An adaptive hp-DG-FE Method for Elliptic Problems. Convergence and Optimality in the 1D Case

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Dedicated to the memory of Professor Ben-yu Guo

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Abstract

We propose and analyze an hp-adaptive DG-FEM algorithm, termed **hp-ADFEM**, and a realization of it in one space dimension which is convergent, instance optimal, and h- and p-robust. The procedure consists of iterating two routines: one hinges on Binev's algorithm for the adaptive hp-approximation of a given function, and finds a near-best hp-approximation of the current discrete solution and data to a desired accuracy; the other one improves the discrete solution to a finer but comparable accuracy, by iteratively applying Dörfler marking and h-refinement.

1 Introduction

The design and analysis of adaptive hp-type finite element methods for elliptic problems is significantly more challenging than for h-type methods. Indeed, as demonstrated e.g. by some examples given in [6, Sect.1], one should include in the adaptive procedure the possibility of stepping back from an early choice between h-refinement and p-enrichment: while the true structure of the solution reveals itself along the iterations, one should be able to re-distribute the allocated degrees of freedom between h- and p-resolution. The existence of (rather) pathological situations has not prevented the development of practical hp-adaptive algorithms that work (see e.g. [9] and the references therein), but in most cases these procedures are not supported by a sound mathematical theory, which assesses the optimality, and even the convergence, of the method (unless a-priori assumptions on the structure of the solution are made).

The crucial issue is an approximation problem: how can we build an hp-finite element space of minimal dimension in which a given function can be approximated with a prescribed accuracy? A constructive answer to this question has been given by P. Binev in the past few years (see [5]), who designed a greedy hp-algorithm, which is incremental with respect to the dimension and has instance optimality properties (see Sect. 2.3).

With a good answer to such an approximation problem, one may think of recursively applying the hp-adaptive algorithm to a sequence of Galerkin discrete solutions of the elliptic problem, built in a way to get closer and closer to the exact solution. This idea

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has been implemented in [6], where a general framework for adaptive hp-discretizations has been devised, and an adaptive algorithm termed **hp-AFEM** has been proposed, which guarantees convergence and instance optimality of the sequence of generated Galerkin solutions. The algorithm is both h- and p-optimal in one space dimension, whereas in higher dimensions p-robustness is lost, partly due to the need of going from the non-conforming meshes produced by Binev's algorithm to the conforming ones needed in a continuous Galerkin method, and partly due to the use of a residual-based error estimator (the latter obstruction may be removed by resorting to equilibrated flux estimators, as done in [7]).

Since Binev's algorithm produces non-conforming meshes and discontinuous approximations, it is quite natural to associate to it a Discontinuous, rather than a Continuous, Galerkin discretization of the elliptic problem. The purpose of this paper is to take a step forward in this direction. In particular, hereafter we propose an hp-adaptive DG-FEM algorithm, termed **hp-ADFEM**, and a realization of it in one space dimension which is convergent, instance optimal, and h- and p-robust. No restriction on the relative size of neighboring elements, nor on the polynomial degrees used on them, is required. In building a discrete solution that matches a prescribed accuracy, we extend to the hp-case the approach developed in [4] for h-type DG methods, using in the analysis several results on hp-type a posteriori error estimators (see e.g. [8] and the references therein). The multi-dimensional case is currently under investigation [1]; while our general convergence theorem holds in any dimension, proving p-robustness seems to require a grading property in the distribution of polynomial degrees over the partition, which is not guaranteed by the algorithm proposed in [5].

The paper is organized as follows. In Sect. 2, we introduce our general framework for the hp-approximation of a given function, and we present Binev algorithm. Sect. 3 describes the hp-DG discretizations that we consider, and collects some of their properties. Sect. 4 contains the general convergence and instance optimality result, based on the concatenation of Binev's algorithm and a procedure to compute DG-solutions with polynomial data, matching a prescribed tolerance. Finally, in Sect. 5 we illustrate a possible realization of this procedure, which is based on the classical SOLVE $\to$ ESTIMATE $\to$ MARK $\to$ REFINE paradigm.

The following notation will be used thoughout the paper. By $A \lesssim B$ we will mean that A can be bounded by a multiple of B, independently of parameters which A and B may depend on. Likewise, $A \simeq B$ is defined as both $A \lesssim B$ and $B \lesssim A$.

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2 *hp*-partitions and *hp*-approximations

Let Ω be a bounded open interval of the real line. In view of the hp-adaptive discretization of a boundary-value problem therein, we introduce some notation concerning partitions in Ω and function spaces built on them.

2.1 Partitions of the domain

We assume that we are given an essentially disjoint initial partition $\mathcal{K}_0$ of $\bar{\Omega}$ into finitely many closed subintervals, which will be the initial geometric elements; the initial subdivision may depend upon the data of the problem at hand. Then, we apply subsequent dyadic subdivisions, by halving each element K that we encounter into two closed subintervals K' and K'' of equal size, the 'children' of K, such that $K = K' \cup K''$ and $|K' \cap K''| = 0$.

The set $\mathfrak{K}$ of all these geometric elements forms an infinite binary 'master tree', having as its roots the elements of the initial partition of $\bar{\Omega}$. A subtree of the master tree is a finite subset of $\mathfrak{K}$ that contains all roots and for each element in the subset both its parent and its sibling are in the subset. The leaves of a subtree form an essentially disjoint partition of $\bar{\Omega}$. The set of all such geometric partitions, or 'h-partitions', will be denoted as $\mathbb{K}$. For $\mathcal{K}, \widetilde{\mathcal{K}} \in \mathbb{K}$, we call $\widetilde{\mathcal{K}}$ a refinement of $\mathcal{K}$, and denoted as $\mathcal{K} \leq \widetilde{\mathcal{K}}$, when any $K \in \widetilde{\mathcal{K}}$ is either in $\mathcal{K}$ or has an ancestor in $\mathcal{K}$.

Starting from an h-partition $\mathcal{K} \in \mathbb{K}$, we obtain an *hp-partition* $\mathcal{D}$ by associating an integer $p \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ to each element $K \in \mathcal{K}$. This integer will represent a polynomial degree, which will identify certain finite dimensional spaces of polynomial functions defined in K. A pair $D = (K_D, p_D) \in \mathfrak{K} \times \mathbb{N}_0$ formed by a geometric element K_D and an integer p_D will be termed an *hp-element*. Thus, a collection $\mathcal{D} = \{D = (K_D, d_D)\}$ of *hp*-elements is an *hp*-partition provided $\mathcal{K}(\mathcal{D}) := \{K_D : D \in \mathcal{D}\} \in \mathbb{K}$; the latter will be the associated h-partition. The collection of all *hp*-partitions is denoted as $\mathbb{D}$. Since $p+1$ is the dimension of the space $\mathbb{P}_p(K)$ of the univariate polynomials of degree $\leq p$ in K, we define the *dimension* of the *hp*-partition $\mathcal{D}$ as the integer

$$\#\mathcal{D} := \sum_{D \in \mathcal{D}} (p_D + 1).$$

For $\mathcal{D}, \widetilde{\mathcal{D}} \in \mathbb{D}$, we call $\widetilde{\mathcal{D}}$ a refinement of $\mathcal{D}$, and write $\mathcal{D} \leq \widetilde{\mathcal{D}}$, when both $\mathcal{K}(\mathcal{D}) \leq \mathcal{K}(\widetilde{\mathcal{D}})$, and $d_{\widetilde{D}} \geq d_D$, for any $D \in \mathcal{D}$, $\widetilde{D} \in \widetilde{\mathcal{D}}$ with K_D being either equal to $K_{\widetilde{D}}$ or an ancestor of $K_{\widetilde{D}}$.

2.2 Approximation spaces on *hp*-partitions

Let Z be a normed space of vector-valued functions $z : \Omega \to \mathbb{R}^m$ ($m \geq 1$), which is relevant for our application. For any geometric element $K \in \mathfrak{K}$, let Z_K be the space collecting the restrictions $z_{|K}$ to K of all functions $z \in Z$. Then, for any geometric partition $\mathcal{K} \in \mathbb{K}$, we define

$$Z_{\mathcal{K}} := \{z : \Omega \to \mathbb{R}^m \ : \ z_{|K} \in Z_K \ \forall K \in \mathcal{K}\} = \prod_{K \in \mathcal{K}} Z_K \, ; \tag{1}$$

obviously, $Z \subseteq Z_{\mathcal{K}}$. In the sequel, we will work with functions that belong to $Z_{\mathcal{K}}$ for some partition $\mathcal{K} \in \mathbb{K}$; therefore, we set

$$\mathcal{Z} := \bigcup_{\mathcal{K} \in \mathbb{K}} Z_{\mathcal{K}}.$$

We assume that for any $K \in \mathfrak{K}$, the space Z_K contains all polynomial functions of any degree, and this set of functions is dense in Z_K. Then, for $p \in \mathbb{N}_0$ we assume we have chosen finite dimensional spaces $Z_{K,p} \subset Z_K$ of polynomial functions on K of degree related to p, satisfying $Z_{K,p} \subset Z_{K,p+1}$ and $Z_{K,p} \subset Z_{K',p} \times Z_{K'',p}$ (K' and K'' being the children of K). For any $D = (K_D, p_D) \in \mathfrak{K} \times \mathbb{N}_0$, we set $Z_D := Z_{K_D, p_D}$. Then, given an *hp*-partition $\mathcal{D}$, we define

$$Z_{\mathcal{D}} := \{z : \Omega \to \mathbb{R}^m \ : \ z_{|K_D} \in Z_D \ \forall D \in \mathcal{D}\} = \prod_{D \in \mathcal{D}} Z_D \, , \tag{2}$$

which obviously satisfies $Z_{\mathcal{D}} \subset Z_{\mathcal{K}(\mathcal{D})}$. We will use the notation $z_{\mathcal{D}}$ to indicate a function in $Z_{\mathcal{D}}$. Note that no interelement continuity is imposed in the definition of $Z_{\mathcal{D}}$. Also note that the dimension of $Z_{\mathcal{D}}$ is proportional to the cardinality $\#\mathcal{D}$.

For all $D \in \mathfrak{K} \times \mathbb{N}_0$, we assume a local projector $Q_D : \mathcal{Z} \to Z_D$, and a local error functional $e_D = e_D(z) \geq 0$, that, for any $z \in \mathcal{Z}$ gives a measure for some function of the distance between $z_{|K_D}$ and its local approximation $z_D := Q_D(z)$. We assume that this error functional is non-increasing under both 'h-refinements' and 'p-enrichments', in the sense that

$$\begin{aligned} e_{D'} + e_{D''} \le e_D \text{ when } K_{D'}, K_{D''} \text{ are the children of } K_D, \text{ and } p_{D'} = p_{D''} = p_D; \\ e_{D'} \le e_D \text{ when } K_{D'} = K_D \text{ and } p_{D'} \ge p_D. \end{aligned} \tag{3}$$

Given any hp-partition $\mathcal{D} \in \mathbb{D}$, we define the global projector $Q_{\mathcal{D}} : \mathcal{Z} \to Z_{\mathcal{D}}$ as $Q_{\mathcal{D}}(z) := (z_D)_{D \in \mathcal{D}}$, and the global error functional

$$\mathrm{E}_{\mathcal{D}}(z) := \sum_{D \in \mathcal{D}} e_D(z), \tag{4}$$

which is a measure for the distance between z and its projection $z_{\mathcal{D}} := Q_{\mathcal{D}}(z)$. Note that (3) is equivalent to

$$\mathrm{E}_{\widetilde{\mathcal{D}}}(z) \le \mathrm{E}_{\mathcal{D}}(z) \quad \forall \widetilde{\mathcal{D}} \ge \mathcal{D}. \tag{5}$$

2.3 The instance optimal *hp*-approximation algorithm

Herafter, we present the greedy algorithm proposed by P. Binev [5] to produce a near-best adaptive hp-approximation of a function $z \in \mathcal{Z}$, based on the associated local error functionals $e_D = e_D(z)$ and global error functional $\mathrm{E}_{\mathcal{D}} = \mathrm{E}_{\mathcal{D}}(z)$ introduced above.

Denote by $R \ge 1$ the cardinality of the initial geometric partition K_0. Using property (3), Binev's algorithm builds a sequence of hp-partitions $\mathcal{D}_N$, $N \ge R$, satisfying $\#\mathcal{D}_N = N$; the construction is incremental, in that going from $\mathcal{D}_N$ to $\mathcal{D}_{N+1}$ one exploits the work already done to build $\mathcal{D}_N$. The main feature of the algorithm is its instance optimality, expressed as follows.

Theorem 2.1 ([5])**.** *For $n \ge R$ let*

$$\sigma_n := \inf_{\#\mathcal{D} \le n} E_{\mathcal{D}}$$

be the smallest error achievable with an hp-partition of cardinality $\le n$. Then, the hp-partitions $\mathcal{D}_N$ produced by Binev's algorithm yield error functionals $E_{\mathcal{D}_N}$ satisfying the bounds

$$\mathrm{E}_{\mathcal{D}_N} \le \frac{2N}{N-n+1} \sigma_n \quad \forall n \le N. \qquad \square \tag{6}$$

Binev's construction can be easily used to produce an instance optimal hp-partition for which the error functional is below a given threshold.

Corollary 2.1 ([6])**.** *Let $B > 1$ arbitrary. Given $\varepsilon > 0$, let $\mathcal{D} \in \mathbb{D}$ be the first partition in Binev's sequence for which $E_{\mathcal{D}}^{\frac{1}{2}} \le \varepsilon$. Then, setting $b = \frac{1}{2}(1 - \frac{1}{B}) < 1$, it holds*

$$\#\mathcal{D} \le B \,\#\hat{\mathcal{D}}$$

for all partitions $\hat{\mathcal{D}} \in \mathbb{D}$ satisfying $E_{\hat{\mathcal{D}}}^{\frac{1}{2}} \le b\varepsilon$. $\square$

This result motivates the introduction of the following routine, which will constitute one of the two major building blocks of our proposed hp-adaptive algorithm.

- $[\mathcal{D}, z_\mathcal{D}] := \textbf{hp-NEARBEST}(\varepsilon, z)$

  The routine **hp-NEARBEST** takes as input $\varepsilon > 0$, and $z \in \mathcal{Z}$, and outputs $\mathcal{D} \in \mathbb{D}$ as well as $z_\mathcal{D} \in Z_\mathcal{D}$ such that $\mathrm{E}_\mathcal{D}(z)^{\frac{1}{2}} \le \varepsilon$ and, for some constants $0 < b < 1 < B$, $\#\mathcal{D} \le B\,\#\widehat{\mathcal{D}}$ for any $\widehat{\mathcal{D}} \in \mathbb{D}$ with $\mathrm{E}_{\widehat{\mathcal{D}}}(z)^{\frac{1}{2}} \le b\varepsilon$.

The approximation $z_\mathcal{D}$ of the input z is just the element-wise projection given by the operator $Q_\mathcal{D}$ associated with the partition $\mathcal{D}$, i.e., we set

$$z_\mathcal{D} := Q_\mathcal{D}(z). \tag{7}$$

3 Discontinuous Galerkin *hp*-discretizations

We are interested in solving numerically the model boundary-value problem

$$Au = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \tag{8}$$

with $Au := -(\nu u_x)_x + \xi u$, where $\nu \in L^\infty(\Omega)$ satisfies $\operatorname{essinf}_\Omega \nu > 0$, $\xi \in L^2(\Omega)$ satisfies $\xi \ge 0$ a.e. in Ω, $f \in L^2(\Omega)$. We actually assume that ν, ξ are piecewise-H^1 functions, precisely that $\nu_{|K}, \xi_{|K} \in H^1(K)$ for each element K of the initial partition $\mathcal{K}_0$ introduced in Sect. 2.1; we will write $\nu, \xi \in H^1(\Omega; \mathcal{K}_0)$. It will be convenient to refer to a triple $g := (\nu, \xi, f)$ as to a "data" of our problem; we thus have $g \in G(\Omega) := (H^1(\Omega;\mathcal{K}_0))^2 \times L^2(\Omega)$. The solution $u \in H^1_0(\Omega)$ of Problem (8) for given data g will be indicated by $u(g)$.

The following notation will be useful in the design of a DG discretization of our problem. For any element $K \in \mathfrak{K}$, let $(v, w)_K$ denote the L^2-inner product in K, with corresponding norm $\|v\|_K$. For any geometric partition $\mathcal{K} \in \mathbb{K}$, let us set

$$V_\mathcal{K} := \{v \in L^2(\Omega) : v_{|K} \in H^1(K)\ \forall K \in \mathcal{K}\}. \tag{9}$$

For $v \in V_\mathcal{K}$, it will be convenient to denote by $\tilde{v}_x$ the function in $L^2(\Omega)$ such that $(\tilde{v}_x)_{|K} = (v_{|K})_x$ for all $K \in \mathcal{K}$; thus, $\|\tilde{v}_x\|^2_\Omega = \sum_{K\in\mathcal{K}} \|(v_{|K})_x\|^2_K$. Let us denote by $\mathcal{E}_\mathcal{K}$ the set of all endpoints of elements in $\mathcal{K}$, and let us define the jumps and averages of a piecewise smooth function ϕ on $\mathcal{K}$ as follows: if $e \in \mathcal{E}_\mathcal{K}$ is shared by two contiguous elements K^- and K^+, then we set

$$[\![\phi]\!]_e := \phi_{|K^-}(e) - \phi_{|K^+}(e), \qquad \{\!\{\phi\}\!\}_e := \frac{1}{2}(\phi_{|K^-}(e) + \phi_{|K^+}(e)),$$

whereas if e is the left/right boundary point of Ω, we set $[\![\phi]\!]_e = +/-\phi(e)$ and $\{\!\{\phi\}\!\}_e = \phi(e)$.

For any *hp*-partition $\mathcal{D} \in \mathbb{D}$ let us set

$$V_\mathcal{D} := \{v \in L^2(\Omega) : v_{|K_D} \in \mathbb{P}_{p_D}(K_D) \ \ \forall D \in \mathcal{D}\} \subset V_{\mathcal{K}(\mathcal{D})}. \tag{10}$$

If $D \in \mathcal{D}$, let $h_D := |K_D|$ denote the size of the element K_D. If $e \in \mathcal{E}_\mathcal{D} := \mathcal{E}_{\mathcal{K}(\mathcal{D})}$, we define the weight

$$\sigma_{\mathcal{D},e} := \max\big(\frac{p^2_{D^-}}{h_{D^-}}, \frac{p^2_{D^+}}{h_{D^+}}\big) \tag{11}$$

if $e \in K_{D^-} \cap K_{D^+}$, and $\sigma_{\mathcal{D},e} := \frac{p^2_D}{h_D}$ if $e \in \partial\Omega \cap K_D$.

It is convenient to introduce the inner product $(\phi, \psi)_{\mathcal{E}_\mathcal{D}} := \sum_{e\in\mathcal{E}_\mathcal{D}} \phi_e \psi_e$ in $\mathbb{R}^{|\mathcal{E}_\mathcal{D}|}$ between two quantities $\phi = (\phi_e)$ and $\psi = (\psi_e)$ indexed in $\mathcal{E}_\mathcal{D}$. The corresponding norm will be denoted by $\|\phi\|_{\mathcal{E}_\mathcal{D}}$.

At this point, we are ready to introduce the symmetric bilinear form $a_\mathcal{D}$ defined on $V_\mathcal{D} \times V_\mathcal{D}$ as

$$\begin{aligned} a_\mathcal{D}(w, v) := {} & (\nu\, \tilde{w}_x, \tilde{v}_x)_\Omega + (\xi\, w, v)_\Omega \\ & - (\{\!\{\nu\, w_x\}\!\}, [\![v]\!])_{\mathcal{E}_\mathcal{D}} - (\{\!\{\nu\, v_x\}\!\}, [\![w]\!])_{\mathcal{E}_\mathcal{D}} + \gamma\,(\sigma_\mathcal{D}[\![w]\!], [\![v]\!])_{\mathcal{E}_\mathcal{D}}\ , \end{aligned} \tag{12}$$

where $\gamma > 0$ is a sufficiently large stabilization parameter, as well as the DG-norm defined on $V_{\mathcal{D}}$ as

$$\|v\|_{\mathcal{D}} := \Big((\nu\, \tilde{v}_x, \tilde{v}_x)^2_\Omega \; + \gamma \, \| \, \sigma_{\mathcal{D}}^{1/2} [\![v]\!] \, \|^2_{\mathcal{E}_{\mathcal{D}}} \Big)^{\frac{1}{2}} . \tag{13}$$

It is well-known (see [2, 3]) that $a_{\mathcal{D}}$ is a continuous form with respect to the DG-norm, and it is coercive provided γ is chosen large enough, with coercivity and continuity constants independent of $\mathcal{D}$; in the sequel, we will assume that this condition is satisfied.

Since $a_{\mathcal{D}}$ depends on the choice of coefficients ν and ξ, and since in the adaptive algorithm we will consider a sequence of DG discretizations with changing (piecewise polynomial) data, sometimes we will prefer the more precise notation $a_{\mathcal{D}}(w, v; \nu, \xi)$ to indicate the right-hand side of (12).

Problem 8 with data $g = (\nu, \xi, f) \in G(\Omega)$ is then discretized by the following Simmetric Interior Penalty Discontinuous-Galerkin method ([2]):

$$u_{\mathcal{D}} \in V_{\mathcal{D}} \quad : \quad a_{\mathcal{D}}(u_{\mathcal{D}}, v_{\mathcal{D}}; \nu, \xi) = (f, v_{\mathcal{D}})_\Omega \qquad \forall v_{\mathcal{D}} \in V_{\mathcal{D}}. \tag{14}$$

We will write $u_{\mathcal{D}} = u_{\mathcal{D}}(g)$ when we want to stress the dependence of $u_{\mathcal{D}}$ upon the given data g.

3.1 Approximation spaces and error functionals

Hereafter, we specify the choice of approximation spaces and error functionals, introduced in a general setting in Sect. 2.2, that is tailored to the discretization problem of interest.

Since we will deal with approximations of a specific solution of Problem 8, and approximations of the corresponding data, our functions z will be of the form $z = (v, g) = (v, \nu, \xi, f)$. Then, a natural choice for the "base" space Z is $Z = H^1(\Omega) \times G(\Omega) = H^1(\Omega) \times (H^1(\Omega; \mathcal{K}_0))^2 \times L^2(\Omega)$. Note that for $\mathcal{K} \in \mathbb{K}$, the local spaces Z_K that form the global space $Z_{\mathcal{K}}$ according to (1) are given by $Z_K = (H^1(K))^3 \times L^2(K)$.

For any element $K \in \mathfrak{K}$ and integer $p \in \mathbb{N}_0$, we set

$$Z_{K,p} = V_{K,p} \times G_{K,p} \qquad \text{with} \quad V_{K,p} = \mathbb{P}_p(K) \quad \text{and} \quad G_{K,p} = \mathbb{P}_{p+1}(K) \times \mathbb{P}_{p+1}(K) \times \mathbb{P}_{p-1}(K).$$

Then, for any $\mathcal{D} \in \mathbb{D}$, we define $Z_{\mathcal{D}}$ according to (2); it is easily seen that $Z_{\mathcal{D}} =: V_{\mathcal{D}} \times G_{\mathcal{D}}$, where $V_{\mathcal{D}}$ has been already introduced in (10). We will write $z_{\mathcal{D}} = (v_{\mathcal{D}}, g_{\mathcal{D}}) = (v_{\mathcal{D}}, \nu_{\mathcal{D}}, \xi_{\mathcal{D}}, f_{\mathcal{D}})$ for the generic element in $Z_{\mathcal{D}}$.

In order to define the projectors Q_D, let $\Pi^0_{K,p} : L^2(K) \to \mathbb{P}_p(K)$ be the L^2-orthogonal projector, and let $\Pi^1_{K,p} : H^1(K) \to \mathbb{P}_p(K)$ be the projector such that

$$(\Pi^1_{K,p} v)_x = \Pi^0_{K,p} v_x \qquad \text{and} \qquad \int_K \Pi^1_{K,p} v = \int_K v, \qquad \forall v \in H^1(K).$$

The latter definition can be extended to functions v that are just piecewise-H^1 on K, by replacing v_x with $\tilde{v}_x$ in the L^2-projection. Then, for $z = (v, g) = (v, \nu, \xi, f) \in \mathcal{Z}$ and $D = (K_D, p_D)$ we set

$$Q_D(z) = (\Pi^1_{K_D, p_D} v_{|K_D}, \Pi^1_{K_D, p_D+1} \nu_{|K_D}, \Pi^1_{K_D, p_D+1} \xi_{|K_D}, \Pi^0_{K_D, p_D-1} f_{|K_D}).$$

The corresponding local error functional is defined as

$$e_D(z) := e_{1,D}(v) + \frac{1}{\kappa^2} \mathrm{osc}^2_D(g) =: e_{1,D}(v) + \frac{1}{\kappa^2} \big(e_{1,D}(\nu) + e_{1,D}(\xi) + e_{0,D}(f) \big), \tag{15}$$

where for $\varphi = v, \nu, \xi$

$$e_{1,D}(\varphi) := \| (\mathrm{I} - \Pi^0_{K_D, p_D}) (\tilde{\varphi}_x)_{|K_D} \|^2_{K_D}, \qquad e_{0,D}(f) := \frac{h_D}{p_D} \, \| (\mathrm{I} - \Pi^0_{K_D, p_D}) f_{|K_D} \|^2_{K_D},$$

and $\kappa > 0$ is a (sufficiently small) penalization parameter to be chosen later on.

Finally, for a given hp-partition $\mathcal{D} \in \mathbb{D}$, the global projector $Q_{\mathcal{D}} : \mathcal{Z} \to Z_{\mathcal{D}}$ and the global error functional $\mathrm{E}_{\mathcal{D}}(z) = \mathrm{E}_{\mathcal{D}}(v, g)$ are defined as in Sect. 2.2 (see (4)).

We now establish some properties involving the functional $\mathrm{E}_{\mathcal{D}}$, that will be useful in the sequel.

Property 3.1. *There exists a constant $C_0 > 0$ such that for any $z = (v, \nu, \xi, f) \in \mathcal{Z}$ and for any partition $\mathcal{D} \in \mathbb{D}$ one has*

$$\|\nu - \nu_{\mathcal{D}}\|_{L^\infty(\Omega)} + \|\xi - \xi_{\mathcal{D}}\|_{L^\infty(\Omega)} \leq C_0 \kappa \, \mathrm{E}_{\mathcal{D}}(z)^{\frac{1}{2}},$$

where $z_{\mathcal{D}} = (v_{\mathcal{D}}, \nu_{\mathcal{D}}, \xi_{\mathcal{D}}, f_{\mathcal{D}}) = Q_{\mathcal{D}}(z)$.

Proof. For any $\mathcal{D} \in \mathbb{D}$ and any $D \in \mathcal{D}$, set $\psi := (\nu - \nu_{\mathcal{D}})_{|K_D}$. Since by construction ψ vanishes at a point in K_D, we have for any $x \in K_D$

$$|\psi(x)| \leq h_D^{1/2} \|\psi_x\|_{K_D} \leq |\Omega| \, e_{1,D}(\nu)^{1/2},$$

from which the bound for $\|\nu - \nu_{\mathcal{D}}\|_{L^\infty(\Omega)}$ easily follows. The coefficient ξ can be treated similarly. □

At this point, let us fix once and for all the data of interest $g_\star = (\nu_\star, \xi_\star, f_\star) \in G(\Omega)$ for Problem (8), and let $u_\star := u(g_\star)$ be the corresponding solution.
Let us set $\nu_0 := \operatorname{ess\,inf}_\Omega \nu_\star > 0$.

Assumption 3.1. *Let $\mathcal{D}_0$ denote the root partition $\mathcal{K}_0$ endowed with polynomials of degree 1 in each element. Setting $z_0 := (0, g_\star) \in \mathcal{Z}$, we assume that $\mathcal{D}_0$ is chosen to satisfy*

$$C_0 \kappa \, \mathrm{E}_{\mathcal{D}_0}(z_0)^{\frac{1}{2}} \leq \frac{\nu_0}{\lambda}, \qquad \text{where} \quad \lambda := 2 + \frac{1}{\sqrt{2}} |\Omega|.$$

Recalling (5), this assumption together with Property 3.1 guarantees that for any $\mathcal{D} \in \mathbb{D}$ (which trivially satisfies $\mathcal{D} \geq \mathcal{D}_0$), Problem 8 with approximate data $\nu_{\mathcal{D}}$ and $\xi_{\mathcal{D}}$ is coercive in $H_0^1(\Omega)$, precisely one has

$$(\nu_{\mathcal{D}} v_x, v_x)_\Omega + (\xi_{\mathcal{D}} v, v)_\Omega \geq \frac{\nu_0}{2} \|v_x\|_\Omega^2 \qquad \forall v \in H_0^1(\Omega). \tag{16}$$

This easily follows using the bound $\|v\|_\Omega \leq \frac{1}{2\sqrt{2}} |\Omega| \, \|v_x\|_\Omega$.

The following result is fundamental for establishing the convergence of our adaptive algorithm.

Proposition 3.1. *i) There exists a constant $C_\star > 0$ with the following property: for all $\mathcal{D} \in \mathbb{D}$ and all $z \in \mathcal{Z}$ of the form $z = (v, g_\star)$, let $z_{\mathcal{D}} = (v_{\mathcal{D}}, g_{\mathcal{D}}) := Q_{\mathcal{D}}(z)$, and let $u(g_{\mathcal{D}}) \in H_0^1(\Omega)$ be the solution of Problem 8 with data $g_{\mathcal{D}}$; then, it holds*

$$\|u_\star - u(g_{\mathcal{D}})\|_{H_0^1(\Omega)} \leq C_\star \kappa \, \mathrm{E}_{\mathcal{D}}(z_0)^{\frac{1}{2}} \leq C_\star \kappa \, \mathrm{E}_{\mathcal{D}}(z)^{\frac{1}{2}}, \tag{17}$$

where κ is the penalization parameter introduced in (15).

ii) For all $\mathcal{D} \in \mathbb{D}$, $v \in V_{\mathcal{K}(\mathcal{D})}$, $w \in H_0^1(\Omega)$ and $g \in G(\Omega)$, it holds

$$|\, \mathrm{E}_{\mathcal{D}}(v, g)^{\frac{1}{2}} - \mathrm{E}_{\mathcal{D}}(w, g)^{\frac{1}{2}} | \leq \|v - w\|_{\mathcal{D}} \, . \tag{18}$$

The proof follows step by step the proof of Proposition 3 in [6], to which we refer.

4 The adaptive algorithm *hp*-ADFEM

As anticipated in the Introduction, the algorithm we propose consists in alternating between a stage in which a new *hp*-partition is found, which is near-optimal for the current accuracy, and a stage in which this partition is further refined to guarantee a higher accuracy for the corresponding DG discrete solution; the data used in the latter stage to define the DG problem are approximations of the exact data, provided by the former stage.

The first stage will be accomplished by a call to the routine **hp-NEARBEST** introduced in Sect. 2.3. The second stage will be realized through a routine **DG-SOLVE** that we present now, postponing to Sect. 5 the detailed description of the underlying algorithm and the analysis of its properties. Essentially, starting from a given *hp*-partition and a corresponding data approximation, several DG problems are solved on subsequently refined partitions, whose generation is driven by an a posteriori error estimator, until a contraction property guarantees that the discretization error is brought below a prescribed threshold. In this stage, optimality is not an issue for the output partition, provided its cardinality remains comparable to that of the input partition.

- $[\bar{\mathcal{D}}, \bar{u}]$:=**DG-SOLVE** $(\varepsilon, \mathcal{D}, z_{\mathcal{D}})$

  The routine **DG-SOLVE** takes as input $\varepsilon > 0$, $\mathcal{D} \in \mathbb{D}$, and $z_{\mathcal{D}} = (v_{\mathcal{D}}, g_{\mathcal{D}}) \in Z_{\mathcal{D}}$. It outputs $\bar{\mathcal{D}} \in \mathbb{D}$ with $\mathcal{D} \leq \bar{\mathcal{D}}$ and $\bar{u} := u_{\bar{\mathcal{D}}}(g_{\mathcal{D}}) \in V_{\bar{\mathcal{D}}}$ such that $\|u(g_{\mathcal{D}}) - \bar{u}\|_{\bar{\mathcal{D}}} \leq \varepsilon$.

We recall that $u_{\bar{\mathcal{D}}}(g_{\mathcal{D}})$ denotes the solution of the following DG problem (see (14)): for $g_{\mathcal{D}} = (\nu_{\mathcal{D}}, \xi_{\mathcal{D}}, f_{\mathcal{D}}) \in G_{\mathcal{D}}$,

$$u_{\bar{\mathcal{D}}} \in V_{\bar{\mathcal{D}}} \quad : \quad a_{\bar{\mathcal{D}}}(u_{\bar{\mathcal{D}}}, v_{\bar{\mathcal{D}}}; \nu_{\mathcal{D}}, \xi_{\mathcal{D}}) = (f_{\mathcal{D}}, v_{\bar{\mathcal{D}}})_\Omega \qquad \forall v_{\bar{\mathcal{D}}} \in V_{\bar{\mathcal{D}}}. \tag{19}$$

The input function $v_{\mathcal{D}} \in V_{\mathcal{D}}$ may be used in the algorithm to define the starting point of the adaptive iterations.

Assumption 4.1. *Let $b < 1 < B$ be the constants that appear in the statement of the instance optimality property for the routine* **hp-NEARBEST**. *We assume that the penalization parameter κ in* (15) *is chosen small enough, so that it holds*

$$C_\star \kappa < b.$$

We are ready to present our algorithm **hp-ADFEM**. Let us introduce the parameters and the input data.

Parameters: two real numbers $\eta \in (0, 1)$, $\omega > 0$ satisfying

$$C_\star \kappa < b(1 - \eta) \qquad \text{and} \qquad \omega \in \left(\frac{1}{b}, \frac{1 - \eta}{C_\star \kappa} \right).$$

(Note that such a choice of ω is equivalent to $b\omega - 1 > 0$ and $C_\star \kappa \omega + \eta < 1$, which are two quantities that will appear below.)

Input data: $g_\star \in G(\Omega)$, $\varepsilon_0 > 0$, and $\bar{u}_0 \in V_{\bar{\mathcal{D}}_0}$ for some $\bar{\mathcal{D}}_0 \in \mathbb{D}$ such that $\|u_\star - \bar{u}_0\|_{\bar{\mathcal{D}}_0} \leq \varepsilon_0$.

Algorithm hp-ADFEM$(\varepsilon_0, \bar{u}_0, g_\star)$

`for` $i = 1, 2, \ldots$ `do`

  $[\mathcal{D}_i, (v_{\mathcal{D}_i}, g_{\mathcal{D}_i})]$:=**hp-NEARBEST**$(\omega \varepsilon_{i-1}, (\bar{u}_{i-1}, g_\star))$

  $[\bar{\mathcal{D}}_i, \bar{u}_i]$:=**DG-SOLVE** $(\eta \varepsilon_{i-1}, \mathcal{D}_i, (v_{\mathcal{D}_i}, g_{\mathcal{D}_i}))$

  $\varepsilon_i := (C_\star \kappa\, \omega + \eta) \varepsilon_{i-1}$

`end do`

Theorem 4.1. *Under Assumptions 3.1 and 4.1, the sequences* $(\bar{u}_i)$, $(\mathcal{D}_i)$ *produced by* **hp-ADFEM** *satisfy the following properties:*

$$\|u_\star - \bar{u}_i\|_{\bar{\mathcal{D}}_i} \le \varepsilon_i \quad \forall i \ge 0, \qquad \mathrm{E}_{\mathcal{D}_i}(u_\star, g_\star)^{\frac{1}{2}} \le (\omega+1)\varepsilon_{i-1} \quad \forall i \ge 1, \tag{20}$$

and

$$\#\mathcal{D}_i \le B\#\mathcal{D} \quad \text{for any } \mathcal{D} \in \mathbb{D} \text{ with } \mathrm{E}_{\mathcal{D}}(u_\star, g_\star)^{\frac{1}{2}} \le (b\omega - 1)\varepsilon_{i-1}. \tag{21}$$

Proof. The bound $\|u_\star - \bar{u}_0\|_{\bar{\mathcal{D}}_0} \le \varepsilon_0$ is valid by assumption. For $i \ge 1$, the tolerances used for **hp-NEARBEST** and **DG-SOLVE**, together with (17) show that

$$\begin{aligned} \|u_\star - \bar{u}_i\|_{\bar{\mathcal{D}}_i} &\le \|u_\star - u(g_{\mathcal{D}_i})\|_{H^1_0(\Omega)} + \|u(g_{\mathcal{D}_i}) - \bar{u}_i\|_{\bar{\mathcal{D}}_i} \\ &\le C_\star \kappa\, \mathrm{E}_{\mathcal{D}_i}(\bar{u}_{i-1}, g_\star)^{\frac{1}{2}} + \mu\varepsilon_{i-1} \le (C_\star\kappa\omega + \mu)\varepsilon_{i-1} = \varepsilon_i. \end{aligned} \tag{22}$$

The first statement follows for all $i \ge 0$. Using this and (18) implies the second assertion

$$\mathrm{E}_{\mathcal{D}_i}(u_\star, g_\star)^{\frac{1}{2}} \le \mathrm{E}_{\mathcal{D}_i}(\bar{u}_{i-1}, g_\star)^{\frac{1}{2}} + \|u_\star - \bar{u}_{i-1}\|_{\bar{\mathcal{D}}_{i-1}} \le (\omega+1)\varepsilon_{i-1} \qquad \forall i \ge 1.$$

Finally, let $\mathcal{D} \in \mathbb{D}$ with $\mathrm{E}_{\mathcal{D}}(u_\star, g_\star)^{\frac{1}{2}} \le (b\omega - 1)\varepsilon_{i-1}$. Then, again by (18), $\mathrm{E}_{\mathcal{D}}(\bar{u}_{i-1}, g_\star)^{\frac{1}{2}} \le b\omega\varepsilon_{i-1}$ and so $\#\mathcal{D}_i \le B\#\mathcal{D}$ because of the optimality property of **hp-NEARBEST**. ◻

The main result of Theorem 4.1 can be summarized by saying that **hp-ADFEM** is *instance optimal* for reducing $\mathrm{E}_{\mathcal{D}}(u_\star, g_\star)$ over $\mathcal{D} \in \mathbb{D}$.

5 The routine DG-SOLVE

The purpose of this section is the description and analysis of a realization of the routine **DG-SOLVE**. It is based on an iterative procedure of the form SOLVE → ESTIMATE → MARK → REFINE, in which ESTIMATE uses a residual-type estimator, whereas REFINE applies a dyadic splitting of each marked element while preserving the polynomial degree. The procedure satisfies a contraction property, which guarantees the reduction of a suitable "error" by a fixed amount at each iteration. Our construction is strongly inspired by [4], whose arguments are hereafter extended to cover the *hp*-case.

In the sequel, the input partition $\mathcal{D}$ will be denoted by $\mathcal{D}_{\text{in}}$, whereas the symbol $\mathcal{D}$ will be used to denote any refinement of $\mathcal{D}_{\text{in}}$ generated by the procedure. Similarly, the input function will be denoted by $z_{\text{in}} = (v_{\text{in}}, g_{\text{in}})$. To avoid cumbersome notation, we will actually write $g_{\text{in}} =: g = (\nu, \xi, f)$, but we will recall that g is a piecewise polynomial approximation on the input partition $\mathcal{D}_{\text{in}}$ of the given data $g_\star = (\nu_\star, \xi_\star, f_\star) \in G(\Omega)$. Coherently, the exact solution of Problem (8) with input data g will be denoted by $u = u(g)$, whereas for any hp-partition $\mathcal{D} \le \mathcal{D}_{\text{in}}$, $u_{\mathcal{D}} = u_{\mathcal{D}}(g)$ will be the solution of the corresponding DG Problem (14).

For the analysis of the procedure, following [3], we extend the definition of the DG form $a_{\mathcal{D}}$ given in (12) on $V_{\mathcal{D}} \times V_{\mathcal{D}}$ to the infinite dimensional space $V_{\mathcal{K}(\mathcal{D})} \times V_{\mathcal{K}(\mathcal{D})}$ (recall (9)). To this end, we introduce the lifting operator $L_{\mathcal{D}} : V_{\mathcal{K}(\mathcal{D})} \to V_{\mathcal{D}}$ such that for all $w \in V_{\mathcal{K}(\mathcal{D})}$

$$L_{\mathcal{D}} w \in V_{\mathcal{D}} \ : \ (\nu v, L_{\mathcal{D}} w)_\Omega = (\{\!\{\nu\, v\}\!\}, [\![w]\!])_{\mathcal{E}_{\mathcal{D}}} \quad \forall v \in V_{\mathcal{D}}. \tag{23}$$

Then, on $V_{\mathcal{K}(\mathcal{D})} \times V_{\mathcal{K}(\mathcal{D})}$ we define the bilinear form

$$\begin{aligned} a_{\mathcal{D}}(w, v) :=\ & (\nu\, \tilde{w}_x, \tilde{v}_x)_\Omega + (\xi\, w, v)_\Omega \\ & - (\nu\, \tilde{w}_x, L_{\mathcal{D}} v)_{\mathcal{E}_{\mathcal{D}}} - (\nu\, \tilde{v}_x, L_{\mathcal{D}} w)_{\mathcal{E}_{\mathcal{D}}} + \gamma\, (\sigma_{\mathcal{D}} [\![w]\!], [\![v]\!])_{\mathcal{E}_{\mathcal{D}}}\ , \end{aligned} \tag{24}$$

which is readily seen to coincide with (12) on $V_{\mathcal{D}} \times V_{\mathcal{D}}$.

The lifting operator satisfies the following stability bound.

Property 5.1. *There exists a constant $C_1 > 0$ independent of $\mathcal{D}$ such that*

$$\|L_{\mathcal{D}} w\|_\Omega \le C_1 \|\sigma_{\mathcal{D}}^{1/2} [\![w]\!]\|_{\mathcal{E}_{\mathcal{D}}} \qquad \forall w \in V_{\mathcal{K}(\mathcal{D})}. \tag{25}$$

Proof. If K is any interval of length h and e is one of its endpoints, the inverse inequality $|\phi(e)| \lesssim \frac{p}{h^{1/2}} \|\phi\|_K$ holds for any $\phi \in \mathbb{P}_p(K)$. Then, the result easily follows by choosing $v = L_{\mathcal{D}} w$ in (23). ◻

Using (25), one proves the existence of a constant $\gamma_0 > 0$ independent of $\mathcal{D}$ such that for any $\gamma \ge \gamma_0$ the bilinear form $a_{\mathcal{D}}$ is continuous and coercive in $V_{\mathcal{K}(\mathcal{D})}$ with respect to the DG norm $\|v\|_{\mathcal{D}}$, uniformly in $\mathcal{D}$. For future references, let us denote by $0 < \alpha_* \le \alpha^*$ the coercivity and continuity constants. Since $a_{\mathcal{D}}$ is symmetric, it defines an inner product in $V_{\mathcal{K}(\mathcal{D})}$; the corresponding norm will be denoted by $\|v\|_{a,\mathcal{D}}$ and is uniformly equivalent to the DG norm $\|v\|_{\mathcal{D}}$ introduced in (13).

It is well-known that while the DG-solution $u_{\mathcal{D}} \in V_{\mathcal{D}}$ satisfies the variational equations

$$a_{\mathcal{D}}(u_{\mathcal{D}}, v_{\mathcal{D}}) = (f, v_{\mathcal{D}})_\Omega \qquad \forall v_{\mathcal{D}} \in V_{\mathcal{D}}, \tag{26}$$

the exact solution $u \in H_0^1(\Omega)$ need not satisfy $a_{\mathcal{D}}(u, v) = (f, v)_\Omega$ for *all* $v \in V_{\mathcal{K}(\mathcal{D})}$ (*inconsistency* of the DG formulation). However, we do have the partial consistency property

$$a_{\mathcal{D}}(u, v) = (f, v)_\Omega \qquad \forall v \in H_0^1(\Omega). \tag{27}$$

This motivates the introduction of the conforming subspace $V_{\mathcal{D}}^c := V_{\mathcal{D}} \cap H_0^1(\Omega)$. Then, by subtraction of (26) from (27), we obtain the *partial orthogonality* property

$$a_{\mathcal{D}}(u - u_{\mathcal{D}}, v_{\mathcal{D}}) = 0 \qquad \forall v_{\mathcal{D}} \in V_{\mathcal{D}}^c. \tag{28}$$

It is useful for the sequel to introduce the orthogonal decomposition

$$V_{\mathcal{D}} = V_{\mathcal{D}}^c \oplus V_{\mathcal{D}}^\perp, \tag{29}$$

where $V_{\mathcal{D}}^\perp$ is the orthogonal complement of $V_{\mathcal{D}}^c$ with respect to the inner product $a_{\mathcal{D}}(w, v)$. Any $v_{\mathcal{D}} \in V_{\mathcal{D}}$ will be split according to (29) as $v_{\mathcal{D}} = v_{\mathcal{D}}^c + v_{\mathcal{D}}^\perp$.

Property 5.2. *There exists a constant $C_2 > 0$ independent of $\mathcal{D}$ for which the following bound on the DG discretization error holds:*

$$\|u - u_{\mathcal{D}}\|_{\mathcal{D}} \le C_2 \left(\inf_{w_{\mathcal{D}} \in V_{\mathcal{D}}^c} \|u - w_{\mathcal{D}}\|_{H_0^1(\Omega)} + \|u_{\mathcal{D}}^\perp\|_{\mathcal{D}} \right).$$

Proof. For any $w_{\mathcal{D}} \in V_{\mathcal{D}}^c$, using (28) we have

$$\begin{aligned} a_{\mathcal{D}}(u_{\mathcal{D}} - w_{\mathcal{D}}, u_{\mathcal{D}} - w_{\mathcal{D}}) &= a_{\mathcal{D}}(u_{\mathcal{D}} - w_{\mathcal{D}}, u_{\mathcal{D}}^c - w_{\mathcal{D}}) + a_{\mathcal{D}}(u_{\mathcal{D}} - w_{\mathcal{D}}, u_{\mathcal{D}}^\perp) \\ &= a_{\mathcal{D}}(u - w_{\mathcal{D}}, u_{\mathcal{D}}^c - w_{\mathcal{D}}) + a_{\mathcal{D}}(u_{\mathcal{D}}^\perp, u_{\mathcal{D}} - w_{\mathcal{D}}) \\ &= a_{\mathcal{D}}(u - w_{\mathcal{D}}, u_{\mathcal{D}} - w_{\mathcal{D}}) - a_{\mathcal{D}}(u_{\mathcal{D}}^\perp, u - u_{\mathcal{D}}), \end{aligned}$$

whence, by the coercivity and continuity of the form $a_{\mathcal{D}}$,

$$\|u_{\mathcal{D}} - w_{\mathcal{D}}\|_{\mathcal{D}}^2 \lesssim \|u - w_{\mathcal{D}}\|_{\mathcal{D}} \|u_{\mathcal{D}} - w_{\mathcal{D}}\|_{\mathcal{D}} + \|u_{\mathcal{D}}^\perp\|_{\mathcal{D}} \|u - u_{\mathcal{D}}\|_{\mathcal{D}}.$$

We conclude by the triangle inequality. ◻

We also introduce an approximation operator $\mathbb{I}_{\mathcal{D}} : V_{\mathcal{K}(\mathcal{D})} \to V_{\mathcal{D}}^c$ that will be useful in the sequel. For any $D \in \mathcal{D}$, set $K_D =: [e_l, e_r]$ and let $\mathcal{P}_D : H^1(K_D) \to \mathbb{P}_{p_D}(K_D)$ be defined as follows:

$$(\mathcal{P}_D v)(x) := v(e_l) + \int_{e_l}^{x} (\Pi^0_{K_D, p_D - 1} v_x)(s)\, ds$$

(recall that Π^0 means L^2-orthogonal projection). Furthermore, consider the Legendre Gauss-Lobatto grid in K_D containing p_D+1 nodes, and let ψ_{D,e_l} and ψ_{D,e_r} denote the Lagrange basis functions of degree p_D on this grid, associated with the boundary nodes. Then, we define $(\mathbb{I}_{\mathcal{D}}v)_{|K_D} := \mathcal{I}_D v_{|K_D}$, where

$$\mathcal{I}_D v := \mathcal{P}_D v - \tau_{e_l}[\![v]\!]_{e_l}\psi_{D,e_l} + \tau_{e_r}[\![v]\!]_{e_r}\psi_{D,e_r} \tag{30}$$

with $\tau_e = 1$ if $e \in \partial\Omega$, $\tau_e = \frac{1}{2}$ otherwise. Checking that $\mathbb{I}_{\mathcal{D}}v \in V_{\mathcal{D}}^c$ is straightforward.

Property 5.3. *The following error estimates hold for any* $v \in H_0^1(\Omega)$*:*

$$\|(v - \mathbb{I}_{\mathcal{D}}v)\omega_D^{-1/2}\|_{K_D} \le \frac{1}{(p_D(p_D+1))^{1/2}}\|v_x\|_{K_D}, \qquad \|(\mathbb{I}_{\mathcal{D}}v)_x\|_{K_D} \le \|v_x\|_{K_D}, \tag{31}$$

where ω_D *is the quadratic bubble function in* K_D*, defined as* $\omega_D(x) = (x-e_l)(e_r-x)$*.*

The following error estimates hold for any $v \in V_{\mathcal{D}}$*:*

$$\|v - \mathbb{I}_{\mathcal{D}}v\|_{K_D} \lesssim \frac{h_D^{1/2}}{p_D}([\![v]\!]_{e_l} + [\![v]\!]_{e_r}), \qquad \|(v - \mathbb{I}_{\mathcal{D}}v)_x\|_{K_D} \lesssim \frac{p_D}{h_D^{1/2}}([\![v]\!]_{e_l} + [\![v]\!]_{e_r}). \tag{32}$$

The latter inequality implies the bound

$$\|\tilde{v}_x - (\mathbb{I}_{\mathcal{D}}v)_x\|_\Omega \ \lesssim \ \|\sigma_{\mathcal{D}}^{1/2}[\![v]\!]\|_{\mathcal{E}_{\mathcal{D}}} \qquad \forall v \in V_{\mathcal{D}}. \tag{33}$$

Proof. The first inequality in (31) can be found in [10], whereas the second one is just the stability of the orthogonal projection. The inequalities (32) easily follow from the bounds $\|\psi_{D,e}\|_{K_D} \simeq \frac{h_D^{1/2}}{p_D}$ and $\|(\psi_{D,e})_x\|_{K_D} \lesssim \frac{p_D^2}{h_D}\|\psi_{D,e}\|_{K_D}$. □

Corollary 5.1. *There exists a constant* $C_3 > 0$ *independent of* $\mathcal{D}$ *such that for any* $v = v^c \oplus v^\perp \in V_{\mathcal{D}} = V_{\mathcal{D}}^c \oplus V_{\mathcal{D}}^\perp$ *one has*

$$\|v^\perp\|_{\mathcal{D}} \le C_3\,\gamma^{1/2}\|\sigma_{\mathcal{D}}^{1/2}[\![v]\!]\|_{\mathcal{E}_{\mathcal{D}}}$$

Proof. One has

$$\|v^\perp\|_{\mathcal{D}} \simeq \|v^\perp\|_{a,\mathcal{D}} = \inf_{w \in V_{\mathcal{D}}^c}\|v - w\|_{a,\mathcal{D}} \simeq \inf_{w \in V_{\mathcal{D}}^c}\|v - w\|_{\mathcal{D}} \le \|v - \mathbb{I}_{\mathcal{D}}v\|_{\mathcal{D}},$$

then one concludes by (33). □

5.1 The residual estimator

Given any $v \in V_{\mathcal{D}}$ and any $D \in \mathcal{D}$, let us define the local residual

$$\mathrm{res}_D(v) := (f - Av)_{|K_D};$$

for any $e \in \partial K_D$, let us define the jump of the flux at e

$$J_e(v) = [\![\,\nu v_x\,]\!]_e.$$

Then, the (squared) local error estimator is defined as follows

$$\eta_D^2(v) := \frac{1}{p_D(p_D+1)}\|\mathrm{res}_D(v)\,\omega_D^{1/2}\|_{K_D}^2 + \sum_{e \in \partial K_D}\sigma_{\mathcal{D},e}^{-1}J_e^2(v),$$

where ω_D denotes the quadratic bubble function introduced in Property 5.3 above. The (squared) global error estimator is

$$\eta_{\mathcal{D}}^2(v) := \sum_{D\in\mathcal{D}} \eta_D^2(v),$$

whereas its restriction to a subset $\mathcal{D}' \subseteq \mathcal{D}$ of elements will be denoted by

$$\eta_{\mathcal{D}}^2(v;\mathcal{D}') := \sum_{D\in\mathcal{D}'} \eta_D^2(v).$$

We show that $\eta_{\mathcal{D}}(u_{\mathcal{D}})$ is a reliable estimator for our DG problem in two steps.

Proposition 5.1. *There exists a constant $C_4 > 0$ independent of $\mathcal{D}$ such that*

$$a_{\mathcal{D}}(u-u_{\mathcal{D}}, u-u_{\mathcal{D}}) \le C_4 \left(\eta_{\mathcal{D}}^2(u_{\mathcal{D}}) + \gamma \,\|\sigma_{\mathcal{D}}^{1/2}[\![u_{\mathcal{D}}]\!]\|_{\mathcal{E}_{\mathcal{D}}}^2\right).$$

Proof. We adapt the proof of [4], Lemma 3.1, to our *hp* setting. Let us split the DG solution as $u_{\mathcal{D}} = u_{\mathcal{D}}^c + u_{\mathcal{D}}^{\perp}$ and let us set $e := u - u_{\mathcal{D}}$ and $w := u - u_{\mathcal{D}}^c \in H_0^1(\Omega)$, so that $e = w - u_{\mathcal{D}}^{\perp}$. Then, recalling (27) and (28),

$$\begin{aligned} a_{\mathcal{D}}(e,e) &= a_{\mathcal{D}}(e,w) - a_{\mathcal{D}}(e,u_{\mathcal{D}}^{\perp}) = a_{\mathcal{D}}(e, w-\mathbb{I}_{\mathcal{D}}w) - a_{\mathcal{D}}(e,u_{\mathcal{D}}^{\perp}) \\ &= (f, w-\mathbb{I}_{\mathcal{D}}w)_\Omega - a_{\mathcal{D}}(u_{\mathcal{D}}, w-\mathbb{I}_{\mathcal{D}}w) - a_{\mathcal{D}}(e,u_{\mathcal{D}}^{\perp}), \end{aligned}$$

Integrating back by parts, we get

$$a_{\mathcal{D}}(u_{\mathcal{D}}, w-\mathbb{I}_{\mathcal{D}}w) = \sum_{D\in\mathcal{D}} (Au_{\mathcal{D}}, w-\mathbb{I}_{\mathcal{D}}w)_{K_D} + (L_{\mathcal{D}}u_{\mathcal{D}}, \nu(w-\mathbb{I}_{\mathcal{D}}w)_x)_\Omega,$$

whence

$$a_{\mathcal{D}}(e,w) = \sum_{D\in\mathcal{D}} (\mathrm{res}_D(u_{\mathcal{D}}), w-\mathbb{I}_{\mathcal{D}}w)_{K_D} + (L_{\mathcal{D}}u_{\mathcal{D}}, \nu(w-\mathbb{I}_{\mathcal{D}}w)_x)_\Omega.$$

Writing $(\mathrm{res}_D(u_{\mathcal{D}}), w-\mathbb{I}_{\mathcal{D}}w)_{K_D} = (\mathrm{res}_D(u_{\mathcal{D}})\omega_D^{1/2}, (w-\mathbb{I}_{\mathcal{D}}w)\omega_D^{-1/2})_{K_D}$ and using (31) as well as (25), we obtain

$$a_{\mathcal{D}}(e,w) \le (\eta_{\mathcal{D}}(u_{\mathcal{D}}) + C_1\|\sigma_{\mathcal{D}}^{1/2}[\![u_{\mathcal{D}}]\!]\|_{\mathcal{E}_{\mathcal{D}}})\|w_x\|_\Omega,$$

where the last norm can be bounded using the coercivity of the form $a_{\mathcal{D}}$:

$$\|w_x\|_\Omega = \|w\|_{\mathcal{D}} \le \|e\|_{\mathcal{D}} + \|u_{\mathcal{D}}^{\perp}\|_{\mathcal{D}} \le \alpha_*^{1/2} a_{\mathcal{D}}(e,e)^{1/2} + \|u_{\mathcal{D}}^{\perp}\|_{\mathcal{D}}.$$

By Young's inequality, we obtain for a suitable constant $C > 0$

$$a_{\mathcal{D}}(e,w) \le \frac{1}{4} a_{\mathcal{D}}(e,e) + C\big(\eta_{\mathcal{D}}^2(u_{\mathcal{D}}) + \|u_{\mathcal{D}}^{\perp}\|_{\mathcal{D}}^2 + \|\sigma_{\mathcal{D}}^{1/2}[\![u_{\mathcal{D}}]\!]\|_{\mathcal{E}_{\mathcal{D}}}^2\big).$$

It remains to bound the term $a_{\mathcal{D}}(e,u_{\mathcal{D}}^{\perp})$, which is easily done using the continuity of $a_{\mathcal{D}}$:

$$a_{\mathcal{D}}(e,u_{\mathcal{D}}^{\perp}) \le a_{\mathcal{D}}(e,e)^{1/2} a_{\mathcal{D}}(u_{\mathcal{D}}^{\perp},u_{\mathcal{D}}^{\perp})^{1/2} \le a_{\mathcal{D}}(e,e)^{1/2}(\alpha^*)^{1/2}\|u_{\mathcal{D}}^{\perp}\|_{\mathcal{D}} \le \frac{1}{4}a_{\mathcal{D}}(e,e) + \alpha^*\|u_{\mathcal{D}}^{\perp}\|_{\mathcal{D}}^2.$$

We obtain the desired result by invoking Corollary 5.1. ∎

Proposition 5.2. *There exists a constant $C_5 > 0$ independent of $\mathcal{D}$ such that for any γ large enough, say $\gamma \geq \gamma_1 \geq \gamma_0$, one has*

$$\gamma \, \|\sigma_{\mathcal{D}}^{1/2} [\![u_{\mathcal{D}}]\!] \|_{\mathcal{E}_{\mathcal{D}}} \leq C_5 \eta_{\mathcal{D}}(u_{\mathcal{D}}).$$

Proof. Here, we adapt the proof of [4], Lemma 3.3, to our *hp* setting. By the coercivity of the form $a_{\mathcal{D}}$ applied to $u_{\mathcal{D}} - \mathbb{I}_{\mathcal{D}} u_{\mathcal{D}}$, we have

$$\gamma \, \|\sigma_{\mathcal{D}}^{1/2} [\![u_{\mathcal{D}}]\!] \|^2_{\mathcal{E}_{\mathcal{D}}} \leq \alpha_*^{-1} a_{\mathcal{D}}(u_{\mathcal{D}} - \mathbb{I}_{\mathcal{D}} u_{\mathcal{D}}, u_{\mathcal{D}} - \mathbb{I}_{\mathcal{D}} u_{\mathcal{D}}) \tag{34}$$

since $[\![\mathbb{I}_{\mathcal{D}} u_{\mathcal{D}}]\!] = 0$. For simplicity, let us set $w := u_{\mathcal{D}} - \mathbb{I}_{\mathcal{D}} u_{\mathcal{D}}$ and $v := \mathbb{I}_{\mathcal{D}} u_{\mathcal{D}} \in H^1_0(\Omega)$. Then,

$$a_{\mathcal{D}}(w, w) = (f, w)_\Omega - a_{\mathcal{D}}(v, w)$$

and, using $L_{\mathcal{D}} v = 0$ several times, we have

$$\begin{aligned} a_{\mathcal{D}}(v, w) &= (\nu v_x, \tilde{w}_x)_\Omega + (\xi v, w)_\Omega - (L_{\mathcal{D}} u_{\mathcal{D}}, \nu v_x)_\Omega \\ &= (\nu \tilde{u}_{\mathcal{D},x}, \tilde{w}_x)_\Omega + (\xi u_{\mathcal{D}}, w)_\Omega - \|\nu^{1/2} \tilde{w}_x\|^2_\Omega - \|\xi^{1/2} w\|^2_\Omega - (L_{\mathcal{D}} u_{\mathcal{D}}, \nu v_x)_\Omega. \end{aligned}$$

Using in this identity

$$(\nu v_x, \tilde{w}_x)_\Omega = -\sum_{D \in \mathcal{D}} ((\nu u_{\mathcal{D},x})_x, w)_{K_D} + ([\![\nu u_{\mathcal{D},x}]\!], \{\!\{w\}\!\})_{\mathcal{E}_{\mathcal{D}}} + ([\![w]\!], \{\!\{\nu u_{\mathcal{D},x}\}\!\})_{\mathcal{E}_{\mathcal{D}}}$$

and observing that $([\![w]\!], \{\!\{\nu u_{\mathcal{D},x}\}\!\})_{\mathcal{E}_{\mathcal{D}}} = (L_{\mathcal{D}} w, \nu \tilde{u}_{\mathcal{D},x})_\Omega$, we obtain

$$\begin{aligned} a_{\mathcal{D}}(w, w) = \sum_{D \in \mathcal{D}} &(\mathrm{res}_D(u_{\mathcal{D}}), w)_{K_D} + (J_{\mathcal{D}}(u_{\mathcal{D}}), \{\!\{w\}\!\})_{\mathcal{E}_{\mathcal{D}}} \\ &+ \|\nu^{1/2} \tilde{w}_x\|^2_\Omega + \|\xi^{1/2} w\|^2_\Omega + (L_{\mathcal{D}} u_{\mathcal{D}}, \nu \tilde{w}_x)_\Omega. \end{aligned} \tag{35}$$

By (32) we have

$$\|w\|_{K_D} \leq \sum_{e \in \partial K_D} \frac{h_D^{1/2}}{p_D} | \, [\![u_{\mathcal{D}}]\!]_e \, | = \sum_{e \in \partial K_D} \frac{h_D^{1/2}}{p_D} \sigma_{\mathcal{D},e}^{-1/2} \sigma_{\mathcal{D},e}^{1/2} | \, [\![u_{\mathcal{D}}]\!]_e \, | \leq \frac{h_D}{p_D^2} \sum_{e \in \partial K_D} \sigma_{\mathcal{D},e}^{1/2} | \, [\![u_{\mathcal{D}}]\!]_e \, |,$$

whence

$$\begin{aligned} (\mathrm{res}_D(u_{\mathcal{D}}), w)_{K_D} &\leq \frac{h_D}{p_D^2} \|\mathrm{res}_D(u_{\mathcal{D}})\|_{K_D} \sum_{e \in \partial K_D} \sigma_{\mathcal{D},e}^{1/2} | \, [\![u_{\mathcal{D}}]\!]_e \, | \\ &\lesssim \frac{1}{p_D} \|\mathrm{res}_D(u_{\mathcal{D}}) \omega_D^{1/2}\|_{K_D} \sum_{e \in \partial K_D} \sigma_{\mathcal{D},e}^{1/2} | \, [\![u_{\mathcal{D}}]\!]_e \, | \leq \eta_D(u_{\mathcal{D}}) \sum_{e \in \partial K_D} \sigma_{\mathcal{D},e}^{1/2} | \, [\![u_{\mathcal{D}}]\!]_e \, |, \end{aligned}$$

where we have used the inverse inequality $\|\phi\|_{K_D} \lesssim \frac{p_D}{h_D} \|\phi \, \omega_D^{1/2}\|_{K_D}$ which holds for all polynomials of degree $\simeq p_D$, since $\mathrm{res}_D(u_{\mathcal{D}})$ is such a polynomial. Thus, we obtain

$$\sum_{D \in \mathcal{D}} (\mathrm{res}_D(u_{\mathcal{D}}), w)_{K_D} \lesssim \eta_{\mathcal{D}}(u_{\mathcal{D}}) \, \|\sigma_{\mathcal{D}}^{1/2} [\![u_{\mathcal{D}}]\!] \|_{\mathcal{E}_{\mathcal{D}}}.$$

Concerning the second term on the right-hand side of (35), we observe that by construction of $\mathbb{I}_{\mathcal{D}} u_{\mathcal{D}}$, one has $w(e) = \frac{1}{2} [\![u_{\mathcal{D}}]\!]_e$ at any internal inter-element point e, whereas $w(e) = 0$ at the boundary points of Ω. Thus,

$$\begin{aligned} (J_{\mathcal{D}}(u_{\mathcal{D}}), \{\!\{w\}\!\})_{\mathcal{E}_{\mathcal{D}}} &\lesssim \sum_{e \in \mathcal{E}_{\mathcal{D}}} |J_e(u_{\mathcal{D}})| \, | \, [\![u_{\mathcal{D}}]\!]_e \, | = \sum_{e \in \mathcal{E}_{\mathcal{D}}} \sigma_{\mathcal{D},e}^{-1/2} |J_e(u_{\mathcal{D}})| \, \sigma_{\mathcal{D},e}^{1/2} | \, [\![u_{\mathcal{D}}]\!]_e \, | \\ &\leq \eta_{\mathcal{D}}(u_{\mathcal{D}}) \, \|\sigma_{\mathcal{D}}^{1/2} [\![u_{\mathcal{D}}]\!] \|_{\mathcal{E}_{\mathcal{D}}}. \end{aligned}$$

Finally, using (32) and (25), the three last terms on the right-hand side of (35) can be bounded by $C\|\sigma_{\mathcal{D}}^{1/2}[\![u_{\mathcal{D}}]\!]\|^2_{\mathcal{E}_{\mathcal{D}}}$. Substituting all the previous bounds in (34), we obtain

$$\gamma\,\|\sigma_{\mathcal{D}}^{1/2}[\![u_{\mathcal{D}}]\!]\|^2_{\mathcal{E}_{\mathcal{D}}} \lesssim \big(\eta_{\mathcal{D}}(u_{\mathcal{D}})\,\|\sigma_{\mathcal{D}}^{1/2}[\![u_{\mathcal{D}}]\!]\|_{\mathcal{E}_{\mathcal{D}}} + \|\sigma_{\mathcal{D}}^{1/2}[\![u_{\mathcal{D}}]\!]\|^2_{\mathcal{E}_{\mathcal{D}}}\big),$$

where the constant implied by the symbol $\lesssim$ is independent of γ. Therefore, choosing γ large enough, we get the desired result. ∎

Corollary 5.2. *There exists a constant $C_6 > 0$ independent of $\mathcal{D}$ such that for any $\gamma \geq \gamma_1$, one has*

$$a_{\mathcal{D}}(u - u_{\mathcal{D}}, u - u_{\mathcal{D}}) \leq C_6\, \eta^2_{\mathcal{D}}(u_{\mathcal{D}}). \qquad ∎$$

5.2 The adaptive iterations

The routine **DG-SOLVE** iterates the mapping

$$(\mathcal{D}, u_{\mathcal{D}}, \eta_{\mathcal{D}}(u_{\mathcal{D}})) \;\to\; (\mathcal{D}_*, u_{\mathcal{D}_*}, \eta_{\mathcal{D}_*}(u_{\mathcal{D}_*}))\,, \tag{36}$$

where $\mathcal{D}_*$ is a refinement of $\mathcal{D}$ obtained by first applying a Dörfler marking to the elements of $\mathcal{D}$ based on the error estimator $\eta_{\mathcal{D}}(u_{\mathcal{D}})$, and then performing a dyadic subdivision to the marked elements and its neighbors.

To be precise, let $\vartheta \in (0,1)$ be the Dörfler parameter. Let us order the local error estimators $\eta_D(u_{\mathcal{D}})$, $D \in \mathcal{D}$, by decreasing value, and let us choose a set $\mathcal{M} \subseteq \mathcal{D}$ of minimal cardinality for which

$$\eta_{\mathcal{D}}(u_{\mathcal{D}}; \mathcal{M}) \geq \vartheta\, \eta_{\mathcal{D}}(u_{\mathcal{D}}). \tag{37}$$

Let $\partial\mathcal{M} \subseteq \mathcal{D}$ denote the set of elements D that share an interface with an element in $\mathcal{M}$. Then, we replace each $D = (K_D, p_D) \in \mathcal{M} \cup \partial\mathcal{M}$ by the two elements $D' = (K'_D, p_D)$ and $D'' = (K''_D, p_D)$, where K'_D and K''_D are the two children of K_D. Thus, the new partition $\mathcal{D}_*$ is defined by

$$\mathcal{D}_* = \{D', D'' : D \in \mathcal{M} \cup \partial\mathcal{M}\} \;\cup\; \{D : D \in \mathcal{D} \setminus (\mathcal{M} \cup \partial\mathcal{M})\}. \tag{38}$$

Our aim is to prove that a suitable combination of (squared) DG error and error estimator, i.e.,

$$\|u - u_{\mathcal{D}}\|^2_{a,\mathcal{D}} + \beta\, \eta^2_{\mathcal{D}}(u_{\mathcal{D}})$$

for some $\beta > 0$, is reduced by a fixed rate $\varrho \in (0,1)$ in performing the mapping (36). The proof, which extends [4] to our *hp*-setting, will be based on the following results.

Lemma 5.1. *There exists a constant $C_7 > 0$ independent of $\mathcal{D}$ such that for any real $\lambda \in (0,1)$, one has*

$$\eta^2_{\mathcal{D}_*}(u_{\mathcal{D}_*}) \leq (1+\lambda)(1 - \frac{\vartheta^2}{2})\, \eta^2_{\mathcal{D}}(u_{\mathcal{D}}) + \frac{C_7}{\lambda} \|u_{\mathcal{D}_*} - u_{\mathcal{D}}\|^2_{\mathcal{D}_*}.$$

Proof. We first establish a few results about the Lipschitz continuity of the local error estimators. Assume that $v, w \in V_{\mathcal{D}}$ and let $D \in \mathcal{D}$. By Minkowski's inequality,

$$|\eta_D(v) - \eta_D(w)| \leq \left(\frac{1}{p_D^2} \|(\mathrm{res}_D(v) - \mathrm{res}_D(w))\, \omega_D^{1/2}\|^2_{K_D} + \sum_{e \in \partial K_D} \sigma^{-1}_{\mathcal{D},e} |J_e(v) - J_e(w)|^2 \right)^{1/2}.$$

One has

$$
\begin{aligned}
\|(\mathrm{res}_D(v) - \mathrm{res}_D(w))\,\omega_D^{1/2}\|_{K_D} &\leq \|(\nu(v-w)_x)_x\,\omega_D^{1/2}\|_{K_D} + \|\xi(v-w)\,\omega_D^{1/2}\|_{K_D} \\
&\lesssim p_D\|\nu(v-w)_x\|_{K_D} + h_D\|\xi(v-w)\|_{K_D} \\
&\lesssim p_D\|(v-w)_x\|_{K_D} + h_D\|(v-w)\|_{K_D},
\end{aligned}
$$

where we have used the inverse inequality $\|\phi_x\,\omega_D^{1/2}\|_{K_D} \lesssim p_D\|\phi\|_{K_D}$, which holds for all polynomial ϕ of degree $\simeq p_D$ in K_D, as well as the bound $\|\omega_D^{1/2}\|_{L^\infty(K_D)} \leq h_D$.

On the other hand, for each $e \in \partial K_D$, let us denote by D' the element in $\mathcal{D}$ sharing the interface e with D. Then,

$$
\begin{aligned}
|J_e(v) - J_e(w)| &\leq |\nu(v-w)_{x|K_D}(e)| + |\nu(v-w)_{x|K_{D'}}(e)| \\
&\lesssim |(v-w)_{x|K_D}(e)| + |(v-w)_{x|K_{D'}}(e)| \\
&\lesssim \frac{p_D}{h_D^{1/2}}\|(v-w)_x\|_{K_D} + \frac{p_{D'}}{h_{D'}^{1/2}}\|(v-w)_x\|_{K_{D'}} \\
&\leq \sigma_{\mathcal{D},e}^{1/2}\left(\|(v-w)_x\|_{K_D} + \|(v-w)_x\|_{K_{D'}}\right),
\end{aligned}
$$

where we have used the inverse inequality $|\psi(e)| \lesssim \frac{p_D}{h_D^{1/2}}\|\psi\|_{K_D}$, which holds for all polynomial ψ of degree $\simeq p_D$ in K_D. We conclude that

$$
|\eta_D(v) - \eta_D(w)| \lesssim \mathcal{N}_D(v-w), \quad \text{with} \;\; \mathcal{N}_D^2(\phi) := \sum_{D'} \|\phi_x\|_{K_D}^2 + \frac{h_D^2}{p_D^2}\|\phi\|_{K_D}^2,
$$

where summation is extended to all $D' \in \mathcal{D}$ such that $K_{D'} \cap K_D$ is nonempty; this implies

$$
\eta_D^2(v) \leq (1+\lambda)\,\eta_D^2(w) + \frac{C}{\lambda}\mathcal{N}_D^2(v-w) \tag{39}
$$

for a suitable constant $C > 0$ independent of D.

We now apply these bounds, with $v = u_{\mathcal{D}_*}$ and $w = u_{\mathcal{D}}$, to the partition (38) generated by the refinement procedure. If $D \in \mathcal{M}$, let D_m, $m = 1, 2$ be the two children in which D is split. We have $\omega_{D_m}(x) \leq \frac{1}{2}\omega_D(x)$ for all $x \in D_m$. By definition of refinement, we have $h_{D_m} = \frac{1}{2}h_D$ as well as $h_{D'_m} = \frac{1}{2}h_{D'}$ for any neighborhood $D' \in \mathcal{D}$ of D, which implies $\sigma_{\mathcal{D}_*,e}^{-1} \leq \frac{1}{2}\sigma_{\mathcal{D},e}^{-1}$ for any $e \in \partial K_D$. Hence, we immediately have $\sum_{m=1}^2 \eta_{D_m}^2(u_{\mathcal{D}}) \leq \frac{1}{2}\eta_D^2(u_{\mathcal{D}})$ and $\sum_{m=1}^2 \mathcal{N}_{D_m}(u_{\mathcal{D}_*} - u_{\mathcal{D}}) \leq \mathcal{N}_D(u_{\mathcal{D}_*} - u_{\mathcal{D}})$, whence

$$
\sum_{m=1}^2 \eta_{D_m}^2(u_{\mathcal{D}_*}) \leq \frac{1}{2}(1+\lambda)\,\eta_D^2(u_{\mathcal{D}}) + \frac{C}{\lambda}\mathcal{N}_D^2(u_{\mathcal{D}_*} - u_{\mathcal{D}}).
$$

If $D \in \partial\mathcal{M}$, we can only say that $\sigma_{\mathcal{D}_*,e}^{-1} \leq \sigma_{\mathcal{D},e}^{-1}$ for any $e \in \partial K_D$, whence

$$
\sum_{m=1}^2 \eta_{D_m}^2(u_{\mathcal{D}_*}) \leq (1+\lambda)\,\eta_D^2(u_{\mathcal{D}}) + \frac{C}{\lambda}\mathcal{N}_D^2(u_{\mathcal{D}_*} - u_{\mathcal{D}}).
$$

Finally, for any unsplit $D \in \mathcal{D} \setminus (\mathcal{M} \cup \partial\mathcal{M})$, we just have

$$
\eta_D^2(u_{\mathcal{D}_*}) \leq (1+\lambda)\,\eta_D^2(u_{\mathcal{D}}) + \frac{C}{\lambda}\mathcal{N}_D^2(u_{\mathcal{D}_*} - u_{\mathcal{D}}).
$$

Summing-up all contributions and using the marking condition, we obtain

$$
\begin{aligned}
\eta_{\mathcal{D}_*}^2(u_{\mathcal{D}_*}) &\leq (1+\lambda)\left(\eta_{\mathcal{D}}^2(u_{\mathcal{D}}) - \frac{1}{2}\eta_{\mathcal{D}}^2(u_{\mathcal{D}};\mathcal{M})\right) + \frac{C}{\lambda}\sum_{D\in\mathcal{D}} \mathcal{N}_D^2(u_{\mathcal{D}_*} - u_{\mathcal{D}}) \\
&\leq (1+\lambda)(1 - \frac{\vartheta^2}{2})\eta_{\mathcal{D}}^2(u_{\mathcal{D}}) + \frac{C}{\lambda}\sum_{D\in\mathcal{D}} \mathcal{N}_D^2(u_{\mathcal{D}_*} - u_{\mathcal{D}}).
\end{aligned}
$$

It remains to prove that $\sum_{D\in\mathcal{D}} \mathcal{N}_D^2(u_{\mathcal{D}_*}-u_{\mathcal{D}}) \lesssim \|u_{\mathcal{D}_*}-u_{\mathcal{D}}\|_{\mathcal{D}_*}^2$. Setting now $w := u_{\mathcal{D}_*}-u_{\mathcal{D}}$, we have

$$\sum_{D\in\mathcal{D}} \mathcal{N}_D^2(w) = \|\tilde{w}_x\|_\Omega^2 + \sum_{D\in\mathcal{D}} \frac{h_D^2}{p_D^2}\|w\|_{K_D}^2 .$$

Writing, for a.e. $x\in\Omega$,

$$w(x) = \sum_{e\in\mathcal{E}_{\mathcal{D}_*}, e<x} [\![w]\!]_e + \int_{\min\Omega}^x \tilde{w}_x(s)\,ds = \sum_{e\in\mathcal{E}_{\mathcal{D}_*}, e<x} \sigma_{\mathcal{D}_*,e}^{-1/2}\sigma_{\mathcal{D}_*,e}^{-1/2}[\![w]\!]_e + \int_{\min\Omega}^x \tilde{w}_x(s)\,ds\,,$$

we have

$$w^2(x) \lesssim \Big(\sum_{e\in\mathcal{E}_{\mathcal{D}_*}} \sigma_{\mathcal{D}_*,e}^{-1}\Big) \sum_{e\in\mathcal{E}_{\mathcal{D}_*}} \sigma_{\mathcal{D}_*,e}[\![w]\!]_e^2 \;+\; |\Omega|\,\|\tilde{w}_x\|_\Omega^2.$$

Since $\sum_{e\in\mathcal{E}_{\mathcal{D}_*}} \sigma_{\mathcal{D}_*,e}^{-1} \le |\Omega|$, we easily obtain the desired bound. □

Lemma 5.2. *There exists a constant $C_8>0$ independent of $\mathcal{D}$ such that for any real $\delta\in(0,1)$ and any $\gamma\ge\gamma_1$, one has*

$$\|u-u_{\mathcal{D}_*}\|_{a,\mathcal{D}_*}^2 \le (1+\delta)\|u-u_{\mathcal{D}}\|_{a,\mathcal{D}}^2 - \frac{\alpha_*}{2}\|u_{\mathcal{D}_*}-u_{\mathcal{D}}\|_{\mathcal{D}_*}^2 + \frac{C_8}{\delta\gamma}\left(\eta_{\mathcal{D}_*}^2(u_{\mathcal{D}_*})+\eta_{\mathcal{D}}^2(u_{\mathcal{D}})\right).$$

Proof. Let us set $w_* := u-u_{\mathcal{D}_*}$, $w := u-u_{\mathcal{D}}$, $d := u_{\mathcal{D}_*}-u_{\mathcal{D}}$, $d^c := u^c_{\mathcal{D}_*}-u^c_{\mathcal{D}}$ and $d^\perp := u^\perp_{\mathcal{D}_*}-u^\perp_{\mathcal{D}}$. Observing that $a_{\mathcal{D}_*}(w_*,d^c)=0$ by the partial orthogonality property (28), one easily gets

$$\|w_*\|_{a,\mathcal{D}_*}^2 = a_{\mathcal{D}_*}(w_*,w_*) = a_{\mathcal{D}_*}(w_*+d^c,w_*+d^c) - a_{\mathcal{D}_*}(d^c,d^c).$$

Using $u_{\mathcal{D}} = u^c_{\mathcal{D}}+u^\perp_{\mathcal{D}}$ and $u_{\mathcal{D}_*} = u^c_{\mathcal{D}_*}+u^\perp_{\mathcal{D}_*}$, one has $w_*+d^c = w-d^\perp$, whence

$$\begin{aligned} a_{\mathcal{D}_*}(w_*+d^c,w_*+d^c) &= a_{\mathcal{D}_*}(w,w) - 2a_{\mathcal{D}_*}(w,d^\perp) + a_{\mathcal{D}_*}(d^\perp,d^\perp) \\ &\le \|w\|_{a,\mathcal{D}_*}^2 + 2(\alpha^*)^{1/2}\|w\|_{a,\mathcal{D}_*}\|d^\perp\|_{\mathcal{D}_*} + \alpha^*\|d^\perp\|_{\mathcal{D}_*}^2, \end{aligned}$$

where we have used the uniform continuity of the form $a_{\mathcal{D}_*}$ with respect to the DG-norm. Using the uniform coercivity and the triangle inequality, we get

$$a_{\mathcal{D}_*}(d^c,d^c) \ge \alpha_*\|d^c\|_{\mathcal{D}_*}^2 \ge \alpha_*\left(\frac{1}{2}\|d\|_{\mathcal{D}_*}^2 - \|d^\perp\|_{\mathcal{D}_*}^2\right).$$

Collecting these inequalities and using Young's inequality, we obtain

$$\|w_*\|_{a,\mathcal{D}_*}^2 \le (1+\delta)\|w\|_{a,\mathcal{D}_*}^2 - \frac{\alpha_*}{2}\|d\|_{\mathcal{D}_*}^2 + \frac{C}{\delta}\|d^\perp\|_{\mathcal{D}_*}^2. \tag{40}$$

At this point, we observe that $\|u^\perp_{\mathcal{D}}\|_{\mathcal{D}_*}^2 \le 2\|u^\perp_{\mathcal{D}}\|_{\mathcal{D}}^2$. Indeed, $\|u^\perp_{\mathcal{D}}\|_{\mathcal{D}_*}^2 = \|(u^\perp_{\mathcal{D}})^\sim_x\|_\Omega^2 + \gamma\sum_{e\in\mathcal{E}_{\mathcal{D}_*}} \sigma_{\mathcal{D}_*,e}[\![u^\perp_{\mathcal{D}}]\!]_e^2$, but the jumps of $u^\perp_{\mathcal{D}}$ occur only at the interfaces $e\in\mathcal{E}_{\mathcal{D}}$, and $\sigma_{\mathcal{D}_*,e}\le 2\sigma_{\mathcal{D},e}$ by definition of the refinement strategy. Thus, using Corollary 5.1, we get

$$\|d^\perp\|_{\mathcal{D}_*}^2 \;\lesssim\; \|u^\perp_{\mathcal{D}_*}\|_{\mathcal{D}_*}^2 + \|u^\perp_{\mathcal{D}}\|_{\mathcal{D}}^2 \;\lesssim\; \gamma\|\sigma_{\mathcal{D}_*}^{1/2}[\![u_{\mathcal{D}_*}]\!]\|_{\mathcal{E}_{\mathcal{D}_*}}^2 + \gamma\|\sigma_{\mathcal{D}}^{1/2}[\![u_{\mathcal{D}}]\!]\|_{\mathcal{E}_{\mathcal{D}}}^2. \tag{41}$$

It remains to replace $\|w\|_{a,\mathcal{D}_*}^2$ by $\|w\|_{a,\mathcal{D}}^2$. To this end, let us write

$$a_{\mathcal{D}_*}(w,w) = a_{\mathcal{D}}(w,w) + 2(L_{\mathcal{D}_*}w,\nu\tilde{w}_x)_\Omega - 2(L_{\mathcal{D}}w,\nu\tilde{w}_x)_\Omega - \gamma\|\sigma_{\mathcal{D}}^{1/2}[\![w]\!]\|_{\mathcal{E}_{\mathcal{D}}}^2 + \gamma\|\sigma_{\mathcal{D}_*}^{1/2}[\![w]\!]\|_{\mathcal{E}_{\mathcal{D}_*}}^2.$$

Using Property 5.1 and the coercivity of the form $a_{\mathcal{D}}$, one gets

$$(L_{\mathcal{D}_*}w, \nu\tilde{w}_x)_\Omega \lesssim \|\sigma_{\mathcal{D}_*}^{1/2}[\![w]\!]\|_{\mathcal{E}_{\mathcal{D}_*}} a_{\mathcal{D}}(w,w)^{1/2} \lesssim \|\sigma_{\mathcal{D}}^{1/2}[\![u_{\mathcal{D}}]\!]\|_{\mathcal{E}_{\mathcal{D}}} a_{\mathcal{D}}(w,w)^{1/2}.$$

A similar bound holds for $(L_{\mathcal{D}}w, \nu\tilde{w}_x)_\Omega$. Therefore, using once more Young's inequality, we arrive at

$$\|w\|_{a,\mathcal{D}_*}^2 \le (1+\delta)\|w\|_{a,\mathcal{D}}^2 + \frac{C}{\delta}\gamma\|\sigma_{\mathcal{D}}^{1/2}[\![u_{\mathcal{D}}]\!]\|_{\mathcal{E}_{\mathcal{D}}}^2. \tag{42}$$

Replacing (41)-(42) into (40), we obtain

$$\begin{aligned}\|u-u_{\mathcal{D}_*}\|_{a,\mathcal{D}_*}^2 \quad &\le \quad (1+\delta)^2\|u-u_{\mathcal{D}}\|_{a,\mathcal{D}}^2 - \frac{\alpha_*}{2}\|u_{\mathcal{D}_*}-u_{\mathcal{D}}\|_{\mathcal{D}_*}^2 \\ &\quad + \frac{C}{\delta}\gamma\left(\|\sigma_{\mathcal{D}_*}^{1/2}[\![u_{\mathcal{D}_*}]\!]\|_{\mathcal{E}_{\mathcal{D}_*}}^2 + \|\sigma_{\mathcal{D}}^{1/2}[\![u_{\mathcal{D}}]\!]\|_{\mathcal{E}_{\mathcal{D}}}^2\right).\end{aligned}$$

The desired result follows from Proposition 5.2, after replacing δ by $\delta/3$. ∎

We are ready to establish the main result of this section.

Theorem 5.1. *Consider the mapping* (36) *defined above. There exist constants $\beta > 0$ and $\varrho \in (0,1)$, independent of $\mathcal{D}$, such that, choosing $\gamma > 0$ large enough in the definition* (24)*, one has*

$$\|u-u_{\mathcal{D}_*}\|_{a,\mathcal{D}_*}^2 + \beta\,\eta_{\mathcal{D}_*}^2(u_{\mathcal{D}_*}) \le \varrho\left(\|u-u_{\mathcal{D}}\|_{a,\mathcal{D}}^2 + \beta\,\eta_{\mathcal{D}}^2(u_{\mathcal{D}})\right).$$

Proof. Let us simplify our notation by setting $E_*^2 := \|u-u_{\mathcal{D}_*}\|_{a,\mathcal{D}_*}^2$, $E^2 := \|u-u_{\mathcal{D}}\|_{a,\mathcal{D}}^2$, $e_*^2 := \|u_{\mathcal{D}_*}-u_{\mathcal{D}}\|_{\mathcal{D}_*}^2$ and $\eta_*^2 := \eta_{\mathcal{D}_*}^2(u_{\mathcal{D}_*})$, $\eta^2 := \eta_{\mathcal{D}}^2(u_{\mathcal{D}})$. Then, the inequalities of Lemmas 5.2-5.1 read as follows:

$$\begin{aligned}E_*^2 \quad &\le \quad (1+\delta)E^2 - \frac{\alpha_*}{2}e_*^2 + \frac{C_8}{\delta\gamma}(\eta_*^2+\eta^2) \\ \eta_*^2 \quad &\le \quad (1+\lambda)(1-\frac{\vartheta^2}{2})\,\eta^2 + \frac{C_7}{\lambda}e_*^2.\end{aligned}$$

Thus, for any real $\beta > 0$,

$$\begin{aligned}E_*^2 + \beta\eta_*^2 \quad &\le \quad (1+\delta)E^2 - \frac{\alpha_*}{2}e_*^2 + \left(\beta + \frac{C_8}{\delta\gamma}\right)\eta_*^2 + \frac{C_8}{\delta\gamma}\eta^2 \\ &\le \quad (1+\delta)E^2 - \frac{\alpha_*}{2}e_*^2 + \left(\beta + \frac{C_8}{\delta\gamma}\right)\left((1+\lambda)(1-\frac{\vartheta^2}{2})\,\eta^2 + \frac{C_7}{\lambda}e_*^2\right) + \frac{C_8}{\delta\gamma}\eta^2.\end{aligned}$$

Writing $1-\frac{\vartheta^2}{2} = \left(1-\frac{\vartheta^2}{4}\right) - \frac{\vartheta^2}{4}$ and using $E^2 \le C_6\eta^2$ from Corollary (5.2), we easily obtain for $\gamma \ge \gamma_1$

$$\begin{aligned}E_*^2 + \beta\eta_*^2 \quad &\le \quad \left[(1+\delta) - \left(\beta + \frac{C_8}{\delta\gamma}\right)\frac{1+\lambda}{C_6}\frac{\vartheta^2}{4}\right]E^2 \;+\; \left[\left(\beta + \frac{C_8}{\delta\gamma}\right)\frac{C_7}{\lambda} - \frac{\alpha_*}{2}\right]e_*^2 \\ &\quad + \left[(1+\lambda)\left(1-\frac{\vartheta^2}{4}\right) + \frac{C_8}{\beta\delta\gamma}\left(1+(1+\lambda)\left(1-\frac{\vartheta^2}{4}\right)\right)\right]\beta\eta^2 \\ &=: \quad \varrho_1 E^2 + \varrho_2\, e_*^2 + \varrho_3\,\beta\eta^2.\end{aligned}$$

At this point, we first choose λ sufficiently small to have $(1+\lambda)\left(1-\frac{\vartheta^2}{4}\right) < 1$. Next, we choose δ sufficiently small to have $\varrho_1 < 1$ for $\gamma = \gamma_1$, hence for any $\gamma \ge \gamma_1$. Then, the parameter $\beta > 0$ is determined by imposing $\varrho_2 = 0$, which is possible provided γ is large enough, say $\gamma \ge \gamma_2 \ge \gamma_1$. Finally, for γ even larger, say $\gamma \ge \gamma_3 \ge \gamma_2$, the second addend in ϱ_3 can be made so small that $\varrho_3 < 1$. In conclusion, the desired result holds for all $\gamma \ge \gamma_3$ with $\varrho := \max(\varrho_1, \varrho_3)$. ∎

Corollary 5.3. *Denote by* $\{(\mathcal{D}_k, u_{\mathcal{D}_k}, \eta_{\mathcal{D}_k}(u_{\mathcal{D}_k})) : k \geq 0\}$ *the sequence produced by iterating the mapping* (36) *from the input partition* $\mathcal{D}_0 := \mathcal{D}_{in}$. *Then,*

$$\|u - u_{\mathcal{D}_k}\|^2_{\mathcal{D}_k} \leq \alpha_*^{-1} \varrho^k \big(\|u - u_{\mathcal{D}_0}\|^2_{a,\mathcal{D}_0} + \beta\, \eta^2_{\mathcal{D}_0}(u_{\mathcal{D}_0}) \big). \qquad \square$$

The latter result guarantees that the target accuracy $\|u - u_{\mathcal{D}_k}\|^2_{\mathcal{D}_k} \leq \varepsilon^2$ of **DG-SOLVE** can be matched provided the iterations are stopped at a sufficiently large k. In particular, if there exists a constant $C_9 > 0$ such that

$$\|u - u_{\mathcal{D}_0}\|^2_{a,\mathcal{D}_0} + \beta\, \eta^2_{\mathcal{D}_0}(u_{\mathcal{D}_0}) \;\leq\; C_9\, \varepsilon^2, \tag{43}$$

then the number K of iterations in **DG-SOLVE** is bounded independently of ε. In this case, since the mapping (36) at most doubles the cardinality of the partition, i.e., $|\mathcal{D}_*| \leq 2|\mathcal{D}|$, we conclude that the cardinality of the output partition $\mathcal{D}_{\text{out}} := \mathcal{D}_K$ is uniformly bounded by the cardinality of the input partition $\mathcal{D}_{\text{in}}$, precisely

$$|\mathcal{D}_{\text{out}}| \leq 2^K |\mathcal{D}_{\text{in}}|.$$

Remark 5.1. (*Arithmetic complexity*) According to [5], if $N := \#\mathcal{D}$ denotes the cardinality of the current *hp*-partition, the arithmetic complexity of **hp-NEARBEST** is $O(N^2)$ (or $O(N \log N)$ in some specific situations). On the other hand, **DG-SOLVE** performs a bounded numbers of solutions of DG problems, which can be achieved in linear complexity. $\square$

5.3 Initialization

Let us discuss a possible strategy to fulfill (43). Recall that we enter **DG-SOLVE** at iteration i of **hp-ADFEM** with input partition $\mathcal{D}_i$ and data $g_{\mathcal{D}_i}$. This means that, with the notation of **hp-ADFEM**, condition (43) reads

$$\|u(g_{\mathcal{D}_i}) - u_{\mathcal{D}_i}\|^2_{a,\mathcal{D}_i} + \beta\, \eta^2_{\mathcal{D}_i}(u_{\mathcal{D}_i}) \;\leq\; C_9\, \varepsilon_i^2. \tag{44}$$

The first term on the left-hand side can be bounded from above by using the uniform continuity of the form $a_{\mathcal{D}_i}$ and the bounds given in Property 5.2, Corollary 5.1 and Proposition 5.2. This yields

$$\|u(g_{\mathcal{D}_i}) - u_{\mathcal{D}_i}\|^2_{a,\mathcal{D}_i} + \beta\, \eta^2_{\mathcal{D}_i}(u_{\mathcal{D}_i}) \leq C_{10} \inf_{w_{\mathcal{D}_i} \in V^c_{\mathcal{D}_i}} \|u(g_{\mathcal{D}_i}) - w_{\mathcal{D}_i}\|^2_{H^1_0(\Omega)} + C_{11} \eta^2_{\mathcal{D}_i}(u_{\mathcal{D}_i})$$

for constants $C_{10}, C_{11} > 0$ independent of $\mathcal{D}_i$. We now show that the infimum on the right-hand side can be bounded by a multiple of ε_i^2.

Property 5.4. *There exists a constant* $C_{12} > 0$ *independent of* $\mathcal{D}_i$ *such that*

$$\inf_{w_{\mathcal{D}_i} \in V^c_{\mathcal{D}_i}} \|u(g_{\mathcal{D}_i}) - w_{\mathcal{D}_i}\|_{H^1_0(\Omega)} \leq C_{12} \varepsilon_i$$

Proof. For simplicity, set again $u := u(g_{\mathcal{D}_i})$. Then, for any $w_{\mathcal{D}_i} \in V^c_{\mathcal{D}_i}$, let us write $u - w_{\mathcal{D}_i} = (u - u_\star) + (u_\star - \bar{u}_{i-1}) + (\bar{u}_{i-1} - w_{\mathcal{D}_i})$. Using (17), we get

$$\|u - u_\star\|_{H^1_0(\Omega)} = \|u(g_\star) - u(g_{\mathcal{D}_i})\|_{H^1_0(\Omega)} \leq C_\star \kappa\, \mathrm{E}_{\mathcal{D}_i}(\bar{u}_{i-1}, g_\star)^{\frac{1}{2}} \leq C_\star \kappa\, \omega \varepsilon_{i-1}. \tag{45}$$

On the other hand, recalling (20), we have

$$\|u_\star - \bar{u}_{i-1}\|_{\bar{\mathcal{D}}_{i-1}} \leq \varepsilon_{i-1}. \tag{46}$$

Let us define $w_{\mathcal{D}_i}$ as follows. Set $\psi := (\bar{u}_{i-1})^{\sim}_{x} \in L^2(\Omega)$ and let $q \in L^2(\Omega)$ be the piecewise polynomial function such that $q_{|K_D} = \Pi^0_{K_D,p_D-1}\psi_{|K_D}$ for all $D \in \mathcal{D}_i$. Notice that, recalling the definition (15), we have

$$\|\psi - q\|^2_\Omega = \sum_{D\in\mathcal{D}_i} \|\psi - q\|^2_{K_D} \leq \sum_{D\in\mathcal{D}_i} e_D(\bar{u}_{i-1}, g_\star) = \mathrm{E}_{\mathcal{D}_i}(\bar{u}_{i-1}, g_\star) \leq \omega^2 \varepsilon^2_{i-1}.$$

On the other hand, it holds

$$\int_\Omega q = \int_\Omega \psi = \sum_{D\in\bar{\mathcal{D}}_{i-1}} \int_{K_D} \bar{u}_{i-1,x} = -\sum_{e\in\mathcal{E}_{\bar{\mathcal{D}}_{i-1}}} [\![\bar{u}_{i-1}]\!]_e = -\sum_{e\in\mathcal{E}_{\bar{\mathcal{D}}_{i-1}}} \sigma^{-1/2}_{\bar{\mathcal{D}}_{i-1},e}\sigma^{1/2}_{\bar{\mathcal{D}}_{i-1},e}[\![\bar{u}_{i-1}]\!]_e,$$

whence

$$\left(\int_\Omega q\right)^2 \leq \Big(\sum_{e\in\mathcal{E}_{\bar{\mathcal{D}}_{i-1}}} \sigma^{-1}_{\bar{\mathcal{D}}_{i-1},e}\Big) \|\sigma^{1/2}_{\bar{\mathcal{D}}_{i-1}}[\![\bar{u}_{i-1}]\!]\|^2_{\mathcal{E}_{\bar{\mathcal{D}}_{i-1}}} \leq |\Omega|\, \|\sigma^{1/2}_{\bar{\mathcal{D}}_{i-1}}[\![\bar{u}_{i-1}]\!]\|^2_{\mathcal{E}_{\bar{\mathcal{D}}_{i-1}}} \leq \frac{|\Omega|}{\gamma}\, \varepsilon^2_{i-1}$$

by (46). Therefore, if we set

$$w_{\mathcal{D}_i}(x) = \int_{x_0}^{x} q(s)\, ds - (x - x_0)\int_\Omega q$$

where $x_0 = \min\Omega$, we realize $w_{\mathcal{D}_i} \in V^c_{\mathcal{D}_i}$ and $\|(\bar{u}_{i-1})^{\sim}_{x} - w_{\mathcal{D}_i,x}\|_\Omega \leq C\varepsilon_{i-1}$. This concludes the proof, since $\varepsilon_{i-1} \simeq \varepsilon_i$. □

By Property 5.4, we get the bound

$$\|u(g_{\mathcal{D}_i}) - u_{\mathcal{D}_i}\|^2_{a,\mathcal{D}_i} + \beta\,\eta^2_{\mathcal{D}_i}(u_{\mathcal{D}_i}) \leq C_{13}\varepsilon^2_i + C_{11}\eta^2_{\mathcal{D}_i}(u_{\mathcal{D}_i}).$$

At this point, we may proceed as follows. Assume that we have chosen, once and for all, an absolute constant $\hat{C} > 0$. We check the validity of

$$\eta^2_{\mathcal{D}_i}(u_{\mathcal{D}_i}) \leq \hat{C}\varepsilon^2_i.$$

- In the affirmative case, $u_{\mathcal{D}_i}$ does satisfy condition (44), and we can start the iterations of **DG-SOLVE**.
- In the negative case, we discard $u_{\mathcal{D}_i}$ and compute $\hat{u}^c_{\mathcal{D}_i} \in V^c_{\mathcal{D}_i}$, the (continuous) Galerkin approximation of $u(g_{\mathcal{D}_i})$ on the partition $\mathcal{D}_i$. For such an approximation, it is known that the residual estimator is both reliable and efficient; hence, resorting once more to Property 5.4,

  $$\eta_{\mathcal{D}_i}(\hat{u}^c_{\mathcal{D}_i}) \;\simeq\; \|u(g_{\mathcal{D}_i}) - \hat{u}^c_{\mathcal{D}_i}\|_a \;\simeq\; \|u(g_{\mathcal{D}_i}) - \hat{u}^c_{\mathcal{D}_i}\|_{H^1_0(\Omega)} \leq C_{12}\varepsilon_i.$$

  Therefore, condition (44) is satisfied with $u_{\mathcal{D}_i}$ replaced by $\hat{u}^c_{\mathcal{D}_i}$, and we start the iterations of **DG-SOLVE** from this approximation.

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