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*Original*

Finite beam elements based on Legendre polynomial expansions and node-dependent kinematics for the global-local analysis of composite structures / Li, G.; de Miguel, A. G.; Pagani, A.; Zappino, E.; Carrera, E.. - In: EUROPEAN JOURNAL OF MECHANICS. A, SOLIDS. - ISSN 0997-7538. - 74:(2019), pp. 112-123.  
[10.1016/j.euromechsol.2018.11.006]

*Availability:*

This version is available at: 11583/2722018 since: 2019-01-07T11:55:28Z

*Publisher:*

Elsevier

*Published*

DOI:10.1016/j.euromechsol.2018.11.006

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<http://dx.doi.org/10.1016/j.euromechsol.2018.11.006>

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# Finite beam elements based on Legendre polynomial expansions and node-dependent kinematics for the global-local analysis of composite structures

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## Abstract

This article presents an approach to obtain refined beam models with optimal numerical efficiency. Hierarchical Legendre Expansions (HLE) and Node-dependent Kinematics (NDK) are used in combination to build efficient global-local FE models. By relating the kinematic assumptions to the selected FE nodes, kinematic refinement local to the nodes can be realized, and global-local models can be conveniently constructed. Without using any coupling approach or superposition of displacement field, beam models with NDK have compact and coherent formulations. Meanwhile, HLE is used in the local zone for the enrichment of the beam cross-sections to satisfy the requirement for high solution accuracy, leaving the global model with lower-order kinematic assumptions. Through the numerical investigation on slender laminated structures, it is demonstrated that the computational costs can be reduced significantly without losing numerical accuracy.

*Keywords:* refined beam theories, Carrera Unified Formulation, hierarchical Legendre expansions, node-dependent kinematics, finite element

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## 1. Introduction

Composite structures are widely used in modern engineering nowadays, especially in the aerospace industry. Nevertheless, their heterogeneous properties give rise to significant challenges to numerical modeling. On the history of structural mechanics, a great variety of 1D  
5 models for slender structures have been proposed. Classical theories such as Euler-Bernoulli

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beam and Timoshenko beam (Timoshenko, 1922) are broadly applied in numerical methods, though they fail to give a precise approximation of the transverse stresses over the cross-sections of slender structures. Refined theories were suggested to overcome such a drawback. Vlasov (1984) suggested the warping functions to capture the cross-sectional warping of thin-walled structures. Schardt (1966) presented a Generalized Beam Theory (GBT) by expanding the displacement field with reference to the middle plane of the thin-walled beam. An interesting review of several of them was presented by Kapania and Raciti (1989a,b). Notably, the higher-order shear deformation theory proposed by Reddy (1984a,b) has been widely adopted in mechanical modeling as by Chikh et al. (2017) and successfully applied to the analysis of functionally graded materials as reported by Ebrahimi and Barati (2017), Ebrahimi and Farazmandnia (2017) and structures with graphene sheets by Ebrahimi and Shafiei (2017). A hyperbolic shear deformation theory was presented by Mahi et al. (2015).

Generally, the accuracy of the models can be improved by increasing the order of the mathematical functions used to describe the deformation of the beam cross-sections. Refined beam models can be developed in an asymptotic/axiomatic expansion approach (Carrera and Petrolo, 2011), for example, by using Mac Laurin's polynomials. Meanwhile, the increased number of expansions lead to raised degrees of freedom and more complex governing equations. Carrera (2002) proposed the Unified Formulation (CUF), which allows the governing equations of refined models to be attained in a compact and unified manner through the so-called fundamental nuclei (FNs). By increasing the order of the polynomials expanded on each cross-section, better approximation accuracy is promisingly to be achieved. In numerical analysis, the mathematical models can be refined until the prescribed accuracy is achieved. Also, by using general approximation functions, the stretching effects of laminated structures can be accounted, which might be important for some flexural responses as commented by Draiche et al. (2016).

Based on CUF, a variety of refined beam theories were applied to implement efficient beam finite elements (Carrera et al., 2014). CUF can incorporate both series expansions and interpolation polynomials to build refined beam models. A variety of refined theories can be implemented in the framework of CUF, including the classical Euler-Bernoulli and Timoshenko beams as well as the higher-order shear deformation theories. Equivalent Single Layer (ESL) models compute the integrals of the energy terms over the cross-section domain as a whole, and suit theories based on series expansions, such as Taylor, trigonometric, and hyperbolic series and so forth. Such models were put into practice by Carrera et al. (2013a) and Filippi et al. (2016). For refined beam elements using Layer-wise (LW) models, 2D-type discretization is used on the

cross-sectional domain for enrichment purposes. Since LW models can account for the physical  
40 boundaries of each layer, the heterogeneity of the laminates can be appropriately considered.  
Different sets of polynomials can be used as assumed deformations of the cross-sections, such  
as the Lagrange-type Carrera et al. (2014) and the Chybeshev-type (Filippi et al., 2015) poly-  
nomials. Recently, the hierarchical Legendre polynomial expansions (HLE) were introduced as  
well for the refinement of kinematic assumption of beam models, as reported by Carrera et al.  
45 (2017a). The adopted hierarchical functions for quadrilateral domains were inspired by Szabó  
and Babuška (1991). Such hierarchical functions can trace back to the work of Peano (1976),  
Szabó and Mehta (1978) and Zienkiewicz et al. (1983). In HLE models, the polynomial degree  
remains as an independent input parameter, which makes a re-meshing on the cross-sections  
unnecessary. Besides, HLE proves to be a useful tool in describing the exact geometrical bound-  
50 aries of the cross-section domains for the refinement of the modes, as discussed by Pagani et al.  
(2016).

The refinement of mathematical assumptions can improve the solution accuracy, but also  
leads to an increased number of degrees of freedom in FE models, and possibly makes the solu-  
tion computationally expensive. A local kinematic refinement can help to reach a compromise  
55 between the desired accuracy and solution expenses. Local refinements can be defined on spe-  
cific layers according to a global-local superposition hypothesis (Li and Liu, 1995, 1997). The  
basic idea is to superimpose an LW displacement assumption defined on a specific layer to a  
global component of ESL type. The underlying method is a multiple assumed displacement field  
approach. Further investigations based this method were carried out by Chen and Wu (2005),  
60 Chen and Si (2013), Khalili et al. (2014), and Lezgy-Nazargah et al. (2011). This approach is  
further used to build adequate models that can facilitate the modeling of delamination, as put  
forward by Williams (1999), Mourad et al. (2008), and Versino et al. (2014, 2015). An alter-  
native method was suggested by Carrera et al. (2017d), who introduced through-the-thickness  
variable kinematic capabilities to refined shell models. In their proposed method, the kinematic  
65 assumptions were directly refined on the chosen layers as LW models, and the other layers will  
be grouped and modeled as equivalent layers. D’Ottavio et al. (2016) suggested a similar con-  
cept which was named as Sublaminated Generalized Unified Formulation (S-GUF). In these local  
refinement approaches, though the LW kinematics have to be used over the entire planar domain  
of the laminates, the requirement for 3D finite elements in ply grouping method (Chang et al.,  
70 1990, Jones et al., 1984, Pagano and Soni, 1983, Sun and Liao, 1990) can be avoided.

A different local refinement scheme regards to the mesh discretization. The most direct

way is the  $h$ -version refinement, which increases the density of the mesh grids. Adaptive mesh-refinement was proposed to regenerate the mesh in the desired area based on an error estimator (Zienkiewicz and Zhu, 1987, Zhu and Zienkiewicz, 1988). Alternatively,  $p$ -version refinement  
75 increases the polynomial order of the element shape functions (Babuška et al., 1981, Surana et al., 2001, Szabó et al., 2004), consequently the numerical convergence performance can be improved. By augmenting the mesh density and element order at the same time, the  $h$ - $p$ -version method combines the advantages of these two approaches (Babuška and Guo, 1988, Oden et al., 1989, Zienkiewicz et al., 1989, Reddy, 1993). The  $s$ -version refinement (Fish, 1992,  
80 Fish and Markolefas, 1992) improves the solution accuracy by superimposing an additional set of independent meshes on the existing FE model, which is also referred to as the mesh superposition technique. Still, this concept is based on the idea of multiple assumed displacement fields. Exploiting this approach, Reddy and Robbins (1994) and Robbins and Reddy (1996) suggested a so-called variable kinematic theory, which superimposed an ESL displacement field on a layer-  
85 wily defined displacement field. Meanwhile, by employing the  $s$ -version refinement method, locally refined mesh with variable kinematics can be overlapped on the global mesh in which ESL assumptions are used. Consequently, the mathematical kinematic refinement and the mesh discretization refinement were both considered.

A variety of methods to couple an adequate global model to a locally refine one were proposed.  
90 By using Lagrangian multipliers to enforce the displacement compatibility at domain interfaces, the global model can be connected to a local one (Prager, 1967, Aminpour et al., 1995, Brezzi and Marini, 2005, Carrera et al., 2013b). This method is also known as the multi-point constraints or the three-field formulations. A multi-line approach was suggested by Carrera and Pagani (2013, 2014) and Carrera et al. (2017b) for refined beam models, in which beam models with differ-  
95 ent orders were used for different layers along the beam-lines, and the interfacial displacement compatibility was ensured through Lagrange multipliers. The Arlequin method, proposed by Dhia (1998) and Dhia and Rateau (2005), can couple two models with incompatible kinematics and different mesh discretization through Lagrangian multipliers in an overlapping zone. This method has been adopted by many researchers in the analysis of multi-layered structures, like  
100 Biscani et al. (2011, 2012a,b), He et al. (2011), and Hu et al. (2008, 2010), to name but a few. A so-called eXtended Variational Formulation (XVF) with two Lagrange multipliers fields was proposed for the coupling of non-overlapping domains with different mathematical assumptions (Blanco et al., 2008, Wenzel et al., 2014). In a typical one-way sequential global-local method, the independent local model is driven by the displacement on the boundaries taken from a previ-

105 ously solved global problem (Muheim Thompson and Hayden Griffin JR, 1990). A drawback of  
this method is that the influence of the local model on the global model is ignored. As a remedy,  
iterative procedures were then proposed to achieve the equilibrium and compatibility at model  
interfaces (Whitcomb and Woo, 1993a,b, Mao and Sun, 1991). In the meanwhile, the iterative  
procedures usually consume extra computational resources. Further efforts towards the devel-  
110 opment of two-way loose global-local coupling approaches were also reported by Hühne et al.  
(2016) and Akterskaia et al. (2018).

Carrera and Zappino (2017) suggested an innovative approach for the construction of FE  
models that can accommodate strong local effects in a natural and straightforward manner,  
which was named as Node-dependent Kinematics (NDK). By relating cross-section functions to  
115 the desired FE nodes, the kinematic assumptions attached to different nodes will contribute to  
the element deformation capabilities through the shape functions. Elements with miscellaneous  
nodal mathematical models can form a transition zone, bridging the refined local model to a  
global model with low-order kinematics. Without using any additional coupling approaches nor  
extra superposition techniques, NDK allows the construction of a simultaneous multi-kinematic  
120 global-local FE model to be built conveniently. Thus, the compactness of the governing equations  
is retained. NDK has been applied to build global-local models of multi-layered structures for  
1D (Carrera et al., 2018a) and 2D (Zappino et al., 2017, Carrera et al., 2017c, Valvano and  
Carrera, 2017) simulation. As a versatile approach, NDK was also used in the FE modeling of  
piezo-patches (Carrera et al., 2017e, 2018b, Zappino et al., 2018).

125 In the present work, HLE is used as the displacement assumptions to generate refined beam  
models and used in the framework of NDK. The combination of HLE kinematic models and NDK  
lead to significantly improved numerical efficiency as well as convenience in the construction of  
global-local FE models. Such an approach enables one to refine the kinematics locally at any  
desirable node and improve the accuracy by simply increasing the polynomial degree of the  
130 hierarchical functions. The related formulations are presented in the following sections. The  
effectiveness of the proposed method is demonstrated through numerical examples on multi-  
layered beam structures.

## 2. Refined beam element based on CUF

For a slender laminated structure shown in Figure 1, Let us consider that the longitudinal  
direction is aligned along the  $y$  direction, the cross-section domain lies in the  $(x, z)$  plane. The

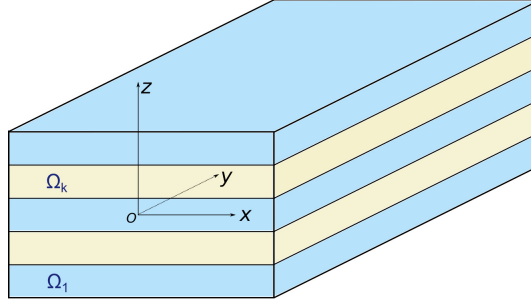


Figure 1: Reference system and notation of a laminated beam.

strain and stress components are herein arranged as:

$$\boldsymbol{\epsilon}^T = \{\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{xz}, \epsilon_{yz}, \epsilon_{yx}\} \quad (1)$$

$$\boldsymbol{\sigma}^T = \{\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xz}, \sigma_{yz}, \sigma_{yx}\} \quad (2)$$

where the strain vector are related to the displacements through the differential operator matrix  $\mathbf{D}$  as:

$$\boldsymbol{\epsilon} = \mathbf{D}\mathbf{u} \quad (3)$$

For problems with infinitesimal strains,  $\mathbf{D}$  in an explicit form is:

$$\mathbf{D} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \quad (4)$$

Meanwhile, the stresses and strains can be related through the constitutive equations:

$$\boldsymbol{\sigma} = \tilde{\mathbf{C}}\boldsymbol{\epsilon} \quad (5)$$

135 in which  $\tilde{\mathbf{C}}$  is the matrix of material coefficients rotated from the material system to the analysis coordinate system shown in Figure 1.

In the framework of CUF, beam models are refined through the cross-section functions

$F_\tau(x, z)$ , which lead to the following expression of the displacement field:

$$\mathbf{u}(x, y, z) = \mathbf{u}_\tau(y)F_\tau(x, z), \quad \tau = 1, \dots, M \quad (6)$$

where  $\mathbf{u}_\tau(y)$  are the axial displacement unknown vectors, and  $M$  is the total number of expansions used in the cross-section functions  $F_\tau(x, z)$ . In FE discretization, the axial displacement vector can be approximated with Lagrangian shape functions and nodal unknowns as follows:

$$\mathbf{u}_\tau(y) = N_i(y)\mathbf{u}_{i\tau} \quad i = 1, \dots, N_n \quad (7)$$

in which  $N_i(y)$  are the shape functions, and  $N_n$  the number of nodes within an element,  $\mathbf{u}_{i\tau}$  the nodal unknowns. Thus, the complete expression of FE displacement functions formulated according to CUF can be written as:

$$\mathbf{u}(x, y, z) = N_i(y)F_\tau(x, z)\mathbf{u}_{i\tau}, \quad \tau = 1, \dots, M; \quad i = 1, \dots, N_n \quad (8)$$

It should be noted that, with the help of Einstein's summation convention, the displacement functions can be expressed in a compact form. The sub-indexes play an important role in describing various beam theories. CUF can account for the two modeling frameworks of laminated structures, namely ESL and LW models as illustrated in Figure 2. Beam theories based on higher-order Taylor series expansion (TE), according to the afore-described formulation, can be written as:

$$\begin{aligned} u_x &= u_{x_1} + xu_{x_2} + zu_{x_3} + x^2u_{x_4} + xzu_{x_5} + z^2u_{x_6} \\ u_y &= u_{y_1} + xu_{y_2} + zu_{y_3} + x^2u_{y_4} + xzu_{y_5} + z^2u_{y_6} \\ u_z &= u_{z_1} + xu_{z_2} + zu_{z_3} + x^2u_{z_4} + xzu_{z_5} + z^2u_{z_6} \end{aligned} \quad (9)$$

where:

$$F_1 = 1, \quad F_2 = x, \quad F_3 = z, \quad F_4 = x^2, \quad F_5 = xz, \quad F_6 = z^2 \quad (10)$$

For the Lagrange interpolation polynomial expansions (LE) defined on a quadrilateral domain  $(s, r)$ , a model based on four interpolation points (LE4) can be expressed as:

$$\begin{aligned} F_1 &= \frac{1}{4}(1 - \xi)(1 - \eta); & F_2 &= \frac{1}{4}(1 + \xi)(1 - \eta); \\ F_3 &= \frac{1}{4}(1 + \xi)(1 + \eta); & F_4 &= \frac{1}{4}(1 - \xi)(1 + \eta). \end{aligned} \quad (11)$$

in which  $s, r \in [-1, 1]$ , and  $F_1(-1, -1) = 1$ ,  $F_2(1, -1) = 1$ ,  $F_3(1, 1) = 1$ ,  $F_4(-1, 1) = 1$ . LE-type cross-section functions expanded on nine points (LE9) can be defined accordingly.

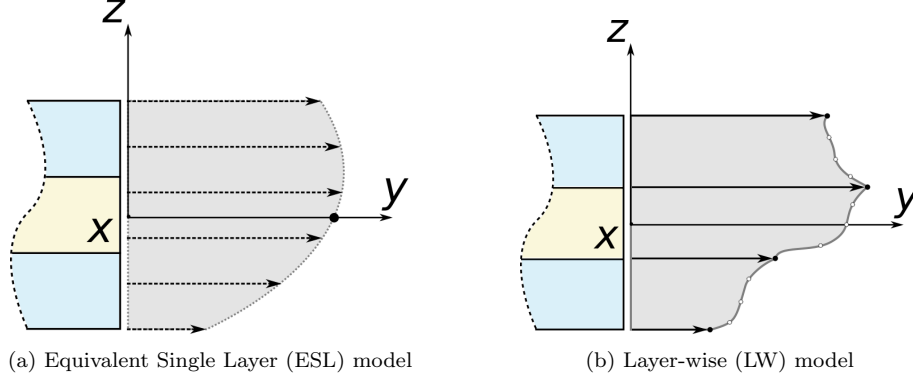


Figure 2: Two types of models for multi-layered structures.

### 3. Hierarchical Legendre Expansions (HLE) as cross-section functions

140 The cross-section functions can also be defined by Hierarchical Legendre Expansions (HLE).  
 Inspired by the work of Szabó and Babuška (1991) and Szabó et al. (2004), HLE was employed for  
 the refinement of beam models first by Pagani et al. (2016). Such type of cross-section functions  
 treat the polynomial degree  $p$  as an independent variable. The functions for a quadrilateral  
 domain  $(r, s)$ , defined for  $[-1, 1]$ , can be classified into vertex modes, side modes, and internal  
 145 modes, as shown in Figure 3.

**Vertex modes:** These functions are defined as linear interpolations over the quadrilateral domain:

$$F_\tau(r, s) = \frac{1}{4}(1 - r_\tau r)(1 - s_\tau s) \quad \tau = 1, 2, 3, 4 \quad (12)$$

where  $r_\tau$  and  $s_\tau$  stand for the local isoparametric coordinates of point  $\tau$  in a quadrilateral sub-domain with four points.

**Side modes:** Correspond to the edge-featuring modes, which are defined as:

$$\begin{aligned} F_\tau(r, s) &= \frac{1}{2}(1 - s)\phi_m(r) & \tau &= 5, 9, 13, 18, \dots \\ F_\tau(r, s) &= \frac{1}{2}(1 + r)\phi_m(s) & \tau &= 6, 10, 14, 19, \dots \\ F_\tau(r, s) &= \frac{1}{2}(1 + s)\phi_m(r) & \tau &= 7, 11, 15, 20, \dots \\ F_\tau(r, s) &= \frac{1}{2}(1 - r)\phi_m(s) & \tau &= 8, 14, 16, 21, \dots \end{aligned} \quad (13)$$

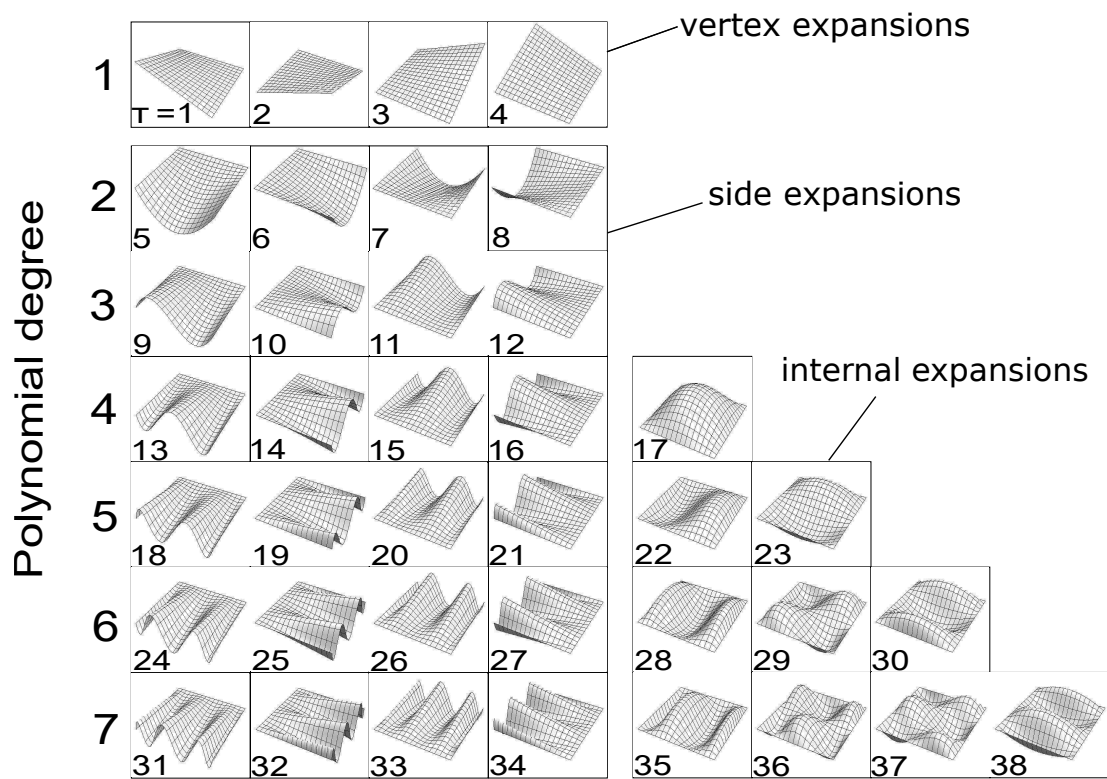


Figure 3: Hierarchical Legendre Expansions (HLE) as cross-section functions of refined beam models, with reference to Szabó and Babuška (1991).

where  $\phi_m$  is expressed as follows:

$$\phi_m(r) = \sqrt{\frac{2m-1}{2}} \int_{-1}^r L_{m-1}(x) dx = \frac{L_m(r) - L_{m-2}(r)}{\sqrt{4m-2}} \quad m = 2, 3, \dots \quad (14)$$

**Internal modes:** Describe the deformation shapes happening on the internal surface which will vanish on the edges and vertexes, which are:

$$F_\tau(r, s) = \phi_m(r)\phi_n(s) \quad m, n \geq 2; \quad \tau = 17, 22, 23, 28, 29, 30, \dots \quad (15)$$

Since the set of functions for  $p-1$  are contained in those for  $p$ , these type of functions are described as *hierarchical*. For a more detailed description, the reader is referred to Carrera et al. (2017a). The four vertexes are used to define the border of the quadrilateral domain on the cross-section of a beam model. In the LW framework, refined beam models using HLE can be formulated. Moreover, HLE can avoid the work in the re-allocation of interpolation points and the consequent re-definition of the functions. In a sense, HLE combines the advantages of Taylor series, i.e. hierarchical kinematics, and Lagrange interpolation polynomials, i.e. non-local distribution of unknowns.

#### 4. Beam elements with Node-Dependent Kinematics (NDK)

In CUF-type displacement functions as in Equation 8, the cross-sections can be further related to its “anchoring” nodes  $i$ , leading to the following expression:

$$\mathbf{u}(x, y, z) = N_i(y)F_\tau^i(x, z)\mathbf{u}_{i\tau}, \quad \tau = 1, \dots, M_i; \quad i = 1, \dots, N_n \quad (16)$$

Equation 16 describes a family of 1D FE models with NDK. In such elements, each node can possess individually defined kinematics over the cross-section, then be interpolated by means of the nodal shape functions  $N_i$  over the element axial domain. As an example, Figure 4 shows a four-node beam with individual displacement assumptions on each node, and a kinematic transition is realized within this element. In this approach, a local kinematic refinement on the nodal level can be conveniently carried out.

The governing equations of NDK FE models can be derived from the principle of virtual displacements (PVD). For elastic bodies in static equilibrium, one has:

$$\delta L_{int} = \delta L_{ext} \quad (17)$$

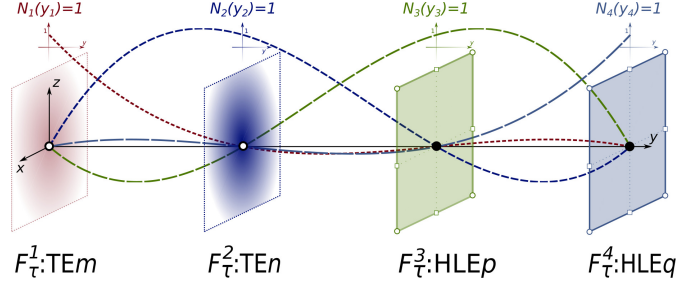


Figure 4: A B4 element with node-dependent kinematics.

where  $\delta L_{int}$  stands for internal work caused by the virtual deformations, and  $\delta L_{ext}$  represents the work done on the virtual displacements by the external forces.  $\delta L_{int}$  can be expressed as:

$$\delta L_{int} = \int_V \delta \boldsymbol{\epsilon}^T \boldsymbol{\sigma} dV \quad (18)$$

By invoking CUF-type displacement functions Equation 16, the geometric relations in Equation 3, and constitutive equations Equation 5, the following expression can be obtained:

$$\delta L_{int} = \delta \mathbf{u}_{js}^T \cdot \int_V N_j F_s^j \mathbf{D}^T \tilde{\mathbf{C}} \mathbf{D} F_{\tau}^i N_i dV \cdot \mathbf{u}_{i\tau} = \delta \mathbf{u}_{js}^T \cdot \mathbf{K}_{ij\tau s} \cdot \mathbf{u}_{i\tau} \quad (19)$$

where  $\mathbf{K}_{ij\tau s}$  is the fundamental nucleus (FN) of stiffness matrix for NDK FE models. The explicit expression of  $\mathbf{K}_{ij\tau s}$  reads:

$$\mathbf{K}_{ij\tau s} = \int_V N_j F_s^j \mathbf{D}^T \tilde{\mathbf{C}} \mathbf{D} F_{\tau}^i N_i dV \quad (20)$$

The virtual work  $\delta L_{ext}$  done by the external load  $\mathbf{p}$  is:

$$\delta L_{ext} = \int_V \delta \mathbf{u}^T \mathbf{p} dV \quad (21)$$

The above equation can be further written in the form of CUF as:

$$\delta L_{ext} = \delta \mathbf{u}_{js}^T \int_V N_j F_s^j \mathbf{p} dV = \delta \mathbf{u}_{js}^T \mathbf{P}_{js} \quad (22)$$

where  $\mathbf{P}^{js}$  represents the FN of the load vector. Hence, the governing equation for 1D FE models with NDK can be obtained as follows:

$$\delta \mathbf{u}_{js} : \mathbf{K}_{ij\tau s} \cdot \mathbf{u}_{i\tau} = \mathbf{P}_{js} \quad (23)$$

For FE models with NDK, the assembly of the stiffness matrix and load vector can be carried out in a convenient and unified manner in the framework of CUF, as elaborated by Carrera and Zappino (2017) and Carrera et al. (2018a).  
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## 5. Numerical results and discussion

In this section, the capabilities of NDK when used in combination with HLE cross-section kinematics are investigated through two numerical examples:

- A simply supported sandwich beam under local pressure;
- 170 • A two-layered cantilever beam subjected to four points loads.

The first example demonstrates the capturing of local pressure concentration with NDK, and the second one presents the probing of structural responses in the region of interest. The accuracy of the solutions is compared against the computational consumption. The choice of the transition zone and the kinematics in the outlying area, as well as their influences on the efficiency, are discussed.  
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### 5.1. A simply supported sandwich beam under local pressure

A sandwich beam under local pressure is considered, which comprises two composite faces and a soft core as shown in Figure 5. The structure has the length  $b = 10\text{mm}$ , width  $a = 2\text{mm}$ , and total height  $h = 2\text{mm}$ , with layers of thickness  $0.1h/0.8h/0.1h$ . The material properties are as detailed in Table 1. Numerical studies on this case was also reported by Wenzel et al. (2014) and Zappino et al. (2017). In the present work, by making use of the symmetry features, a half of the structure is modeled. For the refined HLE beam elements used, the cross-section is meshed as presented in Figure 6, in which the three sub-domains are approximated by the same set of HLE $p$  cross-section functions, respectively. According to the results in Table 2, FE model with 20 B4 elements along the axial direction can give a satisfactory approximation. The HLE refinement is first assessed by increasing the polynomial order  $p$  until 7. Then FE models constructed with NDK are employed in the analysis. The obtained displacements and stresses are summarized in Table 2. Solutions achieved with pure TE and LE kinematics are listed for comparison.  
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In Table 2, with the increase of the polynomial order of the kinematic assumption on the beam cross-section, the numerical results converge gradually. In terms of  $\sigma_{zz}$ , the theoretical solution is -1 MPa on the loaded surface, and all the HLE models can achieve fairly good accuracy. The  
190

Table 1: Material properties used on the sandwich beam.

	$E_{11}$ [GPa]	$E_{22}$ [GPa]	$E_{33}$ [GPa]	$\nu_{12}$	$\nu_{13}$	$\nu_{23}$	$G_{12}$ [GPa]	$G_{13}$ [GPa]	$G_{23}$ [GPa]
Face	131.1	6.9	6.9	0.32	0.32	0.49	3.588	3.088	2.3322
Core	$0.2208 \times 10^{-3}$	$0.2001 \times 10^{-3}$	2.76	0.99	0.00003	0.00003	$16.56 \times 10^{-3}$	0.5451	0.4554

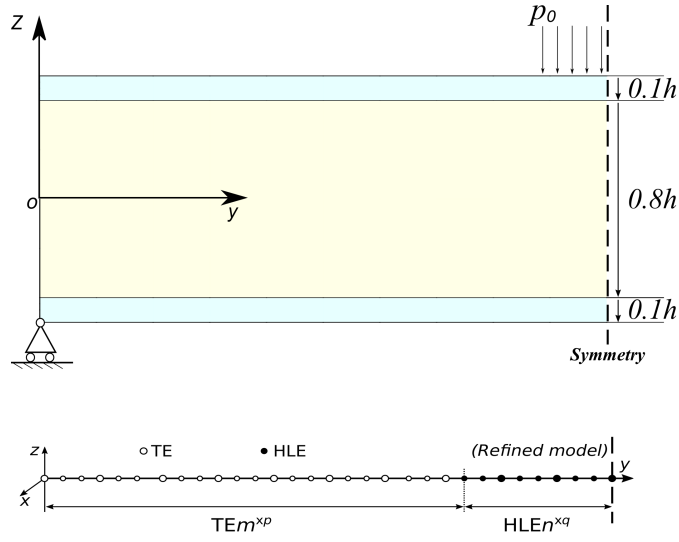


Figure 5: Geometry and FE model of the sandwich beam.

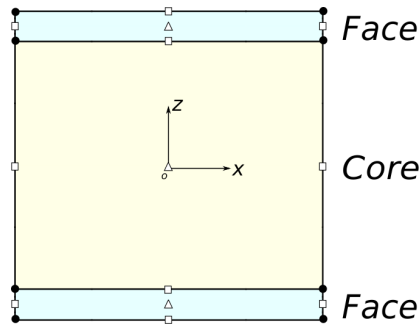


Figure 6: Mesh on the cross-section of the sandwich beam: 3 sub-domains, each with HLE $p$ .

through-the-thickness variation of  $\sigma_{yz}$  obtained with HLE kinematics of different orders are as shown in Figure 7(a). Due to the low stiffness, the core has much lower stress gradients than the faces of the sandwich. It can be observed that, HLE2 fails to capture the variation of  $\sigma_{yz}$  through the two faces of the sandwich. From HLE3 to HLE7,  $\sigma_{yz}$  shows converged distribution through the sandwich thickness, and zero transverse shear stress on the free surfaces is progressively approached. In fact, HLE3 can already satisfy the accuracy requirement of engineering practice. The relative error of  $\sigma_{yz}$  given by different kinematics (with respect to HLE7 solution) are plotted versus the degrees of freedom in Figure 7(b). Even if the curve of relative error is not monotonically decreasing, the overall trend exhibits a convergence pattern. On the other hand, this curve shows the possibility of improving the accuracy by further increasing the polynomial order, yet it may not be necessary considering the computational efforts.

Table 2: Displacement and stress evaluation on the sandwich beam under local pressure.

Mesh	Kinematics	$-w[10^{-3}\text{mm}]$ $(0, \frac{b}{2}, -\frac{h}{2})$	$-\sigma_{yy}[\text{MPa}]$ $(0, \frac{b}{2}, \frac{h}{2})$	$-\sigma_{yz}[\text{MPa}]$ $(\frac{a}{2}, \frac{9b}{20}, \frac{9h}{20})$	$-\sigma_{zz}[\text{MPa}]$ $(0, \frac{b}{2}, \frac{h}{2})$	DOFs
B4×10	HLE2	2.467	17.72	0.8339	1.097	1674
B4×20	HLE2	2.467	17.77	0.8164	1.015	3294
B4×20	HLE3	2.469	18.33	1.062	1.048	5124
B4×20	HLE4	2.469	18.14	1.083	0.9965	7503
B4×20	HLE5	2.469	18.11	1.060	0.9884	10431
B4×20	HLE6	2.469	18.24	1.079	0.9789	13908
B4×20	HLE7	2.469	18.24	1.094	0.9771	17934
B4×20	TE1	1.515	7.300	1.163	0.9547	549
B4×20	TE3	2.309	17.52	0.8777	1.502	1830
B4×20	TE5	2.338	17.25	0.8815	0.7839	3843
B4×20	TE1 <sup>×49</sup> -HLE7 <sup>×12</sup>	1.547	15.63	1.437	0.9791	3969
B4×20	TE1 <sup>×31</sup> -HLE7 <sup>×30</sup>	1.820	18.07	1.112	0.9772	9099
B4×20	TE3 <sup>×31</sup> -HLE7 <sup>×30</sup>	2.376	18.23	1.096	0.9771	9750
B4×20	TE5 <sup>×31</sup> -HLE7 <sup>×30</sup>	2.388	18.23	1.096	0.9771	10773
B4×20	TE7 <sup>×31</sup> -HLE7 <sup>×30</sup>	2.419	18.24	1.095	0.9771	12168
Zappino et al. (2017)(2D)		2.471	18.11	1.180	0.9989	37479

HLE7 is further used together with TE kinematics to build FE models through NDK. In the region containing and near the loaded zone, HLE7 is applied on the beam cross-section, and the rest of the beam is modeled with TE-type theories. The FE models with NDK are indicated by TE $m^{\times p}$ -HLE7 $^{\times q}$ , where  $m$  signifies the order of the TE model used, while  $p$  and  $q$  represent the numbers of nodes adopting the corresponding kinematics. For the FE models with 20 B4 elements, there are 61 nodes in total along the beam axis. As explained in previous sections, in the proposed approach, the transition zone in the global-local model covers the range of one

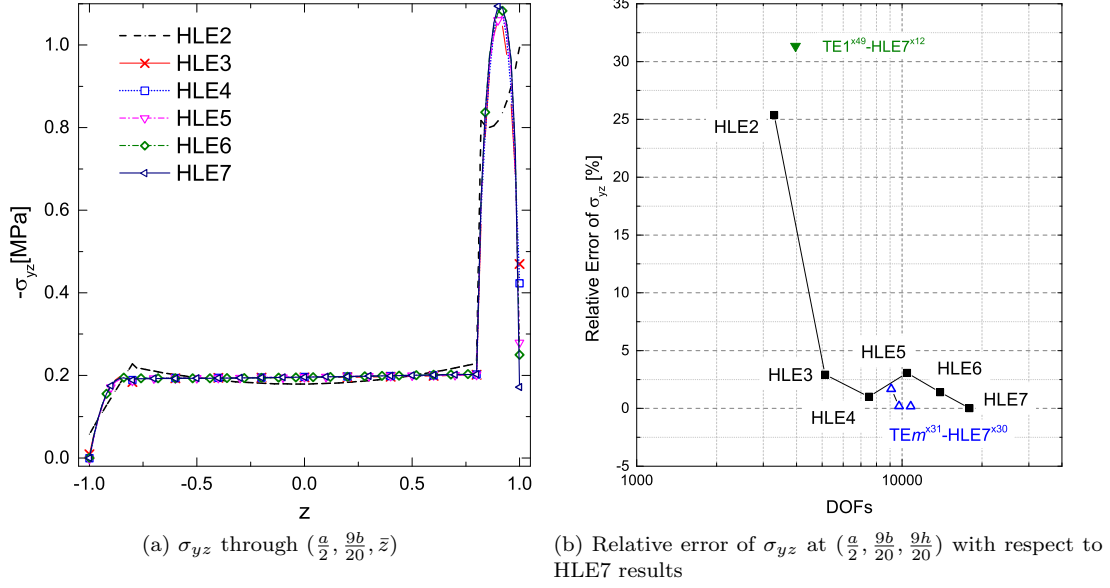


Figure 7: Evaluation of  $\sigma_{yz}$  on the sandwich beam under local pressure.

element. In this section, two locations for the transition zone are examined. Transition zone  $\alpha$  is located around 75% length of the beam which leads to NDK model  $TEm^{\times 49}$ -HLE7 $^{\times 12}$ . While transition zone  $\beta$  is placed near 50% of the length range and corresponds to models denoted by  $TEm^{\times 31}$ -HLE7 $^{\times 30}$ .

215 Concerning the  $\sigma_{yz}$  as illustrated in Figure 7(b), the NDK models with the transition zone  $\beta$  yield comparable accuracy with a pure HLE7 model at a reduced number of degrees of freedom, but results achieved by  $TEm^{\times 49}$ -HLE7 $^{\times 12}$  is far from satisfaction. From the comparison of the displacement and stress evaluation in Table 2 and Figure 8, it can be observed that  $TE1^{\times 31}$ -HLE7 $^{\times 30}$  has better accuracy than  $TE1^{\times 49}$ -HLE7 $^{\times 12}$  within the faces. This reality implies that  
 220 transition zone  $\beta$ , which is further away from the zone with local effects, is more proper than transition zone  $\alpha$ . In the meanwhile, in  $TEm^{\times 31}$ -HLE7 $^{\times 30}$  models, the increase of TE kinematic order  $m$  further helps to improve the solution accuracy at the expense of extra computational effort. Regarding the transverse shear stress  $\sigma_{yz}$ , given that TE5 fails to capture its variation well,  $TE1^{\times 31}$ -HLE7 $^{\times 30}$  already leads to results with fairly good accuracy. At the same time,  
 225 compared with FE model with pure HLE7 kinematics, a 49% reduction in the number of degrees of freedom is also achieved by  $TE1^{\times 31}$ -HLE7 $^{\times 30}$ . It should be noted that the cost of the reduced computational consumption is some loss in the accuracy of the displacement solution.

Figure 9 shows the variation of  $w$ ,  $\sigma_{yy}$  and  $\sigma_{yz}$  along the beam axial direction. Notably, an

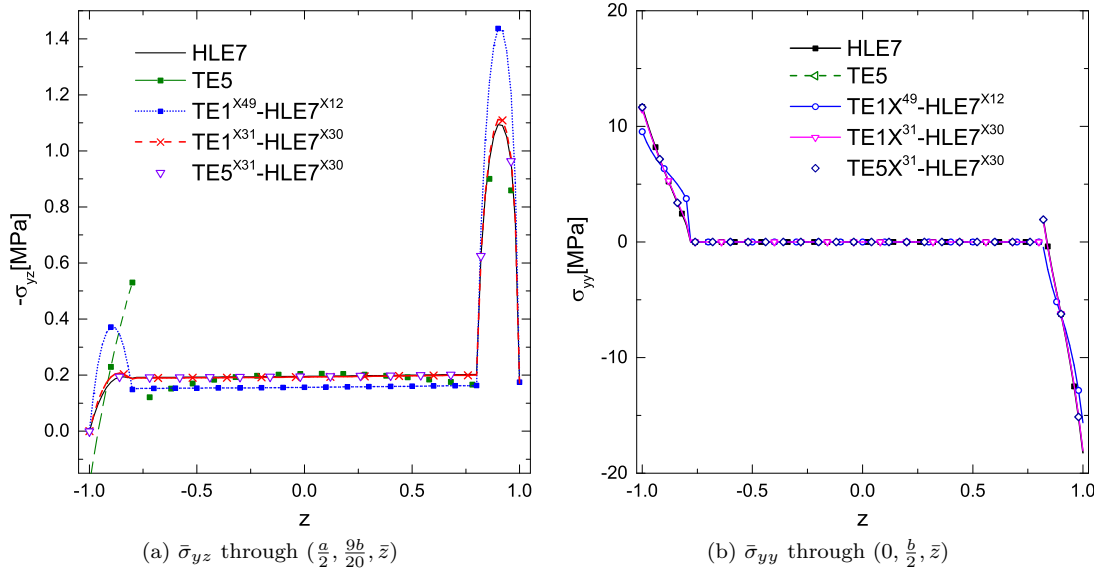
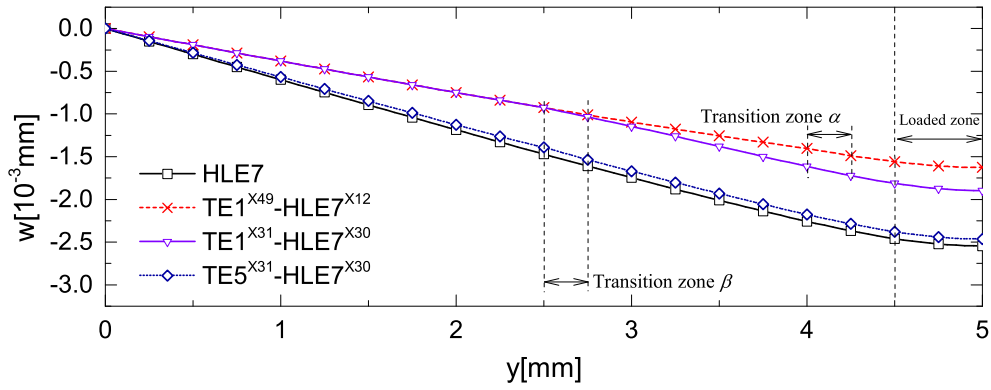


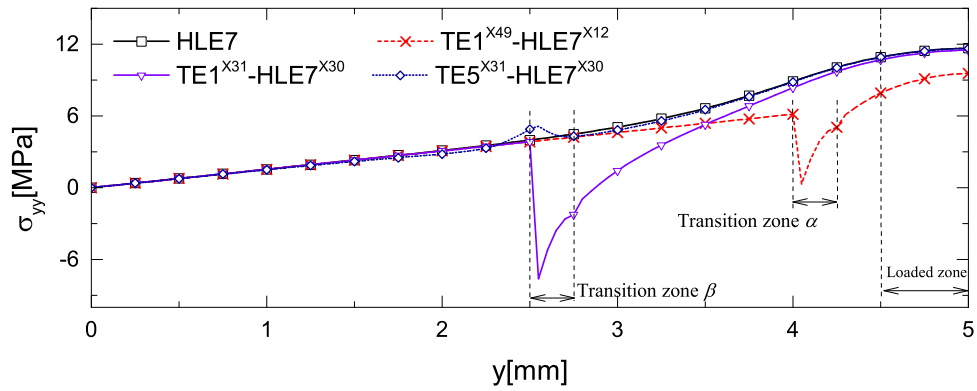
Figure 8: Through-the-thickness variation of  $\bar{\sigma}_{yz}$  and  $\bar{\sigma}_{yy}$  on the sandwich beam under local pressure.

oscillation exists in the stress distribution within and nearby the transition zone, although no  
 230 transition effects are observed in the displacement solutions. Considering  $\sigma_{yy}$  and  $\sigma_{yz}$  in the  
 loaded zone, by taking HLE7 solutions as references, TE1<sup>X31</sup>-HLE7<sup>X30</sup> leads to better results  
 compared with the other models. This fact shows that transition zone  $\beta$  is more appropriately  
 chosen than transition zone  $\alpha$ . Meanwhile, with the refinement of the TE theories, the stress  
 oscillation can be mitigated considerably, and the accuracy of both the displacement and stresses  
 235 is improved.

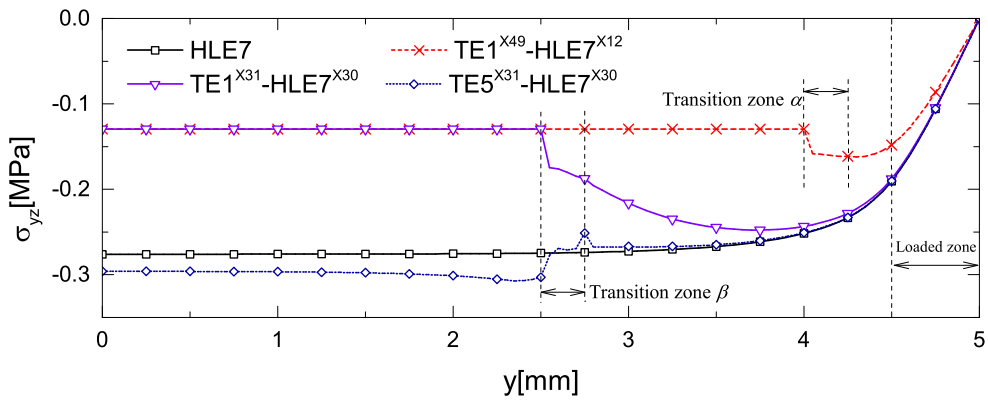
The contour of  $\sigma_{yz}$  and  $\sigma_{zz}$  obtained with model HLE7 and TE5<sup>X31</sup>-HLE7<sup>X30</sup> are compared  
 in Figure 10 and Figure 11, respectively. As a result of the local pressure, significant stress  
 concentration occurs on the right-hand side, especially within the upper face of the sandwich.  
 The stress oscillation can also be observed in the vicinity of the transition zone. Such effects in  
 240 models with incompatible kinematics were also reported by (Wenzel et al., 2014) about eXtended  
 Variational Formulation, and by (Zappino et al., 2017, Carrera et al., 2018a) in NDK approaches.  
 Even though, in the local region including the loaded zone, the stress fields obtained with the  
 two models agree well with each other. In conclusion, compared with the pure HLE7 model, the  
 NDK model TE5<sup>X31</sup>-HLE7<sup>X30</sup> is capable of capturing satisfactory displacement and stress field  
 245 with a much fewer number of degrees of freedom.



(a)  $\bar{w}$  along  $(0, y, 0)$



(b)  $\bar{\sigma}_{yy}$  along  $(0, y, -\frac{h}{2})$



(c)  $\bar{\sigma}_{yz}$  along  $(0, y, 0)$

Figure 9: Variation of  $\sigma_{yy}$  and  $\sigma_{yz}$  along the axial direction of the sandwich beam.

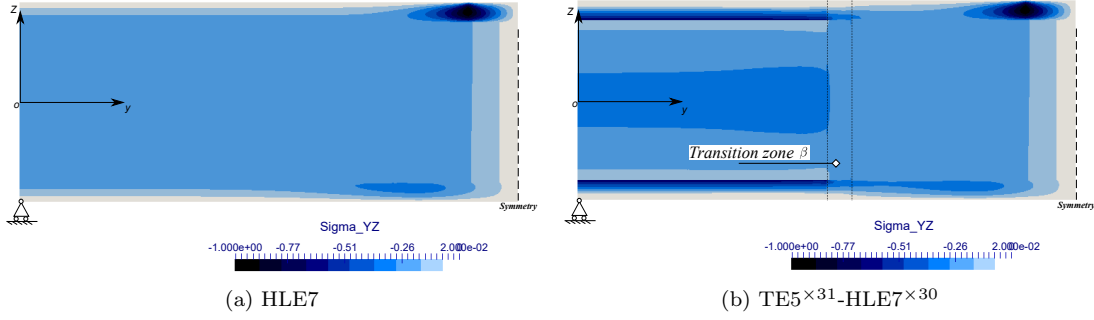


Figure 10: Contour plot of  $\sigma_{yz}$  on surface  $(\frac{a}{2}, y, z)$  of the sandwich beam under local pressure.

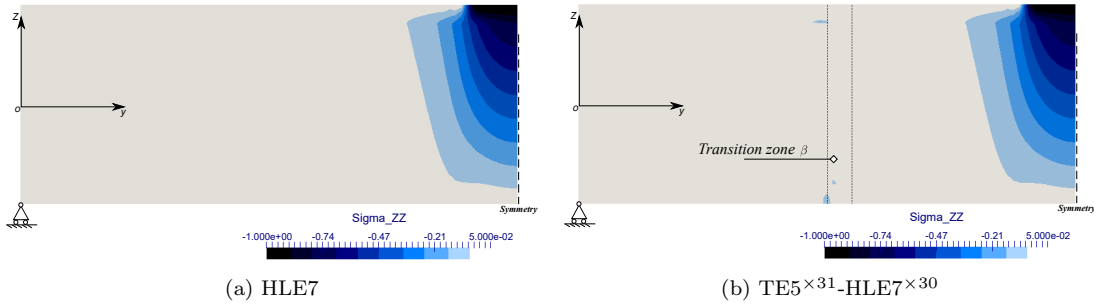


Figure 11: Contour plot of  $\sigma_{zz}$  on surface  $(\frac{a}{2}, y, z)$  of the sandwich beam under local pressure.

### 5.2. A two-layered cantilever beam under four points loads

As the second example, a cantilever beam with two layers is analyzed using the NDK approach. The beam is clamped on one end and subjected to four point loads at the vertexes on the loading end. Geometrical features of the structure are shown in Figure 12, with length  $b = 0.09\text{m}$ , width  $a = 0.001\text{m}$ , and height  $h = 0.01\text{m}$ . The two layers are of equal thickness ( $t = h/2$ ), both with the longitudinal direction along the beam axial direction  $y$ . The lower layer is made of Material 1, and the upper one of Material 2. Elastic properties of the materials are listed in Table 3, in which  $L$  and  $T$  stand the longitudinal and transverse direction of the fibers, respectively. The structure is discretized into a number of B4 elements as in shown Figure 12.

Models with complete HLE kinematics are first analyzed, in which the polynomial order  $p$  is increased until a numerical convergence is achieved. Then, to reduce the computational costs, in the area distant from the loaded region on the clamped side, TE kinematics is introduced, leading to NDK models  $\text{TE1}^{\times 49}\text{-HLE7}^{\times 12}$  and  $\text{TE1}^{\times 31}\text{-HLE7}^{\times 30}$ . The superscripts represent the number of nodes with the corresponding kinematics.  $\text{TE1}^{\times 49}\text{-HLE7}^{\times 12}$  has the transition zone

260 near 75% position along the axis away from the clamped end, and  $TE1^{\times 31}$ -HLE7 $\times 30$  has it near the mid-span position.

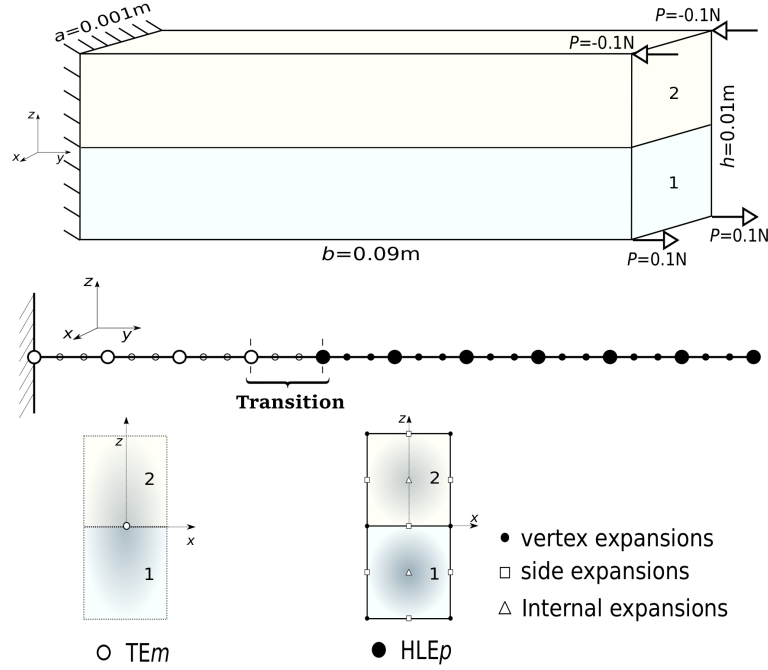


Figure 12: Geometry features and FE model of the two-layered cantilever beam (not to scale).

Table 3: Properties of the materials used for the two-layered cantilever beam.

	$E_L$ [GPa]	$E_T$ [GPa]	$\nu_{LT}$	$G_{LT}$ [GPa]
Material-1	30	1	0.25	0.5
Material-2	5	1	0.25	0.5

From the results summarized in Table 4, it can be noticed that with 20 B4 elements using pure HLE kinematics, the numerical convergence can be reached when HLE5 is employed. Besides, the transverse shear stress  $\sigma_{yz}$  is the critical case concerning the convergence. The convergence process can also be observed from the variation of  $\sigma_{yz}$  through the thickness along  $(0, \frac{8b}{9}, \bar{z})$ , as shown in Figure 13. The  $\sigma_{yz}$  shows a complex distribution due to the anisotropy and thickness effects. The obtained solutions are in good agreement with those given by the ABAQUS 3D model, which uses  $4 \times 180 \times 32$  ( $x \times y \times z$ ) quadratic brick elements with reduced integration (C3DR20).

270 If TE1 kinematics is employed on the left-hand side of the structure, a considerable reduction in the total degrees of freedom can be achieved, which is 65% for  $TE1^{\times 31}$ -HLE7 $\times 30$ , and 49%

for  $\text{TE1}^{\times 49}\text{-HLE7}^{\times 12}$ . According to the results in Table 4 and the stress variation in Figure 14,  $\text{TE1}^{\times 31}\text{-HLE7}^{\times 30}$  has better accuracy compared to model  $\text{TE1}^{\times 49}\text{-HLE7}^{\times 12}$  regarding the transverse shear stress  $\sigma_{yz}$ . These effects also confirm that transition zone  $\beta$  is a more decent choice.

275 Though, both of the two models lead to reasonable evaluations. In engineering practice, if the transition zone lies outside the critical region, it may not be worthy of extra efforts to take the stress oscillation into account.

Table 4: Displacement and stress evaluation on the two-layered cantilever beam.

Mesh	Kinematics	$w[10^{-3}\text{mm}]$ $(0, b, 0)$	$\sigma_{yy}[\text{KPa}]$ $(0, \frac{8b}{9}, -\frac{h}{2})$	$\sigma_{yz}[\text{KPa}]$ $(0, \frac{8b}{9}, -\frac{h}{4})$	DOFs
B4×10	HL2	9.041	236.6	2.563	1209
B4×20	HL2	9.036	234.0	2.610	2379
B4×20	HL3	9.082	245.1	4.518	3660
B4×20	HL4	9.065	236.4	4.432	5307
B4×20	HL5	9.075	233.4	4.972	7320
B4×20	HL6	9.063	233.8	4.986	9699
B4×20	HL7	9.074	234.8	4.972	12444
B4×20	TE1	9.053	215.1	0.000	549
B4×20	$\text{TE1}^{\times 49}\text{-HLE7}^{\times 12}$	9.120	234.0	4.294	2889
B4×20	$\text{TE1}^{\times 31}\text{-HLE7}^{\times 30}$	9.117	234.8	4.970	6399
ABAQUS (3D)		9.071	235.3	4.963	337251

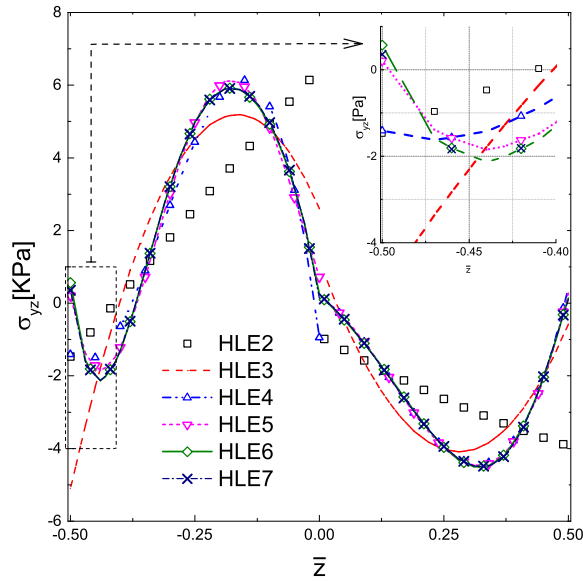


Figure 13: Variation of  $\sigma_{yz}$  along  $(0, \frac{8b}{9}, \bar{z})$  on the two-layered cantilever beam, obtained with HLE models.

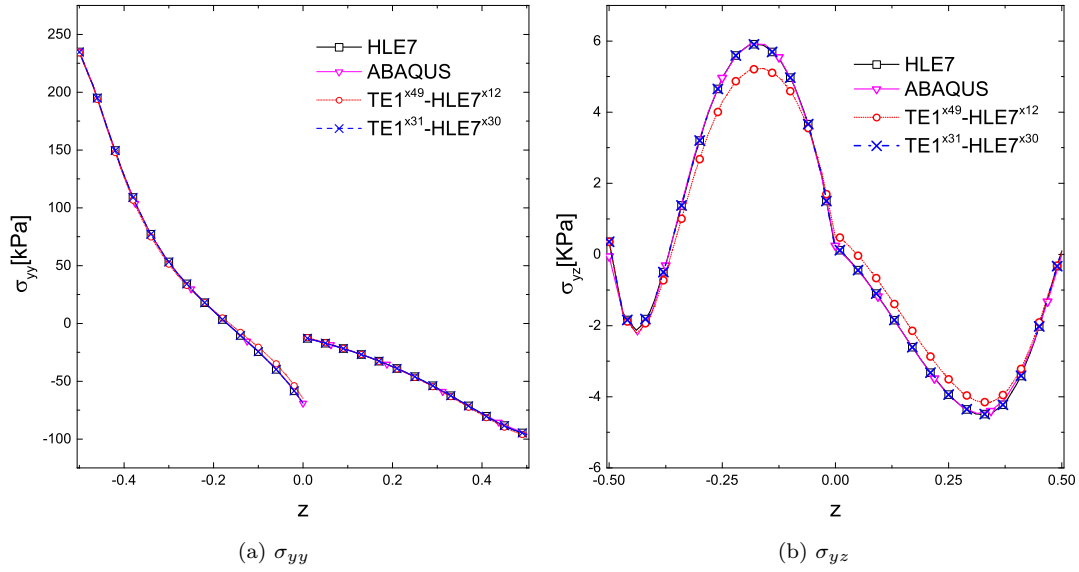


Figure 14: Variation of  $\sigma_{yy}$  and  $\sigma_{yz}$  along  $(0, \frac{8b}{9}, \bar{z})$  on the two-layered cantilever beam, obtained with various models.

## 6. Conclusions

This work presents a class of refined 1D FE models with node-dependent kinematics for the global-local analysis of composite laminated beam structures. Hierarchical Legendre Expansions (HLE) are adopted as cross-section functions for the local refinement on the nodal level. By treating the polynomial degree  $p$  as an input parameter, and assigning refined kinematics to the desirable nodes in the local zone of interest, a series of FE models can be built conveniently when the FE meshes have been chosen. Such an approach can help to improve the numerical efficiency in engineering simulations and simplifies the modeling procedure. It can be highlighted that:

- Node-dependent kinematics provides a solution to integrate the accuracy of LW models and the low computational cost of ESL models and therefore, provides optimal beam models;
- The combination of HLE and NDK improves the computational efficiency of FE models for the analysis of multi-layered slender structures;
- Based on CUF, the compactness of the FE formulations is assured by using no additional coupling nor superposition;

- The presented approach allows the local kinematic refinement to be carried out without changing the FE mesh.

295 As future work, implementing an adaptive nodal-kinematic refinement routine will help to further enhance the efficiency of FE models with the least user intervention. And, more realistic cases of engineering interest can be considered.

## 7. Acknowledgment

This research work has been carried out within the project FULLCOMP (FULLy analysis, 300 design, manufacturing, and health monitoring of COMposite structures), funded by the European Union Horizon 2020 Research and Innovation program under the Marie Skłodowska Curie grant agreement No. 642121. E. Carrera acknowledges the Russian Science Foundation under grant No. 18-19-00092.

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