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# ON THE $p$ -ADIC DENSENESS OF THE QUOTIENT SET OF A POLYNOMIAL IMAGE

PIOTR MISKA, NADIR MURRU, AND CARLO SANNA

ABSTRACT. The *quotient set*, or *ratio set*, of a set of integers  $A$  is defined as

$$R(A) := \{a/b : a, b \in A, b \neq 0\}.$$

We consider the case in which  $A$  is the image of  $\mathbb{Z}^+$  under a polynomial  $f \in \mathbb{Z}[X]$ , and we give some conditions under which  $R(A)$  is dense in  $\mathbb{Q}_p$ . Then, we apply these results to determine when  $R(S_m^n)$  is dense in  $\mathbb{Q}_p$ , where  $S_m^n$  is the set of numbers of the form  $x_1^n + \dots + x_m^n$ , with  $x_1, \dots, x_m \geq 0$  integers. This allows us to answer a question posed in [Garcia *et al.*,  $p$ -adic quotient sets, *Acta Arith.* **179**, 163–184]. We end leaving an open question.

## 1. INTRODUCTION

The *quotient set*, also known as *ratio set*, of a set of integers  $A$  is defined as

$$R(A) := \left\{ \frac{a}{b} : a, b \in A, b \neq 0 \right\}.$$

The question of when  $R(A)$  is dense in  $\mathbb{R}^+$  is a classical topic and has been studied by many researchers (see, e.g., [1, 2, 3, 7, 8, 9, 11, 15]).

Recently, some authors approached the study of the denseness of  $R(A)$  in the field of  $p$ -adic numbers  $\mathbb{Q}_p$ . Garcia and Luca [6] proved that the quotient set of the Fibonacci numbers is dense in  $\mathbb{Q}_p$ , and Sanna [12] extended this result to the  $k$ -generalized Fibonacci numbers. In [5], the denseness of  $R(A)$  in  $\mathbb{Q}_p$  is studied when  $A$  is the set of values of a Lucas sequence, the set of positive integers which are sum of  $k$  squares, respectively  $k$  cubes, or the union of two geometric progressions. Moreover, Miska and Sanna [10] proved that, given any partition  $A_1, \dots, A_k$  of  $\mathbb{Z}^+$ , for all prime numbers  $p$  but at most  $\lfloor \log_2 k \rfloor$  exceptions at least one of  $R(A_1), \dots, R(A_k)$  is dense in  $\mathbb{Q}_p$ .

In this paper, we focus on the study of the denseness of  $R(A)$  in  $\mathbb{Q}_p$  when  $A$  is the image of  $\mathbb{Z}^+$  under a polynomial  $f \in \mathbb{Z}[X]$ . For the sake of notation, we put  $R_f := R(f(\mathbb{Z}^+))$  for any function  $f : \mathbb{Z} \rightarrow \mathbb{Q}_p$ . The following easy lemma provides a necessary condition under which  $R_f$  is dense in  $\mathbb{Q}_p$ .

**Lemma 1.1.** *Let  $f : \mathbb{Z}_p \rightarrow \mathbb{Q}_p$  be a continuous function. If  $R_f$  is dense in  $\mathbb{Q}_p$ , then  $f$  has a zero in  $\mathbb{Z}_p$ .*

*Proof.* Since  $R_f$  is dense in  $\mathbb{Q}_p$ , there exists a sequence of integers  $(x_n)_{n \geq 0}$  such that  $f(x_n) \rightarrow 0$  (in the  $p$ -adic topology) as  $n \rightarrow \infty$ . By the compactness of  $\mathbb{Z}_p$ , there exists a subsequence  $(x_{n_k})_{k \geq 0}$  converging to some  $x_\infty \in \mathbb{Z}_p$ . Since  $f$  is continuous, we get  $f(x_\infty) = 0$ , as desired.  $\square$

Our first result is a sufficient condition under which  $R_f$  is dense in  $\mathbb{Q}_p$ . We postpone its proof to Section 2.

**Theorem 1.2.** *Let  $f : \mathbb{Z}_p \rightarrow \mathbb{Q}_p$  be an analytic function and let  $z_1, z_2 \in \mathbb{Z}_p$  be two (not necessarily distinct) zeros of  $f$  of multiplicities  $\mu_1, \mu_2$ , respectively. If  $\mu_1, \mu_2$  are coprime, then  $R_f$  is dense in  $\mathbb{Q}_p$ .*

As an immediate consequence we have the following corollary.

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**Corollary 1.3.** *If  $f : \mathbb{Z}_p \rightarrow \mathbb{Q}_p$  is an analytic function with a simple zero in  $\mathbb{Z}_p$ , then  $R_f$  is dense in  $\mathbb{Q}_p$ .*

The above results make possible to completely characterize the linear and quadratic polynomials  $f$  for which  $R_f$  is dense in  $\mathbb{Q}_p$ .

**Proposition 1.4.** *Let  $f \in \mathbb{Z}[X]$  be a polynomial of degree 1 or 2. Then,  $R_f$  is dense in  $\mathbb{Q}_p$  if and only if  $f$  has a simple zero in  $\mathbb{Z}_p$ .*

*Proof.* When  $f$  has degree 1, the thesis follows immediately from Lemma 1.1 and Corollary 1.3. Assume  $f$  has degree 2. If  $f$  has a simple zero in  $\mathbb{Z}_p$ , then  $R_f$  is dense in  $\mathbb{Q}_p$  by Corollary 1.3. On the other hand, if  $f$  has no simple zeros in  $\mathbb{Z}_p$ , then we have two cases. In the first case,  $f$  has no zeros in  $\mathbb{Z}_p$ . Then, by Lemma 1.1,  $R_f$  is not dense in  $\mathbb{Q}_p$ . In the second case,  $f$  has a zero in  $\mathbb{Z}_p$  with multiplicity 2, i.e.,  $f(x) = a(x - z)^2$ , for some  $a, z \in \mathbb{Z}_p$  with  $a \neq 0$ . Consequently,  $R_f$  is not dense in  $\mathbb{Q}_p$ , since the  $p$ -adic valuation of each element of  $R_f$  is divisible by 2.  $\square$

For polynomials of higher degrees, we can not exploit Lemma 1.1 and Corollary 1.3 to determine if  $R_f$  is dense in  $\mathbb{Q}_p$ . For instance, consider the case of a polynomial of degree 3 with a double root in  $\mathbb{Z}_p$  and the other root not in  $\mathbb{Z}_p$ . However, if we consider polynomials having all their roots in  $\mathbb{Z}_p$ , then we have the following result.

**Proposition 1.5.** *Let  $f \in \mathbb{Z}[X]$  be a nonconstant polynomial splitting in  $\mathbb{Z}_p$  and of degree less than 31. Then,  $R_f$  is not dense in  $\mathbb{Q}_p$  if and only if there exists an integer  $n > 1$  which divides the multiplicity of each root of  $f$ .*

*Proof.* Let  $\mu_1, \dots, \mu_s$  be the multiplicities of the roots of  $f$ . If there exists an integer  $n > 1$  dividing all  $\mu_1, \dots, \mu_s$ , then  $f = ag^n$ , for some  $a \in \mathbb{Z} \setminus \{0\}$  and  $g \in \mathbb{Z}[X]$ . Consequently,  $R_f$  is not dense in  $\mathbb{Q}_p$ , since the  $p$ -adic valuation of each element of  $R_f$  is divisible by  $n$ . Now suppose that there exists no integer  $n > 1$  dividing all  $\mu_1, \dots, \mu_s$ . We shall prove that  $\gcd(\mu_i, \mu_j) = 1$  for some  $i, j$ . In this way, by Theorem 1.2, it follows that  $R_f$  is dense in  $\mathbb{Q}_p$ . For the sake of contradiction, assume  $\gcd(\mu_i, \mu_j) > 1$  for all  $i, j$ . In particular, we have  $s \geq 3$ , and that each  $\mu_i$  has at least two distinct prime factors. Also, at least one of  $\mu_1, \dots, \mu_s$  is odd. Without loss of generality, we can assume  $\mu_1$  odd. Thus  $\mu_1 \in \{15, 21\}$ , and at least one of  $\mu_2, \dots, \mu_s$  is not divisible by 3. Without loss of generality, we can assume  $\mu_2$  not divisible by 3. Thus  $\mu_2 \in \{10, 14\}$ . Since  $\mu_3$  has at least two distinct prime factors,  $\mu_3 \geq 6$  and consequently  $\deg f = \mu_1 + \dots + \mu_s > 30$ , absurd.  $\square$

*Remark 1.6.* Proposition 1.5 is optimal in the sense that there exists a polynomial  $f \in \mathbb{Z}[X]$  of degree 31, splitting in  $\mathbb{Z}_p$ , with the greatest common divisor of the multiplicities of its roots equal to 1, but such that  $R_f$  is not dense in  $\mathbb{Q}_p$ . Indeed, consider

$$f(X) = (X + 1)^6(X + 2)^{10}(X + 3)^{15}.$$

Then, for  $p > 2$  (respectively  $p = 2$ ) the  $p$ -adic valuation of each element of  $f(\mathbb{Z}^+)$  is of the form  $6n$ ,  $10n$ , or  $15n$  (respectively  $10n$ ,  $6n + 15$ , or  $15n + 6$ ), for some integer  $n \geq 0$ . Therefore, no element of  $R_f$  has  $p$ -adic valuation equal to 1 (respectively 2), and  $R_f$  is not dense in  $\mathbb{Q}_p$ .

*Remark 1.7.* Using the same reasonings as in the proof of Proposition 1.5, one can prove a slightly more general statement: Given  $f = gh$ , where  $g, h \in \mathbb{Z}[X]$  are such that  $g$  splits in  $\mathbb{Z}_p$ ,  $1 \leq \deg g \leq 30$ , and the  $p$ -adic valuation of  $h$  is constant, we have that  $R_f$  is not dense in  $\mathbb{Q}_p$  if and only if there does not exist an integer  $n > 1$  dividing all the multiplicities of the roots of  $g$ .

For integers  $m, n \geq 2$ , define the set

$$S_m^n := \{x_1^n + \dots + x_m^n : x_1, \dots, x_m \in \mathbb{Z}_{\geq 0}\}.$$

The authors of [5] considered  $n = 2, 3$  and proved the following results [5, Theorems 4.1 and 4.2]. (Actually, there is a small error, here corrected, in [5, Theorem 4.2], see Remark 1.15 below.)

**Theorem 1.8.** *For all prime numbers  $p$ , we have:*

- (a)  $R(S_2^n)$  is dense in  $\mathbb{Q}_p$  if and only if  $p \equiv 1 \pmod{4}$ .
- (b)  $R(S_m^2)$  is dense in  $\mathbb{Q}_p$  for all integers  $m \geq 3$ .
- (c)  $R(S_m^3)$  is dense in  $\mathbb{Q}_p$  for all integers  $m \geq 2$ .

For all integers  $n, b \geq 2$ , let  $\gamma(n, b)$  denote the smallest positive integer  $g$  such that for every  $a \in \mathbb{Z}$  the equation

$$(1) \quad X_1^n + \cdots + X_g^n \equiv a \pmod{b}$$

has a solution. Furthermore, let  $\theta(n, b)$  be the smallest positive integer  $g$  such that for  $a = 0$  the equation (1) has a solution with at least one of  $X_1, \dots, X_g$  coprime with  $b$ . The quantities  $\gamma(n, b)$ ,  $\theta(n, b)$  have been studied in regard to analogs of Waring's problem modulo  $p$  (see, e.g., [13, 14]).

We give an effective criterion to establish if  $R(S_m^n)$  is dense in  $\mathbb{Q}_p$ . We postpone its proof to Section 3.

**Theorem 1.9.** *Let  $m, n \geq 2$  be integers, let  $p$  be a prime number, and put  $k := \nu_p(n)$ .*

- (a) *If  $m \geq \theta(n, p^{2k+1})$ , then  $R(S_m^n)$  is dense in  $\mathbb{Q}_p$ .*
- (b) *If  $m < \theta(n, p^{2k+1})$  and  $(n, p) \notin \{(2, 2), (4, 2), (8, 2), (16, 2)\}$ , then  $R(S_m^n)$  is not dense in  $\mathbb{Q}_p$ .*
- (c)  $R(S_m^2)$  is dense in  $\mathbb{Q}_2$  if and only if  $m \geq 3$ .
- (d)  $R(S_m^4)$  is dense in  $\mathbb{Q}_2$  if and only if  $m \geq 8$ .
- (e)  $R(S_m^8)$  is dense in  $\mathbb{Q}_2$  if and only if  $m \geq 16$ .
- (f)  $R(S_m^{16})$  is dense in  $\mathbb{Q}_2$  if and only if  $m \geq 64$ .

*Example 1.10.* Let us consider the denseness of  $R(S_m^6)$  in  $\mathbb{Q}_{11}$ . In order to apply Theorem 1.9, we have to compute  $\theta(6, 11)$ . The nonzero sixth powers modulo 11 are 1, 3, 4, 5, and 9. Hence, the minimum positive integer  $g$  such that the equation  $X_1^6 + \cdots + X_g^6 \equiv 0 \pmod{11}$  has a solution, with at least one of  $X_1, \dots, X_g$  not divisible by 11, is  $\theta(6, 11) = 3$ . Consequently, by points (a) and (b) of Theorem 1.9, we have that  $R(S_m^6)$  is dense in  $\mathbb{Q}_{11}$  if and only if  $m \geq 3$ .

*Example 1.11.* Let us consider the denseness of  $R(S_m^{10})$  in  $\mathbb{Q}_2$ . In order to apply Theorem 1.9, we have to compute  $\theta(10, 8)$ . We have  $x^{10} \equiv 1 \pmod{8}$  for each odd integer  $x$ . Hence, it follows easily that  $\theta(10, 8) = 8$ . Consequently, by points (a) and (b) of Theorem 1.9, we have that  $R(S_m^{10})$  is dense in  $\mathbb{Q}_2$  if and only if  $m \geq 8$ .

For  $m = 2$ , we have the following corollary.

**Corollary 1.12.** *Let  $n \geq 2$  be an integer, let  $p$  be a prime number, and put  $k = \nu_p(n)$ . Then  $R(S_2^n)$  is dense in  $\mathbb{Q}_p$  if and only if  $-1$  is an  $n$ th power modulo  $p^{2k+1}$ . In particular,  $R(S_2^n)$  is dense in  $\mathbb{Q}_p$  whenever  $n$  is odd.*

*Proof.* First, assume  $p = 2$  and  $n \in \{2, 4, 8, 16\}$ . Then, it can be easily checked that  $-1$  is not an  $n$ th power modulo  $p^{2k+1}$ . By Theorem 1.8,  $R(S_2^n)$  is not dense in  $\mathbb{Q}_p$  and, since  $S_2^n \subseteq S_2^2$ , we get that  $R(S_2^n)$  is not dense in  $\mathbb{Q}_p$ . Now assume  $(n, p) \notin \{(2, 2), (4, 2), (8, 2), (16, 2)\}$ . By Theorem 1.9, we have that  $R(S_2^n)$  is dense in  $\mathbb{Q}_p$  if and only if there exist integers  $0 \leq x_1, x_2 < p^{2k+1}$ , not both divisible by  $p$ , such that  $x_1^n + x_2^n$  is divisible by  $p^{2k+1}$ . It is easy to see that this last condition is equivalent to the  $-1$  being an  $n$ th power modulo  $p^{2k+1}$ .  $\square$

In [5, Problem 4.3] it is asked about the denseness in  $\mathbb{Q}_p$  of  $R(S_m^4)$  and  $R(S_m^5)$ . From Corollary 1.12, we have that  $R(S_m^5)$  is dense in  $\mathbb{Q}_p$  for all integers  $m \geq 2$  and prime numbers  $p$ . Regarding  $R(S_m^4)$ , the situation is more complicated. Theorem 1.9(d) already covers the case  $p = 2$ . For  $p > 2$  we have the following result.

**Theorem 1.13.** *For all prime numbers  $p > 2$ , we have:*

- (a)  $R(S_2^4)$  is dense in  $\mathbb{Q}_p$  if and only if  $p \equiv 1 \pmod{8}$ .
- (b)  $R(S_3^4)$  is dense in  $\mathbb{Q}_p$  if and only if  $p \neq 5, 29$ .
- (c)  $R(S_4^4)$  is dense in  $\mathbb{Q}_p$  if and only if  $p \neq 5$ .
- (d)  $R(S_m^4)$  is dense in  $\mathbb{Q}_p$  for all integers  $m \geq 5$ .

*Proof.* By Corollary 1.12,  $R(S_2^4)$  is dense in  $\mathbb{Q}_p$  if and only if  $-1$  is a fourth power modulo  $p$ . In turn, this is well known to be equivalent to  $p \equiv 1 \pmod{8}$ . Hence, (a) is proved. Substituting  $a = -1$  into (1), the bound  $\theta(n, b) \leq \gamma(n, b) + 1$  follows. From [13, Theorem 3'], we have that  $\gamma(4, p) = 2$  for all prime numbers  $p > 41$ . Hence,  $\theta(4, p) \leq 3$  for all prime numbers  $p > 41$ . Then, a computation shows that  $\theta(4, p) \leq 3$  for all prime numbers  $p \neq 5, 29$ . Precisely,  $\theta(4, 5) = 5$  and  $\theta(4, 29) = 4$ . Now the claims (b), (c), and (d) follow from Theorem 1.9.  $\square$

We leave the following general question to the readers.

**Question 1.14.** *Given a prime number  $p$  and a polynomial  $f \in \mathbb{Z}[X]$ , is there an effective criterion to establish if  $R_f$  is dense in  $\mathbb{Q}_p$ ? What about multivariate polynomials?*

*Remark 1.15.* In [5, Theorem 4.2] it is stated that  $R(S_2^3)$  is not dense in  $\mathbb{Q}_3$ . This is not correct, since  $R(S_2^3)$  is dense in  $\mathbb{Q}_3$  in light of Corollary 1.12. The mistake in the proof of [5, Theorems 4.2] is when, at point (b2), it is asserted that: “If  $x/y \in R(S_2^3)$  is sufficiently close to 3 in  $\mathbb{Q}_3$ , then  $\nu_3(x) = \nu_3(y) + 1$ . Without loss of generality, we may suppose that  $\nu_3(x) = 1$  and  $\nu_3(y) = 0$ .” This is not true, because if  $y$  is the sum of two cubes, then there is no guarantee that  $y/3^{\nu_3(y)}$  is still the sum of two cubes. For instance, if  $y = 1^3 + 5^3$  then  $y/3^{\nu_3(y)} = 14$  is not the sum of two cubes.

**Notation.** For each prime number  $p$ , let  $\nu_p$  denote the usual  $p$ -adic valuation, with the convention  $\nu_p(0) := +\infty$ . For integers  $a$  and  $m > 0$ , we write  $(a \bmod m)$  for the unique integer  $r \in ]-b/2, b/2]$  such that  $a - r$  is divisible by  $m$ .

## 2. PROOF OF THEOREM 1.2

We have to prove that for all  $r \in \mathbb{Q}_p$  and  $u > 0$  there exist  $x_1, x_2 \in \mathbb{Z}^+$  such that  $f(x_2) \neq 0$  and

$$\nu_p\left(\frac{f(x_1)}{f(x_2)} - r\right) > u.$$

Clearly, since  $\mathbb{Q}_p^*$  is dense in  $\mathbb{Q}_p$ , it is enough to consider  $r \neq 0$ . Furthermore, since  $\mathbb{Z}^+$  is dense in  $\mathbb{Z}_p$  and  $f$  is continuous, we can assume, less restrictively,  $x_1, x_2 \in \mathbb{Z}_p$ . By hypothesis, for  $i = 1, 2$ , we have  $f(X) = (X - z_i)^{\mu_i} g_i(X)$ , where  $g_i : \mathbb{Z}_p \rightarrow \mathbb{Q}_p$  is an analytic function such that  $g_i(z_i) \neq 0$ . Put  $x_i := y_i p^{k_i} + z_i$ , for  $i = 1, 2$ , where  $y_1, y_2 \in \mathbb{Z}_p \setminus \{0\}$  and  $k_1, k_2 \in \mathbb{Z}^+$  will be chosen later. Without loss of generality, we can assume  $\nu_p(g_1(z_1)) \leq \nu_p(g_2(z_2))$ . Thus, setting  $G := g_2(z_2)/g_1(z_1)$ , we have  $G \in \mathbb{Z}_p \setminus \{0\}$ . Since  $g_1, g_2$  are continuous, for sufficiently large  $k_1, k_2$  we have

$$(2) \quad \nu_p\left(G \cdot \frac{g_1(x_1)}{g_2(x_2)} - 1\right) > u - \nu_p(r),$$

In particular, it is implicit that  $g(x_2) \neq 0$  and consequently  $f(x_2) \neq 0$ . We fix  $k_1, k_2$  such that

$$k_1 \mu_1 - k_2 \mu_2 = \nu_p(r),$$

and (2) holds. This is possible thanks to the condition  $\gcd(\mu_1, \mu_2) = 1$ . Indeed, by Bézout's lemma, the quantity  $k_1 \mu_1 - k_2 \mu_2$  can be equal to any integer with  $k_1$  and  $k_2$  arbitrarily large (if  $k_1 \mu_1 - k_2 \mu_2 = a$ , then  $(k_1 + K \mu_2) \mu_1 - (k_2 + K \mu_1) \mu_2 = a$ , for any integer  $K$ ).

Again by Bézout's lemma, there exist integers  $h_1, h_2 \geq 0$  such that  $h_1 \mu_1 - h_2 \mu_2 = 1$ . We set  $y_i = s^{h_i}$ , for  $i = 1, 2$ , where  $s := p^{-\nu_p(r)} r G$ . Note that  $y_1, y_2 \in \mathbb{Z}_p \setminus \{0\}$ , as required.

Hence, we have

$$\begin{aligned} \frac{f(x_1)}{f(x_2)} &= \frac{(x_1 - z_1)^{\mu_1}}{(x_2 - z_2)^{\mu_2}} \cdot \frac{g_1(x_1)}{g_2(x_2)} = p^{k_1\mu_1 - k_2\mu_2} \cdot \frac{y_1^{\mu_1}}{y_2^{\mu_2}} \cdot \frac{g_1(x_1)}{g_2(x_2)} \\ &= p^{\nu_p(r)} \cdot s^{h_1\mu_1 - h_2\mu_2} \cdot \frac{g_1(x_1)}{g_2(x_2)} = p^{\nu_p(r)} \cdot s \cdot \frac{g_1(x_1)}{g_2(x_2)} = rG \cdot \frac{g_1(x_1)}{g_2(x_2)}, \end{aligned}$$

so that, recalling (2), we get

$$\nu_p\left(\frac{f(x_1)}{f(x_2)} - r\right) = \nu_p\left(r\left(G \cdot \frac{g_1(x_1)}{g_2(x_2)} - 1\right)\right) > u,$$

as desired.

### 3. PROOF OF THEOREM 1.9

(a) Suppose that there exist integers  $0 \leq x_1, \dots, x_m < p^{2k+1}$ , not all divisible by  $p$ , such that  $x_1^n + \dots + x_m^n$  is divisible by  $p^{2k+1}$ . Up to reordering  $x_1, \dots, x_m$ , we can assume that  $p \nmid x_1$ . Put  $f(X) = X^n + x_2^n + \dots + x_m^n$ , so that  $f'(X) = nX^{n-1}$ . In particular, all the roots of  $f$  are simple. Since  $p \nmid x_1$ , we have

$$\nu_p(f(x_1)) \geq 2k + 1 > 2k = 2\nu_p(f'(x_1)),$$

so that, by Hensel's lemma [4, Ch. 4, Lemma 3.1],  $f$  has a simple root in  $\mathbb{Z}_p$ . Hence, by Corollary 1.3,  $R_f$  is dense in  $\mathbb{Q}_p$ . Clearly,  $R_f \subseteq R(S_m^n)$ , so that  $R(S_m^n)$  is dense in  $\mathbb{Q}_p$ .

(b) Suppose that there are no integers  $x_1, \dots, x_m$  as before, and that

$$(3) \quad (n, p) \notin \{(2, 2), (4, 2), (8, 2), (16, 2)\}.$$

We shall prove that  $4k + 1 < n$ . For the sake of contradiction, suppose  $4k + 1 \geq n$ . Since  $n \geq 2$ , we have  $k \geq 1$ . Also, we have  $4k + 1 \geq p^k$ , which implies  $p \leq 5$ . Now, taking into account (3), it can be readily checked that

$$(n, p) \in \{(3, 3), (9, 3), (5, 5)\}.$$

But  $3^3 \mid (1^3 + 8^3)$ ,  $3^5 \mid (1^9 + 26^9)$ , and  $5^3 \mid (1^5 + 24^5)$ , contradicting the nonexistence of  $x_1, \dots, x_m$ .

Let  $y_1, \dots, y_m \geq 0$  be integers, not all equal to zero. Put  $\mu := \min\{\nu_p(y_i) : i = 1, \dots, m\}$ ,  $I := \{i : \nu_p(y_i) = \mu\}$ , and  $J := \{1, \dots, m\} \setminus I$ . Also, put  $z_i := y_i/p^\mu$  for  $i \in I$ , so that  $z_i$  is an integer not divisible by  $p$ . The nonexistence of  $x_1, \dots, x_m$  implies that

$$(4) \quad \nu_p\left(\sum_{i \in I} z_i^n\right) \leq 2k.$$

Therefore, since  $2k < n$ , we have

$$\nu_p\left(\sum_{i \in I} y_i^n\right) = \mu n + \nu_p\left(\sum_{i \in I} z_i^n\right) \leq \mu n + 2k < (\mu + 1)n \leq \nu_p\left(\sum_{j \in J} y_j^n\right),$$

and consequently

$$\nu_p(y_1^n + \dots + y_m^n) = \nu_p\left(\sum_{i \in I} y_i^n\right) = \mu n + \nu_p\left(\sum_{i \in I} z_i^n\right),$$

which in turn, by (4), implies that

$$(\nu_p(y_1^n + \dots + y_m^n) \bmod n) \in \{0, \dots, 2k\}.$$

Thus, for each  $a \in R(S_m^n) \setminus \{0\}$  we have

$$(\nu_p(a) \bmod n) \in \{-2k, \dots, 2k\},$$

that is, the  $p$ -adic valuations of the nonzero elements of  $R(S_m^n)$  belong to at most  $4k + 1$  residue classes modulo  $n$ . Since  $4k + 1 < n$ , at least one residue class modulo  $n$  is missing and, a fortiori,  $R(S_m^n)$  is not dense in  $\mathbb{Q}_p$ .

(c) The claim follows immediately from Theorem 1.8.

From now on, assume  $n = 2^k$ , with  $k \in \{2, 3, 4\}$ . Let  $T_m^n$  be the topological closure of  $S_m^n$  in  $\mathbb{Q}_2$ . Clearly, we have

$$T_m^n = \{x_1^n + \cdots + x_m^n : x_1, \dots, x_m \in \mathbb{Z}_2\}.$$

It is a standard exercise showing that the nonzero  $n$ th powers of  $\mathbb{Z}_2^*$  are exactly the elements of the form  $1 + 4ny$ , with  $y \in \mathbb{Z}_2$ . As a consequence,

$$T_1^n = \{2^{nv}(1 + 4ny) : v \in \mathbb{Z}_{\geq 0}, y \in \mathbb{Z}_2\} \cup \{0\}.$$

Let  $v_1, v_2 \geq 0$ ,  $j \geq 1$  be integers and  $y_1, y_2 \in \mathbb{Z}_2$ . If  $v_1 = v_2$ , then

$$2^{nv_1}(j + 4ny_1) + 2^{nv_2}(1 + 4ny_2) = 2^{nv_1}(j + 1 + 4nz),$$

where  $z := y_1 + y_2 \in \mathbb{Z}_2$ . If  $v_1 < v_2$ , then

$$2^{nv_1}(j + 4ny_1) + 2^{nv_2}(1 + 4ny_2) = 2^{nv_1}(j + 4nz),$$

where  $z := y_1 + 2^{n(v_2-v_1)-k-2}(1 + 4ny_2) \in \mathbb{Z}_2$ , since  $n = 2^k \geq k + 2$ . If  $v_1 > v_2$ , then

$$2^{nv_1}(j + 4ny_1) + 2^{nv_2}(1 + 4ny_2) = 2^{nv_2}(1 + 4nz),$$

where  $z := 2^{n(v_1-v_2)-k-2}(j + 4ny_1) + y_2 \in \mathbb{Z}_2$ , again since  $n \geq k + 2$ .

Therefore, it follows easily by induction on  $m$  that

$$(5) \quad T_m^n = \{2^{nv}(j + 4ny) : v \in \mathbb{Z}_{\geq 0}, j \in \{1, \dots, m\}, y \in \mathbb{Z}_2\} \cup \{0\}.$$

(d) On the one hand, using (5), it can be checked quickly that  $15 \notin R(T_7^4)$ . Hence,  $R(S_7^4)$  is not dense in  $\mathbb{Q}_2$ . On the other hand, we have

$$2^{4v+r}(1 + 2y) = \frac{2^{4v}(8 + 16y)}{2^{4 \cdot 0}(2^{3-r} + 16 \cdot 0)} \in R(T_8^4),$$

for all  $v \in \mathbb{Z}_{\geq 0}$ ,  $r \in \{0, 1, 2, 3\}$ , and  $y \in \mathbb{Z}_2$ . Hence,  $\mathbb{Z}_p \subseteq R(T_8^4)$  and, since  $R(T_8^4)$  is closed by inversion, we get that  $R(T_8^4) = \mathbb{Q}_p$ . Thus  $R(S_8^4)$  is dense in  $\mathbb{Q}_p$ .

(e) On the one hand, by (5), the 2-adic valuation of each nonzero element of  $T_{15}^8$  is congruent to 0, 1, 2, or 3 modulo 8. Hence,  $R(T_{15}^8)$  contains no element with 2-adic valuation equal to 4, and consequently  $R(S_{15}^8)$  is not dense in  $\mathbb{Q}_2$ . On the other hand, we have

$$2^{8v+r}(1 + 2y) = \frac{2^{8v}(16 + 32y)}{2^{8 \cdot 0}(2^{4-r} + 32 \cdot 0)} \in R(T_{16}^8)$$

and

$$2^{8v+r+4}(1 + 2y) = \frac{2^{8(v+1)}(2^r + 32 \cdot 0)}{2^{8 \cdot 0}(16 + 32 \frac{-y}{1+2y})} \in R(T_{16}^8)$$

for all  $v \in \mathbb{Z}_{\geq 0}$ ,  $r \in \{0, 1, 2, 3, 4\}$ , and  $y \in \mathbb{Z}_2$ . Hence,  $\mathbb{Z}_p \subseteq R(T_{16}^8)$  and, since  $R(T_{16}^8)$  is closed by inversion, we get that  $R(T_{16}^8) = \mathbb{Q}_p$ . Thus  $R(S_{16}^8)$  is dense in  $\mathbb{Q}_p$ .

(f) On the one hand, by (5), the 2-adic valuation of each nonzero element of  $T_{63}^{16}$  is congruent to 0, 1, 2, 3, 4, or 5 modulo 16. Hence,  $R(T_{63}^{16})$  contains no element with 2-adic valuation equal to 6, and consequently  $R(S_{63}^{16})$  is not dense in  $\mathbb{Q}_2$ . On the other hand,  $2^9$  divides  $5^{16} + 1^{16} + \cdots + 1^{16}$  (63 times  $1^{16}$ ). Hence, by point (a), we get that  $R(S_{64}^{16})$  is dense in  $\mathbb{Q}_2$ .

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