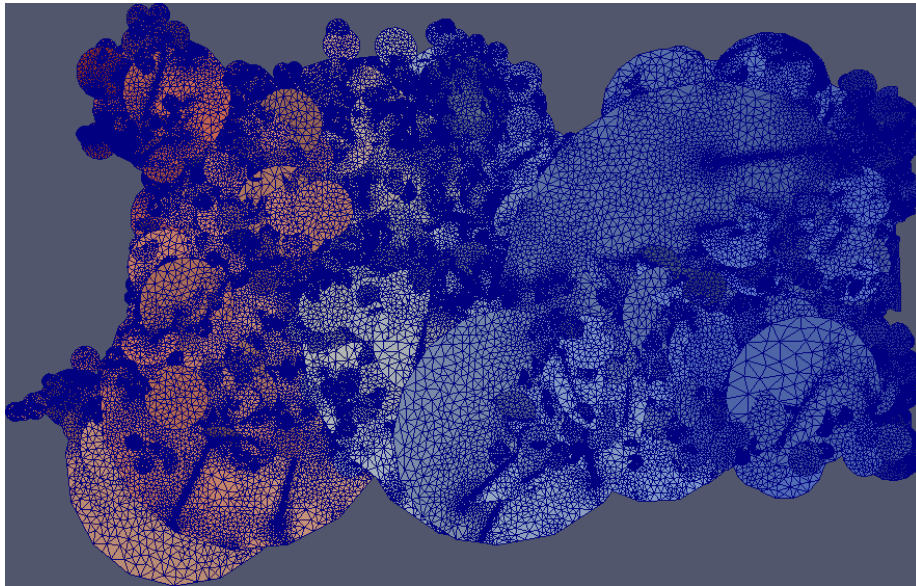
231 7.2.4. *Frac15K, adaptive and uniform refinement*

232 In this section we compare the convergence behaviour of the adaptive mesh re-
233 finement and of the uniform mesh refinement when applied to the same DFN of the
234 previous subsection. The initial mesh on each fracture contains about 16 elements.
235 During the solution process the maximal number of CG iteration is always reached. In
236 Figure 12a we report the initial mesh generated on the DFN for adaptive and uniform
237 refinement, and in Figure 12b a detail of the mesh is reported from where we can see
238 the chaotic intersections of fractures. In Figure 13a we compare the behaviour of the
239 error estimator during the adaptive and uniform refinement process. We also report the

Figure 9: Frac15K. DFN with meshes at some refining steps

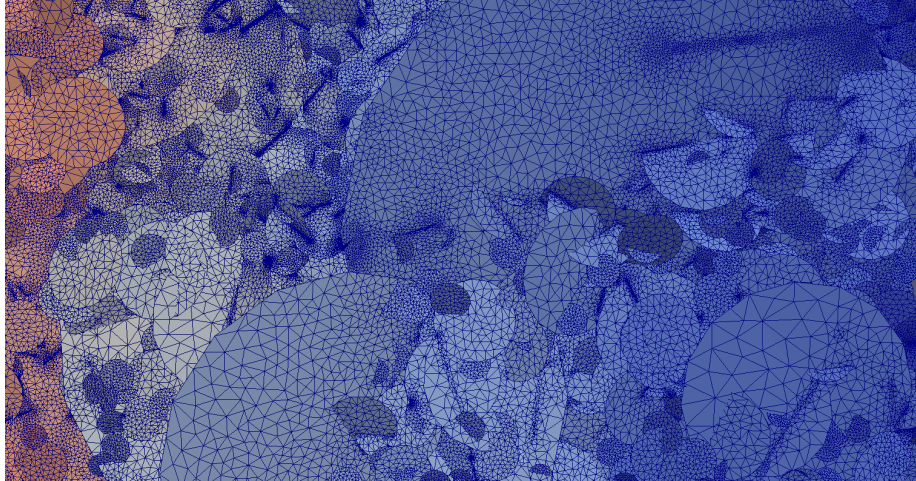


(a) Initial mesh

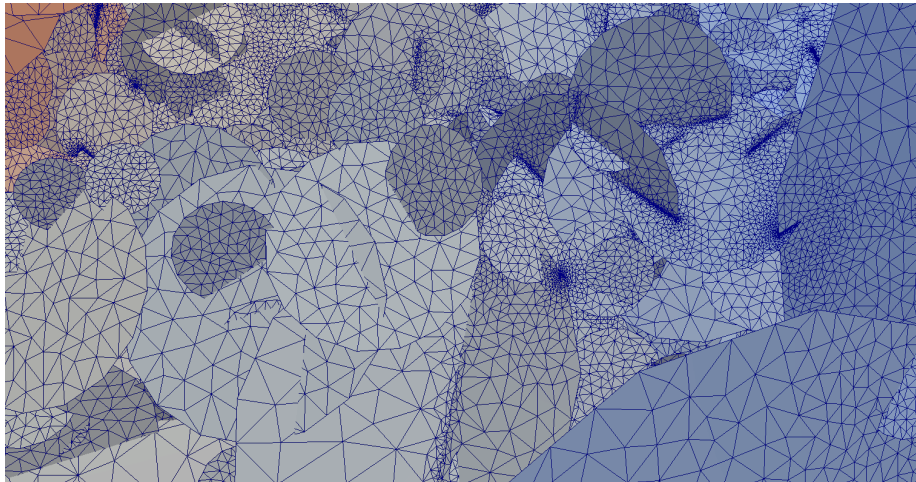


(b) Step 8

Figure 10: Frac15K. Details of the mesh, refining step 8



(a)



(b)

Figure 11: Frac15K. Plots

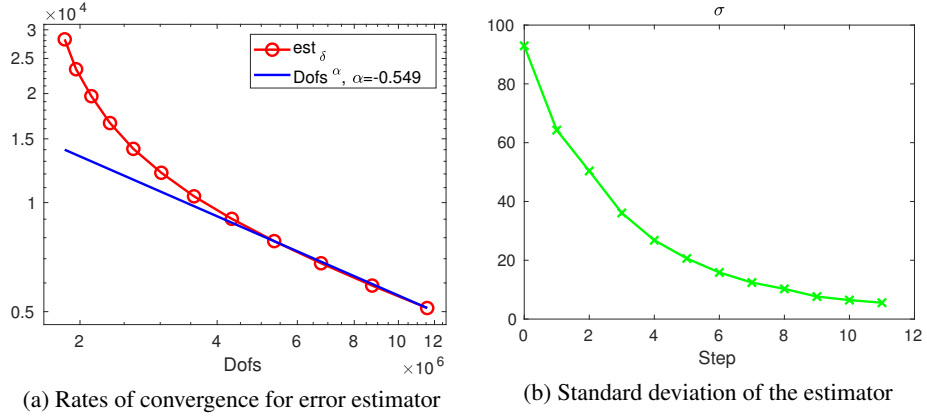


Table 7: Frac15K, Data during the refinement process

Step	Ω -Cells	Γ -Cells	Dofs	σ	CG-It	J
0	2417952	211478	1851290	9.29E+01	80000	3.93E-01
1	2593115	221175	1961046	6.43E+01	80000	4.11E-01
2	2842022	234468	2118259	5.04E+01	80000	4.06E-01
3	3182041	251833	2327579	3.61E+01	80000	3.90E-01
4	3660033	274509	2619066	2.68E+01	80000	3.68E-01
5	4329339	302330	3016777	2.06E+01	80000	3.42E-01
6	5256393	336908	3558969	1.58E+01	80000	3.04E-01
7	6557532	380421	4307937	1.24E+01	77055	2.69E-01
8	8364479	436335	5337434	1.03E+01	74740	2.32E-01
9	10893729	507994	6762737	7.65E+00	72113	1.99E-01
10	14462152	602244	8757293	6.46E+00	71689	1.68E-01
11	19527267	726470	11565176	5.56E+00	70432	1.39E-01

Table 8: Frac15K, Data during adaptive refinement process in Section 7.2.4

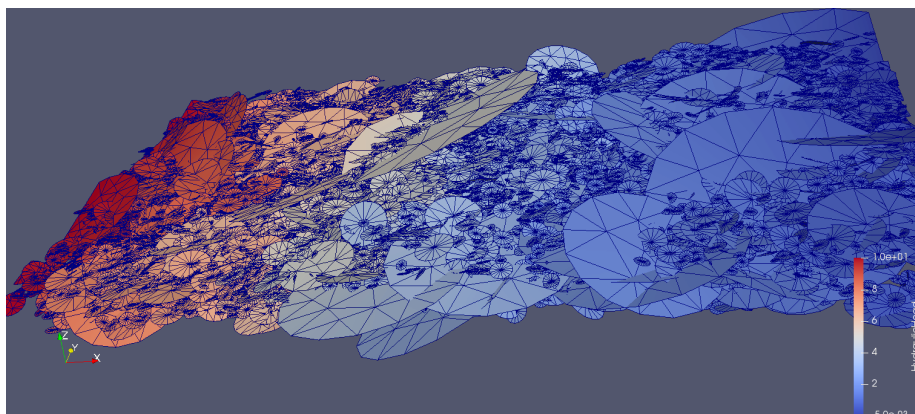
Step	Ω -Cells	Γ -Cells	Dofs	σ	est_δ	J
0	242855	145888	549087	6.64E+02	6.98E+04	2.79E-01
1	261204	145648	557797	3.68E+02	4.52E+04	2.99E-01
2	298859	146330	578051	2.18E+02	3.61E+04	3.50E-01
3	355061	148626	612975	1.43E+02	2.99E+04	3.71E-01
4	435454	152500	662769	1.01E+02	2.55E+04	3.95E-01
5	547776	158279	733433	7.92E+01	2.22E+04	4.00E-01
6	703220	166511	830831	5.33E+01	1.92E+04	4.05E-01
7	929697	178259	972138	3.93E+01	1.67E+04	3.92E-01
8	1255878	194066	1172353	2.83E+01	1.46E+04	3.76E-01
9	1730104	215214	1459305	2.20E+01	1.27E+04	3.51E-01
10	2416585	243498	1868478	1.57E+01	1.11E+04	3.23E-01
11	3417205	280966	2456115	1.26E+01	9.62E+03	2.91E-01
12	4864293	330821	3294938	9.92E+00	8.34E+03	2.56E-01
13	6971745	396246	4499103	7.56E+00	7.22E+03	2.20E-01
14	10032775	483200	6227824	5.85E+00	6.22E+03	1.86E-01
15	14500876	599711	8725258	5.98E+00	5.36E+03	1.54E-01
16	20935334	752245	12284501	4.43E+00	4.60E+03	1.26E-01
17	30236802	956686	17388891	4.06E+00	3.95E+03	1.01E-01

240 slopes of the fitting of the error estimators and we can see that the adaptive refinement
 241 is always very close to the optimal rate of convergence using linear finite elements,
 242 whereas the uniform refinement is displaying a reduced rate of convergence. From
 243 Figures 13a-13c we can appreciate the very good behaviour of the adaptive method
 244 with respect to uniform refinement in term of error reduction, as well as in term of
 245 equidistribution of the error and functional reduction.

246 We note that the reduced rate of convergence of the estimator in the case of uni-
 247 form refinement can be ascribed to the global low regularity of the solution, that does
 248 not belong to $H^2(\Omega)$. In particular, the asymptotic rates of convergence are compatible
 249 with the regularity of the solution, that belongs to $H^{\frac{3}{2}-\epsilon}(\Omega)$, $\forall \epsilon > 0$, due to the lack
 250 of smoothness around trace tips. This behaviour cannot be circumvented also using
 251 meshes conforming to the intersections. Applying adaptive approaches, we are per-
 252 forming a non-linear approximation [44] that depends on the higher Besov regularity
 253 of the solution, and this is compatible with the maximal rate of convergence possible
 254 with linear finite elements ($\alpha \sim 0.5$), that is well approximated in Figure 13a. Further-
 255 more, we note that in the pre-asymptotic regime the error decrease is much stronger in
 256 the adaptive approach with respect to the uniform refinement.

257 Figure 13d compares the computational time required on each uniform mesh with
 258 the cumulative time required to reach and perform each step of the adaptive process.
 259 We remark that in the current implementation we recompute all the trace-mesh inter-
 260 sections: this can be avoided in case of nested refined meshes. In Tables 8-9 we report
 261 the values of the most significant quantities to describe the refinement process.

Figure 12: Frac15K. Details of the mesh used for the test in Section 7.2.4



(a)

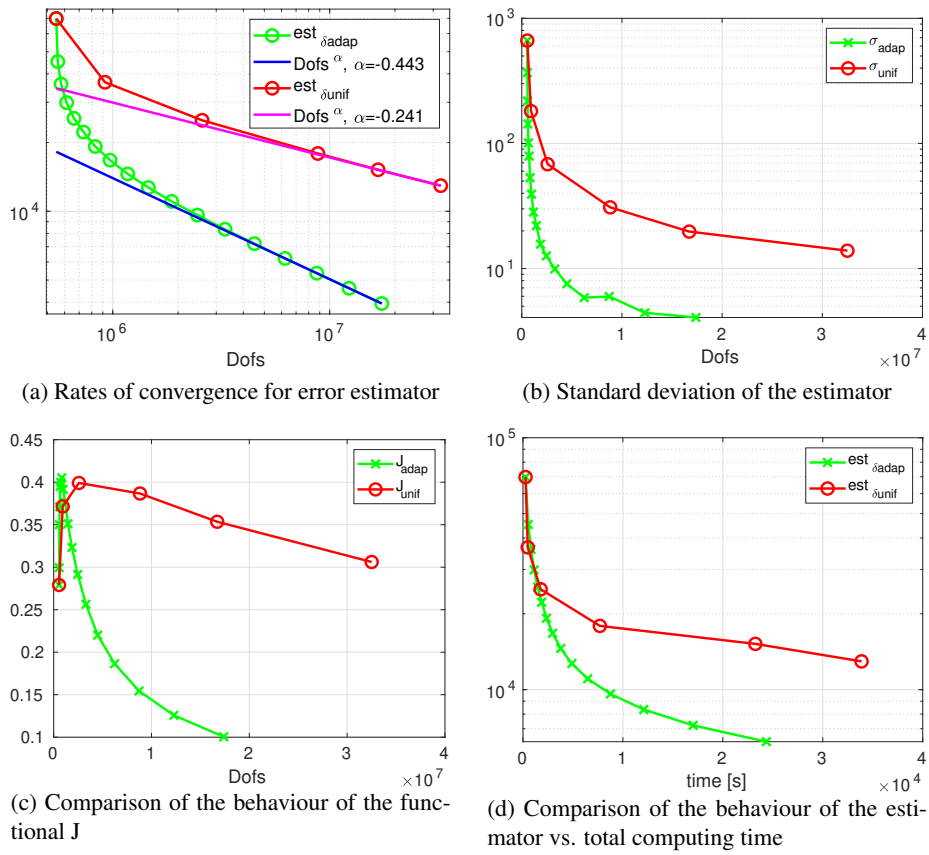


(b)

Table 9: Frac15K, Data during uniform refinement process in Section 7.2.4

Step	Ω -Cells	Γ -Cells	Dofs	σ	est_δ	J
0	242855	145888	549087	6.64E+02	6.98E+04	2.79E-01
1	894372	168924	921096	1.81E+02	3.67E+04	3.71E-01
2	3699937	243136	2581970	6.82E+01	2.50E+04	3.99E-01
3	15084448	404039	8826748	3.08E+01	1.79E+04	3.87E-01
4	29978097	542013	16718991	1.97E+01	1.52E+04	3.53E-01
5	60191364	740664	32510129	1.39E+01	1.30E+04	3.06E-01

Figure 13: Frac15K. Comparison between adaptive and uniform refinements



262 8. Conclusions

263 An adaptive algorithm based on a residual a posteriori error estimate is devised for
264 the computation of the hydraulic head distribution in a Discrete Fracture Network. The
265 approach reduces the error under a certain percentage of the error at the first step of the
266 adaptive process and equidistributes it in space. Numerical tests verify the equivalence
267 of the true error and the error estimate and show the capability to capture the relevant
268 features of the flux automatically.

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