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ON THE p -ADIC VALUATION OF HARMONIC NUMBERS

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ABSTRACT. For any prime number p , let J_p be the set of positive integers n such that p divides the numerator of the n -th harmonic number H_n . An old conjecture of Eswarathasan and Levine states that J_p is finite. We prove that for $x \geq 1$ the number of integers in $J_p \cap [1, x]$ is less than $129p^{2/3}x^{0.765}$. In particular, J_p has asymptotic density zero. Furthermore, we show that there exists a subset S_p of the positive integers, with logarithmic density greater than 0.273, and such that for any $n \in S_p$ the p -adic valuation of H_n is equal to $-\lfloor \log_p n \rfloor$.

1. INTRODUCTION

For each positive integer n , let

$$H_n := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

be the n -th harmonic number. The arithmetic properties of harmonic numbers have been studied since a long time. For example, Wolstenholme [7] proved in 1862 that for any prime number $p \geq 5$ the numerator of H_{p-1} is divisible by p^2 ; while in 1915, Taisinger [6, p. 3115] showed that H_n is never an integer for $n > 1$.

For each prime number p , let J_p be the set of positive integers n such that the numerator of H_n is divisible by p . Eswarathasan and Levine [4] conjectured that J_p is finite for all primes p , and provided a method to compute the elements of J_p . If J_p is finite, then, after sufficient computation, their method gives a proof that it is finite. They computed $J_2 = \emptyset$, $J_3 = \{2, 7, 22\}$, $J_5 = \{4, 20, 24\}$, and

$$J_7 = \{6, 42, 48, 295, 299, 337, 341, 2096, 2390, 14675, 16731, 16735, 102728\}.$$

Boyd [2], using some p -adic expansions, improved the algorithm of Eswarathasan and Levine, and determined J_p for all primes $p \leq 547$, but 83, 127, and 397; confirming that J_p is finite for those prime numbers. Notably, he showed that J_{11} has 638 elements, the largest being an integer of 31 digits. Boyd gave also an heuristic model predicting that J_p is always finite and that its cardinality is $\#J_p = O(p^2(\log \log p)^{2+\varepsilon})$. However, the conjecture of Eswarathasan and Levine is still open.

We write $J_p(x) := J_p \cap [1, x]$, for $x \geq 1$. Our first result is the following.

Theorem 1.1. *For any prime number p and any $x \geq 1$, it holds*

$$\#J_p(x) < 129p^{2/3}x^{0.765}.$$

In particular, J_p has asymptotic density zero.

For any prime number p , let $\nu_p(\cdot)$ be the usual p -adic valuation over the rational numbers. Boyd [2, Proposition 3.3] proved the following lemma.

Lemma 1.2. *For any prime p , the set J_p is finite if and only if $\nu_p(H_n) \rightarrow -\infty$, as $n \rightarrow +\infty$.*

Therefore, the study of J_p is strictly related to the negative growth of the p -adic valuation of H_n . It is well-known and easy to prove that $\nu_2(H_n) = -\lfloor \log_2 n \rfloor$. (Hereafter, $\lfloor x \rfloor$ denotes the greatest integer not exceeding the real number x .) Moreover, Kamano [5, Theorem 2] proved

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that $\nu_3(H_n)$ can be determined easily from the expansion of n in base 3. Note that, since obviously $\nu_p(k) \leq \lfloor \log_p n \rfloor$ for any $k \in \{1, \dots, n\}$, we have the lower bound

$$(1) \quad \nu_p(H_n) \geq -\lfloor \log_p n \rfloor.$$

Our next result shows that in (1) the equality holds quite often. We recall that the logarithmic density of a set of positive integers S is defined as

$$\delta(S) := \lim_{x \rightarrow +\infty} \frac{1}{\log x} \sum_{n \in S \cap [1, x]} \frac{1}{n},$$

whenever this limit exists.

Theorem 1.3. *For any prime number p , there exists a set S_p of positive integers, with logarithmic density $\delta(S_p) > 0.273$, and such that $\nu_p(H_n) = -\lfloor \log_p n \rfloor$ for each $n \in S_p$.*

2. PROOF OF THEOREM 1.1

For any prime p , define the sequence of sets $J_p^{(1)}, J_p^{(2)}, \dots$ as follow:

$$\begin{aligned} J_p^{(1)} &:= \{n \in \{1, \dots, p-1\} : p \mid H_n\}, \\ J_p^{(k+1)} &:= \{pn + r : n \in J_p^{(k)}, r \in \{0, \dots, p-1\}, p \mid H_{pn+r}\} \quad \forall k \geq 1. \end{aligned}$$

First, we need the following lemma.

Lemma 2.1. *For all prime numbers p , it holds $J_p^{(k)} = J_p \cap [p^{k-1}, p^k[$, for each integer $k \geq 1$. In particular, $J_p = \bigcup_{k=1}^{\infty} J_p^{(k)}$.*

Proof. From [4, Eq. 2.5] we know that if n is a positive integer and $r \in \{0, \dots, p-1\}$, then $pn + r \in J_p$ implies that $n \in J_p$. Therefore, the claim follows quickly by induction on k . \square

Now we prove a result regarding the number of elements of J_p in a short interval.

Lemma 2.2. *For any prime p , and any real numbers x and y , with $1 \leq y < p$, we have*

$$\#(J_p \cap [x, x+y]) < \frac{3y^{2/3}}{2} + 1.$$

Proof. Set $c := \#(J_p \cap [x, x+y])$. If $c \leq 1$, then there is nothing to prove. Hence, suppose $c \geq 2$ and let $n_1 < \dots < n_c$ be the elements of $J_p \cap [x, x+y]$. Moreover, define $d_i := n_{i+1} - n_i$, for any $i = 1, \dots, c-1$. Given a positive integer d , consider the polynomial

$$(2) \quad f_d(X) := (X+1)(X+2) \cdots (X+d).$$

Taking the logarithms of both sides of (2) and deriving, we obtain the identity

$$\frac{f'_d(X)}{f_d(X)} = \frac{1}{X+1} + \frac{1}{X+2} + \cdots + \frac{1}{X+d}.$$

Thus for any $i = 1, \dots, c-1$ we have

$$\frac{f'_{d_i}(n_i)}{f_{d_i}(n_i)} = \frac{1}{n_i+1} + \frac{1}{n_i+2} + \cdots + \frac{1}{n_{i+1}} = H_{n_{i+1}} - H_{n_i} \equiv 0 \pmod{p},$$

so that $f'_{d_i}(n_i) \equiv 0 \pmod{p}$. Since $f'_d(X)$ is a non-zero polynomial of degree $d-1$, there are at most $d-1$ solutions modulo p of the equation $f'_d(X) \equiv 0 \pmod{p}$. Therefore, for any $z \geq 1$, on the one hand we have

$$(3) \quad \#\{i : d_i \leq z\} = \sum_{1 \leq d \leq z} \#\{i : d_i = d\} \leq \sum_{1 \leq d \leq z} (d-1) < \frac{z^2}{2}.$$

On the other hand,

$$(4) \quad \#\{i : d_i > z\} < \frac{1}{z} \sum_{i=1}^{c-1} d_i = \frac{n_c - n_1}{z} \leq \frac{y}{z}$$

In conclusion, by summing (3) and (4), we get

$$c - 1 = \#\{i : d_i \leq z\} + \#\{i : d_i > z\} < \frac{z^2}{2} + \frac{y}{z}$$

and the claim follows taking $z = y^{1/3}$. \square

We are ready to prove Theorem 1.1. If $p < 83$ then, from the values of $\#J_p$ computed by Boyd [2, Table 2], one can check that $\#J_p/p^{2/3} < 129$, so the claim is obvious. Hence, suppose $p \geq 83$, and put $A := \frac{3}{2}(p-1)^{2/3} + 1$. By the definition of the sets $J_p^{(k)}$, and since Lemma 2.2, we get that

$$\#J_p^{(1)} = \#(J_p \cap [1, p-1]) < A,$$

while

$$\#J_p^{(k+1)} = \sum_{n \in J_p^{(k)}} \#(J_p \cap [pn, pn+p-1]) < \#J_p^{(k)} \cdot A,$$

hence it follows by induction that $\#J_p^{(k)} < A^k$.

Now let s be the positive integer determined by $p^{s-1} \leq x < p^s$. Note that $p^s \notin J_p$, indeed $\nu_p(H_{p^s}) = -s$ (this is a particular case of Lemma 3.2 in the next section). Thanks to Lemma 2.1 and the previous considerations, we have

$$\begin{aligned} \#J_p(x) &\leq \#J_p(p^s) = \#J_p(p^s - 1) = \sum_{k=1}^s \#(J_p \cap [p^{k-1}, p^k]) = \sum_{k=1}^s \#J_p^{(k)} \\ &< \sum_{k=1}^s A^k < \frac{A^2}{A-1} \cdot A^{s-1} = \frac{A^2}{A-1} \cdot (p^{s-1})^{\log_p A} < \frac{A^2}{A-1} \cdot x^{0.765} < 129p^{2/3}x^{0.765}, \end{aligned}$$

since $p^{s-1} \leq x$, while it can be checked quickly that $\log_p A < 0.765$. The proof is complete.

3. PROOF OF THEOREM 1.3

For any integer $b \geq 2$ and any $d \in \{1, \dots, b-1\}$, let $F_b(d)$ be the set of positive integers that have the most significant digit of their base b expansion equal to d . The set $F_b(d)$ has not an asymptotic density, however $F_b(d)$ has a logarithmic density. In fact, $F_b(d)$ satisfies a kind of Benford's law [1], as shown by the following lemma.

Lemma 3.1. *For all integers $b \geq 2$ and $d \in \{1, \dots, b-1\}$, it holds $\delta(F_b(d)) = \log_b(1 + 1/d)$.*

Proof. See [3]. \square

Write $J_p^* := \{1, \dots, p-1\} \setminus J_p^{(1)}$.

Lemma 3.2. *For p prime, $d \in J_p^*$, and $n \in F_p(d)$, it holds $\nu_p(H_n) = -\lfloor \log_p n \rfloor$.*

Proof. Since $n \in F_p(d)$, we can write $n = p^k d + r$, where $k := \lfloor \log_p n \rfloor$ and $r < p^k$ is a non-negative integer. Hence,

$$(5) \quad H_n = \sum_{\substack{m=1 \\ p^k \nmid m}}^n \frac{1}{m} + \sum_{j=1}^d \frac{1}{p^k j} = \sum_{\substack{m=1 \\ p^k \nmid m}}^n \frac{1}{m} + \frac{H_d}{p^k}.$$

On the one hand, it is clear that the last sum in (5) has p -adic valuation greater than $-k$. On the other hand, we have $\nu_p(H_d/p^k) = -k$, since $d \in J_p^*$ and so $p \nmid H_d$.

In conclusion, $\nu_p(H_n) = -k$ as desired. \square

Now we can prove Theorem 1.3. Define the set S_p as

$$S_p := \bigcup_{d \in J_p^*} F_p(d).$$

It follows immediately from Lemma 3.2 that $\nu_p(H_n) = -\lfloor \log_p n \rfloor$, for each $n \in S_p$. Moreover, since the sets $F_p(d)$ are disjoint, and thanks to Lemma 3.1, we have

$$(6) \quad \begin{aligned} \delta(S_p) &:= \sum_{d \in J_p^*} \delta(F_p(d)) = \sum_{d \in J_p^*} \log_p \left(1 + \frac{1}{d} \right) \geq \sum_{d = \#J_p^{(1)} + 1}^{p-1} \log_p \left(1 + \frac{1}{d} \right) \\ &= \log_p \left(\frac{p}{\#J_p^{(1)} + 1} \right) = 1 - \frac{\log(\#J_p^{(1)} + 1)}{\log p}. \end{aligned}$$

Suppose $p \geq 1013$. By Lemma 2.2 we have

$$\#J_p^{(1)} = \#(J_p \cap [1, p-1]) < \frac{3}{2}(p-2)^{2/3} + 1,$$

hence from (6) we get

$$\delta(S_p) > 1 - \frac{\log\left(\frac{3}{2}(p-2)^{2/3} + 2\right)}{\log p} > 0.273.$$

At this point, the proof is only a matter of computation. The author used the Python programming language (since it has native support for arbitrary-sized integers) to compute the numerators of the harmonic numbers H_n , up to $n = 1012$. Then he determined $\#J_p^{(1)}$ for each prime number $p < 1013$, and using (6) he checked that the inequality $\delta(S_p) > 0.273$ holds. This required only a few seconds on a personal computer.

4. CONCLUDING REMARKS

From the proof of Theorem 1.1, it is clear that with our methods one cannot obtain an upper bound better than $\#J_p(x) < Cp^{2/3}x^{2/3+\varepsilon}$, for some $C, \varepsilon > 0$. Similarly, in the statement of Theorem 1.3 a logarithmic density greater than $1/3 - \varepsilon$ cannot be achieved.

One way to obtain better results could be an improvement of Lemma 2.2, we leave this as an open question for the readers.

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