

Coupling among deformation, fluid flow, structural reorganisation and fibre reorientation in fibre-reinforced, transversely isotropic biological tissues

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(Article begins on next page)

1 Coupling among deformation, fluid flow,
2 structural reorganisation and fibre reorientation in
3 fibre-reinforced, transversely isotropic biological tissues

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9 **Abstract**

We highlight some mechanical aspects of the coupling among deformation, fluid flow, structural evolution, and reorientation of fibres in fibre-reinforced, hydrated, soft biological tissues. For our purposes, we elaborate a model in which the tissue’s interstitial fluid is inviscid and obeys Darcy’s law, and the solid constituents are transversely isotropic, hyperelastic materials. Within this setting, we consider two different types of remodelling: One consists of the reorientation of the fibres, while the other one is the manifestation, at the tissue scale, of structural rearrangements representable in terms of inelastic distortions. Our focus is on the interplay between the latter ones and the fibre reorientation. In our model, such interplay is a consequence of the constitutive framework, which resolves explicitly the space variability of a parameter, the “fibre mean angle”, that determines the direction along which the fibres tend to align themselves. Our main results concern the description of a Mandel-like stress tensor, which drives the inelastic distortions when the fibre mean angle is distributed inhomogeneously throughout the tissue, and of a diffusion-like tensor depending on the inelastic distortions, which guides the evolution of the fibre mean angle.

10 *Keywords:* Biological Tissues, Remodelling, Fibre Reorientation, Inelastic
11 Distortions, Anisotropic Materials

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13 1. Introduction

14 The study of fibre-reinforced composite materials is of great interest in
15 Biomechanics, since it permits to understand various aspects of the mechani-
16 cal behaviour of biological tissues. In the literature, there are works dedicated
17 to fundamental questions, e.g. [1, 2, 3, 4], studies that infer the elastic and
18 hydraulic properties of a tissue on the basis of micro-scale information, e.g.
19 [5, 6, 7, 8, 9, 10, 11], and studies devoted to the formulation of computational
20 methods and algorithms (see e.g. [12, 13, 14, 15, 16, 17]).

21 For fibre reinforced tissues, it is essential to provide a robust theoretical
22 background to study their growth, structural reorganisation, and damage (see
23 e.g. [18, 19, 20, 21, 22, 23]), and to relate such processes to the evolution of
24 the material properties. This knowledge, indeed, is helpful for predicting the
25 behaviour of injured or diseased tissues, and it may supply indications in the
26 design of engineered tissues.

27 With these motivations, we propose to improve and extend the model pre-
28 sented in [24], where the reorientation of fibres was studied in a transversely
29 isotropic fibre-reinforced tissue, with fibres aligned according to a prescribed
30 probability density. Such probability density was parameterised by an angle
31 denominated “fibre mean angle” and determining, at each material point,
32 the direction of the most probable fibre alignment.

33 In the present work, there are three relevant differences with respect to
34 [24]. The *first* and major difference is that we now account for plastic-
35 like distortions and study their influence on the reorientation of the collagen
36 fibres by adhering to the formalism introduced in [25]. Plastic-like distortions
37 are meant to describe the onset and progression of irreversible strains in
38 the tissue, which may arise in response to diseases or injuries [26], or the
39 reorganisation of the tissue’s extracellular matrix, as is the case for cellular
40 aggregates and tumour spheroids [27, 28, 29]. In the literature, the concept
41 of inelastic distortions is often related to that of residual stresses, an issue
42 typically investigated with the aid of the Bilby-Kröner-Lee decomposition
43 of the deformation gradient tensor. A rather different point view, however,
44 has been recently proposed in [30], where a study on the impact of residual
45 stresses on the mechanical behaviour of tissues is presented. The *second*
46 difference is related to the rationale with which the concept of *target angle*
47 is accounted for. We recall that the “target angle” is a preferred angle that,
48 depending on the deformation or stress state in the tissue, contributes to
49 direct the evolution of the fibre mean angle. In fact, it can be thought of as

50 the generator of an external force that drives the fibre mean angle towards
 51 the value determined by the interactions of the fibres with the environment
 52 in which they are embedded. After mentioning the approaches proposed, for
 53 example, in [12, 31, 32, 33], we select for our purposes a modification of the
 54 target angle put forward in [12]. The *third* difference is a re-definition of
 55 the constitutive framework and, in particular, of the free energy density of
 56 the Allen-Cahn type [24], which models the reorientation of the fibres and
 57 constituted the crux of [24] (see Sect. 4.2).

58 The most significant contribution of our work is the enrichment of the
 59 constitutive framework through the definition of two “non-standard” terms
 60 in the total free energy density of the system, W_ν . One of these terms,
 61 denoted by W_{Grad} , is said to be the “gradient part” of W_ν since it features
 62 the material gradient of the fibre mean angle, \mathbf{q} . The energy density W_{Grad}
 63 keeps track, already at the constitutive level, of the *explicit* dependence of
 64 \mathbf{q} on material points [24]. Thus, such dependence is not inherited from the
 65 quantities involved in the evolution equation of \mathbf{q} . Rather, it is accounted for
 66 *a priori* by enrolling $\text{Grad}\mathbf{q}$ among the constitutive arguments of W_ν . This
 67 gives rise to a generalised force that, by embodying the inhomogeneity of \mathbf{q} ,
 68 contributes to drive the evolution of \mathbf{q} itself. As a consequence, the coupling
 69 of \mathbf{q} with the dynamics of the plastic-like distortions introduce a novelty with
 70 respect to [24].

71 The other non-standard term in W_ν is referred to as the “structural part”
 72 and is denoted by W_{str} . In our view, it represents the potential energy that
 73 pertains to a given distribution of \mathbf{q} , and its existence is postulated *a pri-*
 74 *ori*, regardless of the fact that the tissue is deformed elastically or distorted
 75 inelastically. In fact, W_{str} can be non trivial also in the absence of deforma-
 76 tion and plastic-like distortions, although we do allow for its coupling with
 77 these kinematic variables. The way in which this is done here is another
 78 novelty of our work, for we strongly modify the coupling previously defined
 79 in [24]. Moreover, we compare our concept of structural energy with the one
 80 introduced in [31], within a setting rather different from ours. This issue is
 81 addressed in Sect. 7.

82 The proposed constitutive framework leads to the key point of this work:
 83 The coupling among the kinematic variables is such that the dynamics of the
 84 system can be depicted as a “game among three players”, i.e., the motion,
 85 the plastic-like distortions, and the fibre mean angle. The way in which they
 86 interact with one another is highlighted in Sect. 7.

87 In our model, plastic-like distortions are assumed to be set off, for in-

88 stance, when the tissue undergoes irreversible strains [26], when the cells of
89 the tissue redistribute their adhesion bonds, or when the tissue’s extracel-
90 lular matrix rearranges the cross-links forming its structure [29]. In these
91 cases, the solid constituent of the tissue experiences transformations that
92 cannot be described in terms of shape changes, and that necessitate, thus,
93 new descriptors. As suggested in [34], such descriptors should be regarded
94 as independent kinematic variables that represent the structural degrees of
95 freedom of the tissue. Within this picture, and by regarding the tissue as a
96 deformable porous medium permeated by an interstitial fluid, our goal is to
97 describe the interactions among deformation, fluid flow, and the aforemen-
98 tioned structural changes, emphasising the coupling between the plastic-like
99 distortions and the fibre reorientation.

100 The remainder of this work is organised as follows. In Sect. 2 we enunciate
101 the fundamental hypotheses of our model. In Sect. 3 we introduce all the
102 model equations. In Sect. 4 we explain in detail the constitutive framework.
103 In Sect. 5 we study the Dissipation Inequality. In Sect. 6 we comment on the
104 results of our simulations. Finally, in Sect. 7 we present in detail our main
105 theoretical achievements, and in Sect. 8 we summarise the key-points of our
106 work, and we outline a possible future research.

107 **2. Modelling hypotheses**

108 We regard the tissue under study as a mixture comprising a solid and
109 a fluid. The solid represents a porous medium and is assumed to feature a
110 matrix and reinforcing collagen fibres. The matrix is composed of biologi-
111 cal polymers and tissue cells. The fluid consists of water and several other
112 chemical substances. In spite of its major role on the tissue’s dynamics, in
113 this study we neglect the presence of chemical substances other than water.
114 On the one hand, this modelling choice precludes the resolution of the phe-
115 nomena related to the tissue’s chemistry. On the other hand, however, it is
116 capable of accounting for a strong entanglement among the flow of the fluid,
117 the deformation of the tissue, the reorganisation of its internal structure, and
118 the reorientation of the reinforcing fibres, while containing computational
119 costs. Moreover, the results predicted by our model can be used as inputs
120 for studying the evolution of chemical agents, when the coupling between
121 their dynamics and the aforementioned processes is weak enough.

122 The mathematical model discussed in this work rests on the following
123 main hypotheses: (i) the solid is hyperelastic and the fluid macroscopically

124 inviscid; (ii) both constituents are intrinsically incompressible, so that the
 125 change of volume of the tissue as a whole is due to the variation of porosity
 126 (since the saturation condition applies, such variation is expressed through
 127 the variation of the volumetric fraction of the solid and of the fluid); (iii) the
 128 dynamics of the fluid adheres to Darcy’s law; (iv) all body forces acting on
 129 the solid are negligible, with the exception of those describing the momen-
 130 tum exchange with the fluid; (v) growth is not accounted for in the model,
 131 so that the fluid and the solid locally preserve their mass.

132

133 3. Theoretical background

134 According to the hypotheses itemised above, the flow of the fluid and the
 135 deformation of the considered tissue are accounted for by the mass balance
 136 law and the linear momentum balance law for the tissue as a whole, i.e.,

$$\dot{J} = \text{Div} [\mathbf{K} \text{Grad} p], \quad \text{in } \mathcal{B} \times \mathcal{I}, \quad (1a)$$

$$\text{Div} [-Jp \mathbf{g}^{-1} \mathbf{F}^{-T} + \mathbf{P}_{\text{sc}}] = \mathbf{0}, \quad \text{in } \mathcal{B} \times \mathcal{I}. \quad (1b)$$

137 In (1a) and (1b), \mathbf{F} is the deformation gradient tensor of the solid, $J =$
 138 $\det \mathbf{F} > 0$ is said to be the *volume ratio*, p is the fluid pressure, \mathbf{K} is re-
 139 ferred to as the *material* permeability tensor of the tissue [9], \mathbf{g} is the met-
 140 ric tensor associated with the three-dimensional Euclidean space \mathcal{S} , \mathbf{P}_{sc} is
 141 the constitutive part of the first Piola-Kirchhoff stress tensor of the solid,
 142 $\mathcal{B} \subset \mathcal{S}$ is a region of space that can be taken as reference for the con-
 143 sidered solid-fluid mixture, and \mathcal{I} is the interval of time over which the
 144 tissue is observed. We recall that \mathbf{F} is the tangent map of the deformation
 145 $\chi : \mathcal{B} \times \mathcal{I} \rightarrow \mathcal{S}$ [35], which defines the configuration of the solid at time
 146 $t \in \mathcal{I}$, i.e., $\chi(\mathcal{B}, t) = \mathcal{C}_s(t) \in \mathcal{S}$ (see Appendix).

147 We first consider the reorganisation of the tissue due to the production of
 148 inelastic distortions. Since these are generally incompatible [34], i.e., they are
 149 not expressible as the gradient of a deformation, their descriptor should be at
 150 least a non-integrable second-order tensor field over \mathcal{B} . Following the same
 151 line of thought as Elastoplasticity, we indicate such tensor field by \mathbf{F}_p , where
 152 the subscript “p” stands for “plastic-like distortions”. Then, in accordance
 153 with [34], we introduce a set of generalised forces dual of the virtual velocity
 154 associated with \mathbf{F}_p , and we distinguish between the internal and the external
 155 forces of this kind, denoted by $\mathbf{Y}_\nu^{\text{int}}$ and $\mathbf{Y}_\nu^{\text{ext}}$, respectively. Hence, by invoking

156 the Principle of Virtual Powers, we obtain the force balance [34]

$$\mathbf{Y}_\nu^{\text{int}} = \mathbf{Y}_\nu^{\text{ext}}, \quad \text{in } \mathcal{B} \times \mathcal{I}, \quad (2)$$

157 where the subscript “ ν ” means that $\mathbf{Y}_\nu^{\text{int}}$ and $\mathbf{Y}_\nu^{\text{ext}}$ are defined with respect to
 158 a relaxed state of the solid, which is said to be “natural state” [36]. Finally,
 159 using the jargon of [34], we remark that Eq. (2) is consistent with a “*grade*
 160 *zero*” theory, in which no gradient of \mathbf{F}_p is accounted for.

161 A rationale for \mathbf{F}_p is provided by the Bilby-Kröner-Lee (BKL) multiplica-
 162 tive decomposition of \mathbf{F} (see [37] for a review), i.e., $\mathbf{F} = \mathbf{F}_e \mathbf{F}_p$, where \mathbf{F}_e is
 163 said to be the tensor of elastic distortions. For every $X \in \mathcal{B}$ and $t \in \mathcal{I}$,
 164 we set $\mathbf{F}_p(X, t) : T_X \mathcal{B} \rightarrow \mathcal{N}_t(X)$ and $\mathbf{F}_e(X, t) : \mathcal{N}_t(X) \rightarrow T_{\chi(X, t)} \mathcal{S}$, where
 165 $T_X \mathcal{B}$ is the tangent space attached to $X \in \mathcal{B}$, $\mathcal{N}_t(X)$ is the image of $T_X \mathcal{B}$
 166 through $\mathbf{F}_p(X, t)$ [38], and $T_{\chi(X, t)} \mathcal{S}$ is the tangent space of \mathcal{S} in $x = \chi(X, t)$.
 167 For future use, we also introduce the tangent bundles $T\mathcal{B} = \sqcup_{X \in \mathcal{B}} T_X \mathcal{B}$ and
 168 $T\mathcal{S} = \sqcup_{x \in \mathcal{S}} T_x \mathcal{S}$ associated with \mathcal{B} and \mathcal{S} , respectively. Note that we
 169 adopt the letter “ \mathcal{N} ” in $\mathcal{N}_t(X)$ in order to highlight that the vectors of
 170 $\mathcal{N}_t(X)$ are in the natural state, i.e., they are relaxed. Accordingly, $\mathcal{N}_t(X)$
 171 describes a stress-free state of the material at X , as depicted in Fig. 1 (note
 172 that the above notation has been recently used in [39]).

173 We denote by \mathbf{g} and \mathbf{G} the metric tensors associated with \mathcal{S} and \mathcal{B} ,
 174 respectively, so that the Cauchy-Green deformation tensor, $\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F} =$
 175 $\mathbf{F}^T \mathbf{g} \mathbf{F}$, reduces to $\mathbf{C} = \mathbf{G}$ in the absence of deformation [35]. In addition,
 176 we introduce the metric tensor $\boldsymbol{\eta}$, associated with the tissue’s natural state,
 177 which allows to define the tensors $\mathbf{C}_p = \mathbf{F}_p^T \cdot \mathbf{F}_p = \mathbf{F}_p^T \boldsymbol{\eta} \mathbf{F}_p$ and $\mathbf{B}_p = \mathbf{C}_p^{-1} =$
 178 $\mathbf{F}_p^{-1} \boldsymbol{\eta}^{-1} \mathbf{F}_p^{-T}$. We keep $\boldsymbol{\eta}$ formally different from \mathbf{g} and \mathbf{G} , although, in some
 179 cases, it could be taken equal to one of those (see e.g. [33]). For future use,
 180 we also define the right elastic Cauchy-Green tensor $\mathbf{C}_e = \mathbf{F}_e^T \cdot \mathbf{F}_e = \mathbf{F}_e^T \mathbf{g} \mathbf{F}_e$.

181 We now turn to the reorientation of the reinforcing fibres. As reported
 182 in [31, 24, 40], the alignment of the fibres in the tissue is governed by a
 183 probability density that depends on a given set of scalar parameters. The
 184 variation of these parameters is responsible for the reorientation of the fibres.
 185 In our model (see Sect. 4), we select one parameter only, which we indicate
 186 with \mathbf{q} and employ to describe the kinematics of the fibres. In particular, \mathbf{q}
 187 acquires the meaning of “fibre mean angle”. Analogously to the reasoning
 188 that has led us to (2), we consider both internal and external generalised
 189 forces dual of the (scalar) virtual velocity \mathbf{v} associated with \mathbf{q} . In this case,
 190 however, since we aim at resolving explicitly the point dependence of \mathbf{q} ,

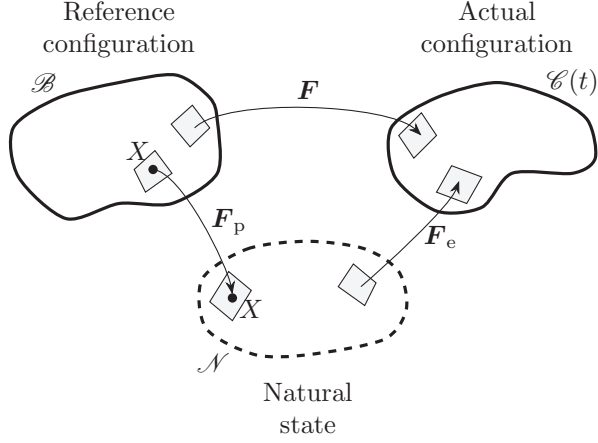


Figure 1: Graphical representation of the Bilby-Kröner-Lee (BKL) multiplicative decomposition of the deformation gradient tensor \mathbf{F} , i.e., $\mathbf{F} = \mathbf{F}_e \mathbf{F}_p$

191 we also need to account for the kinematic descriptor $\text{Grad } \mathbf{q}$, along with
 192 its virtual counterpart $\text{Grad } \mathbf{v}$. Then, by employing again the Principle of
 193 Virtual Powers, and restricting it for brevity only to the sub-problem of the
 194 fibre reorientation, we find

$$\int_{\mathcal{B}} \{y^{(0)} \mathbf{v} + \mathbf{y}^{(1)} \text{Grad } \mathbf{v}\} = \int_{\mathcal{B}} h^{(0)} \mathbf{v} + \int_{\partial \mathcal{B}_N} h^{(1)} \mathbf{v}, \quad (3)$$

195 where $y^{(0)}$ and $\mathbf{y}^{(1)}$ are a scalar and a vector-like internal force, defined as
 196 the dual entities of \mathbf{v} and $\text{Grad } \mathbf{v}$, respectively, $h^{(0)}$ is an external force, $h^{(1)}$
 197 is an external contact force, $\partial \mathcal{B}_N$ is the Neumann boundary of $\partial \mathcal{B}$, and the
 198 virtual velocity \mathbf{v} is assumed to vanish identically on the Dirichlet portion of
 199 $\partial \mathcal{B}$, i.e., on $\partial \mathcal{B}_D = \partial \mathcal{B} \setminus \partial \mathcal{B}_N$. Equation (3) leads to the balance laws

$$y^{(0)} - \text{Div } \mathbf{y}^{(1)} = h^{(0)}, \quad \text{in } \mathcal{B} \times \mathcal{I}, \quad (4a)$$

$$\mathbf{y}^{(1)} \cdot \mathbf{N} = h^{(1)}, \quad \text{on } \partial \mathcal{B}_N \times \mathcal{I}. \quad (4b)$$

200 Upon setting, in the case of isochoric plastic-like distortions,

$$\mathcal{R}_v^{\text{ext}} \equiv h^{(0)}, \quad (5a)$$

$$\mathcal{R}_v^{\text{int}} \equiv y^{(0)} - \text{Div } \mathbf{y}^{(1)}, \quad (5b)$$

201 we can rephrase (4a) as

$$\mathcal{R}_\nu^{\text{int}} = \mathcal{R}_\nu^{\text{ext}}, \quad \text{in } \mathcal{B} \times \mathcal{I}, \quad (6)$$

202 thereby generalising the results in [33, 40]. Equations (1a), (1b), (2), reformulated for the case of isochoric plastic-like distortions, and (6) describe the
 203 dynamics of the system under study. Their solution determines the model
 204 dynamics of the system under study. Their solution determines the model
 205 unknowns, identified with p , χ , \mathbf{F}_p , and \mathbf{q} . Among those, a *true* configura-
 206 tion of the solid is obtained by specifying the triple of descriptors $(\chi, \mathbf{F}_p, \mathbf{q})$.
 207 In the sequel, we refer to \mathbf{q} and \mathbf{F}_p as to *remodelling variables*, and to $\mathbf{Y}_\nu^{\text{int}}$,
 208 $\mathbf{Y}_\nu^{\text{ext}}$, $\mathcal{R}_\nu^{\text{int}}$, and $\mathcal{R}_\nu^{\text{ext}}$ as to generalised *remodelling forces*.

209 4. Constitutive laws

210 To constitutively characterise the fibre-reinforced medium under study,
 211 we assign a free energy density consisting of two terms, both of which are
 212 written per unit volume of the material in its natural state, i.e. [24],

$$W_\nu = W_{\text{std}} + W_{\text{rem}}. \quad (7)$$

213 The term W_{std} takes into account the hyperelastic behaviour of the solid
 214 material, and relies on a mechanical model of fibre-reinforced media, in which
 215 the fibres are oriented statistically [6, 7, 9]. In this respect, we denote the
 216 corresponding strain energy density by W_{std} , where the subscript “std” stands
 217 for “standard”. The other term, W_{rem} , is not standard and it has been
 218 introduced in order to specifically account for remodelling [31, 24, 41]. The
 219 energy density W_{rem} is assumed to exist independently of deformation and,
 220 in fact, it is conceived as the energetic contribution that characterises each
 221 possible directional distribution of the fibres in the tissue. For this reason,
 222 W_{rem} may be nontrivial also in the undeformed configuration of the tissue
 223 [24].

224 4.1. “Standard” Constitutive laws

225 Following [6, 7, 9, 24, 40], we define W_{std} as a function of \mathbf{C}_e and \mathbf{q} , i.e.,
 226 we set $W_{\text{std}} = \hat{W}_{\text{std}}(\mathbf{C}_e, \mathbf{q})$, with

$$\hat{W}_{\text{std}}(\mathbf{C}_e, \mathbf{q}) = \Phi_{\text{sv}} \hat{U}(J_e) + \Phi_{\text{0sv}} \hat{W}_0(\mathbf{C}_e) + \Phi_{\text{1sv}} \left[\hat{W}_{\text{1i}}(\mathbf{C}_e) + \hat{W}_{\text{1a}}(\mathbf{C}_e, \mathbf{q}) \right]. \quad (8)$$

227 In (8), $\Phi_{sv} = J_e \phi_s$, $\Phi_{0sv} = J_e \phi_{0s}$, and $\Phi_{1sv} = J_e \phi_{1s}$ are the volumetric fractions
 228 of the solid as a whole, of the matrix, and of the fibres, respectively, all
 229 expressed per unit volume of the solid in the natural state, and $J_e = \sqrt{\det \mathbf{C}_e}$
 230 is the volume ratio associated with the elastic distortions; $\hat{U}(J_e)$ is a *penalty*
 231 term, introduced to prevent the tissue from the occurrence of compaction
 232 [9, 42]; $\hat{W}_0(\mathbf{C}_e)$ is the strain energy density of the solid matrix, assumed to
 233 be isotropic; $\hat{W}_{1i}(\mathbf{C}_e)$ and $\hat{W}_{1a}(\mathbf{C}_e, \mathbf{q})$ are the *isotropic* and the *anisotropic*
 234 strain energy densities of the fibres, respectively. The latter one is defined
 235 through the directional average [3, 7, 9, 14, 43, 44]

$$\hat{W}_{1a}(\mathbf{C}_e, \mathbf{q}) = \left\langle\left\langle \hat{\mathcal{W}}_{1a}(\mathbf{C}_e, \mathbf{m}) \right\rangle\right\rangle(\mathbf{q}), \quad (9)$$

236 where $\left\langle\left\langle \bullet \right\rangle\right\rangle$ is the operator of directional averaging, $\hat{\mathcal{W}}_{1a}(\mathbf{C}_e, \mathbf{m})$ is the trans-
 237 versely isotropic strain energy density of a single fibre, and \mathbf{m} is a field of
 238 unit vectors individuating the direction of space along which the fibres are
 239 locally oriented.

240 Possible explicit constitutive expressions for $\hat{U}(J_e)$, $\hat{W}_0(\mathbf{C}_e)$, $\hat{W}_{1i}(\mathbf{C}_e)$, and
 241 $\hat{\mathcal{W}}_{1a}(\mathbf{C}_e, \mathbf{m})$ are given by [9, 45]

$$\hat{U}(J_e) = \alpha_0 \frac{[J_e - J_{cr}]^{2q}}{[J_e - \Phi_{sv}]^r} \mathcal{H}(J_{cr} - J_e), \quad (10a)$$

$$\begin{aligned} \hat{W}_0(\mathbf{C}_e) &= \check{W}_0(I_{1e}, I_{2e}, I_{3e}) \\ &= \alpha_0 \left\{ \frac{\exp(\alpha_1[I_{1e} - 3] + \alpha_2[I_{2e} - 3])}{I_{3e}^{\alpha_3}} - 1 \right\}, \end{aligned} \quad (10b)$$

$$\begin{aligned} \hat{W}_{1i}(\mathbf{C}_e) &= \check{W}_{1i}(I_{1e}, I_{2e}, I_{3e}) \\ &= \alpha_{i0} \left\{ \frac{\exp(\alpha_{i1}[I_{1e} - 3] + \alpha_{i2}[I_{2e} - 3])}{I_{3e}^{\alpha_{i3}}} - 1 \right\}, \end{aligned} \quad (10c)$$

$$\hat{\mathcal{W}}_{1a}(\mathbf{C}_e, \mathbf{m}) = \check{\mathcal{W}}_{1a}(I_{4e}) = \check{\mathcal{V}}_{1a}(I_{4e}) \mathcal{H}(I_{4e} - 1), \quad (10d)$$

$$\check{\mathcal{V}}_{1a}(I_{4e}) = \frac{k_1}{2k_2} \left\{ \exp(k_2[I_{4e} - 1]^2) - 1 \right\}. \quad (10e)$$

242 The constitutive expression of $\check{\mathcal{V}}_{1a}$ is taken from [33, 46]. We remark that,
 243 although we are aware of the importance of the invariant $I_{5e} = \mathbf{C}_e^2 : (\mathbf{m} \otimes \mathbf{m})$
 244 in the formulation of the anisotropic part of the strain energy density (see
 245 e.g. [47, 48]), we retained here the form (10d) of $\hat{\mathcal{W}}_{1a}(\mathbf{C}_e, \mathbf{m})$ because, since
 246 it has been used in other works, it allows us for an easier comparison with

247 the results obtained in the presence of remodelling.

248 In (10a)–(10d), $I_{1e} = \text{tr}(\mathbf{C}_e)$, $I_{2e} = \frac{1}{2}[I_{1e}^2 - \text{tr}(\mathbf{C}_e^2)]$, and $I_{3e} = J_e^2 = \det \mathbf{C}_e$
 249 are the principal invariants of \mathbf{C}_e , I_{4e} is given by $I_{4e} = \mathbf{C}_e : (\mathbf{m} \otimes \mathbf{m})$, $\mathcal{H}(I_{4e} - 1)$
 250 is the Heaviside function, J_{cr} is a critical value of J_e below which $\hat{U}(J_e)$ is
 251 active (see [9] for details), and $\{\alpha_0, \alpha_1, \alpha_2\}$, $\{\alpha_{i0}, \alpha_{i1}, \alpha_{i2}\}$, and $\{k_1, k_2\}$ are
 252 sets of material parameters (see Table 1). In particular, α_0 , α_{i0} and k_1 have
 253 physical units of energy per unit volume. As prescribed in [41], we enforce
 254 the conditions $\alpha_3 = \alpha_1 + 2\alpha_2$ and $\alpha_{i3} = \alpha_{i1} + 2\alpha_{i2}$. By attaching the *set of*
 255 *all space directions*

$$\mathbb{S}_X^2 \mathcal{B} = \{\mathbf{m}_X : \|\mathbf{m}_X\| = 1\} \quad (11)$$

256 to each $X \in \mathcal{B}$, and by defining the bundle $\mathbb{S}^2 \mathcal{B} = \sqcup_{X \in \mathcal{B}} \mathbb{S}_X^2 \mathcal{B}$, the field of
 257 unit vectors \mathbf{m} can be re-defined as $\mathbf{m} : \mathcal{B} \rightarrow \mathbb{S}^2 \mathcal{B}$, so that $\mathbf{m}(X) = \mathbf{m}_X \in$
 258 $\mathbb{S}_X^2 \mathcal{B}$. Finally, the directional average (9),

$$\hat{W}_{1a}(\mathbf{C}_e, \mathbf{q}) = \left\langle \hat{\mathcal{W}}_{1a}(\mathbf{C}_e, \mathbf{m}) \right\rangle (\mathbf{q}) := \int_{\mathbb{S}^2 \mathcal{B}} \hat{\mathcal{W}}_{1a}(\mathbf{C}_e, \mathbf{m}) \Psi(\mathbf{m}, \mathbf{q}), \quad (12)$$

259 is computed with respect to a given probability density Ψ (see e.g. [7, 3,
 260 14]). Here, Ψ is assumed to depend only on the direction of the local fibre
 261 orientation and on the remodelling variable, \mathbf{q} . However, in more general
 262 contexts, it can depend on several other parameters. It is important to
 263 remark that, in this work, it is taken transversely isotropic with respect to a
 264 direction \mathbf{m}_0 for the tissue as a whole. To justify this assumption, we consider
 265 a specimen of tissue of cylindric shape, and we assume that the symmetry
 266 axis of the cylinder coincides with \mathbf{m}_0 .

267 We remark that $\hat{\mathcal{W}}_{1a}(\mathbf{C}_e, \mathbf{m})$ depends on \mathbf{m} through the structure tensor
 268 $\mathbf{a} := \mathbf{m} \otimes \mathbf{m}$ and, since \mathbf{a} is invariant under the transformation $\mathbf{m} \mapsto -\mathbf{m}$,
 269 it also holds that $\hat{\mathcal{W}}_{1a}(\mathbf{C}_e(X, t), \mathbf{m}_X) = \hat{\mathcal{W}}_{1a}(\mathbf{C}_e(X, t), -\mathbf{m}_X)$, for all $X \in \mathcal{B}$
 270 and for all times.

271 While the strain energy density of a single fibre, $\hat{\mathcal{W}}_{1a}(\mathbf{C}_e, \mathbf{m})$, is trans-
 272 versely isotropic with respect to \mathbf{m} , the directional average (9) models a
 273 material that is transversely isotropic with respect to \mathbf{m}_0 . To guarantee
 274 this property, for all $X \in \mathcal{B}$ in the natural state, we first choose a triad
 275 $\{\mathbf{e}_\alpha(X)\}_{\alpha=1}^3$ of basis unit vectors, with $\mathbf{e}_3(X)$ parallel to \mathbf{m}_0 . Then, we in-
 276 troduce the polar coordinates $(\vartheta, \varphi) \in [0, \pi] \times [0, 2\pi[$, so that \mathbf{m}_X reads

$$\mathbf{m}_X \equiv \check{\mathbf{m}}_X(\vartheta, \varphi) = \sin \vartheta \cos \varphi \mathbf{e}_1 + \sin \vartheta \sin \varphi \mathbf{e}_2 + \cos \vartheta \mathbf{e}_3, \quad (13)$$

Symbol	Definition	Units	Symbol	Definition	Units
r	0.50	[—]	$\bar{\Phi}_{0s\nu}$	$0.046 + 0.038\xi - 0.062\xi^2$	[—]
q	2.00	[—]	$\bar{\Phi}_{1s\nu}$	$0.204 - 0.138\xi + 0.062\xi^2$	[—]
α_0	0.125	[MPa]	Γ	$1 \cdot 10^4$	[Pa s]
$\alpha_1 = \alpha_{i1}$	0.778	[—]	D_0	$1 \cdot 10^{-4}$	[N(rad) ⁻¹]
$\alpha_2 = \alpha_{i2}$	0.111	[—]	ζ_0	0.50	[(MPa s) ⁻¹]
α_{i0}	7.59	[MPa]	e_ν	$(1 - \Phi_{s\nu})/\Phi_{s\nu}$	[—]
J_{cr}	$0.1 + \bar{\Phi}_{s\nu}(\xi)$	[—]	$e_\nu^{(0)}$	4.0	[—]
k_1	13.00	[kPa]	κ_0	0.0848	[—]
k_2	12.20	[—]	m_0	4.6380	[—]
σ_y	0.002	[MPa]	$k_{0\nu}^{(0)}$	$3.7729 \cdot 10^{-3}$	[mm ⁴ (N s) ⁻¹]
$\bar{\Phi}_{s\nu}$	$\bar{\Phi}_{0s\nu} + \bar{\Phi}_{1s\nu}$	[—]	\mathcal{A}_0	$(k_1/k_2)(4.387\xi^{2.228} + 1)$	[kPa]

Table 1: Parameters used in the energy densities (10a)–(10e). See [9, 45], and the references therein, for the values in the first seven rows on the left. Note that $\xi = \frac{X^3}{L}$.

277 and we enforce the condition

$$\Psi(\mathbf{m}_X, \mathbf{q}) = \Psi(\check{\mathbf{m}}_X(\vartheta, \varphi), \mathbf{q}) \equiv \check{\Psi}(\vartheta, \mathbf{q}). \quad (14)$$

278 Since the probability density $\check{\Psi}(\vartheta, \mathbf{q})$ re-defined in (14) is independent of φ ,
279 the directional average (9) has to be transversely isotropic with respect to
280 \mathbf{m}_0 . Several functional forms can be used to express $\check{\Psi}(\vartheta, \mathbf{q})$. For example,
281 it can be a pseudo-Gaussian distribution [31, 24, 40, 49], defined by

$$\check{\Psi}_{PG}(\vartheta, \mathbf{q}) \equiv \hat{\Psi}_{PG}(\vartheta, \mathbf{q}, \omega) = \frac{\hat{\gamma}(\vartheta, \mathbf{q}, \omega)}{\mathcal{N}(\mathbf{q}, \omega)}, \quad (15a)$$

$$\hat{\gamma}(\vartheta, \mathbf{q}, \omega) = \exp\left(-\frac{(\vartheta - \mathbf{q})^2}{2\omega^2}\right), \quad (15b)$$

$$\mathcal{N}(\mathbf{q}, \omega) = 2\pi \int_0^{\pi/2} \hat{\gamma}(\vartheta, \mathbf{q}, \omega) \sin \vartheta \, d\vartheta. \quad (15c)$$

282 In (15a)–(15c), $\omega^2 > 0$ is the variance of the pseudo-Gaussian distribution,
283 $\mathcal{N}(\mathbf{q}, \omega)$ is the normalisation factor, and the remodelling angle \mathbf{q} is the angle,
284 taken from \mathbf{m}_0 , that denotes the semi-aperture of a cone of fibres with the
285 apex in X . The angle \mathbf{q} is conceived in such a way that $\check{\mathbf{m}}(\mathbf{q}, \varphi)$ represents
286 the set of most probable directions of fibre alignment, with φ varying in

287 $\in [0, 2\pi[$. We remark that, according to the definitions (15a)–(15c), the
 288 values of ϑ that are admissible for $\check{\Psi}_{\text{PG}}$ range in $[0, \pi/2]$. For this reason,
 289 also \mathbf{q} is allowed to vary within the same interval only.

290 Other forms of the probability density can be found e.g. in [31, 44]

291 4.2. “Non-Standard” constitutive laws

292 The energy density associated with remodelling is given in the form [24]

$$\hat{W}_{\text{rem}}(\mathbf{F}_e, \mathbf{F}_p, \mathbf{q}, \text{Grad } \mathbf{q}) = \Phi_{\text{Isv}} \left[\hat{W}_{\text{str}}(\mathbf{C}_e, \mathbf{q}) + \hat{W}_{\text{Grad}}(\mathbf{F}_e, \mathbf{F}_p, \text{Grad } \mathbf{q}) \right], \quad (16)$$

293 where $\hat{W}_{\text{str}}(\mathbf{C}_e, \mathbf{q})$ and $\hat{W}_{\text{Grad}}(\mathbf{F}_e, \mathbf{F}_p, \text{Grad } \mathbf{q})$ are referred to as the *structural*
 294 part and the *gradient* part of the strain energy density, respectively. Al-
 295 though (16) has recently been introduced in [24], in the present work the
 296 constitutive expressions of \hat{W}_{str} and \hat{W}_{Grad} are rather different from those
 297 supplied in [24].

298 The first difference concerns $\hat{W}_{\text{str}}(\mathbf{C}_e, \mathbf{q})$, which is assumed here to be
 299 transversely isotropic, and to depend on \mathbf{C}_e only through $\bar{\mathbf{C}}_e = J_e^{-2/3} \mathbf{C}_e$, i.e.,

$$\hat{W}_{\text{str}}(\mathbf{C}_e, \mathbf{q}) = \mathcal{A}_0 \mathcal{P}(\mathbf{q}) \left[1 + \frac{k_2}{k_1} \langle\langle \check{\mathcal{V}}_{1a}(\bar{I}_{4e}) \mathcal{H}(I_{4e} - 1) \rangle\rangle(\mathbf{q}) \right], \quad (17)$$

300 where $\bar{I}_{4e} = \bar{\mathbf{C}}_e : (\mathbf{m} \otimes \mathbf{m}) = J_e^{-2/3} I_{4e}$, \mathcal{A}_0 is a point-dependent material
 301 coefficient, and $\mathcal{P}(\mathbf{q})$ a double-well function of the fibre mean angle [24], i.e.,

$$\mathcal{P}(\mathbf{q}) = \frac{1}{(\pi/4)^4} \mathbf{q}^2 \left(\mathbf{q} - \frac{\pi}{2} \right)^2. \quad (18)$$

302 As noticed above, a more complete constitutive approach would call for ex-
 303 pressing $\check{\mathcal{V}}_{1a}$ as a function of both I_{4e} and I_{5e} [47]. However, since such a
 304 modelling choice does not change the “philosophy” of our work, we opt here
 305 for an easier form of $\check{\mathcal{V}}_{1a}$.

306 The second difference concerns the definition of W_{Grad} , which is assumed
 307 to depend also on the plastic-like distortions through the expression

$$\begin{aligned} W_{\text{Grad}} &= \frac{1}{2} \mathbf{d} : \text{grad } {}^s \mathbf{q} \otimes \text{grad } {}^s \mathbf{q} \\ &= \frac{1}{2} \mathbf{F}_p^{-1} \mathbf{F}_e^{-1} \mathbf{d} \mathbf{F}_e^{-\text{T}} \mathbf{F}_p^{-\text{T}} : \text{Grad } \mathbf{q} \otimes \text{Grad } \mathbf{q}, \end{aligned} \quad (19)$$

308 where we employed the identity $\text{grad } {}^s \mathbf{q}(x, t) = \mathbf{F}^{-\text{T}}(X, t) \text{Grad } \mathbf{q}(X, t)$, with

309 ${}^s\mathbf{q}(\cdot, t) : \mathcal{C}(t) \rightarrow [0, \pi/2]$ being the spatial version of the fibre mean angle.
 310 Among the many possible choices for expressing the second-order tensor \mathbf{d} ,
 311 which has the physical meaning of angular stiffness per unit length [24], we
 312 select $\mathbf{d} = D_0 \mathbf{b}_e$, where $\mathbf{b}_e = \mathbf{F}_e \cdot \mathbf{F}_e^T$ is the elastic left Cauchy-Green deformation
 313 tensor, and the coefficient D_0 is assumed to be constant. In general, D_0
 314 should be a function of material points. However, in this work, we attribute
 315 the dependence on material points to the “effective” coefficient $\Phi_{1s\nu} D_0$, which
 316 features in the definition of W_{rem} , and is obtained by multiplying W_{Grad} by
 317 $\Phi_{1s\nu}$, as done in (16). Upon substituting $\mathbf{d} = D_0 \mathbf{b}_e$ into (19), we can rephrase
 318 W_{Grad} as a function of \mathbf{F}_p and $\text{Grad } \mathbf{q}$, i.e.,

$$W_{\text{Grad}} = \tilde{W}_{\text{Grad}}(\mathbf{F}_p, \text{Grad } \mathbf{q}) = \frac{1}{2} D_0 \mathbf{B}_p : \text{Grad } \mathbf{q} \otimes \text{Grad } \mathbf{q}. \quad (20)$$

319 5. Residual Dissipation Inequality and Remodelling Equations

320 We adapt the study of the dissipation inequality from [24, 50] and, to
 321 avoid lengthy calculations, we report here only the results that are most
 322 important for this work. By exploiting the identity $\mathbf{C}_e = \mathbf{F}_p^{-T} \mathbf{C} \mathbf{F}_p^{-1}$, we can
 323 rephrase the constitutive expression of the overall free energy density W_ν (see
 324 Sect. 4) as a function of \mathbf{C} , \mathbf{F}_p and \mathbf{q} , i.e.,

$$W_\nu = \tilde{W}_\nu(\mathbf{C}, \mathbf{F}_p, \mathbf{q}, \text{Grad } \mathbf{q}). \quad (21)$$

325 By assuming isochoric plastic-like distortions, i.e., $J_p = 1$, we obtain

$$\mathbf{P}_s = -\Phi_{s\nu} p \mathbf{g}^{-1} \mathbf{F}^{-T} + \mathbf{F} \left(2 \frac{\partial \tilde{W}_\nu}{\partial \mathbf{C}} \right), \quad (22a)$$

$$\mathbf{P}_f = -(J - \Phi_{s\nu}) p \mathbf{g}^{-1} \mathbf{F}^{-T}, \quad (22b)$$

326 where \mathbf{P}_s and \mathbf{P}_f are the first Piola-Kirchhoff stress tensors of the solid
 327 and the fluid, respectively. Next, we write $W_{\text{std}} = \tilde{W}_{\text{std}}(\mathbf{C}, \mathbf{F}_p, \mathbf{q})$, $W_{\text{str}} =$
 328 $\tilde{W}_{\text{str}}(\mathbf{C}, \mathbf{F}_p, \mathbf{q})$, and $W_{\text{Grad}} = \tilde{W}_{\text{Grad}}(\mathbf{F}_p, \text{Grad } \mathbf{q})$. Subsequently, we introduce
 329 the Mandel stress tensors

$$\boldsymbol{\Sigma}_{\text{std}} = -\mathbf{F}_p^T \frac{\partial \tilde{W}_{\text{std}}}{\partial \mathbf{F}_p} = \mathbf{C} \left(2 \frac{\partial \tilde{W}_{\text{std}}}{\partial \mathbf{C}} \right), \quad (23a)$$

$$\begin{aligned}\boldsymbol{\Sigma}_{\text{str}} &= -\mathbf{F}_p^T \left(\Phi_{1s\nu} \frac{\partial \tilde{W}_{\text{str}}}{\partial \mathbf{F}_p} \right) = \mathbf{C} \left(2\Phi_{1s\nu} \frac{\partial \tilde{W}_{\text{str}}}{\partial \mathbf{C}} \right) \\ &= \langle\langle f \text{Dev}(\mathbf{C} \mathbf{F}_p^{-1} \mathbf{a} \mathbf{F}_p^{-T}) \rangle\rangle, \end{aligned} \quad (23b)$$

$$\boldsymbol{\Sigma}_{\text{Grad}} = -\mathbf{F}_p^T \left(\Phi_{1s\nu} \frac{\partial \tilde{W}_{\text{Grad}}}{\partial \mathbf{F}_p} \right) = \text{Grad} \mathbf{q} \otimes \left(\Phi_{1s\nu} \frac{\partial \tilde{W}_{\text{Grad}}}{\partial \text{Grad} \mathbf{q}} \right), \quad (23c)$$

330 where the factor f is defined by

$$f = 2\Phi_{1s\nu} \mathcal{A}_0 \mathcal{P}(\mathbf{q}) I_{3e}^{-1/3} k_2 [\bar{I}_{4e} - 1] \exp(k_2 [\bar{I}_{4e} - 1]^2) \mathcal{H}(I_{4e} - 1). \quad (24)$$

331 Hence, we obtain the residual dissipation inequality

$$\begin{aligned}\mathfrak{D}_{\text{res}} &= -\phi_f^{-1} \boldsymbol{\pi}_{\text{fd}} \cdot \mathbf{F} \boldsymbol{\mathcal{Q}} + \left\{ y^{(0)} - \frac{\partial \tilde{W}_\nu}{\partial \mathbf{q}} \right\} \dot{\mathbf{q}} + \left\{ \mathbf{y}^{(1)} - \frac{\partial \tilde{W}_\nu}{\partial \text{Grad} \mathbf{q}} \right\} \text{Grad} \dot{\mathbf{q}} \\ &\quad + \left\{ \mathbf{F}_p^{-T} (\mathbf{F}_p^T \mathbf{Y}_\nu^{\text{int}} + \boldsymbol{\Sigma}_{\text{std}} + \boldsymbol{\Sigma}_{\text{str}} + \boldsymbol{\Sigma}_{\text{Grad}}) \mathbf{F}_p^T \right\} : \mathbf{L}_p \geq 0, \end{aligned} \quad (25)$$

332 where $\boldsymbol{\pi}_{\text{fd}}$ is the force density describing the exchange of linear momentum
333 between the solid and the fluid, and $\boldsymbol{\mathcal{Q}} = J \mathbf{F}^{-1} \mathbf{q}$ is referred to as *material*
334 *filtration velocity*, i.e., the backward Piola-transformation of the filtration
335 velocity $\mathbf{q} = \phi_f [\mathbf{v}_f - \mathbf{v}_s]$.

336 With reference to (23a)–(23c), $\boldsymbol{\Sigma}_{\text{std}}$ can be found in several theories on
337 remodelling available in the literature (see e.g. [21, 22]); $\boldsymbol{\Sigma}_{\text{str}}$ represents
338 a *structural* generalised force that descends from the coupling between the
339 deformation and the evolution of the fibres accounted for by W_{str} ; $\boldsymbol{\Sigma}_{\text{Grad}}$
340 stems from the coupling of the plastic-like distortions with the evolution of
341 the fibres, and is a direct consequence of the introduction of the free energy
342 density W_{Grad} .

343 Tensor $\boldsymbol{\Sigma}_{\text{Grad}}$ can be interpreted as a generalisation of the Korteweg stress
344 tensor. Coherently with W_{Grad} , it represents a generalised configurational
345 force that is power-conjugate to $\mathbf{L}_p = \dot{\mathbf{F}}_p \mathbf{F}_p^{-1}$, and that results from the cou-
346 pling between \mathbf{F}_p and \mathbf{q} . We also remark that, since in our model W_{Grad} is in-
347 dependent of \mathbf{C} , the differentiation of W_{Grad} with respect to \mathbf{C} is null, thereby
348 implying that W_{Grad} does not contribute to the second Piola-Kirchhoff stress
349 tensor of the solid. Therefore, $\boldsymbol{\Sigma}_{\text{Grad}}$ cannot possess the same properties
350 as the Mandel stress tensors $\boldsymbol{\Sigma}_{\text{std}}$ and $\boldsymbol{\Sigma}_{\text{str}}$ defined in (23a) and (23b), re-
351 spectively. For instance, it cannot be written in terms of the product of

352 \mathbf{C} with $(2\Phi_{1s\nu}\partial\tilde{W}_{\text{Grad}}/\partial\mathbf{C})$, and it does not fulfil the symmetry conditions
 353 $\boldsymbol{\Sigma}_{\text{std}}\mathbf{C} = (\boldsymbol{\Sigma}_{\text{std}}\mathbf{C})^{\text{T}}$ and $\boldsymbol{\Sigma}_{\text{str}}\mathbf{C} = (\boldsymbol{\Sigma}_{\text{str}}\mathbf{C})^{\text{T}}$. These stem from the coupling
 354 among \mathbf{F} , \mathbf{F}_{p} , and \mathbf{q} , a coupling that is accounted for by \tilde{W}_{std} and \tilde{W}_{str} ,
 355 but not by W_{Grad} . We notice that $\boldsymbol{\Sigma}_{\text{Grad}}$ satisfies the symmetry conditions
 356 $\mathbf{B}_{\text{p}}\boldsymbol{\Sigma}_{\text{Grad}} = (\mathbf{B}_{\text{p}}\boldsymbol{\Sigma}_{\text{Grad}})^{\text{T}}$ [51].

357 In (25), we perform the identification

$$\mathbf{y}^{(1)} = \frac{\partial\tilde{W}_{\nu}}{\partial\text{Grad}\mathbf{q}} = \Phi_{1s\nu}D_0\mathbf{B}_{\text{p}}\text{Grad}\mathbf{q}, \quad (26)$$

358 which amounts to require that $\mathbf{y}^{(1)}$ has no dissipative contribution, and, by
 359 recalling the definition of $\mathcal{R}_{\nu}^{\text{int}}$ given in (5b), we introduce the dissipative
 360 parts of the internal generalised forces $\mathcal{R}_{\nu}^{\text{int}}$ and $\mathbf{Y}_{\nu}^{\text{int}}$:

$$\mathcal{R}_{\nu,d}^{\text{int}} := y^{(0)} - \frac{\partial\tilde{W}_{\nu}}{\partial\mathbf{q}} = \mathcal{R}_{\nu}^{\text{int}} - \mathcal{E}(\mathbf{q}, \text{Grad}\mathbf{q}), \quad (27a)$$

$$\mathbf{F}_{\text{p}}^{\text{T}}\mathbf{Y}_{\nu,d}^{\text{int}} := \text{dev}(\mathbf{F}_{\text{p}}^{\text{T}}\mathbf{Y}_{\nu}^{\text{int}}) + \text{dev}(\boldsymbol{\Sigma}_{\text{std}} + \boldsymbol{\Sigma}_{\text{str}} + \boldsymbol{\Sigma}_{\text{Grad}}), \quad (27b)$$

361 where $\mathcal{E}(\mathbf{q}, \text{Grad}\mathbf{q})$ is the scalar generalised force given by

$$\mathcal{E}(\mathbf{q}, \text{Grad}\mathbf{q}) := \frac{\partial\tilde{W}_{\nu}}{\partial\mathbf{q}} - \text{Div}\left(\frac{\partial\tilde{W}_{\nu}}{\partial\text{Grad}\mathbf{q}}\right). \quad (28)$$

362 Hence, $\mathfrak{D}_{\text{res}}$ becomes

$$\mathfrak{D}_{\text{res}} = -\phi_{\text{f}}^{-1}\boldsymbol{\pi}_{\text{fd}}\cdot\mathbf{F}\boldsymbol{\mathcal{Q}} + \mathcal{R}_{\nu,d}^{\text{int}}\dot{\mathbf{q}} + \mathbf{F}_{\text{p}}^{-\text{T}}(\mathbf{F}_{\text{p}}^{\text{T}}\mathbf{Y}_{\nu,d}^{\text{int}})\mathbf{F}_{\text{p}}^{\text{T}} : \mathbf{L}_{\text{p}} \geq 0. \quad (29)$$

363 By recalling the force balances (2), reformulated for the case of isochoric
 364 plastic-like distortions, and (6), which allow to substitute $\mathcal{R}_{\nu}^{\text{int}}$ with $\mathcal{R}_{\nu}^{\text{ext}}$ in
 365 (27a) and $\mathbf{Y}_{\nu}^{\text{int}}$ with $\mathbf{Y}_{\nu}^{\text{ext}}$ in (27b), we obtain [24, 33, 50]

$$\mathcal{R}_{\nu,d}^{\text{int}} = \mathcal{R}_{\nu}^{\text{ext}} - \mathcal{E}(\mathbf{q}, \text{Grad}\mathbf{q}), \quad (30a)$$

$$\mathbf{F}_{\text{p}}^{\text{T}}\mathbf{Y}_{\nu,d}^{\text{int}} = \text{dev}(\mathbf{F}_{\text{p}}^{\text{T}}\mathbf{Y}_{\nu}^{\text{ext}}) + \text{dev}(\boldsymbol{\Sigma}_{\text{std}} + \boldsymbol{\Sigma}_{\text{str}} + \boldsymbol{\Sigma}_{\text{Grad}}). \quad (30b)$$

366 If $\mathcal{R}_{\nu,d}^{\text{int}}$ and $\mathbf{Y}_{\nu,d}^{\text{int}}$ can be related constitutively to $\dot{\mathbf{q}}$ and \mathbf{L}_{p} , respectively, (30a)
 367 and (30b) become evolution laws for \mathbf{q} and \mathbf{F}_{p} . For this purpose, we study
 368 the dissipation inequality, and we require here each summand of (29) to be

369 non-negative independently on the other ones [29], i.e.,

$$\mathfrak{D}_{\text{flow}} = -\phi_f^{-1} \boldsymbol{\pi}_{\text{fd}} \cdot \mathbf{F} \mathfrak{Q} \geq 0, \quad (31a)$$

$$\mathfrak{D}_{\text{q}} = \mathcal{R}_{\nu, \text{d}}^{\text{int}} \dot{\mathbf{q}} \geq 0, \quad (31b)$$

$$\mathfrak{D}_{\text{p}} = \mathbf{F}_{\text{p}}^{-\text{T}} (\mathbf{F}_{\text{p}}^{\text{T}} \mathbf{Y}_{\nu, \text{d}}^{\text{int}}) \mathbf{F}_{\text{p}}^{\text{T}} : \mathbf{L}_{\text{p}} \geq 0. \quad (31c)$$

370 First, we consider the inequality $\mathfrak{D}_{\text{flow}} \geq 0$, and, by hypothesising a linear
371 relationship between $\boldsymbol{\pi}_{\text{fd}}$ and \mathfrak{Q} [52, 53], we obtain Darcy's law, i.e.,

$$\mathfrak{Q} = -\mathbf{K} \text{Grad } p. \quad (32)$$

372 Then, to satisfy $\mathfrak{D}_{\text{q}} \geq 0$, we assume $\mathcal{R}_{\nu, \text{d}}^{\text{int}} = \Gamma \dot{\mathbf{q}}$, with Γ being a strictly
373 positive quantity (in general, it suffices that Γ be non-negative).

374 Finally, we turn to \mathfrak{D}_{p} , and we assume that the plastic-like distortions
375 evolve according to a modified rate-independent formulation of plasticity,
376 compatible with an associative normality rule [54]. Moreover, we hypothesise
377 that $\mathbf{Y}_{\nu}^{\text{ext}}$ is identically null [36] and, by performing the change of variable
378 $\mathbf{H} = \mathbf{F}_{\text{p}}^{-1}$ and setting $\boldsymbol{\Lambda} = \dot{\mathbf{H}} \mathbf{H}^{-1}$, we obtain

$$\mathbf{H}^{-\text{T}} \mathbf{Y}_{\nu, \text{d}}^{\text{int}} = \text{dev}(\boldsymbol{\Sigma}_{\text{std}} + \boldsymbol{\Sigma}_{\text{str}} + \boldsymbol{\Sigma}_{\text{Grad}}) \equiv \text{dev} \boldsymbol{\Sigma}_{\text{eff}}, \quad (33a)$$

$$\mathfrak{D}_{\text{p}} = -\boldsymbol{\Sigma}_{\text{eff}} : \boldsymbol{\Lambda} = -(\text{dev} \boldsymbol{\Sigma}_{\text{eff}}) : \boldsymbol{\Lambda} \geq 0, \quad (33b)$$

379 where $\boldsymbol{\Sigma}_{\text{eff}}$ is referred to as the *effective* Mandel-like stress tensor and is the
380 sum of $\boldsymbol{\Sigma}_{\text{std}}$, $\boldsymbol{\Sigma}_{\text{str}}$, and $\boldsymbol{\Sigma}_{\text{Grad}}$. We remark that, because of the constraint
381 $\det \mathbf{H} = 1$, $\boldsymbol{\Lambda}$ is deviatoric and, consequently, it selects only the deviatoric
382 part of $\boldsymbol{\Sigma}_{\text{eff}}$ in \mathfrak{D}_{p} .

383 Next, we use $\boldsymbol{\Sigma}_{\text{eff}}$ to define the *effective* Cauchy-like stress tensor

$$\boldsymbol{\sigma}_{\text{eff}} := J^{-1} \mathbf{g}^{-1} \mathbf{F}^{-\text{T}} \boldsymbol{\Sigma}_{\text{eff}} \mathbf{F}^{\text{T}}. \quad (34)$$

384 We remark that, because of the presence of $\boldsymbol{\Sigma}_{\text{Grad}}$, $\boldsymbol{\Sigma}_{\text{eff}}$ is *not* a true Mandel
385 stress tensor and, analogously, $\boldsymbol{\sigma}_{\text{eff}}$ is *not* a true Cauchy stress tensor. Rather,
386 $\boldsymbol{\sigma}_{\text{eff}}$ only represents the spatial counterpart of $\boldsymbol{\Sigma}_{\text{eff}}$, constructed as shown in
387 (34), but it does not necessarily satisfy the properties that a true Cauchy
388 stress tensor should fulfil. For example, it is not symmetric. Still, we employ
389 $\boldsymbol{\sigma}_{\text{eff}}$ to formulate a yield criterion of the von Mises type. To this end, we

390 introduce the yield function

$$\mathcal{Y} = \|\text{dev}\boldsymbol{\sigma}_{\text{eff}}\|_{\mathbf{g}} - \sqrt{(2/3)}\sigma_y, \quad (35)$$

391 where σ_y is a strictly positive yield stress, and, to comply with the condition
 392 $J_p = 1$, only the deviatoric part of $\boldsymbol{\sigma}_{\text{eff}}$ is considered. We remark that the
 393 norm $\|\text{dev}\boldsymbol{\sigma}_{\text{eff}}\|_{\mathbf{g}}$ is computed with respect to the spatial metric \mathbf{g} , i.e.,

$$\|\text{dev}\boldsymbol{\sigma}_{\text{eff}}\|_{\mathbf{g}} = \sqrt{\mathbf{g} : (\text{dev}\boldsymbol{\sigma}_{\text{eff}})\mathbf{g}(\text{dev}\boldsymbol{\sigma}_{\text{eff}})^{\text{T}}}. \quad (36)$$

394 By expressing the norm $\|\text{dev}\boldsymbol{\sigma}_{\text{eff}}\|_{\mathbf{g}}$ in terms of $\boldsymbol{\Sigma}_{\text{eff}}$, i.e.,

$$\|\text{dev}\boldsymbol{\sigma}_{\text{eff}}\|_{\mathbf{g}} = J^{-1}\|\text{dev}\boldsymbol{\Sigma}_{\text{eff}}\|_{\mathbf{C}}, \quad (37a)$$

$$\|\text{dev}\boldsymbol{\Sigma}_{\text{eff}}\|_{\mathbf{C}} := \sqrt{\mathbf{C}^{-1} : (\text{dev}\boldsymbol{\Sigma}_{\text{eff}})\mathbf{C}(\text{dev}\boldsymbol{\Sigma}_{\text{eff}})^{\text{T}}}, \quad (37b)$$

395 we rephrase \mathcal{Y} in terms of $\boldsymbol{\Sigma}_{\text{eff}}$ and \mathbf{C} , thereby obtaining

$$\mathcal{Y} = \hat{\mathcal{Y}}(\mathbf{C}, \boldsymbol{\Sigma}_{\text{eff}}) = J^{-1}\|\text{dev}\boldsymbol{\Sigma}_{\text{eff}}\|_{\mathbf{C}} - \sqrt{(2/3)}\sigma_y. \quad (38)$$

396 We use (38) to maximise \mathfrak{D}_p over all the possible stresses [55]. For this pur-
 397 pose, we adopt the Karush-Kuhn-Tucker technique [55], along with the mod-
 398 ified dissipation

$$\tilde{\mathfrak{D}}_p(\mathbf{C}, \boldsymbol{\Sigma}_{\text{eff}}, \lambda) = -\text{dev}\boldsymbol{\Sigma}_{\text{eff}} : \boldsymbol{\Lambda} - \lambda\hat{\mathcal{Y}}(\mathbf{C}, \boldsymbol{\Sigma}_{\text{eff}}) \geq 0, \quad (39)$$

399 where λ is a Karush-Kuhn-Tucker (KKT) multiplier, to be determined. The
 400 search for maximisers of $\tilde{\mathfrak{D}}_p(\mathbf{C}, \boldsymbol{\Sigma}_{\text{eff}}, \lambda)$ is accomplished by differentiating $\tilde{\mathfrak{D}}_p$
 401 with respect to $\boldsymbol{\Sigma}_{\text{eff}}$ and λ , and leads to the Karush-Kuhn-Tucker optimality
 402 conditions [55]. Since in this work the yield stress, σ_y , is assumed to be a
 403 given model parameter, such optimality conditions read

$$\frac{\partial \tilde{\mathfrak{D}}_p}{\partial \boldsymbol{\Sigma}_{\text{eff}}}(\mathbf{C}, \boldsymbol{\Sigma}_{\text{eff}}, \lambda) = -\boldsymbol{\Lambda} - \lambda \frac{\partial \hat{\mathcal{Y}}}{\partial \boldsymbol{\Sigma}_{\text{eff}}}(\mathbf{C}, \boldsymbol{\Sigma}_{\text{eff}}) = \mathbf{0}, \quad (40a)$$

$$\lambda \geq 0, \quad \hat{\mathcal{Y}}(\mathbf{C}, \boldsymbol{\Sigma}_{\text{eff}}) \leq 0, \quad \lambda \hat{\mathcal{Y}}(\mathbf{C}, \boldsymbol{\Sigma}_{\text{eff}}) = 0. \quad (40b)$$

404 5.1. Reorientation of the fibres

405 By substituting $\mathcal{R}_{\nu,d}^{\text{int}} = \Gamma \dot{\mathbf{q}}$ into (30a), and writing $\mathcal{E}(\mathbf{q}, \text{Grad } \mathbf{q})$ explicitly,
 406 Eq. (30a) takes on the form

$$\Gamma \dot{\mathbf{q}} = \text{Div} [\Phi_{1s\nu} D_0 \mathbf{B}_p \text{Grad } \mathbf{q}] - \Phi_{1s\nu} \frac{\partial(\hat{W}_{1a} + \hat{W}_{\text{str}})}{\partial \mathbf{q}} + \mathcal{R}_{\nu}^{\text{ext}}. \quad (41)$$

407 The first term on the right-hand-side of (41) contributes to the evolution of
 408 the fibre mean angle by resolving the spatial variability of \mathbf{q} . The coefficient
 409 $\Phi_{1s\nu} D_0$ multiplies the inverse (plastic) metric tensor \mathbf{B}_p , thereby leading to
 410 the tensorial coefficient $\Phi_{1s\nu} D_0 \mathbf{B}_p$. We notice that, in spite of some formal
 411 similarities with a diffusion-reaction equation, (41) describes no diffusion,
 412 since it is not a mass balance, but the evolution of an order parameter [56].

413 To solve (41), we need to provide $\mathcal{R}_{\nu}^{\text{ext}}$. In two previous papers on this
 414 subject [24, 40], one of us reviewed some results presented by other authors,
 415 e.g. [32, 33], who defined the external remodelling force $\mathcal{R}_{\nu}^{\text{ext}}$ by introducing
 416 the concept of *target angle*, \mathbf{q}_T . The target angle is an angle that defines the
 417 direction of space, which we may call *target direction*, along which the fibres
 418 “would like to be aligned”. By definition, the fibres tend to orient themselves
 419 along the target direction and it has been observed that, in a tissue subjected
 420 to mechanical stress and deformation, the target angle depends on stress
 421 [32, 33] or deformation [12, 31].

422 Although the issue of the target angle was discussed in [24, 40], the focus
 423 in those papers was on the particular situations in which no external force
 424 $\mathcal{R}_{\nu}^{\text{ext}}$ was active, i.e., when the condition $\mathcal{R}_{\nu}^{\text{ext}} = 0$ applies in (41). In these
 425 cases, indeed, a “target angle” may be identified with a stationary solution
 426 of (41), i.e., a function \mathbf{q}_{∞} satisfying

$$\text{Div} [\Phi_{1s\nu} D_0 \mathbf{B}_p \text{Grad } \mathbf{q}] - \Phi_{1s\nu} \frac{\partial(\hat{W}_{1a} + \hat{W}_{\text{str}})}{\partial \mathbf{q}} = 0, \quad (42)$$

427 together with time-independent boundary conditions. Since $\partial \hat{W}_{\text{str}} / \partial \mathbf{q}$ does
 428 not vanish when $\mathbf{B}_p = \mathbf{G}^{-1}$ and the tissue is undeformed, (42) admits solu-
 429 tions of sigmoidal shape that interpolate between the zeroes of the double-
 430 well potential $\mathcal{P}(\mathbf{q})$, i.e., $\mathbf{q}_0 = 0$ and $\mathbf{q}_1 = \pi/2$. Always in the absence of
 431 deformation, such profiles can also be obtained as the stationary solutions
 432 of (41), when the initial distribution of \mathbf{q} is a random function of material
 433 points [24].

434 In the case of vanishing D_0 , the energy density W_{Grad} is null, and we
 435 end up with a description of remodelling determined by ordinary differential
 436 equations. In such situations, and for $\mathcal{R}_\nu^{\text{ext}} = 0$, the search for stationary
 437 solutions amounts to seek for the zeros of the equation

$$-\Phi_{1s\nu} \frac{\partial(\hat{W}_{1a} + \hat{W}_{\text{str}})}{\partial \mathbf{q}} = 0. \quad (43)$$

438 In general, however, (43) may admit either no solutions or multiple solutions,
 439 i.e., different target angles. Whereas the existence of multiple stationary so-
 440 lutions to (43) can be a normal fact, because the Cauchy problem

$$\Gamma \dot{\mathbf{q}} = -\Phi_{1s\nu} \frac{\partial(\hat{W}_{1a} + \hat{W}_{\text{str}})}{\partial \mathbf{q}}, \quad (44a)$$

$$\mathbf{q}(X, 0) = \mathbf{q}_{\text{in}}(X), \quad (44b)$$

441 if well-posed, selects a unique solution, the case of no stationary solution may
 442 be unphysical. Similar circumstances may occur when the right-hand-side of
 443 (44a) features only $\partial \hat{W}_{1a} / \partial \mathbf{q}$.

444 By introducing a non-vanishing $\mathcal{R}_\nu^{\text{ext}}$, relating it to the concept of an
 445 *a priori* defined target angle, \mathbf{q}_T , and assuming the existence of a station-
 446 ary limit \mathbf{q}_T^∞ , the unphysical case of no stationary solutions is eliminated at
 447 source. Indeed, it suffices to notice that a stationary angle is attained when
 448 the external force $\mathcal{R}_\nu^{\text{ext}}$ balances the internal ones under the condition $\dot{\mathbf{q}} = 0$.
 449 This implies that the following equality has to be verified [33]

$$\mathcal{R}_\nu^{\text{ext}} = \Phi_{1s\nu} \frac{\partial(\hat{W}_{1a} + \hat{W}_{\text{str}})}{\partial \mathbf{q}} \Big|_{\mathbf{q}=\mathbf{q}_T^\infty}. \quad (45)$$

450 This result can also be generalised to the case in which the target angle is
 451 not stationary, so that Eq. (44a) is rewritten as

$$\Gamma \dot{\mathbf{q}} = -\Phi_{1s\nu} \frac{\partial(\hat{W}_{1a} + \hat{W}_{\text{str}})}{\partial \mathbf{q}} + \Phi_{1s\nu} \frac{\partial(\hat{W}_{1a} + \hat{W}_{\text{str}})}{\partial \mathbf{q}} \Big|_{\mathbf{q}=\mathbf{q}_T}, \quad (46)$$

452 where the term on the right-hand-side is computed for a non-stationary target
 453 angle \mathbf{q}_T , driven by stress or deformation.

454 Even more generally, when the remodelling equation is given by (41), the
 455 external force $\mathcal{R}_\nu^{\text{ext}}$ may be defined as

$$\begin{aligned} \mathcal{R}_\nu^{\text{ext}} &= \mathcal{E}(q_T, \text{Grad } \mathbf{q}_T) \\ &= -\text{Div} [\Phi_{1s\nu} D_0 \mathbf{B}_p \text{Grad } \mathbf{q}_T] + \Phi_{1s\nu} \left. \frac{\partial(\hat{W}_{1a} + \hat{W}_{\text{str}})}{\partial \mathbf{q}} \right|_{\mathbf{q}=\mathbf{q}_T}, \end{aligned} \quad (47)$$

456 thereby obtaining the following generalisation of [24, 33, 40]:

$$\begin{aligned} \Gamma \dot{\mathbf{q}} &= \text{Div} [\Phi_{1s\nu} D_0 \mathbf{B}_p \text{Grad } \mathbf{q}] - \Phi_{1s\nu} \frac{\partial(\hat{W}_{1a} + \hat{W}_{\text{str}})}{\partial \mathbf{q}} \\ &\quad - \text{Div} [\Phi_{1s\nu} D_0 \mathbf{B}_p \text{Grad } \mathbf{q}_T] + \Phi_{1s\nu} \left. \frac{\partial(\hat{W}_{1a} + \hat{W}_{\text{str}})}{\partial \mathbf{q}} \right|_{\mathbf{q}=\mathbf{q}_T}. \end{aligned} \quad (48)$$

457 5.2. Evolution of the plastic-like distortions

458 The explicit computation of the derivative of $\hat{\mathcal{Y}}$ with respect to Σ_{eff} , see
 459 (38), permits to rewrite (40a) as

$$\mathbf{\Lambda} = -J^{-1} \lambda \frac{\mathbf{C}^{-1}(\text{dev } \Sigma_{\text{eff}}) \mathbf{C}}{\|\text{dev } \Sigma_{\text{eff}}\|_{\mathbf{C}}}, \quad (49)$$

460 which implies $\|\mathbf{\Lambda}\|_{\mathbf{C}} \equiv \sqrt{\mathbf{C} : \mathbf{\Lambda} \mathbf{C}^{-1} \mathbf{\Lambda}^{\text{T}}} = J^{-1} \lambda \geq 0$. Moreover, since $\mathbf{\Lambda}$ is
 461 given by $\mathbf{\Lambda} = \dot{\mathbf{H}} \mathbf{H}^{-1}$, (49) can be recast in the form of an evolution equation
 462 for \mathbf{H} or, equivalently, for $\mathbf{F}_p = \mathbf{H}^{-1}$, i.e.,

$$\dot{\mathbf{H}} = \left\{ -J^{-1} \lambda \frac{\mathbf{C}^{-1}(\text{dev } \Sigma_{\text{eff}}) \mathbf{C}}{\|\text{dev } \Sigma_{\text{eff}}\|_{\mathbf{C}}} \right\} \mathbf{H}. \quad (50)$$

463 Within the classical framework of finite Elastoplasticity, the KKT-multiplier
 464 λ is determined by enforcing a condition known as “consistency condition”
 465 [55], which has to be solved together with the flow rule —represented here by
 466 (50)— and the other model equations. Very often, the consistency condition
 467 is solved algorithmically (see e.g. [55]). In this work, however, we propose
 468 a rather different approach, which is motivated by the need of keeping our
 469 calculations at a minimum level of complexity (see Sect. 5.3 for some technical
 470 details on this issue). In fact, we prescribe λ from the outset, and, for our

471 purposes, we define it as

$$\begin{aligned}\lambda &= J\zeta_0\phi_s \left[\|\text{dev}\boldsymbol{\sigma}_{\text{eff}}\|_g - \sqrt{(2/3)}\sigma_y \right]_+ \\ &= \zeta_0\Phi_{sv} \left[J^{-1}\|\text{dev}\boldsymbol{\Sigma}_{\text{eff}}\|_C - \sqrt{(2/3)}\sigma_y \right]_+, \end{aligned} \quad (51)$$

472 where $\zeta_0 > 0$ is a constant model parameter, and $[A]_+ = A$, for $A > 0$,
473 and $[A]_+ = 0$, otherwise. We notice that the equality $\Phi_{sv} = J\phi_s$ is verified,
474 because it holds that $J = J_e$, since the condition $J_p = 1$ applies. Finally, by
475 substituting (51) into (50), we obtain

$$\dot{\mathbf{H}} = - \frac{\zeta_0\Phi_{sv} \left[J^{-1}\|\text{dev}\boldsymbol{\Sigma}_{\text{eff}}\|_C - \sqrt{(2/3)}\sigma_y \right]_+}{J} \frac{\mathbf{C}^{-1}(\text{dev}\boldsymbol{\Sigma}_{\text{eff}})\mathbf{C}}{\|\text{dev}\boldsymbol{\Sigma}_{\text{eff}}\|_C} \mathbf{H}, \quad (52)$$

476 i.e., the ordinary differential equation describing the evolution of \mathbf{H} .

477 Equation (52) looks like an evolution law of the Norton-Hoff type [57] and,
478 with some modifications, might be rated among those. However, compared
479 with that in [57], our (52) features three differences: (i) the full tensor $\text{dev}\boldsymbol{\Sigma}_{\text{eff}}$
480 is considered in lieu of its symmetric part only (see [57] for some remarks
481 on this issue); (ii) the “transformed” generalised stress $\mathbf{C}^{-1}\text{dev}\boldsymbol{\Sigma}_{\text{eff}}\mathbf{C}$, rather
482 than $\text{dev}\boldsymbol{\Sigma}_{\text{eff}}$, is regarded as the driving force for \mathbf{H} ; (iii) our $\boldsymbol{\Sigma}_{\text{eff}}$ contains
483 $\boldsymbol{\Sigma}_{\text{Grad}}$, which is a fundamental character of our framework.

484 We notice that the coefficient λ in (51) has the form of the activation
485 factor featuring in the flow rule of a Perzyna-like model of viscoplasticity
486 [37]. Dimensional analysis shows that the parameter ζ_0 can be expressed as
487 $\zeta_0 = (\tau_c\sigma_c)^{-1}$, where τ_c is the characteristic relaxation time of \mathbf{H} , and σ_c is a
488 reference value of stress. The time scale τ_c is available in the literature, and
489 we choose $\tau_c = 22\text{ s}$, as suggested in [28], where the inelastic behaviour of
490 cellular aggregates is studied by means of a Perzyna-like flow rule. However,
491 there seems to be some freedom in the choice of the reference stress σ_c . In
492 principle, indeed, σ_c could be taken equal to σ_y , if one wants to normalise
493 λ with the yield stress, or it could be defined by combining the material
494 parameters involved in the definition of $\boldsymbol{\sigma}_{\text{eff}}$. In the latter case, one should
495 use parameters, such as D_0 and \mathcal{A}_0 , that, being other than the standard
496 elastic coefficients, are not available in the literature, at least to the best of
497 our knowledge. Thus, we refer here to a value of σ_c that has already been used
498 in [28], within a framework similar to ours. To this end, by comparing (51)

499 with the flow rule in [28], we identify σ_c with $\sigma_c = 2\mu_0\langle\Phi_{sv}\rangle$, where μ_0 is the
500 shear modulus of the matrix, and $\langle\Phi_{sv}\rangle$ is the mean value of the solid phase
501 volumetric ratio. Hence, upon computing $\mu_0 = 2(\alpha_1 + \alpha_2)\alpha_0 \approx 0.222$ MPa
502 [41] and $\langle\Phi_{sv}\rangle = \int_0^1 \Phi_{sv}(\xi)d\xi = 0.2$ (see Table 1), we find $\sigma_c \approx 0.09$ MPa and
503 $\zeta_0 \approx 0.50$ MPa $^{-1}$ s $^{-1}$ (cf. Table 1). Such σ_c is obtained by considering only
504 the isotropic part of the standard energy of our model, whereas considering
505 also the other terms of the energy would lead to higher values of σ_c and,
506 then, to smaller values of ζ_0 . On the other hand, smaller values of σ_c are
507 conceivable, but they could result into too high values of ζ_0 for the problem
508 at hand, thereby leading to unphysical time scales for the evolution of \mathbf{H} .

509 5.3. Summary of the model equations and technical details

510 After enforcing the left polar decomposition of \mathbf{H} , i.e., $\mathbf{H} = \mathbf{V}\mathbf{R}$ [35], we
511 study only the case in which \mathbf{R} reduces to a shifter [35], so that the unknown
512 determining the plastic-like distortions becomes the symmetric, second-order
513 tensor \mathbf{V} . Even though this choice has the disadvantage of restricting the
514 investigation to the case of no plastic-like rotations, it allows to work with \mathbf{V} ,
515 which, being symmetric, is computationally cheaper. In summary, thus, our
516 mathematical model consists of the following set of four, highly non-linear,
517 coupled equations,

$$\dot{J} - \text{Div}(\mathbf{K}\text{Grad } p) = 0, \quad (53a)$$

$$\text{Div}(-Jp\mathbf{g}^{-1}\mathbf{F}^{-T} + \mathbf{P}_{sc}) = \mathbf{0}, \quad (53b)$$

$$\begin{aligned} \Gamma\dot{\mathbf{q}} = & \text{Div}[\Phi_{1sv}D_0\mathbf{B}_p\text{Grad } \mathbf{q}] - \Phi_{1sv}\frac{\partial(\hat{W}_{1a} + \hat{W}_{str})}{\partial\mathbf{q}} \\ & - \text{Div}[\Phi_{1sv}D_0\mathbf{B}_p\text{Grad } \mathbf{q}_T] + \Phi_{1sv}\frac{\partial(\hat{W}_{1a} + \hat{W}_{str})}{\partial\mathbf{q}} \Bigg|_{\mathbf{q}=\mathbf{q}_T}, \end{aligned} \quad (53c)$$

$$\dot{\mathbf{V}} = -\text{sym} \left[\left(\frac{\lambda}{J} \frac{\mathbf{C}^{-1}(\text{dev}\Sigma_{\text{eff}})\mathbf{C}}{\|\text{dev}\Sigma_{\text{eff}}\|_{\mathbf{C}}} \right) \mathbf{V} \right], \quad (53d)$$

518 in the unknowns p , χ , \mathbf{q} , and \mathbf{V} , respectively. Note that we take the sym-
519 metric part of the right-hand-side of (53d) in order to ensure that $\dot{\mathbf{V}}$, and
520 its time discrete form, be symmetric. Moreover, the material permeability is
521 given by [41, 6, 8, 9]

$$\mathbf{K} = k_0 \frac{(J - \Phi_{1sv})^2}{J} \mathbf{C}^{-1} + k_0 \frac{(J - \Phi_{1sv})\Phi_{1sv}}{J} \mathbf{H} \left\langle \left\langle \frac{\mathbf{a}}{I_{4e}} \right\rangle \right\rangle \mathbf{H}^T, \quad (54a)$$

$$k_0 = k_{0\nu} \left[\frac{J - \Phi_{s\nu}}{1 - \bar{\Phi}_{s\nu}} \right]^{\kappa_0} \exp \left(\frac{1}{2} m_0 [J^2 - 1] \right). \quad (54b)$$

522 The material parameters κ_0 and m_0 as well as the expressions of the volu-
 523 metric fractions $\Phi_{s\nu}$, $\Phi_{0s\nu}$, and $\Phi_{1s\nu}$ are reported in Table 1.

524 We solve Eqs. (53a)–(53d) for a cylindric specimen of tissue, of initial
 525 height $L = 1$ mm and initial radius $R = 1.5$ mm, and whose boundary can
 526 be written as $\partial\mathcal{B} = \partial\mathcal{B}_U \sqcup \partial\mathcal{B}_L \sqcup \partial\mathcal{B}_\ell$, where the subscripts “U”, “L”, “ ℓ ”
 527 stand for “upper”, “lower” and “lateral”, respectively. Then, we complete
 528 Eqs. (53a)–(53d) with the following boundary and initial conditions

$$-(\mathbf{K} \text{ Grad } p) \cdot \mathbf{N} = 0, \quad \text{on } \partial\mathcal{B}_U \sqcup \partial\mathcal{B}_L, \quad (55a)$$

$$p = 0, \quad \text{on } \partial\mathcal{B}_\ell, \quad (55b)$$

$$[\chi(X, 0) - \chi(X, t)] \cdot \mathbf{e}_3 = u(t), \quad \text{on } \partial\mathcal{B}_U, \quad (55c)$$

$$\chi(X, t) - \chi(X, 0) = \mathbf{0}, \quad \text{on } \partial\mathcal{B}_L, \quad (55d)$$

$$(-J p \mathbf{g}^{-1} \mathbf{F}^{-T} + \mathbf{P}_{sc}) \cdot \mathbf{N} = \mathbf{0}, \quad \text{on } \partial\mathcal{B}_\ell, \quad (55e)$$

$$(\Phi_{1s\nu} D_0 \mathbf{B}_p \text{ Grad } \mathbf{q}) \cdot \mathbf{N} = \mathbf{0}, \quad \text{on } \partial\mathcal{B}_U \sqcup \partial\mathcal{B}_\ell, \quad (55f)$$

$$\mathbf{q}(X, t) = \mathbf{0}, \quad \text{on } \partial\mathcal{B}_L, \quad (55g)$$

$$\chi(X, 0) = \chi_0(X), \quad \text{in } \mathcal{B}, \quad (55h)$$

$$\mathbf{q}(X, 0) = \mathbf{q}_{\text{hist}}(X), \quad \text{in } \mathcal{B}, \quad (55i)$$

$$\mathbf{V}(X, 0) = \mathbf{G}^{-1}(X), \quad \text{in } \mathcal{B}. \quad (55j)$$

529 In (55a), (55e), and (55f), \mathbf{N} is the field of unit vectors normal to $\partial\mathcal{B}$; in
 530 (55c), the imposed displacement $u(t)$ is given by

$$u(t) = \frac{u_{\max} t}{t_{\text{ramp}}} [\Theta(t) - \Theta(t - t_{\text{ramp}})] + u_{\max} \Theta(t - t_{\text{ramp}}), \quad (56)$$

531 where $\Theta(s) = 1$, for $s \geq 0$, and $\Theta(s) = 0$, for $s < 0$, $u_{\max} = 0.20$ mm is
 532 the maximum imposed displacement, and $t_{\text{ramp}} = 20$ s is the final time of the
 533 loading ramp. In the simulated compression test, u_{\max} is kept constant until
 534 $t_f = 120$ s. In (55h), $\chi_0(X)$ represents the initial placement and, in this work,
 535 it returns the points X of the reference configuration \mathcal{B} . In (55i), $\mathbf{q}_{\text{hist}}(X)$
 536 denotes the initial distribution of the fibre mean angle, and is taken here to

537 be equal to an experimentally observed “histological” profile [49], given by

$$\mathbf{q}_{\text{hist}}(X) = \frac{\pi}{2} \left\{ 1 - \cos \left(\frac{\pi}{2} \left[-\frac{2}{3} \left(\frac{X^3}{L} \right)^2 + \frac{5}{3} \frac{X^3}{L} \right] \right) \right\}, \quad (57)$$

538 where X^3 is the axial coordinate. Finally, the initial value $\mathbf{V}(X, 0)$ is taken in
 539 (55j) equal to the inverse metric tensor associated with \mathcal{B} , which means that
 540 no inelastic distortions occur before the deformation process commences.

541 We remark that (53a)–(53d) are valid in general, in the sense that they
 542 apply to the studied system, under all the specified hypotheses, but without
 543 any specialisation to a particular benchmark problem. In fact, they can be
 544 adopted for a variety of case studies, and to formulate a proof of concept for
 545 testing a proposed model. In our work, we employ (53a)–(53d) for analysing
 546 the coupling among fluid flow, deformation and structural reorganisation of
 547 the matrix, and fibre reorientation in the tissue under study. For this pur-
 548 pose, we solve numerically a well-documented benchmark test consisting in
 549 the unconfined compression of a cylindrical specimen of tissue. The latter
 550 is assumed here to be articular cartilage because of the availability of ex-
 551 perimental data, but the test can also be performed on other tissues. For
 552 the considered test, a sample of tissue is placed between two plates, assumed
 553 to be rigid and impermeable (see (55a)), as shown in 2. The lower plate is
 554 fixed and the specimen is clamped to it, so as to simulate the adhesion of the
 555 cartilage to bone (see (55d)). The upper plate, instead, compresses axially
 556 the sample (see (55c)), in such a way that the deformation remains axial-
 557 symmetric over the whole duration of the simulation. The lateral surface of
 558 the sample is assumed to constitute a free boundary, which means that both
 559 the pressure and the radial stress have to be equal to zero (see (55b) and
 560 (55e)).

561 We also have to impose boundary conditions on the fibre mean angle,
 562 \mathbf{q} . These are specified by (55f) and (55g). The Dirichlet condition (55g)
 563 forces the fibres to remain orthogonal to the bone-cartilage interface for the
 564 whole duration of the simulation. Due to the geometry of the specimen
 565 and the symmetry of the problem, this restriction implies that, on the lower
 566 boundary, the fibres are maintained parallel to the specimen’s symmetry
 567 axis. Furthermore, the Neumann condition (55f) requires that the normal
 568 component of $\mathbf{y}^{(1)} = \Phi_{1sv} D_0 \mathbf{B}_p \text{Grad} \mathbf{q}$ vanishes on the upper and lateral
 569 boundary of the sample. We notice that the coupling between \mathbf{q} and \mathbf{F}_p ,

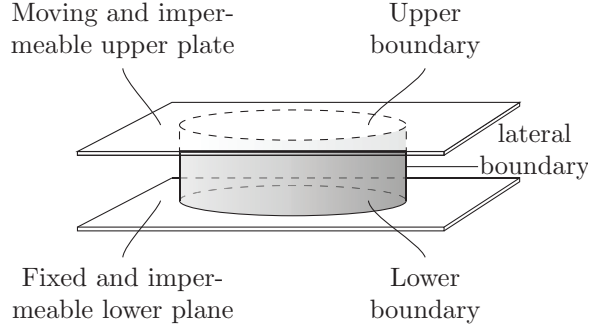


Figure 2: Panel describing the considered benchmark test

570 accounted for by $\mathbf{B}_p = \mathbf{F}_p^{-1} \cdot \mathbf{F}_p^{-T}$, affects the way in which (55f) is satisfied
 571 and, consequently, the way in which \mathbf{q} approaches the boundary. Indeed, in
 572 the absence of plastic-like distortions, i.e., for $\mathbf{B}_p = \mathbf{G}^{-1}$, (55f) requires that
 573 the normal derivative of \mathbf{q} is zero on $\partial\mathcal{B}_U \sqcup \partial\mathcal{B}_\ell$. For $\mathbf{B}_p \neq \mathbf{G}^{-1}$, this result
 574 is no longer true, and the gradient of \mathbf{q} is no longer orthogonal to \mathbf{N} .

575 **Remark 1.** *To clarify the physical meaning of (55f), we recall that, in our*
 576 *model, \mathbf{q} and $\text{Grad}\mathbf{q}$ are kinematic descriptors and, consistently with (3),*
 577 *the vector $\mathbf{y}^{(1)}$ is the internal generalised force conjugated with $\text{Grad}\mathbf{q}$. Thus,*
 578 *$\mathbf{y}^{(1)}$ plays the role of stress and, as anticipated in (4b), $\mathbf{y}^{(1)} \cdot \mathbf{N}$ is the stress*
 579 *component that has to balance the generalised “contact” force $h^{(1)}$, defined on*
 580 *the Neumann boundary of the sample. It follows from these considerations*
 581 *that (55f) rephrases (4b) in the particular case in which no such forces are*
 582 *active, thereby yielding $\mathbf{y}^{(1)} \cdot \mathbf{N} = h^{(1)} = 0$. This amounts to say that $\partial\mathcal{B}_U \sqcup$*
 583 *$\partial\mathcal{B}_\ell$ is a free boundary with respect to \mathbf{q} .*

584 6. Results

585 To perform a comparative study of the various phenomena accounted for
 586 in our work, we consider four different sub-models, which we denominate M1,
 587 M2, M3, and M4.

588 *Model M1 (poroelasticity with $\mathcal{R}_v^{\text{ext}} = 0$).* As reference case, we consider a
 589 deformable poroelastic material, in which the evolution of the fibre direction
 590 is driven by deformation only. Thus, we solve (53a)–(53c), along with (55a)–
 591 (55i). In the computations we set $\mathcal{R}_v^{\text{ext}}$ equal to zero, which amounts to ignore

592 in (53c) all the terms containing the target angle \mathbf{q}_T . We do that with the aim
 593 of providing an estimate of the importance of the target angle on the guidance
 594 of the fibre evolution. Indeed, even in the absence of $\mathcal{R}_\nu^{\text{ext}}$, the inhomogeneity
 595 of the fibre mean angle and the generalised forces $\Phi_{1s\nu}\partial(\hat{W}_{1a} + \hat{W}_{\text{str}})/\partial\mathbf{q}$ are
 596 capable of triggering the evolution of the fibres. By dealing with a poroelastic
 597 model, \mathbf{V} is kept equal to its initial value, \mathbf{G}^{-1} , thereby switching off the
 598 evolution of the plastic-like distortions.

599 *Model M2 (poroelasticity with $\mathcal{R}_\nu^{\text{ext}} \neq 0$).* This case is the completion of the
 600 model M1, as fibre re-orientation is also driven by the target angle. To this
 601 end, we solve the same set of equations and initial and boundary conditions
 602 as implemented in M1. In M2, however, all the terms appearing in (53c) are
 603 activated, and \mathbf{q}_T is computed as

$$\begin{aligned} \mathbf{q}_T &= \arctan \left(\frac{1}{C_{e33}} \left[\frac{1}{2\pi} \int_0^{2\pi} \mathbf{C}_e : \mathbf{e}_R(\varphi) \otimes \mathbf{e}_R(\varphi) d\varphi \right] \right) \\ &= \arctan \left(\frac{\frac{1}{2}[C_{e11} + C_{e22}]}{C_{e33}} \right), \end{aligned} \quad (58)$$

604 where $\mathbf{e}_R(\varphi) = \cos \varphi \mathbf{e}_1 + \sin \varphi \mathbf{e}_2$ is a unit vector orthogonal to the specimen's
 605 symmetry axis, and oriented radially. Note that other definitions are possible.
 606 For example, one may define the target angle as a function of stress [12, 31,
 607 32, 33] or as a function of the deformation [31]. The expression of \mathbf{q}_T given
 608 in (58) takes inspiration from [12, 31], and assumes that the target angle
 609 is entirely determined by \mathbf{C}_e . Specifically, the factor $\frac{1}{2}[C_{e11} + C_{e22}]$ is the
 610 in-plane directional average of the radial component of \mathbf{C}_e , while C_{e33} is
 611 the axial component of \mathbf{C}_e . Under the considered loading conditions, (58)
 612 implies that, for increasing radial dilatation and increasing axial contraction,
 613 $\frac{1}{2}[C_{e11} + C_{e22}]/C_{e33}$ tends towards infinity, and \mathbf{q}_T tends towards $\pi/2$. In
 614 this limit, the target angle indicates that the fibres should be preferably
 615 aligned orthogonally to the specimen's symmetry axis. Clearly, the way in
 616 which the fibre mean angle complies with this condition is modulated both
 617 by the deformation and the plastic-like distortions. To us, another physically
 618 relevant situation occurs in the absence of deformation and elastic distortions,
 619 i.e., when (58) prescribes $\mathbf{q}_T = \pi/4$, and (53c) becomes

$$\Gamma \dot{\mathbf{q}} = \text{Div}[\Phi_{1s\nu} D_0 \mathbf{G}^{-1} \text{Grad} \mathbf{q}] - \Phi_{1s\nu} \mathcal{A}_0 \frac{d\mathcal{P}}{d\mathbf{q}}. \quad (59)$$

620 In this case, the concept of target angle \mathbf{q}_T as manifestation of external force
 621 is not explicitly present in (59), and the evolution of \mathbf{q} is self-driven, with the
 622 target angles being identified with the stationary solutions of (59).

623 *Model M3 (full model, with $\mathcal{R}_\nu^{\text{ext}} = 0$).* This case study is complete, since it
 624 requires to solve the whole set of the model equations (53a)–(53d) together
 625 with (55a)–(55g) and (55h)–(55j). However, as done in M1, in the computa-
 626 tions we set $\mathcal{R}_\nu^{\text{ext}}$ equal to zero.

627 *Model M4 (full model).* As for M3, also M4 describes the complete model and
 628 requires the solution of the same list of equations, with the same boundary
 629 and initial conditions. However, in M4 the target angle is accounted for.

630 *Computational aspects.* To determine the numerical solution of our problem,
 631 we perform Finite Element simulations for each of the sub-models M1, M2,
 632 M3, and M4. This requires the weak formulation of (53a)–(53d), the gener-
 633 ation of a grid for the discretisation of \mathcal{B} and $\partial\mathcal{B}$, and the selection of
 634 a time integration scheme. Since the problem is nonlinear, a linearisation
 635 procedure is necessary. In general, the grid is unstructured and the interpo-
 636 lations adopted for p , χ , and \mathbf{q} are different from each other. Equation (53d)
 637 is solved only at the integration points of the finite element discretisation,
 638 for it does not contain partial derivatives of \mathbf{V} with respect to the spatial
 639 variables. Hence, we do not provide any weak form for (53d), nor do we
 640 introduce in this work test functions associated with \mathbf{V} .

641 A Backward Euler scheme of the fifth order is used for the integration
 642 in time of all the model equations and boundary conditions. Moreover, in
 643 each sub-model, the directional averages of the constitutive functions are
 644 computed by employing the Spherical Design Algorithm (see e.g. [14, 58]) as
 645 implemented in [15], i.e., the integrals over $\mathbb{S}^2\mathcal{B}$ are evaluated for each time
 646 step and at each iteration of the Newton method.

647 In our work, the numerical simulations were performed with the aid of
 648 the commercial software COMSOL[®]v5.3. Details about the algorithms used
 649 for the Finite Element solution of a problem involving (53a), (53b), and an
 650 evolution equation similar to (53d) can be found in [29, 59].

651 *Comments to figures.* To sample the data, we took four measuring points,
 652 located along the vertical axis, and with cartesian coordinates $X_L = (0, 0, L)$,
 653 $X_{3L/4} = (0, 0, 3L/4)$, $X_{L/4} = (0, 0, L/4)$, $X_0 = (0, 0, 0)$.

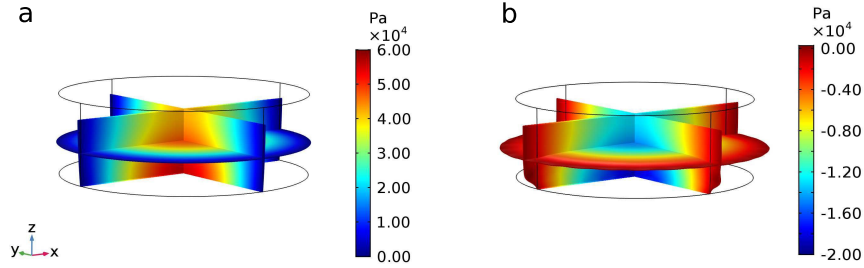


Figure 3: Pressure. 3D contour plots of the pressure for the models M2 (panel a) and M4 (panel b), whilst showing the deformation undergone by the tissue. The models M1 and M3 are not reported since they would lead to no observable difference with respect to M2 and M4, respectively.

654 First of all, we present a three-dimensional view of the deformed tissue
 655 at the end of the loading history. Figure 3 depicts the differences in the
 656 deformation of the sample and in the pressure distribution for the models M2
 657 and M4. The radial displacement of the tissue appears relatively contained in
 658 M2 (Fig. 3a), while it is more pronounced in M4, i.e., when plastic distortions
 659 are active (Fig. 3b). A peculiar characteristic of this case is given by the shape
 660 of the profile of the deformed lateral boundary. Indeed, in M2, such profile
 661 undergoes a gradual deformation from the bottom to the top, whereas in M4
 662 it experiences an abrupt deformation close to the bottom, while it remains
 663 almost parallel to the symmetry axis in the middle and in the upper parts
 664 of the sample. A possible explanation of this phenomenon can be outlined
 665 through the analysis of the fibre mean angle, as shown in Fig. 5.

666 Another peculiarity of Fig. 3 concerns the values attained by the pressure.
 667 In contrast to the elastic case, when plastic-like distortions are accounted for,
 668 the pressure goes lower than zero, thereby leading to a “syringe effect” [25].
 669 To better describe this phenomenon, Fig. 4 presents the time variation of
 670 the pressure in X_0 . No significant differences can be observed for models M1
 671 and M2, in which, after the increase due to the loading ramp, the pressure
 672 monotonically decreases toward zero. On the other hand, for both models
 673 accounting for the plastic-like distortions, i.e., M3 and M4, after a first rapid
 674 increase at the beginning of the loading experiment, we observed a rather
 675 slow increase of the pressure values. Afterwards, when the loading ramp
 676 terminates, we assist to an abrupt pressure drop, that leads to negative
 677 pressure values. This sudden change is then followed by a slow recovery, that
 678 would lead to null pressure in the long term.

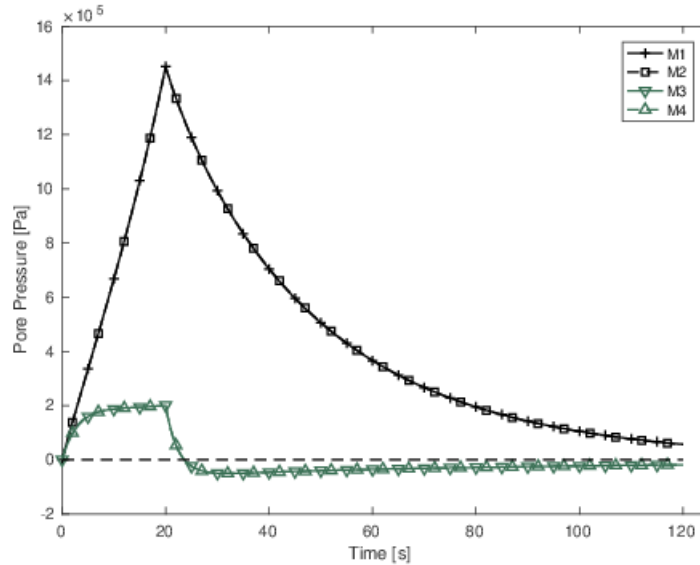


Figure 4: Time-evolution of the pore pressure. For all the implemented models, the temporal evolution of the pore pressure is monitored in X_0 .

679 A key point of this work is the role played by the fibre mean angle and
680 by the target angle. To analyse their evolution we present Fig. 5. The top
681 panels of Fig. 5 depict the evolution of the fibre mean angle, \mathbf{q} , along the
682 symmetry axis, starting from the initial histological profile (55i), to the final
683 fibre distribution obtained within M2 (Fig. 5a) and M4 (Fig. 5b). Note that,
684 thanks to the upper boundary condition 55f, the value of \mathbf{q} corresponding to
685 the upper surface is free to evolve. Interestingly, the greater variations are
686 registered in the plastic case (M4) and, enhanced by the introduction of the
687 gradient term, the variability extends to the tissue beneath. While in the
688 middle-upper portion of the tissue we assist to a smooth change of the fibre
689 mean angle, on the lower part there is quite an abrupt variation from the
690 histological profile. This might be due to the Dirichlet boundary condition
691 on the lower boundary of the specimen.

692 To understand the role of \mathbf{q}_T and to further describe the behaviour of \mathbf{q} ,
693 the temporal evolution of the fibre mean angle is shown in the lower panels
694 of Fig. 5, where the trend of the target angle \mathbf{q}_T is presented alongside the
695 fibre mean angle, evaluated in two different sampling points. Indeed, by
696 comparing M1 with M2, and M3 with M4, it is evident that the introduction

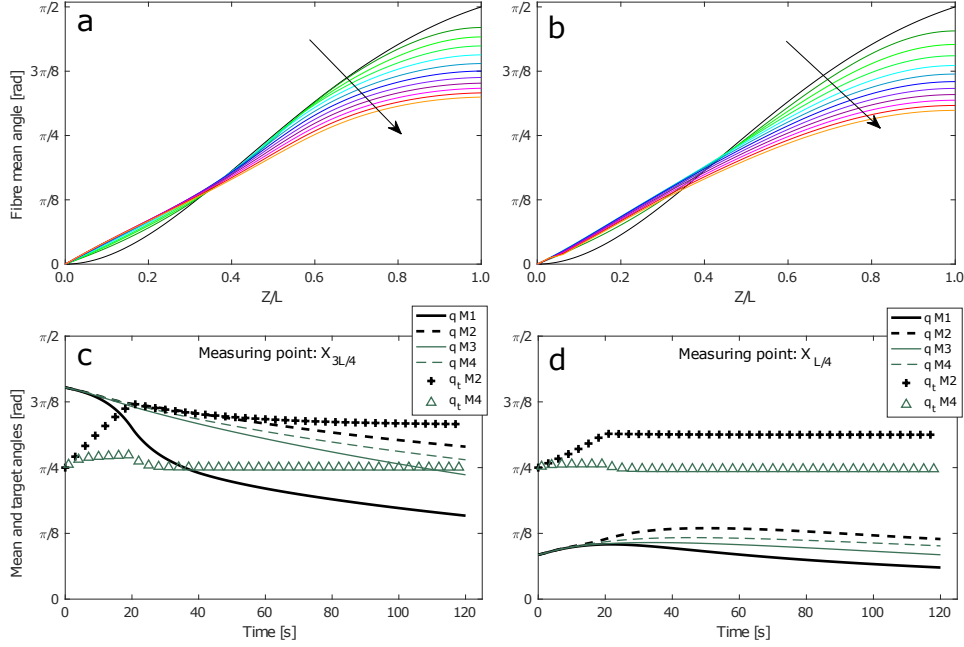


Figure 5: Fibre mean angle. In panels a and b, 10 seconds time-laps recording the evolution of the fibre mean angle along the vertical axis, for M2 (a) and M4 (b); arrows indicate the increase of time. In panels c and d, the temporal evolution of both the fibre mean angle and the target angle, observed in $X_{3L/4}$ (c) and $X_{L/4}$ (d), for all the presented models. The target angle is implemented in M2 and M4 only.

697 of q_T strongly modulates q by controlling, and then by reducing, its variation,
 698 especially in M2. In particular, looking at Fig. 5d, we see how q is driven
 699 upward by the presence of q_T (M2 and M4), especially during the loading
 700 ramp.

701 Comparing Fig. 3 with Fig. 5c and Fig. 5d, we notice that the behaviour
 702 of q influences the way in which the tissue deforms. Indeed, the more the
 703 variation of q is contained in time, the less the sample tends to deform radi-
 704 ally. This behaviour is model dependent and is more evident for M3 and M4
 705 than it is for M1 and M2.

706 The analysis of the target angle is worth of a separate discussion. Once
 707 again, by making reference to Fig. 5c and Fig. 5d there are appreciable dif-
 708 ferences among the elastic and the plastic case studies, concerning both the
 709 evolution and the stationary limit of q_T . The most relevant variations of q_T
 710 can be appreciated in M2, in which the relatively high values of the target

711 angle, reached at the end of the loading ramp, seem to affect the stationary
 712 limit. In this case, different values of \mathbf{q}_T^∞ are recovered at a different depth.
 713 On the other hand, in M4 elastic distortions fade after the loading ramp,
 714 practically leading to the recovery of the stationary value $\pi/4$ throughout
 715 the whole tissue.

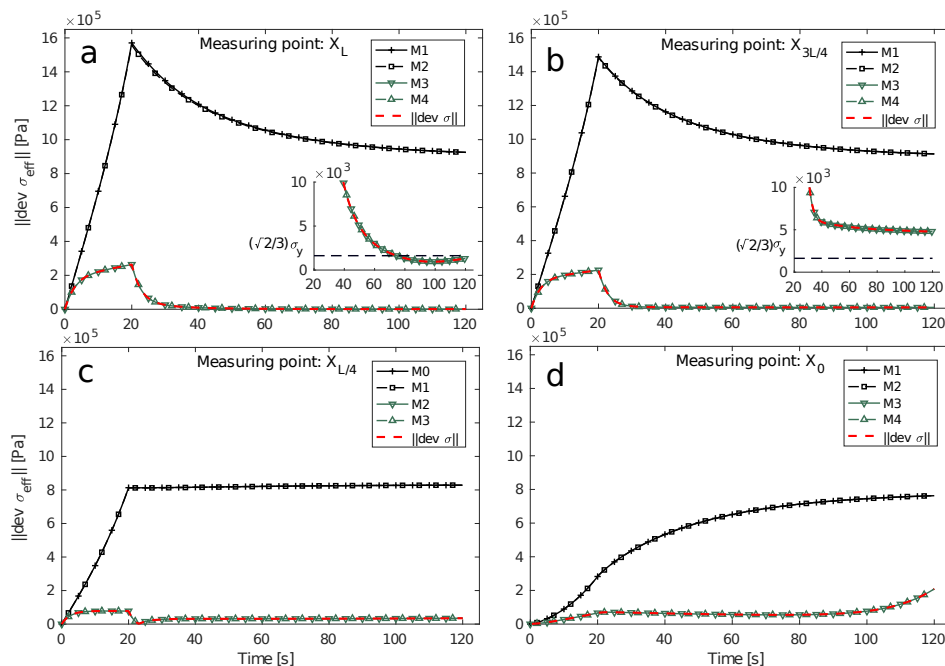


Figure 6: Effective stress tensor. For all the presented models, the norm of the effective stress tensor is evaluated in the measuring points X_L (a), $X_{3L/4}$ (b), $X_{L/4}$ (c) and X_0 (d).

716 To complete the analysis, Fig. 6 depicts the norm of the deviatoric part
 717 of the effective stress tensor, i.e., $\|\text{dev } \sigma_{\text{eff}}\|_{\mathbf{g}}$. With the exception of the bot-
 718 tom of the sample (see Fig. 6d), where the specimen is tied to the tidemark,
 719 $\|\text{dev } \sigma_{\text{eff}}\|_{\mathbf{g}}$ reaches its maximum at the end of the loading ramp. The conse-
 720 quent decrease towards a stationary value is monotonic for the elastic cases
 721 M1 and M2, while it is not for the plastic models M3 and M4 (see Figs. 6a
 722 and 6c). In the insert of Figs. 6a and 6b, $\sqrt{2/3}\sigma_y$ is reported to highlight
 723 when and where plastic-like distortions are de-activated. In Fig. 6a we note
 724 that, after approximately 70 s, the effective stress is below the threshold
 725 $\sqrt{2/3}\sigma_y$, thereby implying a temporary switch-off of the plastic-like distor-
 726 tions. Figure 6 also reports $\|\text{dev } \sigma\|_{\mathbf{g}}$, evaluated for model M4. Although not

727 visible at the length scale selected for our figures, we do measure differences
 728 between the effective stress, $\boldsymbol{\sigma}_{\text{eff}}$, and the “standard” Cauchy stress $\boldsymbol{\sigma}$, which
 729 does not take into account $\boldsymbol{\Sigma}_{\text{Grad}}$. In turn, $\boldsymbol{\Sigma}_{\text{Grad}}$ is influenced by Φ_{1sv} and
 730 \mathbf{B}_p , and it vanishes gradually on the way to X_L , because of the Neumann
 731 zero boundary condition on \mathbf{q} on $\partial\mathcal{B}_U$.

732 7. Discussion

733 The key aspect of our work is the mutual interaction among the motion,
 734 χ , the tensor of plastic-like distortions, \mathbf{F}_p , and the fibre mean angle, \mathbf{q} .
 735 Firstly, we notice that \mathbf{F}_p and \mathbf{q} interact with χ through the constitutive
 736 law expressing \mathbf{P}_{sc} in (53b). Secondly, χ and \mathbf{q} interact with \mathbf{F}_p through the
 737 term between parentheses in (53d). Such interaction manifests itself through
 738 \mathbf{C} and $\boldsymbol{\Sigma}_{\text{eff}}$. Thirdly, the interaction of χ and \mathbf{F}_p with \mathbf{q} finds its expression
 739 in the generalised forces $\Phi_{1sv}[\partial(\hat{W}_{1a} + \hat{W}_{\text{str}})/\partial\mathbf{q}]$. Finally, \mathbf{F}_p and \mathbf{q} interact
 740 with each other through $\Phi_{1sv}D_0\mathbf{B}_p\text{Grad}\mathbf{q}$, i.e., in such a way that only two
 741 players out of three interact.

742 *Role played by the free energy density W_{Grad} .* In the form given in (19), W_{Grad}
 743 constitutes the lowest-order approximation of the self-interaction of the scalar
 744 field \mathbf{q} . The strength of such self-interaction is measured by D_0 . As in [24],
 745 we consider the particular case in which \hat{W}_{Grad} is independent of deforma-
 746 tion, but we do allow it to depend on the plastic-like distortions through \mathbf{B}_p ,
 747 whose presence generates $\boldsymbol{\Sigma}_{\text{Grad}}$. This tensor is purely configurational, and
 748 has no direct geometric counterpart, since it emerges as a consequence of the
 749 coupling between the structural degrees of freedom \mathbf{q} and \mathbf{F}_p . More impor-
 750 tantly, $\boldsymbol{\Sigma}_{\text{Grad}}$ features as a summand of $\boldsymbol{\Sigma}_{\text{eff}}$ among the configurational forces
 751 that drive the evolution law of the plastic-like distortions in (52). Hence,
 752 differently from other models on the subject (see e.g. [33]), in which the
 753 configurational stress that triggers remodelling can be obtained from Cauchy
 754 stress, in our theory we have the configurational force $\boldsymbol{\Sigma}_{\text{Grad}}$ that exists on
 755 its own, and participates to activate the structural reorganisations of the
 756 tissue. In fact, it might be interpreted as the contribution to the structural
 757 reorganisation given by the reorientation of the fibres, i.e., the output of the
 758 interplay between \mathbf{F}_p and \mathbf{q} alone.

759 *Role played by the free energy density W_{str} .* The energy density W_{str} defined
 760 in our model is such that the “structural” contribution to the overall second

761 Piola-Kirchhoff stress tensor, $\mathbf{S}_{\text{str}} = 2\Phi_{1s\nu}(\partial\hat{W}_{\text{str}}/\partial\mathbf{C}_e)$, and the “structural”
762 contribution to the overall elasticity tensor, $\mathbf{C}_{\text{str}} = 4(\partial^2\hat{W}_{\text{str}}/\partial\mathbf{C}_e^2)$ vanishes in
763 the natural state. The function $\mathcal{A}_0\mathcal{P}(\mathbf{q})$ coincides with the structural energy
764 in the natural state, i.e., $\hat{W}_{\text{str}}^{(0)}(\mathbf{q}) = \mathcal{A}_0\mathcal{P}(\mathbf{q})$. We notice that such functional
765 form is adequate for describing large fluctuations of the order parameter \mathbf{q}
766 from the two reference values $\mathbf{q}_0 = 0$ and $\mathbf{q}_1 = \pi/2$, each of which returns the
767 global minimum of $\hat{W}_{\text{str}}^{(0)}$, i.e., $\hat{W}_{\text{str}}^{(0)}(0) = \hat{W}_{\text{str}}^{(0)}(\pi/2) = 0$. As discussed in [24],
768 an example of this behaviour is provided by the articular cartilage used for
769 mechanical tests [49] in which, prior to the application of any loading history,
770 a “histological profile” of the fibre mean angle can be defined [24, 49], which
771 varies throughout the tissue, taking on the values \mathbf{q}_0 and \mathbf{q}_1 at the interface
772 with the bone and at the articular surface, respectively (see e.g. [60]).

773 One may wonder whether the introduction of W_{str} is really necessary and,
774 if it is, why it should have the functional form suggested in this work. To
775 answer these questions, let us first notice that there are studies in which the
776 structural energy is tacitly used. Baaijens et al. [31], for example, prescribe
777 that the fibre mean angle evolves according to the law

$$\dot{\mathbf{q}} = -\frac{1}{\tau}[\mathbf{q} - \mathbf{q}_T], \quad (60)$$

778 where τ is a model parameter describing the system’s relaxation coefficient,
779 \mathbf{q} is the angle that the fibres in a blood vessel form with the symmetry axis,
780 and the target angle, \mathbf{q}_T , determines the preferred alignment of the fibres
781 (in the case of a blood vessel, $2\mathbf{q}$ is the angle between the two families of
782 fibres coiled helically around the vessel). Looking at (60), and comparing it
783 with our (46), which is obtained in the limit of vanishing D_0 , we notice that
784 (60) can be recovered from (46) by neglecting the force $\Phi_{1s\nu}(\partial\hat{W}_{1a}/\partial\mathbf{q})$, and
785 retaining only $\Phi_{1s\nu}(\partial\hat{W}_{\text{str}}/\partial\mathbf{q})$, with the constitutive choice

$$\hat{W}_{\text{str}}(\mathbf{q}) \equiv \hat{W}_{\text{str}}^{\text{quad}}(\mathbf{q}) = \frac{1}{2}\kappa[\mathbf{q} - \mathbf{q}_{\text{ref}}]^2, \quad (61)$$

786 where the superscript “quad” stands for “quadratic”, κ is an angular stiffness
787 density (thus, having units of force per unit area), and \mathbf{q}_{ref} is a reference angle.
788 Indeed, computing the derivative of $\hat{W}_{\text{str}}^{\text{quad}}$ with respect to \mathbf{q} , and substituting
789 the result into (46) yield

$$\Gamma\dot{\mathbf{q}} = -\Phi_{1s\nu}\kappa[\mathbf{q} - \mathbf{q}_T], \quad (62)$$

790 and (60) is re-obtained upon identifying $1/\tau = \Phi_{1s\nu}\kappa/\Gamma$.

791 In the absence of deformation and plastic-like distortions, W_{std} vanishes
 792 identically, regardless of the value taken by \mathbf{q} , and the energetic content of
 793 the tissue is the integral over \mathcal{B} of the remodelling energy density

$$\hat{W}_{\text{rem}}^{(0)}(\mathbf{q}, \text{Grad}\mathbf{q}) = \frac{1}{2}\Phi_{1s\nu}D_0\|\text{Grad}\mathbf{q}\|^2 + \Phi_{1s\nu}\hat{W}_{\text{str}}^{(0)}(\mathbf{q}), \quad (63)$$

794 which is nonzero for \mathbf{q} other than the constant values $\mathbf{q} = \mathbf{q}_0$ and $\mathbf{q} = \mathbf{q}_1$
 795 [24]. Hence, as reported in [24], the “natural state” of the tissue, which cor-
 796 responds to the state of zero mechanical stress, is not necessarily its *ground*
 797 *state*, which is attained when the residual energy density $\hat{W}_{\text{rem}}^{(0)}$ reaches its
 798 global minimum. The ground state, in fact, is individuated by either $\mathbf{q} = \mathbf{q}_0$
 799 or $\mathbf{q} = \mathbf{q}_1$, for which each term on the right-hand-side of (63) is identically
 800 null. In our case, the probability density, $\check{\Psi}(\vartheta, \mathbf{q}_0)$, depicts the situation in
 801 which the fibres are most likely oriented along the tissue’s symmetry axis,
 802 whereas $\check{\Psi}(\vartheta, \mathbf{q}_1)$ describes the case in which the fibres tend to align them-
 803 selves perpendicularly to the symmetry axis. Any other distribution of the
 804 fibre mean angle corresponds to a deviation from the ground state, and is
 805 associated with nontrivial energies. The coefficient \mathcal{A}_0 defines the height of
 806 the energy barrier that has to be overcome to pass from one ground state
 807 configuration, e.g. \mathbf{q}_0 , to the other one, \mathbf{q}_1 , or vice versa. In our model, such
 808 height is assumed to depend only on $\Phi_{1s\nu}$, which is point-dependent. How-
 809 ever, when deformation and plastic-like distortions are active, we allow for
 810 a modulation of \mathcal{A}_0 by means of the terms between brackets in (17). Note
 811 that, since the directional average in (17) depends on \mathbf{q} , the modulation also
 812 represents a self-interaction of the fibre-mean angle.

813 8. Conclusions

814 We proposed two conceptual results that, to the best of our knowledge,
 815 might be regarded as novelties: First, our calculations naturally lead to a
 816 Mandel-like stress tensor, denoted by Σ_{Grad} , which contributes to the onset
 817 and evolution of the plastic-like distortions. These, in turn, contribute to
 818 the evolution of the fibre mean angle through the term $\Phi_{1s\nu}D_0\mathbf{B}_p\text{Grad}\mathbf{q}$.
 819 Secondly, we define a structural energy that generalises some other choices
 820 available in the literature (see e.g. [31]). These results, discussed in detail
 821 in Sect. 7, characterise the interplay between the reorientation of fibres and
 822 plastic-like distortions.

823 As anticipated in Sect. 2, our model can be used, with some modifications,
824 for a generic tissue with fibre-reinforcement and evolving internal structure.
825 The major strength of our model is its flexibility, since it establishes the
826 “mathematical infrastructure” for describing transverse isotropy and for re-
827 solving interactions that are usually not resolved in more “classical” theories
828 (see e.g. [31, 33, 40]). In turn, its major weakness is that it does not account
829 for growth, which is crucial for tissues like cellular aggregates and tumours.

830 Describing growth requires to reformulate the present setting to con-
831 sider different cell populations, include chemical substances, and account for
832 the coupling among stress, structural reorganisation, and variation of mass.
833 These modifications result in the introduction of an evolution equation for
834 the inelastic distortions related to growth, and in one mass balance law for
835 each chemical species and cell population considered in the model. All these
836 equations should be combined with (53a)–(53d), and new interactions should
837 be resolved. These also call for a review of the constitutive framework.

838 Another possible specific problem for which our theory could be useful is
839 “inverse poroelasticity” [61]. Finally, the theory presented in this work could
840 be compared with that developed by Capriz in [62], and this is subject of our
841 current investigations.

842 **Conflict of interest**

843 The Authors declare that they have no conflict of interest.

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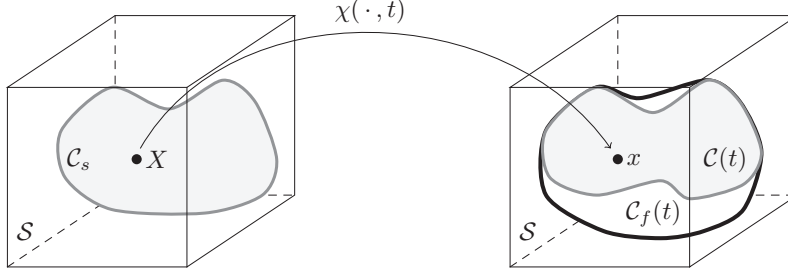


Figure 7: Schematic representation of the considered kinematics of mixtures.

855 Appendix

856 We recall some concepts of the kinematics of biphasic mixtures. To this
857 end, we adopt the theory put forward in [63], and used in [15, 24, 29, 45].
858 Accordingly, the set $\mathcal{B} \subset \mathcal{S}$ is introduced as the reference placement of
859 the solid. Then, $\chi : \mathcal{B} \times \mathcal{I} \rightarrow \mathcal{S}$ denotes a smooth mapping such that
860 $\mathcal{C}_s(t) \equiv \chi(\mathcal{B}, t) \subset \mathcal{S}$ is the configuration of the solid at time $t \in \mathcal{I}$, and
861 $\mathcal{C}_f(t)$ is the portion of \mathcal{S} occupied by the fluid at the same instant of time.
862 Finally, $\mathcal{C}(t) := \mathcal{C}_s(t) \cap \mathcal{C}_f(t) \subset \mathcal{S}$ is the region of space in which the solid-
863 fluid mixture finds itself at $t \in \mathcal{I}$. Even though $\chi(\cdot, t) : \mathcal{B} \rightarrow \mathcal{S}$ is not
864 invertible, the map $\hat{\chi}(\cdot, t) : \mathcal{B} \rightarrow \mathcal{C}_s(t)$, defined by $\hat{\chi}(X, t) = \chi(X, t)$ for all
865 $(X, t) \in \mathcal{B} \times \mathcal{I}$, is invertible and such that $\mathcal{B} = \hat{\chi}^{-1}(\mathcal{C}_s(t), t)$. In general,
866 it occurs that $\hat{\chi}^{-1}(\mathcal{C}(t), t) \subset \mathcal{B}$. However, in all the cases studied in our
867 work, the stronger condition $\hat{\chi}^{-1}(\mathcal{C}(t), t) = \mathcal{B}$ applies, since the identity
868 $\mathcal{C}_s(t) = \mathcal{C}(t)$ is verified for all $t \in \mathcal{I}$ see Fig. 7. For this reason, in Eqs.
869 (1a) and (1b), \mathcal{B} can be viewed as a reference placement for the mixture
870 as a whole. We denote by $T_X \mathcal{B}$ and $T_x \mathcal{S}$ the tangent spaces attached to
871 $X \in \mathcal{B}$ and $x \in \mathcal{S}$, respectively [35], and we identify the deformation
872 gradient tensor of the solid with the tangent map of χ , i.e., $\mathbf{F}(X, t) : T_X \mathcal{B} \rightarrow$
873 $T_x \mathcal{S}$, with $x = \chi(X, t)$. We also introduce the spatial velocity of the solid
874 at $\chi(X, t) \in \mathcal{C}_s(t)$, ${}^s \mathbf{v}_s(\chi(X, t), t)$, and the spatial velocity of the fluid at
875 $x \in \mathcal{C}_f$, ${}^s \mathbf{v}_f(x, t)$. In addition, we define the solid velocity field over \mathcal{B} ,
876 i.e., $\mathbf{v}_s(\cdot, t) : \mathcal{B} \rightarrow T\mathcal{S} = \sqcup_{x \in \mathcal{S}} T_x \mathcal{S}$, through $\mathbf{v}_s(X, t) := {}^s \mathbf{v}_s(x, t)$. For
877 $x \in \mathcal{C}(t) = \mathcal{C}_s(t) \cap \mathcal{C}_f(t)$, there exists $X \in \hat{\chi}^{-1}(\mathcal{C}(t), t)$ such that the equality
878 ${}^s \mathbf{v}_f(x, t) = \mathbf{v}_f(X, t)$ is verified, thereby defining the fluid velocity field over
879 $\hat{\chi}^{-1}(\mathcal{C}(t), t) \subset \mathcal{B}$, i.e., $\mathbf{v}_f(\cdot, t) : \hat{\chi}^{-1}(\mathcal{C}(t), t) \rightarrow T\mathcal{S}$.

880 With each $x \in \mathcal{C}(t)$ we associate the spatial *volumetric fractions* ${}^s \phi_s(x, t)$
881 and ${}^s \phi_f(x, t)$, which measure, respectively, the local volumetric content of

882 solid and fluid with respect to a representative volume of the mixture. Since
883 the mixture is assumed to be saturated, it holds that ${}^s\phi_s(x, t) + {}^s\phi_f(x, t) = 1$,
884 for all $x \in \mathcal{C}(t)$ and for all t . Along with ${}^s\phi_s$ and ${}^s\phi_f$, we also introduce
885 $\phi_s(X, t) = {}^s\phi_s(\chi(X, t), t)$ and $\phi_f(X, t) = {}^s\phi_f(\chi(X, t), t)$, for $X \in \hat{\chi}^{-1}(\mathcal{C}(t), t)$.

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