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# Structured deformations and applications

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The scope of this contribution is to present an overview of the theory of structured deformations of continua and two applications, all of which involve using this rather new formulation of mechanics problems in contexts that are different from one another, thus showing the power and versatility of the theory.

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## 1 Introduction

The necessity of a theoretical apparatus that allows to incorporate multiple scales in the modelling of mechanical deformations became more and more evident as new insight on the microscopic behaviour of material deformations were available, both at the theoretical and at the experimental level. *Structured deformations* [6] respond to this need by providing a multiscale geometry that captures the contributions at the macroscopic level of both smooth geometrical changes and non-smooth geometrical changes at submacroscopic levels. These non-smooth geometrical changes, which are called *disarrangements*, encode the presence of cracks and defects in the continuum.

Structured deformations have been successfully applied in many contexts to model plastic deformations and cracks [4, 7–9]; the theory has been extended to more general contexts, especially by defining second-order structured deformations [12], which permit the inclusion of bending effects in the energy functional [3, 11, 14]. Also relevant are the works [2, 13, 16], which focus on interfacial energies, relevant, among other things, for the study of granular and composite materials (see [10] in this context), as well as [15], where a more general functional setting is investigated.

In Section 2, we present the general functional and energetic setting. In Section 3, we present the results for two problems.

## 2 Functional setup and relaxation of energies

Let  $\Omega \subset \mathbb{R}^N$  be a bounded open subset, which we take as the reference configuration of the body.

**Definition 2.1** (see [6]) A *structured deformation* is a triple  $(\kappa, g, G)$ , where  $\kappa$  is a surface-like subset of  $\Omega$ , and the injective and piecewise differentiable map  $g: \Omega \rightarrow \mathbb{R}^N$  the piecewise continuous tensor field  $G: \Omega \rightarrow \mathbb{R}^{N \times N}$  are such that  $0 < C < \det G(x) \leq \det \nabla g(x)$  at each point  $x \in \Omega$ .

The crucial result in the theory of Del Piero and Owen [6] is the following approximation theorem, stating that each structured deformation can be seen as the limit, in the sense of  $L^\infty$  convergence, of simple deformations.

**Theorem 2.2** (see [6, Theorem 5.8]) *For each structured deformation  $(\kappa, g, G)$  there exists a sequence of injective, piecewise smooth deformations  $f_n$  and a sequence of surface-like subsets  $\kappa_n$  of the body such that  $g = \lim_{n \rightarrow \infty} f_n$ ,  $G = \lim_{n \rightarrow \infty} \nabla f_n$ , and  $\kappa = \bigcup_{n=1}^\infty \bigcap_{j=n}^\infty \kappa_j$ .*

By the convergences in Theorem 2.2, the tensor field  $G$  is not influenced by any discontinuities associated with the  $f_n$ 's. The non-smooth parts of the approximation  $f_n$  determine the *disarrangements tensor*  $M$ , in such a way that the following relationship holds:  $\nabla g = G + M$ . This additive decomposition justifies the names *deformation without disarrangements* and *deformation due to disarrangements* for  $G$  and  $M$ , respectively.

In [5] the theory has been cast in a variational framework, thus making it suitable to treat problems involving energy minimisation. The definition of structured deformation can be given in weaker functions spaces, namely the space of *special functions of bounded variations*  $SBV(\Omega; \mathbb{R}^d)$  and the space of integrable matrix-valued functions  $L^1(\Omega; \mathbb{R}^{d \times N})$ . This has the advantage of formalising the notion of discontinuity of a function  $u$  and to properly define its jump set  $S(u)$  (see [1]).

**Definition 2.3** (see [5]) The space of structured deformations is  $SD(\Omega) := \{(g, G) : g \in SBV(\Omega; \mathbb{R}^d), G \in L^1(\Omega; \mathbb{R}^{d \times N})\}$ .

In view of Definition 2.3, Theorem 2.2 has the following counterpart.

**Theorem 2.4** (see [5, Theorem 2.12]) *Let  $(g, G) \in SD(\Omega)$ . Then there exist  $u_n \in SBV(\Omega; \mathbb{R}^d)$  such that  $u_n \rightarrow g$  in  $L^1(\Omega; \mathbb{R}^d)$ , and  $\nabla u_n \xrightarrow{*} G$  in  $\mathcal{M}(\Omega; \mathbb{R}^{d \times N})$ .*

Let now  $u \in SBV(\Omega; \mathbb{R}^d)$  and consider an initial energy functional defined by

$$E(u) := \int_{\Omega} W(\nabla u(x)) \, dx + \int_{S(u) \cap \Omega} \Psi([u], \nu(u)) \, d\mathcal{H}^{N-1}, \quad (1)$$

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where the densities  $W: \mathbb{R}^{d \times N} \rightarrow [0, +\infty)$  and  $\Psi: \mathbb{R}^d \times \mathbb{S}^{N-1} \rightarrow [0, +\infty)$  satisfy suitable coercivity and growth assumptions. Given a structured deformation  $(g, G) \in SD(\Omega)$ , the relaxation of the energy (1) is

$$I(g, G) := \inf_{u_n \in SBV(\Omega; \mathbb{R}^d)} \left\{ \liminf_{n \rightarrow \infty} E(u_n) : u_n \rightarrow g \text{ in } L^1(\Omega; \mathbb{R}^d), \nabla u_n \overset{*}{\rightharpoonup} G \text{ in } \mathcal{M}(\Omega; \mathbb{R}^{d \times N}), \sup_n \|\nabla u_n\|_{L^p} < \infty \right\}.$$

**Theorem 2.5** (see [5, Theorem 2.17]) *Let  $(g, G) \in SD(\Omega)$  with  $G \in L^p(\Omega; \mathbb{R}^{d \times N})$ . Then*

$$I(g, G) = \int_{\Omega} H(\nabla g, G) \, dx + \int_{S(g) \cap \Omega} h([g], \nu(g)) \, d\mathcal{H}^{N-1}, \quad (2)$$

where the densities  $H$  (derived from both  $W$  and  $\Psi$ ) and  $h$  (derived from  $\Psi$ ) are obtained via cell formulas.

### 3 Two problems

For  $d = N$ , consider *purely interfacial* energy functionals (1), that is, with  $W = 0$ . Let  $E^{|\cdot|}(u) := \int_{S(u) \cap \Omega} |[u] \cdot \nu(u)| \, d\mathcal{H}^{N-1}$  and  $E^{\pm}(u) := \int_{S(u) \cap \Omega} ([u] \cdot \nu(u))^{\pm} \, d\mathcal{H}^{N-1}$ . For  $(g, G) \in SD(\Omega)$ , let  $\mathcal{V}^{|\cdot|}(g, G)$  and  $\mathcal{V}^{\pm}(g, G)$  be the relaxed energies (2).

**Theorem 3.1** (see [2, 13, 16]) *The initial disarrangement densities  $\Psi^{|\cdot|}(\lambda, \nu) := |\lambda \cdot \nu|$  and  $\Psi^{\pm}(\lambda, \nu) := (\lambda \cdot \nu)^{\pm}$  have relaxed disarrangement densities given by  $H^{|\cdot|}(A, B) = |\operatorname{tr}(A - B)|$ ,  $h^{|\cdot|}(\lambda, \nu) = \Psi^{|\cdot|}(\lambda, \nu)$ , and  $H^{\pm}(A, B) = (\operatorname{tr}(A - B))^{\pm}$ ,  $h^{\pm}(\lambda, \nu) = \Psi^{\pm}(\lambda, \nu)$ .*

In the context of *optimal design*, consider a two-component fractured medium with prescribed macroscopic strain. Let  $\chi \in BV(\Omega; \{0, 1\})$  be the characteristic function of a set of finite perimeter (see [1]) describing one of the constituents. The initial energy (1) is tailored to account for the fine structure of the material: given a deformation  $u \in SBV(\Omega; \mathbb{R}^d)$ , consider

$$E(\chi, u) := \int_{\Omega} ((1 - \chi)W^0(\nabla u) + \chi W^1(\nabla u)) \, dx + \int_{\{\chi=0\} \cap S(u) \cap \Omega} \Psi_1^0([u], \nu(u)) \, d\mathcal{H}^{N-1} \\ + \int_{\{\chi=1\} \cap S(u) \cap \Omega} \Psi_1^1([u], \nu(u)) \, d\mathcal{H}^{N-1} + \int_{S(\chi) \cap S(u) \cap \Omega} \Psi_2(\chi^+, \chi^-, u^+, u^-, \nu(u)) \, d\mathcal{H}^{N-1} + |D\chi|(\Omega), \quad (3)$$

Assume that  $W^i$ ,  $\Psi_1^i$ , and  $\Psi_2$  satisfy suitable coercivity and growth assumptions and let  $(\chi, g, G) \in BV(\Omega; \{0, 1\}) \times SD(\Omega)$ ; the relaxed energy is defined by

$$I(\chi, u, G) := \inf_{\substack{\chi_n \in BV(\Omega; \{0, 1\}) \\ u_n \in SBV(\Omega; \mathbb{R}^d)}} \left\{ \liminf_{n \rightarrow \infty} E(\chi_n, u_n) : \chi_n \overset{*}{\rightharpoonup} \chi \text{ in } BV(\Omega; \{0, 1\}), u_n \overset{L^1(\Omega; \mathbb{R}^d)}{\rightharpoonup} u, \nabla u_n \overset{L^p(\Omega; \mathbb{R}^{d \times N})}{\rightharpoonup} G \right\}.$$

**Theorem 3.2** (see [10, Theorem 3.3]) *Let  $(\chi, g, G) \in BV(\Omega; \{0, 1\}) \times SD(\Omega)$ . Then*

$$I(\chi, g, G) = \int_{\Omega} H(\chi, \nabla g, G) \, dx + \int_{S(\chi, g) \cap \Omega} h(\chi^+, \chi^-, g^+, g^-, \nu) \, d\mathcal{H}^{N-1}, \quad (4)$$

where  $H$  and  $h$  are obtained via cell formulas and are characterised by the interplay between the optimisation of sharp interfaces and the diffusion of microscopic cracks.

### References

- [1] L. Ambrosio, N. Fusco, and D. Pallara. Oxford University Press, 2000. ISBN: 9780198502456.
- [2] A. C. Barroso, J. Matias, M. Morandotti, and D. R. Owen. *Math. Mech. Complex Syst.*, **5**(2), 163-189 (2017).
- [3] A. C. Barroso, J. Matias, M. Morandotti, and D. R. Owen. *Arch. Rational Mech. Anal.* (2017). DOI:10.1007/s00205-017-1120-5.
- [4] R. Choksi, G. Del Piero, I. Fonseca, and D. R. Owen. *Math. Mech. Solids.*, **4**, 321-356 (1999).
- [5] R. Choksi and I. Fonseca. *Arch. Rational Mech. Anal.*, **138**, 37-103 (1997).
- [6] G. Del Piero and D. R. Owen. *Arch. Rational Mech. Anal.*, **124**, 99-155 (1993).
- [7] G. Del Piero and L. Truskinovsky. *Int. J. of Solids and Structures*, **38**, 1135-1148 (2001).
- [8] L. Deseri and D. R. Owen. *J. Elast.*, **70**, 197-236 (2003).
- [9] L. Deseri and D. R. Owen. *Int. J. Eng. Sci.* **96**, 111-140 (2015).
- [10] J. Matias, M. Morandotti, and E. Zappale. *J. Math. Anal. Appl.*, **449**, 1094-1132 (2017).
- [11] D. R. Owen. *J. Elasticity*, **127** 115-150 (2017).
- [12] D. R. Owen and R. Paroni. *Arch. Rational Mech. Anal.*, **155**, 215-235 (2000).
- [13] D. R. Owen and R. Paroni. *Arch. Rational Mech. Anal.*, **218**, 1633-1652 (2015).
- [14] R. Paroni. In G. Del Piero and D. R. Owen eds., *CISM 447*, Springer (2004).
- [15] M. Šilhavý. *Math. Mech. Complex Syst.*, **3**, 83-100 (2015).
- [16] M. Šilhavý. *Math. Mech. Complex Syst.*, **5**(2), 191-215 (2017).