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Stockwell-Like Frames for Sobolev Spaces

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Abstract We construct a family of frames describing the norm and seminorm of the space $H^s(\mathbb{R}^d)$. We also characterise Besov spaces modeled on $L^2(\mathbb{R}^d)$. Our work is inspired by the Discrete Orthonormal Stockwell Transform introduced by R.G. Stockwell, which provides a time-frequency localised version of the Fourier basis of $L^2([0,1])$. This approach is a hybrid between Gabor and Wavelet frames. We construct explicit and computable examples of these frames, discussing their properties and comparing them with the existing literature.

Keywords 42C15 · 42C40 · 46E35

1 Introduction

The discretisation of function spaces is an interesting problem both from a pure and applied perspective. One of the leading ideas is that the smoothness or regularity must be characterised via decay or sparsity properties of the associated discrete expansions. For example, it is well known that $H^s([0,1])$ can be characterised by the decay properties of the Fourier coefficients. Similarly, Sobolev spaces $H^s(\mathbb{R})$, and more generally inhomogeneous Besov spaces, can be described using suitable wavelet expansions, see for example [10]. See [6,7, 12–15,18,22,24] for other discrete time-frequency techniques and representations of function spaces related to our research.

Here, we focus on the Stockwell frames associated to a dyadic partition of the frequency domain. In [2], the Stockwell frames were first introduced in connection to the so called α-partitioning, see e.g. [14]. It is well known that α-partitioning and α-modulation spaces with $\alpha = 1$, i.e. with pure dyadic scaling factor $2^j, j \in \mathbb{Z}$, lead to wavelet analysis and Sobolev spaces, see [13,14]. In particular, it can be shown that wavelets can be used to characterise these spaces,

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see [10]. Hence, we investigate the conditions under which our Stockwell frame characterises Sobolev spaces representing their norms and seminorms and also show how to characterise Besov spaces. We introduce a dyadic partition of the frequency domain with a fixed number of possible directions at any scale, similar to what it is usually done with multi-dimensional wavelets, which is suited for both Sobolev and Besov spaces. Our theory can even cover a wider class of decomposition spaces, see [5, 27] and reference therein. In particular, we later introduce a parameter δ which regulates the number of directions we have at each step of the decomposition and describes the different partitions associated with it. This captures the case of finite directions described above while also allowing the number of directions to grow; in this way, one obtains partitions related with different kind of parabolic molecules, see Remark 7 below. We also show that, in some cases, the Stockwell frame detects higher regularity than wavelets. In particular, it is known that $H^s(\mathbb{R}), s < \frac{1}{2}$ can be characterised using Haar wavelets. Our frame associated with the characteristic function improves the result to $s < 1$.

The paper is organised as follows. In Section 2 we introduce the concept of admissible multi-dimensional partition; roughly, a dyadic partition which covers the whole of $\mathbb{R}^d$ (seen as frequency space) and has the uniformly finite intersection property, and a finite number of possible directions in frequency, see Definition 21 below. Giving a suitable function φ, we define the following system of functions

$$\varphi_{\bullet,\lambda} = T_\lambda \varphi(x),\ \varphi_{j,k,\lambda}(x) = T_{2^{-j}\lambda}\left(\frac{1}{2^{jd/2}} \sum_{\eta \in Z_{j,k}} e^{2\pi \mathrm{i}\, \eta t} \varphi(x)\right), \tag{1}$$

depending on a certain number of suitable parameters. The parameter λ represents the space translation, j represents the (parabolic) scaling, while the parameter k the "direction" as we shall show later on. For example, when $d = 1$, we have just two admissible directions $k = \pm$ and we can set

$$Z_{j,+} = 2^j, \dots, 2^{j+1} - 1 \quad \text{and} \quad Z_{j,-} = -2^{j+1} + 1, \dots, -2^j. \tag{2}$$

It is worth explaining why we choose (1) as a candidate to be an L^2-frame.

Consider dimension $d = 1$ and let $Z_{j,\pm}$ be as in (2). In [1] it is proved that the so called DOST-functions,

$$\begin{aligned} &P_{j,k,\tau}(t) = T_{\frac{\tau}{2^j}} \frac{1}{2^{j/2}} \sum_{\eta \in Z_{j,k}} e^{2\pi \mathrm{i}\, \eta t}, \quad j \in \mathbb{N}, \tau = 0, \dots, 2^j - 1, k = \pm, \\ &P_\bullet(t) = 1, \end{aligned} \tag{3}$$

form an orthonormal basis of $L^2([0, 1])$; these functions (3) were first introduced in [25] as a discretisation of the S-transform, defined in [26]. This basis was further studied by Yanwei Wang and Orchard [29] and then extended to dimension $d = 2$ by Drabycz, Stockwell and Mitchell and Brown, Lauzon, Frayne [31]. A first step into the direction of the Stockwell frame developed done by Yan and Zhu [28], where the authors studied general and flexible tiling in time-frequency space in dimension $d = 1$. See also [29] for a numerical perspective and [17, 23, 30] for a more abstract analysis of the S-transform.

Then, the naive idea of the definition of the Stockwell frames comes from the interpretation of Gabor frames

$$T_\lambda \left(e^{2\pi \mathrm{i}\, mt} \varphi(t) \right), \quad \lambda \in \nu\mathbb{Z}, m \in \mathbb{Z},$$

as the uniform translation of $\left\{ e^{2\pi \mathrm{i}\, mt} \right\}_{m\in\mathbb{Z}}$, the usual Fourier basis of $L^2([0, 1])$, localised via a window function φ. Using the DOST basis instead of the Fourier one, and a natural non-stationary translation, we are led exactly to the system of functions defined in (1). Notice that the non-stationary translation is related to the frequency parameter j. Roughly, we refine the space translation as the frequency increases.

In Section 3, we study the system of functions (1) in detail. In particular we prove our main result, Theorem 32, i.e. the system (1) is a frame of $L^2(\mathbb{R}^d)$ which characterises the H^s-Sobolev norm, that is

$$A \|f\|_s^2 \le \sum_{\lambda\in\nu\mathbb{Z}^d} |\langle f, \varphi_{\bullet,\lambda}\rangle|^2 + \sum_{j\in\mathbb{N}, k\in K, \lambda\in\nu\mathbb{Z}^d} 2^{2js} |\langle f, \varphi_{j,k,\lambda}\rangle|^2 \le B \|f\|_s^2 . \tag{4}$$

The requirements, see Definition 31, involve the decay and non vanishing properties of suitable linear combinations of translations of $\widehat{\varphi}$, which is the Fourier Transform of φ. Given $s \ge 0$, it is not difficult to find functions which satisfy such conditions, as we show at the end of the section in Theorem 36. For example, the Gaussian function works for all s.

In Theorem 34, we generalise the construction and prove a similar result for seminorms. As expected, the conditions are stronger than those of the H^s-norm characterisation. As an application, we show that this frame can also characterise Besov spaces $B^s_{2,q}$, with $s \in (0, 1)$ and $q > 1$. We end the section by stating some sufficient conditions for the existence of our frames.

In Section 4, we give some explicit examples of Stockwell-like frames and, in a particular case, we show that the system of functions (1) is indeed an orthonormal basis of $L^2(\mathbb{R})$, characterizing all Sobolev spaces. Moreover, as anticipated, we show how the Stockwell frame compares with Haar-wavelet in terms of Sobolev regularity, showing how the former improves the wavelet result.

In Section 5, we extend the result to isotropic admissible partitions with a growing number of frequency directions.

Notations

We choose the following normalisation for the Fourier Transform

$$(\mathrm{F}\, u)\,(\omega) = \int e^{-2\pi i x\cdot\omega} u(x) dx.$$

We denote by $\|\cdot\|_s$ and $|\cdot|_s$ the H^s-norm and seminorm, respectively. We write $f \lesssim g$, if there exists a constant $C > 0$ such that $f \leq C\,g$. We write $f \asymp g$ if $f \lesssim g$ and $g \lesssim f$. Finally, $\mathrm{d}(\cdot,\cdot)$ represents the euclidean distance. We also use $\lceil\cdot\rceil$ to represent the "ceiling" function.

2 Stockwell-like Frames

First, we define a class of suitable tiling of $\mathbb{R}^d$ in the frequency domain; then we construct frames for the H^s-norm associated to these admissible partitions.

2.1 Admissible partitions of $\mathbb{R}^d$ - the finite directions case

We state here a first definition in arbitrary dimension with finitely many frequency directions; then we give some examples of admissible partitions. Similar concepts are used in [13, 14], and we refer also to the bibliography presented there.

Definition 21 *The family $\{I_{j,k}\}_{j\in\mathbb{N},k\in K_j} \cup I_\bullet$, where K_j is an index set, is called admissible partition if*

i) $I_{j,k}$ *and* $I_\bullet$ *are non empty connected subsets of* $\mathbb{R}^d$*;*
ii) $\bigcup_{j\in\mathbb{N},k\in K_j} I_{j,k} \cup I_\bullet = \mathbb{R}^d$*;*
iii) there exists $N \in \mathbb{N}$ *such that for each* $\left(\bar{j},\bar{k}\right)$ *there are at most* N *indices* (j,k) *such that* $I_{\bar{j},\bar{k}} \cap I_{j,k} \neq \emptyset$ *(uniformly finite intersection);*
iv) $I_\bullet$ *is a neighbourhood of the origin and, for each* $j \in \mathbb{N}$ *and* $k \in K_j$, $I_{j,k} \cap \mathbb{Z}^d \neq \emptyset$*;*
v) there exist $c_{\min}, c_{\max} > 0$ *such that for all* $\omega \in I_{j,k}$

$$c_{\min} < \frac{|\omega|}{2^j} < c_{\max}, \quad \textit{uniformly in } j,k;$$

vi) $|I_{j,k}| \asymp |I_{j,k'}|, k,k' \in K_j$*, where* $|I_{j,k}|$ *is the Lebesgue measure of* $I_{j,k}$*;*
vii) there exists C_K *such that* $|K_j| \lesssim C_K$*, uniformly in* j*.*

Remark 1 *An admissible partition covers $\mathbb{R}^d$ with no holes - i), ii) - with finitely many intersections - iii) - and each set contains at least some point with integer coordinates - iv). The distance from the origin of each elements of the set $I_{j,k}$ is of order 2^j - v). In vi), at the scale j, each set has comparable size. Finally - vii) - the constant C_K represents the possible number of frequency direction at any scale j, which we assume to be uniformly bounded.*

We now describe some examples of admissible partition in different dimensions.

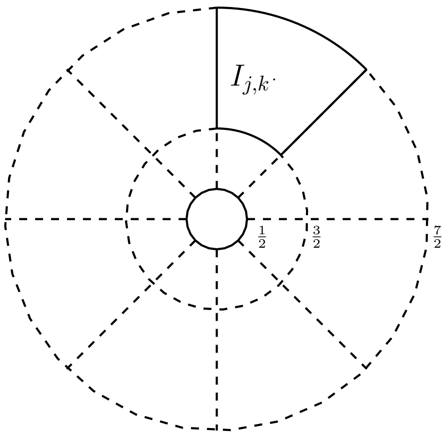


Fig. 1: An example of admissible partition in $d = 2$.

Dimension $d = 1$ In this case we just have two possible directions represented by $k = \pm$. We set

$$\begin{aligned} I_{j,+} &= \left\{\omega \in \mathbb{R}, \omega \in \left[2^j - \frac{1}{2}, 2^{j+1} - \frac{1}{2}\right)\right\}, \\ I_{j,-} &= \left\{\omega \in \mathbb{R}, \omega \in \left(-2^{j+1} + \frac{1}{2}, -2^j + \frac{1}{2}\right]\right\}, \\ I_\bullet &= \left(-\frac{1}{2}, \frac{1}{2}\right). \end{aligned} \tag{5}$$

One can easily see that this partition is admissible. With this particular choice, we show in Section 4 that it is possible to define a Stockwell-like frame which is also an orthonormal basis of $L^2(\mathbb{R})$.

Dimension $d = 2$ We express $\omega \in \mathbb{R}^2$ using polar coordinates (ρ, θ) and set
Set

$$\begin{aligned} I_{j,k} &= \left\{\omega \in \mathbb{R}^2 : 2^j - \frac{1}{2} \leq \rho < 2^{j+1} - \frac{1}{2}, \ \frac{k\pi}{4} \leq \theta < \frac{(k+1)\pi}{4}\right\}, \ k = 0, \dots, 7 \\ I_\bullet &= \left\{\omega \in \mathbb{R}^2 : \rho < \frac{1}{2}\right\}. \end{aligned} \tag{6}$$

As before, one can easily check that this is an admissible partition; we refer to Figure 1 for a plot.

The last example represents a partition which naturally extends to arbitrary dimensions.

2.2 Stockwell-like frames

Consider functions $\varphi, \varphi_\bullet \in L^2(\mathbb{R}^d)$, an admissible partition $\{I_{j,k}\}_{j\in\mathbb{N},k\in K_j} \cup I_\bullet$ and

$$\Gamma = \left\{(j,k,\lambda) \mid j \in \mathbb{N}, k \in K_j, \lambda \in \nu\mathbb{Z}^d\right\}. \tag{7}$$

Let $Z_{j,k} = I_{j,k} \cap \mathbb{Z}^d$, then the Stockwell frame is defined as

$$\varphi_{\bullet,\lambda}(x) = \varphi_\bullet(x-\lambda),\ \varphi_{j,k,\lambda}(x) = T_{2^{-j}\lambda}\left(\frac{1}{2^{jd/2}}\sum_{\eta\in Z_{j,k}} e^{2\pi\mathrm{i}\,\eta\cdot x}\varphi(x)\right),$$

with $(j,k,\lambda) \in \Gamma$, $x \in \mathbb{R}^d$. We choose $2^{jd/2}$ as $\ell^2(\mathbb{Z}^d)$ normalisation, since $|Z_{j,k}| \asymp 2^{jd}$. Finally, we define the Stockwell-like system as

$$\mathcal{S}(\varphi_\bullet, \varphi, \Gamma) = \left\{\varphi_{\bullet,\lambda}, \lambda \in \nu\mathbb{Z}^d\right\} \cup \{\varphi_{j,k,\lambda},\ j,k,\lambda \in \Gamma\}. \tag{8}$$

Applying the Fourier transform one gets

$$\begin{aligned} \widehat{\varphi_{\bullet,\lambda}}(\omega) &= e^{-2\pi\mathrm{i}\,\omega\cdot\lambda}\widehat{\varphi_\bullet}(\omega), \\ \widehat{\varphi_{j,k,\lambda}}(\omega) &= e^{-2\pi\mathrm{i}\,\omega\cdot 2^{-j}\lambda}\left(\frac{1}{2^{jd/2}}\sum_{\eta\in Z_{j,k}}\widehat{\varphi}(\omega-\eta)\right). \end{aligned} \tag{9}$$

If φ is a localisation window, the sum over the integer in $Z_{j,k}$ determines, in the space domain, a higher localisations near the point $2^{-j}\lambda$ as the scale j increases.

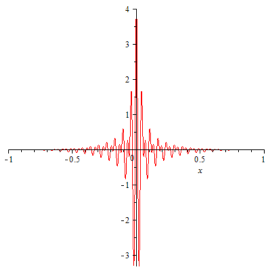
In frequency, it implies that the frame element $\varphi_{j,k,\lambda}$ has a Fourier transform $\widehat{\varphi_{j,k,\lambda}}$ with the support "essentially" in the set $I_{j,k}$. In Figure 2 we plot some frame elements with different windows φ, in the dimension $d = 1$ case. In Figure 3, we plot a frame element and its Fourier transform in the dimension $d = 2$ case using the partition defined above, see (6).

3 Frames of $H^s(\mathbb{R}^d)$

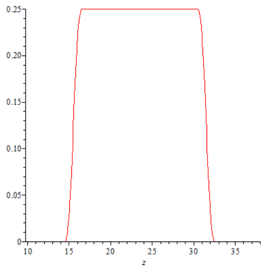
In this section, we introduce conditions under which the system of functions $\{\varphi_{j,k,\lambda}\}_{j,k,\lambda\in\Gamma}$ is a frame representing the H^s-norm. Later, we analyse the H^s-seminorm's case as well.

3.1 H^s-norm

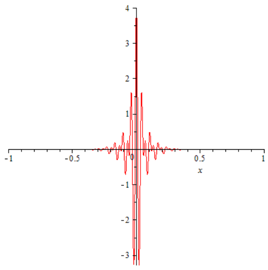
Before going into specific details, let us give a brief explanation of which conditions we need to impose. We require that the frame elements cover in the frequency domain the whole $\mathbb{R}^d$, and we require decay properties at infinity in the frequency domain, in order to obtain Sobolev regularity. Precisely, we have the following definition.



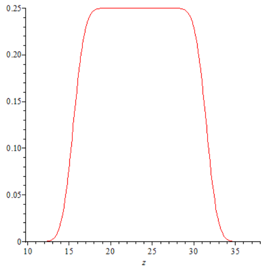
(a) Time, $\varphi_{4,+,0}(t)$, $n = 3$



(b) Frequency, $\widehat{\varphi_{4,+,0}}(\omega)$, $n = 3$



(c) Time, $\varphi_{4,+,0}(t)$, $n = 15$



(d) Frequency, $\widehat{\varphi_{4,+,0}}(\omega)$, $n = 15$

Fig. 2: Real parts of the frame element $\varphi_{j,k,\lambda}(t)$, $\varphi = \mathrm{sinc}^n(t)$, in dimension $d = 1$, both in time and frequency domain. Here, the scale is $2^j, j = 4$. We observe that the frame window in frequency *works* as a characteristic function of $\left[2^j - \frac{1}{2}, 2^{j+1} - \frac{1}{2}\right]$.

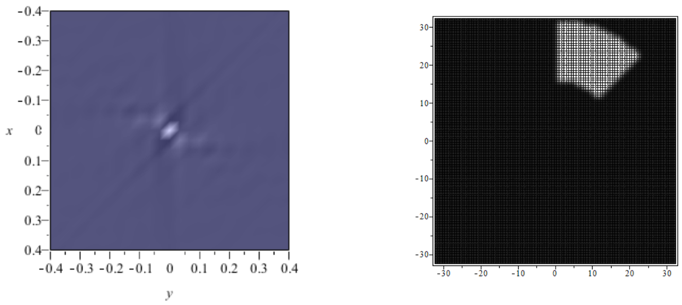


Fig. 3: A frame element obtained using the partition of Figure 1, the scale parameter is $j = 4$ the angular parameter is the same of Figure 1, that is $k = 1$. As localizing window we consider $\varphi(x, y) = \mathrm{sinc}^3(x) \cdot \mathrm{sinc}^3(y)$. On the left is plotted the real part of the frame element and on the right the absolute value of its Fourier transform.

Definition 31 (s-admissibility for H^s-norms, finite case) *Consider an admissible partition $\{I_{j,k}\}_{j,k\in\Gamma} \cup I_\bullet$ and $s \geq 0$; we say that a pair of functions $\varphi_\bullet, \varphi$ is s-admissible with respect to the partition if, given*

$$\Phi_\bullet(\omega) = \widehat{\varphi_\bullet}(\omega), \qquad \Phi_{j,k}(\omega) = \sum_{\eta\in Z_{j,k}} \hat{\varphi}(\omega-\eta), \tag{10}$$

then

i) there exists $\alpha > d/2$

$$|\Phi_\bullet(\omega)| \lesssim \frac{1}{(1+|\omega|)^\alpha}, \quad \text{and} \quad |\Phi_{j,k}(\omega)| \lesssim \begin{cases} \frac{2^{jd/2}}{(1+d(\omega,I_{j,k}))^{\alpha+s}}\,, & \omega \notin I_{j,k} \\ 1\,, & \omega \in \mathbb{R}^d \end{cases}; \tag{11}$$

ii) there exists $a > 0$ such that, for all $\omega \in \mathbb{R}^d$ then

$$|\Phi_\bullet(\omega)| \geq a, \ \text{if}\ \omega \in I_\bullet, \qquad |\Phi_{j,k}(\omega)| \geq a, \ \text{if}\ \omega \in I_{j,k} \tag{12}$$

(the constant a does not depend on j,k).

Remark 2 *The decay hypothesis on $\varphi_\bullet$ implies*

$$\sum_{\lambda\in\nu\mathbb{Z}^d} |\langle T_\lambda\varphi_\bullet, f\rangle|^2 \lesssim \|f\|^2_{L^2(\mathbb{R}^d)}\,,$$

see e.g. [9][Thm 9.2.5 p.206].

We state the main result for our Stockwell-like system. The L^2-case, hereby represented by $s = 0$, has been proved already in [2] under a different and stronger set of hypotheses.

Theorem 32 *Let $s \geq 0$ and consider an s-admissible couple of functions $\varphi_\bullet, \varphi$. Then there exists $\nu_0 > 0$ such that for each $\nu \in (0,\nu_0)$ the system $\mathcal{S}(\varphi_\bullet, \varphi, \Gamma)$ defined in (8) is a frame representing the $H^s(\mathbb{R}^d)$ norm. Precisely, there exist $A, B > 0$ such that for each $f \in H^s(\mathbb{R}^d)$*

$$A\,\|f\|_s^2 \leq \sum_{\lambda\in\nu\mathbb{Z}^d} |\langle f, T_\lambda\varphi_\bullet\rangle|^2 + \sum_{j,k,\lambda\in\Gamma} 2^{2js}\,|\langle f, \varphi_{j,k,\lambda}\rangle|^2 \leq B\,\|f\|_s^2\,.$$

Proof The result follows by Property 1 (upper bound) and Property 2 (lower bound), stated and proved below.

In order to show the boundedness of the coefficients with respect to the Sobolev norm, we introduce the following family of sets

$$E_\bullet = I_\bullet, \quad E_{j,k} = \left\{\omega \in \mathbb{R}^d : \mathrm{d}\,(\omega, I_{j,k}) \leq 2^{j-1}\right\}. \tag{13}$$

Property 1 (Upper Bound) *In the hypothesis of Theorem 32, for any $s \geq 0$ there exist a positive constant C such that*

$$\sum_{j,k,\lambda\in\Gamma} 2^{2js} \left|\langle f, \varphi_{j,k,\lambda}\rangle\right|^2 \leq C \left\|f\right\|_s^2, \tag{14}$$

where C depends on ν, the lattice constant defined in (7).

Proof For each index j set

$$f_{j,k,1}(t) = \mathrm{F}^{-1}\left(\chi_{E_{j,k}(\omega)}\widehat{f}(\omega)\right), \quad f_{j,k,2}(t) = 1 - f_{j,k,1}(t),$$

where $E_{j,k}$ is as in (13). Notice that $E_{j,k} \cap E_{j',k} = \emptyset, \quad \text{if } |j - j'| \geq 2$. By Plancherel Theorem,

$$\begin{aligned}
\sum_{j,k,\lambda\in\Gamma} 2^{2js} \left|\langle f, \varphi_{j,k,\lambda}\rangle\right|^2 &= \sum_{j,k,\lambda\in\Gamma} 2^{2js} \left|\langle \mathrm{F}(f), \mathrm{F}(\varphi_{j,k,\lambda})\rangle\right|^2 \\
&= \sum_{j,k,\lambda\in\Gamma} \left|\langle \mathrm{F}(f)(1+|\omega|)^s, \frac{2^{js}}{(1+|\omega|)^s}\,\mathrm{F}(\varphi_{j,k,\lambda})\rangle\right|^2 \\
&= \sum_{j,k,\lambda\in\Gamma} \left|\langle \mathrm{F}(f)(1+|\omega|)^s, \frac{1}{2^{jd/2}} e^{-2\pi \mathrm{i}\, 2^{-j}\lambda(\cdot)} \varPhi_{j,k}(\cdot)\frac{2^{js}}{(1+|\omega|)^s}\rangle\right|^2 \\
&= \sum_{j,k,\lambda\in\Gamma} \left|\langle \mathrm{F}(f)(1+|\omega|)^s, \mathrm{F}(\widetilde{\varphi_{j,k,\lambda}})\rangle\right|^2, \qquad (15)
\end{aligned}$$

where - cf. Lemma A.4 -

$$\widetilde{\varphi_{j,k,\lambda}}(t) = \frac{1}{2^{jd/2}} T_{2^{-j}\lambda}\,\mathrm{F}^{-1}\left(\frac{2^{js}}{(1+|\omega|)^s}\varPhi_{j,k}(\omega)\right)(t).$$

Splitting $f = f_{j,k,1} + f_{j,k,2}$ for each j we get

$$\begin{aligned}
\sum_{j,k,\lambda\in\Gamma} 2^{2js} \left|\langle f, \varphi_{j,k,\lambda}\rangle\right|^2 \lesssim &\sum_{j,k,\lambda\in\Gamma} \left|\langle \mathrm{F}(f_{j,k,1})(1+|\omega|)^s, \mathrm{F}(\widetilde{\varphi_{j,k,\lambda}})\rangle\right|^2 \\
&+ \sum_{j,k,\lambda\in\Gamma} \left|\langle \mathrm{F}(f_{j,k,2})(1+|\omega|)^s, \mathrm{F}(\widetilde{\varphi_{j,k,\lambda}})\rangle\right|^2.
\end{aligned}$$

Notice that $f_{j,k,1}, f_{j,k,2}$ satisfy the hypothesis of Lemma A.4 and Lemma A.5 respectively, therefore

$$\begin{aligned}
\sum_{j,k,\lambda\in\Gamma} 2^{2js} \left|\langle f, \varphi_{j,k,\lambda}\rangle\right|^2 &\lesssim \frac{1}{\nu}\sum_{j,k}\left(\left\|f_{j,k,1}\right\|_s^2 + 2^{j(d-2\alpha)}\left\|\mathrm{F}\left(f_{j,k,2}\right)(1+|\cdot|)^s\right\|^2\right) \\
&\lesssim \nu^{-1}\|f\|_s^2,
\end{aligned}$$

as desired. Indeed, since the partition $\{I_{j,k}, I_\bullet\}$ is admissible then $E_{j,k}$ have the (uniform) finite intersection property as well, then

$$\sum_{j,k} \|f_{j,k,1}\|_s^2 \le \sum_{j,k} \|f\|^2_{H^s(E_{j,k})} \lesssim \|f\|_s^2 \,,$$

and

$$\sum_{j,k} 2^{j(d-2\alpha)} \left\|\mathrm{F}\left(f_{j,k,2}\right)(1+|\omega|)^s\right\|^2 \le \|f\|_s^2 \sum_{j,k} 2^{j(d-2\alpha)} \lesssim \|f\|_s^2 \,.$$

In the last inequality, we used that the series

$$\sum_{j,k} 2^{j(d-2\alpha)} \tag{16}$$

is convergent, since $d-2\alpha<0$.

Corollary 1 *The analysis operator*

$$\begin{aligned} C : L^2(\mathbb{R}^d) &\longrightarrow \ell^2(\Gamma) \\ f &\longmapsto \{\langle f, \varphi_{j,k,\lambda}\rangle\}_{j,k,\lambda\in\Gamma} \end{aligned}$$

is continuous. Hence, the same is true for the frame operator $S = C^*C$, where C^* is the adjoint operator.

Using the hypothesis on the window functions, we show that there exists a (uniform) lower bound for the H^s-norm.

Property 2 (Lower Bound) *In the hypothesis of Theorem 32, for any $s \ge 0$ there exist $\nu_0 > 0$ and $C > 0$ such that for every $\nu \in (0, \nu_0)$, we have*

$$\sum_{\lambda\in\nu\mathbb{Z}^d} |\langle T_\lambda\varphi_\bullet, f\rangle|^2 + \sum_{j,k,\lambda\in\Gamma} 2^{2js} \left|\langle f, \varphi_{j,k,\lambda}\rangle\right|^2 \ge C \|f\|_s^2 \,, \tag{17}$$

where the constant C depends on ν.

Proof Consider the (modified) frame operator

$$S^s f(x) = S^s_{\varphi_\bullet} f(x) + S^s_\varphi f(x),$$

where

$$S^s_{\varphi_\bullet} f(x) = \sum_\lambda \langle f, \varphi_{\bullet,\lambda}\rangle \varphi_{\bullet,\lambda} \,, \qquad S^s_\varphi f(x) = \sum_{j,k,\lambda\in\Gamma} 2^{2js} \langle f, \varphi_{j,k,\lambda}\rangle \varphi_{j,k,\lambda}(x).$$

We use the representation formula presented in [2, Lemma 4.6], to rewrite

$$\langle S^s f(x), f(x)\rangle = \langle \sum_\sigma \sum_{m\in\mathbb{Z}^d} 4^{\sigma s} T_{\frac{m}{\nu}\beta_\sigma} \left(\overline{\Phi_\sigma \widehat{f}}\right) \Phi_\sigma, \widehat{f}\rangle_{L^2(\mathbb{R}^d)}, \tag{18}$$

where $\sigma \in \{\bullet, (j,k)\}_{j,k\in\Gamma}$ and, with an abuse of notation, we set

$$\beta_\sigma = 2^j, \text{ for } \sigma = (j,k), \qquad \beta_\sigma = 1, \text{ for } \sigma = \bullet.$$

First, for $m = 0$ we apply (50) and obtain

$$\begin{aligned}
&\langle \sum_\sigma 4^{\sigma s} \frac{|\Phi_\sigma|^2}{(1+|\omega|)^{2s}} (1+|\omega|)^s \widehat{f}, (1+|\omega|)^s \widehat{f}\rangle_{L^2(\mathbb{R}^d)} \\
&= \langle \left[\frac{|\Phi_\bullet|^2}{(1+|\omega|)^{2s}} + \sum_{j,k} 4^{\sigma s} \frac{|\Phi_{j,k}|^2}{(1+|\omega|)^{2s}} \right] (1+|\omega|)^s \hat{f}, (1+|\omega|)^s \hat{f}\rangle_{L^2(\mathbb{R}^d)} \\
&\geq C_s a^2 \|f\|_s^2 .
\end{aligned}$$

Hence,

$$\langle S^s f(x), f(x)\rangle \geq a^2 \|f\|_s^2 + \langle \sum_\sigma 4^{\sigma s} \sum_{m\neq 0} T_{\frac{m}{\nu}\beta_\sigma} \left(\overline{\Phi_\sigma \widehat{f}}\right) \Phi_\sigma, \widehat{f}\rangle_{L^2(\mathbb{R}^d)}. \tag{19}$$

We study the last term of the above sum by splitting the different components. Precisely,

$$\langle \sum_\sigma 4^{\sigma s} \sum_{m\neq 0} T_{\frac{m}{\nu}\beta_\sigma} \left(\overline{\Phi_\sigma \widehat{f}}\right) \Phi_\sigma, \widehat{f}\rangle_{L^2(\mathbb{R}^d)} = R_{1,1} + R_{2,1} + R_{1,2} + R_{2,2}$$

where

$$R_{i,j} = \langle \sum_\sigma 4^{\sigma s} \sum_{m\neq 0} T_{\frac{m}{\nu}\beta_\sigma} \left(\overline{\Phi_\sigma \widehat{f_{\sigma,i}}}\right) \Phi_\sigma, \widehat{f_{\sigma,j}}\rangle_{L^2(\mathbb{R}^d)}, \quad i,j = 1,2. \tag{20}$$

and

$$f = f_{\sigma,1} + f_{\sigma,2}, \text{ and } \widehat{f}_{\sigma,1} = \widehat{f}\chi_{E_\sigma},$$

E_σ is defined in (13). We want to show that the terms $R_{1,1}, R_{1,2}, R_{2,1}$ in (20) go to zero as ν does. Precisely, we show that there exists $\nu_0 > 0$ such that for each $\nu < \nu_0$ one has

$$\frac{C_s a^2}{2} \|f\|_s^2 \geq R_{1,1} + R_{2,1} + R_{1,2} . \tag{21}$$

Also, we know that $R_{2,2} \geq 0$. Indeed, we can rewrite $R_{2,2}$ in (20) as

$$\sum_\sigma 2^{2s\sigma} \left| \langle f_{\sigma,2}, \varphi_{\sigma,\lambda}\rangle_{L^2(\mathbb{R}^d)} \right|^2 ,$$

which is quadratic hence positive, as desired.

Step 1 Show that (20), with $i = j = 1$ is identically zero.

Equation (20) vanishes for all $m \neq 0$ if ν is small, since the support of $\widehat{f}_{\sigma,1}$ is

compact.
Indeed, assuming $\nu < \frac{1}{2}$, if $\omega \in E_\sigma$, then

$$\omega - \frac{m}{\nu}\beta_\sigma \notin E_\sigma, \quad m \in \mathbb{Z}^d \setminus \{0\},$$

since the diameter of E_σ is bounded by $2^{\sigma+1}$.
Step 2. Show that the term in (20), with $i = 2, j = 1$ goes to zero as ν does.

We rearrange the sum as

$$\sum_\sigma 4^{s\sigma} \sum_{m\neq 0} \langle \overline{\widehat{f_{\sigma,2}}} \left(\cdot - m\frac{\beta_\sigma}{\nu}\right) \Phi_\sigma, \Phi_\sigma \left(\cdot - m\frac{\beta_\sigma}{\nu}\right) \widehat{f_{\sigma,1}} \rangle_{L^2(\mathbb{R}^d)}.$$

Multiply and dividing by $(1 + |\omega|)^s$, $\left(1 + \left|\omega - m\frac{\beta_\sigma}{\nu}\right|\right)^s$, since $f_{\sigma,1}$ vanishes outside E_σ, we are led to

$$\sum_\sigma 4^{\sigma s} \sum_{m\neq 0} \int_{E_\sigma} \left(1 + \left|\omega - m\frac{\beta_\sigma}{\nu}\right|\right)^s \overline{\widehat{f_{\sigma,2}}} \left(\omega - m\frac{\beta_\sigma}{\nu}\right) \frac{\Phi_\sigma(\omega)}{(1 + |\omega|)^s}$$
$$\frac{\overline{\Phi_\sigma}\left(\omega - m\frac{\beta_\sigma}{\nu}\right)}{\left(1 + \left|\omega - m\frac{\beta_\sigma}{\nu}\right|\right)^s} (1 + |\omega|)^s \overline{\widehat{f_{\sigma,1}(\omega)}} \mathrm{d}\omega.$$

Then, by Cauchy-Schwartz inequality,

$$\langle \sum_\sigma 4^{\sigma s} \sum_{m\neq 0} T_{m\frac{\beta_\sigma}{\nu}} \left(\overline{\Phi_\sigma \widehat{f_{\sigma,2}}}\right) \Phi_\sigma, \widehat{f_{\sigma,1}} \rangle_{L^2(\mathbb{R}^d)} \leq \sum_\sigma \sum_{m\neq 0} c_{\sigma,m}^{1/2} d_{\sigma,m}^{1/2}, \tag{22}$$

where

$$\begin{aligned} c_{\sigma,m} &= \int_{E_\sigma} \left| \overline{\widehat{f_{\sigma,2}}} \left(\omega - m\frac{\beta_\sigma}{\nu}\right)\right|^2 \left(1 + \left|\omega - m\frac{\beta_\sigma}{\nu}\right|\right)^{2s} 4^{\sigma s} \frac{|\Phi_\sigma(\omega)|^2}{(1 + |\omega|)^{2s}} \mathrm{d}\omega \\ &\lesssim \int_{E_\sigma} \left| \overline{\widehat{f_{\sigma,2}}} \left(\omega - m\frac{\beta_\sigma}{\nu}\right)\right|^2 \left(1 + \left|\omega - m\frac{\beta_\sigma}{\nu}\right|\right)^{2s} \mathrm{d}\omega \end{aligned}$$

and

$$\begin{aligned} d_{\sigma,m} &= \int_{E_\sigma} \left|\overline{\widehat{f_{\sigma,1}}}(\omega)\right|^2 (1 + |\omega|)^{2s} 4^{\sigma s} \frac{\left|\Phi_\sigma\left(\omega - m\frac{\beta_\sigma}{\nu}\right)\right|^2}{\left(1 + \left|\omega - m\frac{\beta_\sigma}{\nu}\right|\right)^{2s}} \mathrm{d}\omega \\ &\leq \int_{E_\sigma} \left|\overline{\widehat{f_{\sigma,1}}}(\omega)\right|^2 (1 + |\omega|)^{2s} \frac{|\beta_\sigma|^d}{\left(1 + \mathrm{d}\left(\omega - m\frac{\beta_\sigma}{\nu}, I_\sigma\right)\right)^{2\alpha}} \mathrm{d}\omega \\ &\leq \sup_{\omega\in E_\sigma} \left\{ \frac{|\beta_\sigma|^d}{\left(1 + \mathrm{d}\left(\omega - m\frac{\beta_\sigma}{\nu}, I_\sigma\right)\right)^{2\alpha}} \right\} \int_{E_\sigma} \left|\overline{\widehat{f_{\sigma,1}}}(\omega)\right|^2 (1 + |\omega|)^{2s} \mathrm{d}\omega. \end{aligned}$$

Notice that we have used inequality (38). Since

$$\mathrm{d}\left(\omega - m\frac{\beta_\sigma}{\nu}, I_\sigma\right) \geq \frac{\beta_\sigma}{\nu}\left(|m| - 2\nu\right), \quad \omega \in E_\sigma,$$

if $\nu < \frac{1}{2}$, then $(|m| - 2\nu) > 0$, for any $m \neq 0$. Therefore,

$$d_{\sigma,m} \leq \nu^{2\alpha}\beta_\sigma^{(d-2\alpha)}\frac{1}{\left(|m| - 2\nu\right)^{2\alpha}}\left\|f_{\sigma,1}\right\|_s^2.$$

Hence, using the properties of the scalar product and the norm in the space ℓ^2 with parameters (σ, m), we can write

$$\begin{aligned}\sum_{\sigma,m\neq 0} c_{\sigma,m}^{1/2} d_{\sigma,m}^{1/2} &\lesssim \sum_{\sigma,m\neq 0} c_{\sigma,m}^{1/2}\nu^\alpha\beta_\sigma^{\frac{1}{2}(d-2\alpha)}\frac{1}{\left(|m| - 2\nu\right)^{\alpha}}\left\|f_{\sigma,1}\right\|_s \\ &\lesssim \nu^\alpha\left\|c_{\sigma,m}^{1/2}\beta_\sigma^{\frac{1}{2}(d-2\alpha)}\right\|_{\ell^2}\left\|\frac{1}{\left(|m| - 2\nu\right)^{\alpha}}\left\|f_{\sigma,1}\right\|_s\right\|_{\ell^2}. \qquad (23)\end{aligned}$$

Since $\{I_{j,k}, I_\bullet\}$ is an admissible partition, $\{E_\sigma\}$ is a partition of $\mathbb{R}^d$ as well, with the uniformly finite intersection property, and since $\alpha > d/2$

$$\begin{aligned}\left\|\frac{1}{\left(|m| - 2\nu\right)^{\alpha}}\left\|f_{\sigma,1}\right\|_s\right\|_{\ell^2}^2 &\leq \left(\sum_{m\neq 0}\frac{1}{\left(|m| - 2\nu\right)^{2\alpha}}\right)\sum_\sigma\left\|f_{\sigma,1}\right\|_s^2 \\ &\lesssim \left\|f\right\|_s^2. \qquad (24)\end{aligned}$$

Moreover,

$$\left\|c_{\sigma,m}^{1/2}\beta_\sigma^{\frac{1}{2}(d-2\alpha)}\right\|_{\ell^2}^2 = \sum_\sigma\left(\beta_\sigma^{(d-2\alpha)}\left(\sum_{m\neq 0} c_{\sigma,m}\right)\right) \lesssim \left\|f\right\|_s^2. \qquad (25)$$

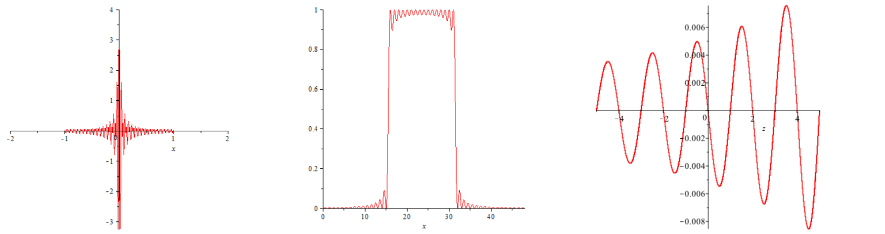
Step 3 Show that the term in (20) with $i = 1, j = 2$ goes to zero as ν does.

This follows from the previous step using a change of variable of integration.

3.2 H^s-seminorm

In Theorem 32 we proved that $F(\varphi_\bullet, \varphi, \Gamma)$ is a frame that describes the H^s-norm, provided the parameter ν is small enough. Going through the proof it is clear that, under the hypothesis of the previous section, it is not possible to describe the H^s-seminorm as well. The problem arises, near the point $\omega = 0$, in the frequency domain. The main reason is that the partition (5) is too coarse near the origin, in particular, the set $I_\bullet$ is too large. Therefore we need sufficiently many vanishing moments in the origin, see [21]. In Figure 4, we show an example of window with such properties.

We first introduce the dilation operator $(\mathrm{D}_a\, f)(x) = a^{-d/2} f\left(\frac{x}{a}\right), \quad a \in \mathbb{R} \setminus$



(a) Time domain (b) Frequency domain (c) Frequency domain, zoom.

Fig. 4: Frame element $\varphi_{j,k,\lambda}(t)$ in both time and frequency. Here, φ is the convolution between $\chi_{[-\frac{1}{2},\frac{1}{2}]}$ and itself, the scale is $j = 4$. Figure (c) shows a zoom in the origin and the vanishing moments of the window.

$\{0\}$, which is unitary on $L^2(\mathbb{R}^d)$. Let $\{I_{j,k} \cup I_\bullet\}$ be an admissible partition in the sense of Definition 21, we consider a new set of indices (compared to Γ in (7)), defined as follows:

$$\Sigma = \left\{(j,k,\lambda) \mid j \in \mathbb{Z}, k \in K_j, \lambda \in \nu\mathbb{Z}^d\right\},$$

and the sets

$$\begin{aligned} I_{-j,k} &= \left\{x \in \mathbb{R}^d \mid 2^{2j}x \in I_{j,k}\right\}, \qquad && j \in \mathbb{N} \setminus \{0\}, \quad k \in K_j, \\ Z_{-j,k} &= Z_{j,k}, && j \in \mathbb{N} \setminus \{0\}, \quad k \in K_j. \end{aligned} \tag{26}$$

Remark 3 *These new sets are just contractions of the original ones. We define these sets in order to refine the analysis on $I_\bullet$. The union of these covers $I_\bullet$ entirely, except the point $\{0\}$ which has zero measure.*

For each φ, we define the new frame

$$\begin{aligned} \varphi_{j,k,\lambda}(x) &= \varphi_{j,k,\lambda}(x), \quad j \in \mathbb{N}, k \in K_j, \\ \varphi_{-j,k,\lambda}(x) &= D_{2^{2j}}\varphi_{j,k,\lambda}(x) = \mathrm{D}_{2^{2j}}\, T_{2^{-j}\lambda}\varphi_{j,k,0}(t), \quad j \in \mathbb{N} \setminus \{0\}, \ k \in K_j. \end{aligned} \tag{27}$$

Recalling (10), the natural expression of $\Phi_{-j,k}(\omega)$ for $j \in \mathbb{N}\setminus\{0\}$ is: $\Phi_{-j,k}(\omega) = \Phi_{j,k}(2^{2j}\omega)$. Finally, we define the Stockwell-like system for the seminorm as

$$\mathcal{G}(\varphi, \Sigma) = \{\varphi_{j,k,\lambda},\ j,k,\lambda \in \Sigma\}. \tag{28}$$

We introduce now the admissibility criteria for the seminorm characterisation.

Definition 33 (s-admissibility for H^s-seminorms, finite case) *Let $s \geq 0$, we say that φ is s-admissible for the Sobolev seminorm with respect to the partition $\{I_{j,k}\}_{j\in\mathbb{Z},k\in K_j}$ if* *i) there exists* $\alpha > d/2$ *such that for all* $j \in \mathbb{N}$

$$|\Phi_{j,k}(\omega)| \lesssim \begin{cases} \frac{2^{jd/2}\min(1,|\omega|)^s}{(1+d(\omega,I_{j,k}))^{\alpha+s}}\,, & \omega \notin I_{j,k} \\ 1\,, & \omega \in \mathbb{R}^d\,; \end{cases} \tag{29}$$

ii) there exists a constant $a > 0$ *such that*

$$|\Phi_{j,k}(\omega)| \geq a, \quad \omega \in I_{j,k}.$$

(the constant a *does not depend on* j, k*).*

We state the counterpart of Theorem 32 for seminorms.

Theorem 34 *Let* $s \geq 0$ *and consider an* s*-admissible couple of functions* $\varphi_\bullet, \varphi$*. Then there exists* $\nu_0 > 0$ *such that for each* $\nu \in (0, \nu_0)$*, the system* $\mathcal{G}(\varphi, \Sigma)$ *defined in* (28) *is a frame representing the* $H^s(\mathbb{R}^d)$ *seminorm. That is, there exist constants* $A, B > 0$ *such that*

$$A\,|f|_s^2 \leq \sum_{j,k,\lambda\in\Sigma} 2^{2js}\,|\langle f, \varphi_{j,k,\lambda}\rangle|^2 \leq B\,|f|_s^2\,. \tag{30}$$

Proof The proof is a consequence of the upper bound and of the lower bound as in Theorem 32. We avoid an explicit proof here. Nonetheless, in the appendix, we provide the counterparts of the properties used in the first part so that the entire proof can be reproduced.

3.3 Characterisation of Besov spaces

We discuss here an important consequence of Theorem 34; precisely we show how to characterise Besov spaces of type $B^s_{2,q}$ with $s \in (0, 1]$ and $q > 1$, using (real) interpolation techniques. We recall that a function f belongs to the Besov space $B^s_{2,q}(\mathbb{R}^d) \subseteq L^2(\mathbb{R}^d)$, if the following seminorm is finite

$$|f|_{B^s_{2,q}} = \left(\int_0^{+\infty} [t^{-s}\omega_2^{(r)}(f,t)]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}},$$

with $r \in \mathbb{N}, r > s$, where

$$\omega_2^{(r)}(f,t) = \sup_{|h|<t} \|\Delta_h^r f(\cdot)\|_2\,,$$

and Δ_h^r is the difference of order r and step h

$$\Delta_h^r f(x) = \sum_{j=1}^{r}(-1)^{r+j}\binom{r}{j} f(x+jh).$$

A norm can be defined as follows

$$\|f\|_{B^s_{2,q}} = \|f\|_2 + |f|_{B^s_{2,q}}\,.$$

Theorem 35 *Consider the Stockwell frame* $\{\varphi_{j,k,\lambda}\}_{j,k,\lambda\in\Sigma}$ *as defined in Section 3.2 and let* $s\in(0,1)\,,q>1$*; then for any* $v\in B^s_{2,q}$ *one has*

$$|v|_{B^s_{2,q}} \asymp \left(\sum_{j\in\mathbb{Z}} 2^{jsq}\left(\sum_{k,\lambda\in\Delta_j}|\langle\varphi_{j,k,\lambda},v\rangle|^2\right)^{q/2}\right)^{1/q}.$$

Proof The result follows from Theorem B.1 and the fact that for $\alpha\in(0,1)$ and $q\in(1,+\infty]$,

$$B^s_{2,q}(\mathbb{R})=(L^2(\mathbb{R}),H^1(\mathbb{R}))_{s,q}$$

(see e.g. [19]).

3.4 Sufficient conditions for Sobolev-like frames

The definition of s-admissible window function involves properties of $\Phi_{j,k}$ rather than of the window function φ. In general, it is very difficult to check the properties of $\Phi_{j,k}$ since it is a sum of translations (recal Definition 31). Nevertheless, it is possible to provide sufficient conditions on φ which guarantee the s-admissibility. In the following we will always consider $\varphi=\varphi_\bullet$.

Theorem 36 *Let* φ *be a function such that*

$$|\widehat{\varphi}(\omega)|\lesssim\frac{1}{(1+\omega)^{s+d+\varepsilon}},\quad \varepsilon>0, \tag{31}$$

for a certain $s\in\mathbb{R}$ *and such that* $\widehat{\varphi}$ *has definite sign and* $|\widehat{\varphi}(\omega)|\geq a$*, for* $\omega\in I_\bullet$ *then the pair* (φ,φ) *is s-admissible for* H^s*-norm.*

Proof Condition (31) implies that $\widehat{\varphi}$ belongs to $L^1(\mathbb{R}^d)$, moreover

$$\begin{aligned}|\Phi_{j,k}(\omega)| &\leq \sum_{\eta\in Z_{j,k}}|\widehat{\varphi}(\omega-\eta)|\lesssim\sum_{\eta\in Z_{j,k}}\frac{1}{(1+|\omega-\eta|)^{s+d+\varepsilon}}\\ &\lesssim\int\frac{1}{(1+|\omega|)^{d+\varepsilon}}d\omega\lesssim 1.\end{aligned}$$

Therefore, $|\Phi_{j,k}(\omega)|$ is uniformly bounded.

In order to prove the decay property, notice again that the issue is to have a uniform bound with respect to j. Since $\Phi_{j,k}(\omega)$ is uniformly bounded, we can prove the inequality when $d(\omega,I_{j,k})=\varepsilon>0$. Under this hypothesis, recalling that, $|Z_{j,k}|\asymp 2^{jd}$ we can write

$$\begin{aligned}|\Phi_{j,k}(\omega)| &\leq \sum_{\eta\in Z_{j,k}}|\widehat{\varphi}(\omega-\eta)|\lesssim\sum_{\eta\in Z_{j,k}}\frac{1}{(1+|\omega-\eta|)^{s+d+\varepsilon}}\\ &\lesssim\sum_{|r|<2^j}\frac{1}{(1+\varepsilon+|r|)^{s+d+\varepsilon}}\end{aligned}$$

If $\varepsilon < 2^{j-1}$, we notice that this sum may be bounded by the corresponding integral and, after a change of variable,

$$|\Phi_{j,k}(\omega)| \lesssim \frac{1}{(1+\varepsilon)^{s+d+\varepsilon}} \int_{|x|<2^j} \frac{1}{\left(1+\frac{x}{1+\varepsilon}\right)^{s+d/2+\varepsilon}}$$
$$\lesssim \frac{(1+\varepsilon)^d}{(1+\varepsilon)^{s+d+\varepsilon}} = \frac{(1+\varepsilon)^{d/2}}{(1+\varepsilon)^{s+d/2+\varepsilon}}$$

due to the convergence of the integral on the whole $\mathbb{R}^d$ space. Thus, upon noticing that $(1+\varepsilon)^{d/2} \lesssim 2^{jd/2}$, one gets

$$|\Phi_{j,k}(\omega)| \lesssim \frac{2^{jd/2}}{(1+\varepsilon)^{s+d/2+\varepsilon}} \leq \frac{2^{jd/2}}{(1+d(\omega, I_{j,k}))^{s+d/2+\varepsilon}}$$

as desired.

On the contrary, if $\varepsilon > 2^{j-1}$, then

$$|\Phi_{j,k}(\omega)| \lesssim \sum_{|r|<2^j} \frac{1}{(1+\varepsilon+|r|)^{s+d+\varepsilon}} \leq \frac{2^{jd}}{(1+\varepsilon)^{s+d+\varepsilon}} \lesssim \frac{2^{jd/2}}{(1+d(\omega, I_{j,k}))^{s+d/2+\varepsilon}},$$

since

$$\frac{2^{jd/2}}{(1+d(\omega, I_{j,k}))^{d/2}} \lesssim 1.$$

To conclude, we need to show that $\Phi_{j,k}$ satisfies condition (12), since for $\Phi_\bullet$ there is nothing to prove. Notice that, by hypothesis,

$$|\Phi_{j,k}(\omega)| = \sum_{\eta \in Z_{j,k}} |\widehat{\varphi}(\omega-\eta)|$$

and that for all $\omega \in I_{j,k}$ there exists $\bar{\eta} \in Z_{j,k}$ such that $\omega - \bar{\eta} \in I_\bullet$, therefore

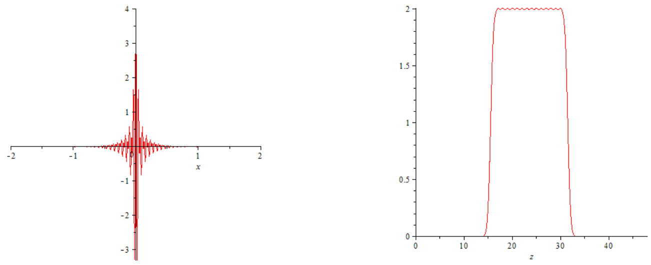
$$|\Phi_{j,k}(\omega)| = \sum_{\eta \in Z_{j,k}} |\varphi(\omega-\eta)| \geq |\varphi(\omega-\bar{\eta})| \geq a.$$

The easiest example of a function satisfying the conditions of Theorem 36 is the Gaussian, see Figure 5.

We can obtain similar sufficient conditions for the seminorm by adding a vanishing condition on non-zero integers. Indeed, summing integer translations of window functions will generate vanishing moments at the origin as required by the s-admissibility condition. We show an example using sinc$(\cdot)$ in Subsection 4.2 below.

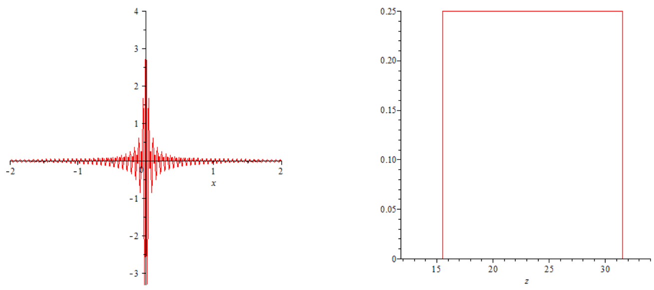
4 Explicit examples of H^s-frames

In this section we provide some explicit examples of frames which discretise the H^s-norm and seminorm.



(a) Time domain, $j = 4$ (b) Frequency domain, $j = 4$

Fig. 5: Time and frequency outlook of a frame element with normalised Gaussian window. On the right the real part in the time domain, on the left the absolute value in the frequency domain.



(a) Time domain (b) Frequency domain

Fig. 6: Frame element $\varphi_{j,k,\lambda}(t)$ in both time and frequency, $\varphi = \operatorname{sinc}(t)$ and $j = 4$.

4.1 Shannon-like basis

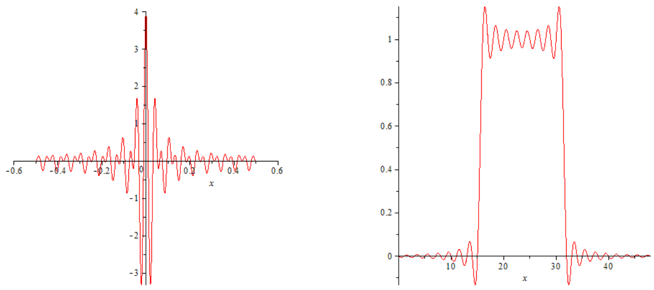
Let us consider $\varphi_\bullet(t) = \varphi(t) = \operatorname{sinc}(t) = \left(\mathrm{F}^{-1}\chi_{\left(-\frac{1}{2},\frac{1}{2}\right)}\right)(t)$ and the partition introduced in (5); see Figure 6. Since the characteristic function has compact support, the couple $\varphi_\bullet, \varphi$ is trivially s-admissible for all $s \geq 0$ both for the Sobolev norm and seminorm, hence a frame for ν small enough.

This example is indeed very peculiar. This system is not only a frame for ν small enough, but it is an orthonormal basis of $L^2(\mathbb{R})$ if we set $\nu = 1$. Indeed, notice that, using the notations of Section 2

$$\Phi_\bullet(\omega) = \chi_{(-\frac{1}{2},\frac{1}{2})}(\omega), \quad \Phi_{j,\pm}(\omega) = \sum_{\eta=\pm 2^j}^{\pm(2^j-1)} \chi_{\left(-\frac{1}{2},\frac{1}{2}\right)}(\omega - \eta) = \chi_{I_{j,\pm}}(\omega).$$

Therefore, if $j \neq j'$ or $k \neq k'$, by Plancherel Theorem

$$\langle \varphi_{j,k,\lambda}, \varphi_{j',k',\lambda'} \rangle = \langle \frac{1}{2^{j/2}} e^{-2\pi \mathrm{i}\cdot\lambda/2^j} \chi_{I_{j,k}}(\cdot), \frac{1}{2^{j'/2}} e^{-2\pi \mathrm{i}\cdot\lambda'/2^{j'}} \chi_{I_{j',k'}}(\cdot) \rangle = 0$$



(a) Time domain

(b) Frequency domain

Fig. 7: Frame element $\varphi_{j,k,\lambda}(t)$ in both time and frequency. Here, $\varphi = \chi_{[-\frac{1}{2},\frac{1}{2}]}$ and $j = 4$.

for all $\lambda, \lambda' \in \mathbb{Z}$. Moreover if $j = j'$ and $k = k'$, by well known properties of Fourier series,

$$\langle \varphi_{j,k,\lambda}, \varphi_{j,k,\lambda'} \rangle = \langle \frac{1}{2^{j/2}} e^{-2\pi \mathrm{i} \cdot \lambda/2^j} \chi_{I_{j,k}}(\cdot), \frac{1}{2^{j/2}} e^{-2\pi \mathrm{i} \cdot \lambda'/2^j} \chi_{I_{j,k}}(\cdot) \rangle = \delta_0,$$

where δ_0 is the Dirac delta. The same holds for $\varphi_{\bullet,\lambda}$ which is also orthogonal to $\varphi_{j,k,\lambda}$ for all j, k, λ.

Remark 4 *The orthonormal basis system we introduced is strictly related to the Shannon basis. As expected the localisation properties in the time domain of this frame are not so strong, due to the lack of such properties of the* sinc *function. Nevertheless, in our setting, we can gain localisation considering powers of* sinc*. That is, we can consider $\varphi(t) = \operatorname{sinc}(t)^n$ and this new window has increasing localisation as n increases, moreover it is always s-admissible since its Fourier transform has compact support (see Figure 2).*

4.2 Haar-like frame

In this subsection we describe in detail an example in dimension $d = 1$ which shows that the conditions of Theorem 36 are sufficient, but not necessary.

Let us consider

$$\varphi(t) = \chi_{(-\frac{1}{2},\frac{1}{2})}(t) = \left(\mathrm{F}^{-1} \operatorname{sinc}(\cdot)\right)(t) = \left(\mathrm{F}^{-1} \frac{\sin(\pi\cdot)}{\pi\cdot}\right)(t). \tag{32}$$

It is clear that φ defined in (32) does not satisfy the sufficient conditions (31). Nevertheless, it will provide a frame. See Figure 7 for the plots of the frame with this particular window function. One can prove that the pair (φ, φ) is s-admissible for the Sobolev seminorm for each $s \in [0, 1)$.

It is enough to prove the decay property for $j \geq 0$.

Let us suppose $k = +$, the case $k = -$ is equivalent. By definition

$$\Phi_{j,+}(\omega) = \sum_{\eta=2^j}^{2^{j+1}-1} \mathrm{sinc}(\omega - \eta) = \sum_{\eta=2^j}^{2^{j+1}-1} \frac{\sin(\pi(\omega-\eta))}{\pi(\omega-\eta)}.$$

If $\omega \in I_{j,+}$ then, by construction, there exists $\bar{\eta} \in Z_{j,+}$ such that $|\omega - \bar{\eta}| \le \frac{1}{2}$, hence

$$\left| \frac{\sin(\pi(\omega-\bar{\eta}))}{\pi(\omega-\bar{\eta})} \right| \le 1.$$

We can write, using trigonometric inequalities,

$$\begin{aligned} |\Phi_{j,+}(\omega)| &= \left| \sum_{\eta=2^j}^{2^{j+1}-1} \frac{\sin(\pi(\omega-\eta))}{\pi(\omega-\eta)} \right| \\ &\le \left| \sum_{m=1}^{\bar{\eta}-2^j} \frac{(-1)^m}{\pi(\omega-\eta+m)} \right| + \left| \sum_{m=1}^{2^{j+1}-\bar{\eta}} \frac{(-1)^m}{\pi(\omega-\eta-m)} \right| + 1. \end{aligned}$$

Since the alternate harmonic series is convergent, we can obtain a uniform bound with respect to j for $\Phi_{j,+}(\omega)$, if $\omega \in I_{j,+}$. Notice that, if $\omega \notin I_{j,+}$ the above inequality still holds; actually one could improve the bound, but this is not important for our purpose. In order to prove (29), in view of the uniform bound of $\Phi_{j,+}$ we can suppose $d(w, I_{j,+}) > 2^{j-1}$, as shown above. Therefore

$$\begin{aligned} |\Phi_{j,+}(\omega)| &= \left| \sum_{m=0}^{2^j-1} \frac{\sin(\pi\omega)(-1)^m}{\pi(\omega-m-2^j)} \right| \\ &\le \left| \sin(\pi\omega) \sum_{m=0}^{2^{j-1}-1} \frac{1}{\pi(\omega-m-2^j)(\omega-m-1-2^j)} \right| \\ &\le 2^j \left| \sin(\pi\omega) \frac{1}{\pi d(\omega, I_{j,+})^2} \right| \lesssim 2^{j/2} \left| \sin(\pi\omega) \frac{1}{\pi d(\omega, I_{j,+})^{\frac{3}{2}}} \right|, \end{aligned}$$

which implies (29) for each $s \in [0, 1)$.
With the same notation as above, notice that $\left| \frac{\sin(\pi(\omega-\bar{\eta}))}{\pi(\omega-\bar{\eta})} \right| \ge \frac{2}{\pi}$. Hence

$$\begin{aligned} |\Phi_{j,+}(\omega)| &\ge \frac{2}{\pi} - \left| \sin(\pi(\omega-\bar{\eta})) \left(\sum_{m=1}^{\bar{\eta}-2^j} \frac{(-1)^m}{\pi(\omega-\eta+m)} + \sum_{m=1}^{2^{j+1}-\bar{\eta}} \frac{(-1)^m}{\pi(\omega-\eta-m)} \right) \right| \\ &\ge \frac{2}{\pi} - \frac{1}{2} > 0. \end{aligned}$$

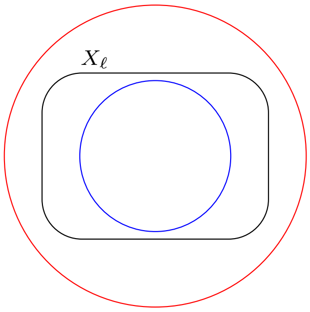


Fig. 8: A simple example of almost-isotropic set. In blue and red the inner and outer ball, respectively; in black the set X_ℓ.

Remark 5 *This example is closely related to the standard Haar basis. It is well known that the Haar basis is suited to represent Sobolev spaces $H^s(\mathbb{R})$ with $s < \frac{1}{2}$, essentially due to the lack of continuity in the time domain which is related to the slow decay at infinity of the function* sinc. *With our approach we can get rid of this problem and reach all $s < 1$, the main reason is that performing sums instead of dilation we are able to exploit the oscillation behavior of the* sinc *function and increase the decay rate of the functions $\Phi_{j,k}$.*

5 Extension to a wider class of partitions

In this section we discuss the extension of the Stockwell-Like frame defined in Section 3 to more general partitions. Specifically, so far we only considered a fixed amount of direction, while here we relax that requirement.

We start with a preliminary definition.

Definition 51 *Let $\{X_\ell\}_{\ell \in L}$, be a family of subsets of $\mathbb{R}^d$ with L index sets at most countable. Let us denote by v_ℓ the volume of each X_ℓ and define two constants c_1, c_2 such that $0 < c_1 < c_2$. Then $\{X_\ell\}_{\ell \in L}$ is an almost-isotropic family of sets if for each $\ell \in L$ there exist a point $x_\ell \in X_\ell$ such that*

$$B_{c_1 v_\ell^{1/d}}(x_\ell) \subseteq X_\ell \subseteq B_{c_2 v_\ell^{1/d}}(x_\ell),$$

uniformly in ℓ, where $B_r(x)$ is the ball of radius r and center x.

See Figure 8 for a simple example of isotropic set.

Remark 6 *With this definition, we guarantee that every point on the boundary of X_ℓ has a comparable distance from the centroid of the set itself, and this distance is roughly $v_\ell^{1/d}$ modulo uniform constants. This requirement is crucial in order to prove technical results such as the generalization of Lemma A.4 and Lemma A.5.*

We now state here a general definition and later give some examples of admissible partitions with growing number of frequency directions.

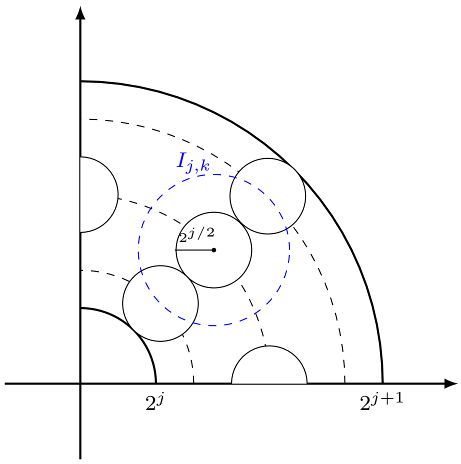


Fig. 9: Isotropic tailing of the frequency plane related to isotropic frames.

Definition 52 *The family* $\{I_{j,k}\}_{j\in\mathbb{N},k\in K_j} \cup I_\bullet$*, where* K_j *is an index set, is an admissible partition if it satisfies i)-vi) of Definition 21 and moreover:*

vii) there exists δ*, with* $0 \leq \delta \leq \min(1, d-1)$*, such that* $|K_j| \asymp 2^{|j|\delta(d-1)}$*;*
viii) $\{I_{j,k}\}_{j,k}$ *is a family of almost isotropic sets, see Definition 51 below.*

Let us give an example of such partitions which is similar to the one presented in [4]. For $j \geq 0$ we let $\{c_{j,k} : j > 0, k \leq n_j\} \subseteq \mathbb{R}^d, d > 1$, for some $n_j > 2$, be a set of points such that:

1. the elements of the family $\{B_{2^{j/2}}(c_{j,k}), k = 1, \ldots, n_j\}$ are pairwise disjoint sets and are contained in the corona $C_j = B_{2^{j+1}}(0) \setminus B_{2^j}(0)$;
2. $C_j \subseteq \bigcup_{k=1}^{n_j} B_{2^{(j+1)/2}}(c_{j,k})$.

We set $I_{j,k} = B_{2^{(j+1)/2}}(c_{j,k})$, see Figure 9. Then, from a volume estimate, it is easy to see that the number of sets n_j inside each corona is $n_j \asymp 2^{jd/2}$. This is clearly an admissible partition which satisfies the requirements with $\delta = \frac{d}{2(d-1)}$.

5.1 Frame generalisation

Let $Z_{j,k} = I_{j,k} \cap \mathbb{Z}^d$ and let $\beta_{j,k} \in \mathbb{R}^d$ be the diameter of $I_{j,k}$. Then the Stockwell system depending on the parameters $(j, k, \lambda) \in \Gamma$, $x \in \mathbb{R}^d$ is defined to be

$$\varphi_{\bullet,\lambda}(x) = \varphi_\bullet(x-\lambda),\ \varphi_{j,k,\lambda}(x) = T_{\beta_{j,k}^{-1}\lambda}\left(\frac{1}{|\beta_{j,k}|^{d/2}} \sum_{\eta\in Z_{j,k}} e^{2\pi \mathrm{i}\, \eta\cdot x}\varphi(x)\right),$$

which can be written in compact form as

$$\mathcal{S}(\varphi_\bullet,\varphi,\Gamma)=\left\{\varphi_{\bullet,\lambda},\lambda\in\nu\mathbb{Z}^d\right\}\cup\{\varphi_{j,k,\lambda},\ j,k,\lambda\in\Gamma\}\,, \tag{33}$$

with

$$\begin{aligned}\widehat{\varphi_{\bullet,\lambda}}(\omega)&=e^{-2\pi\mathrm{i}\,\omega\cdot\lambda}\widehat{\varphi_\bullet}(\omega),\\ \widehat{\varphi_{j,k,\lambda}}(\omega)&=e^{-2\pi\mathrm{i}\,\omega\cdot\beta_{j,k}^{-1}\lambda}\left(\frac{1}{|\beta_{j,k}|^{d/2}}\sum_{\eta\in Z_{j,k}}\widehat{\varphi}(\omega-\eta)\right).\end{aligned} \tag{34}$$

Remark 7 *Consider a scale j, we want to determine the number of points with integer coordinates contained in $I_{j,k}$, i.e. $|Z_{j,k}|$. The number of such points contained in the union $\bigcup_{k\in K_j} I_{j,k}$ is approximately $|K_j|\,|I_{j,k}|\asymp 2^{jd}$. Thus*

$$|Z_{j,k}|\asymp\frac{1}{|K_j|}2^{dj}\asymp 2^{jd-j\delta(d-1)}=2^{jd(1-\delta)+j\delta}.$$

Definition 31 generalises to the following one.

Definition 53 (s-admissibility for H^s-norms - infinite case) *Consider an admissible partition $\{I_{j,k}\}_{j,k\in\Gamma}\cup I_\bullet$ and $s\geq 0$, we say that a pair of functions $\varphi_\bullet,\varphi$ is s-admissible with respect to the partition if, given $\Phi_\bullet,\Phi_{j,k}$ as in* (10), *then*

i) there exists $\alpha>\frac{d}{2(1-\delta\frac{d-1}{d})}$

$$|\Phi_\bullet(\omega)|\lesssim\frac{1}{(1+|\omega|)^\alpha},\qquad and\qquad |\Phi_{j,k}(\omega)|\lesssim\begin{cases}\frac{(\beta_{j,k})^{d/2}}{(1+d(\omega,I_{j,k}))^{\alpha+s}}\,, & \omega\notin I_{j,k}\\ 1\,, & \omega\in\mathbb{R}^d\end{cases}; \tag{35}$$

ii) there exists $a>0$ such that, for all $\omega\in\mathbb{R}^d$ then

$$|\Phi_\bullet(\omega)|\geq a,\ \text{if}\ \omega\in I_\bullet,\qquad |\Phi_{j,k}(\omega)|\geq a,\ \text{if}\ \omega\in I_{j,k} \tag{36}$$

(the constant a does not depend on j,k).

Remark 8 *Notice that the decay rate increases with the number of directions allowed, in particular, we ask $\alpha>d/2$ for $\delta=0$, $\alpha>\frac{2d^2}{d+1}$ for $\delta=\frac{1}{2}$ and $\alpha>d^2/$ if $\delta=1$.*

With these new definitions, we can generalise Theorem 32.

Theorem 54 *Let $s\geq 0$ and consider an s-admissible set of functions $\varphi_\bullet,\varphi$ - cf. Definition 53. Then there exists $\nu_0>0$ such that for each $\nu\in(0,\nu_0)$ the system $\mathcal{S}\left(\varphi_\bullet,\varphi,\Gamma\right)$ defined in* (33) *is a frame representing the $H^s(\mathbb{R}^d)$ norm. Precisely, there exist $A,B>0$ such that for each $f\in H^s(\mathbb{R}^d)$*

$$A\,\|f\|_s^2\leq\sum_{\lambda\in\nu\mathbb{Z}^d}|\langle f,T_\lambda\varphi_\bullet\rangle|^2+\sum_{j,k,\lambda\in\Gamma}2^{2js}\,|\langle f,\varphi_{j,k,\lambda}\rangle|^2\leq B\,\|f\|_s^2\,.$$

Proof The proof of this result follows the one of Theorem 32, since all the technical results extend to the general definition of isotropic partition and s-admissibility. The main difference, which is also the key point of the proof, is to show that the summability of the series in (16) is preserved. This can be shown as follows.

Since $\beta_{j,k}^d = |Z_{j,k}| \lesssim \frac{2^{jd}}{|K_j|}$, then

$$\sum_{j,k} \beta_{j,k}^{d-2\alpha} \lesssim \sum_j |K_j| \beta_{j,k}^{d-2\alpha} \lesssim \sum_j 2^{jd} \left(\beta_{j,k}\right)^{-2\alpha}.$$

Notice that, making δ explicit

$$\beta_{j,k} \asymp 2^{j-\delta j \frac{d-1}{d}}, \qquad \left(\beta_{j,k}\right)^{2\alpha} \asymp 2^{2\alpha j (1-\delta \frac{d-1}{d})}.$$

Finally, the sum converges if

$$2\alpha(1-\delta\frac{d-1}{d}) > d, \qquad \text{i.e. } \alpha > \frac{d}{2(1-\delta\frac{d-1}{d})},$$

as assumed.

Similarly, one can extend the case of the seminorm using the same pattern.

6 Conclusion

In this paper we focused on the Sobolev properties of the Stockwell-like frame in arbitrary dimension obtaining a characterisation of these spaces. Although this is a standard property that many other types of frames share, we believe that the extreme flexibility of this frame opens interesting research paths. Specifically, this frame can be adapted to a variety of different partitions. Hence, we claim that our approach can be used to describe decomposition spaces associated to different partitions of the frequency plane.

It is clear from the result on the Sobolev seminorms, that this frame is very close to the wavelet one. With our frame, it is possible to construct a structure close to the wavelet Multi Resolution Analysis and use that to represent interpolation spaces. As an example, we showed the characterisation of Besov spaces.

In Section 4 we proposed several examples of frame windows; in particular, we considered the powers of the sinc function, that gives arbitrary high regularity while providing great localisation in space and being just a perturbation of the characteristic function of the frequency band in the Fourier domain.

The example of the Haar-like Stockwell frame presented in Section 4, shows that even if we start with the same characteristic window as in the Haar wavelets, we can reach higher regularity.

In Section 5, we provided a generalisation to a frequency tiling that allows isotropic sets with a (parabolic) number of frequency directions paired with the scale j. It is clear that the machinery presented here is not tailored to treat anisotropic partitions; this is a possible topic for future researches.

Among other interesting open problems on this frame, it would be challenging to investigate the density of the Stockwell frame with Gaussian window and compare it with well known results on Gabor frames.

Concerning numerics, we implemented a new algorithm for Stockwell-like frames. We aim to test this on numeric problems and compare it to other existing frames. The code is available for tests and academic use upon request to the authors.

One of the possible applications of our frame is medical imaging, since the Stockwell transform is widely used in that research area, see [3],[20] for recent results on this matter.

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A Technical Results

We show some technical result needed to prove the frame property.

Lemma A.1 *Let $s \geq 0$ and φ such that $\Phi_{j,k}$ - cf.* (10) *- satisfies*

$$|\Phi_{j,k}(\omega)| \lesssim \min\left(1, \frac{2^{jd/2}}{\big(1+d(\omega, I_{j,k})\big)^{\alpha+s}}\right), \tag{37}$$

for some α. Then

$$\frac{2^{js}}{(1+|\omega|)^s}|\Phi_{j,k}(\omega)| \lesssim \min\left(1, \frac{2^{jd/2}}{\big(1+d(\omega, I_{j,k})\big)^{\alpha}}\right). \tag{38}$$

Proof Inequality (38) is trivially verified when $\omega \in I_{j,k}$ since $\omega \asymp 2^j$.

Assume $\omega \notin I_{j,k}$, then we immediately notice that by hypothesis (37)

$$\frac{2^{js}}{(1+|\omega|)^s}\left|\Phi_{j,k}(\omega)\right| \lesssim \frac{2^{js}}{(1+|\omega|)^s\big(1+d(\omega, I_{j,k})\big)^s}\frac{2^{jd/2}}{\big(1+d(\omega, I_{j,k})\big)^{\alpha}}.$$

Now, there exists $\overline{\omega} \in I_{j,k}$ such that $d(\omega, I_{j,k}) = |\omega - \overline{\omega}|$, moreover $|\overline{\omega}| \asymp 2^j$ because the partition is admissible. Hence, by triangular inequality

$$\begin{aligned}\frac{2^{js}}{(1+|\omega|)^s}\left|\Phi_{j,k}(\omega)\right| &\lesssim \frac{2^{js}}{(1+|\omega|)^s(1+|\overline{\omega}-\omega|)^s}\frac{2^{jd/2}}{\big(1+d(\omega, I_{j,k})\big)^{\alpha}}\\ &\lesssim \frac{2^{js}}{(1+|\overline{\omega}|)^s}\frac{2^{jd/2}}{\big(1+d(\omega, I_{j,k})\big)^{\alpha}} \lesssim \frac{2^{jd/2}}{\big(1+d(\omega, I_{j,k})\big)^{\alpha}}.\end{aligned}$$

With the same argument, one can show the following result.

Lemma A.2 *Let $s \geq 0$ and φ such that for all $j \in \mathbb{N}$, $\Phi_{j,k}$ - cf.* (10) *- satisfies*

$$|\Phi_{j,k}(\omega)| \lesssim \begin{cases} \frac{\min(1,|\omega|)^s 2^{jd/2}}{\left(1+d(\omega,I_j)\right)^{\alpha+s}}, & \omega \notin I_{j,k} \\ 1, & \omega \in \mathbb{R}^d \end{cases}, \tag{39}$$

for some $\alpha > 0$. Then

$$\frac{2^{js}}{|\omega|^s}|\Phi_{j,k}(\omega)| \lesssim \min\left(1, \frac{2^{jd/2}}{\left(1+d(\omega,I_{j,k})\right)^{\alpha}}\right) \tag{40}$$

for all $j \in \mathbb{N}$.

As a consequence of Lemma A.1, we have the following result.

Lemma A.3 *Assume the hypothesis* (37) *of Lemma A.1 above and also that $\alpha > d/2$. Then, there exists $b_s \in \mathbb{R}$ such that*

$$\sum_{j,k\in\Gamma} \frac{2^{2js}}{(1+|\omega|)^{2s}}|\Phi_{j,k}(\omega)|^2 \leq b_s, \quad a.e.\ \omega \in \mathbb{R}^d. \tag{41}$$

Proof Given $\omega \in \mathbb{R}^d$, since the partition is admissible, there exists a finite collection of indices $\Gamma_F, |\Gamma_F| < W_1$ such that $\omega \in I_{j,k}, \{j,k\} \in \Gamma_F$. Assume for the moment $W_1 = 1$ and $\omega \in I_{\overline{j},\overline{k}}$. First, we want to estimate

$$\sum_{k\in K_j} \frac{2^{2js}}{(1+|\omega|)^{2s}}|\Phi_{j,k}(\omega)|^2,$$

for $j = \overline{j}$. Clearly, $\omega \asymp 2^{\overline{j}}$, hence $\frac{2^{2js}}{(1+|\omega|)^{2s}}|\Phi_{\overline{j},k}(\omega)|^2 \lesssim 1$. Moreover, since the number of possible directions is bounded by C_K,

$$\sum_{k\in K_{\overline{j}}} \frac{2^{2\overline{j}s}}{(1+|\omega|)^{2s}}|\Phi_{\overline{j},k}(\omega)|^2 \lesssim C_K. \tag{42}$$

With this majorisation, we can treat also the cases of the adjacent coronae, i.e. for $j = \overline{j} \pm 1$. If $j > \overline{j}+1$ or $j < \overline{j}-1$, then $\mathrm{d}(\omega, I_{j,k}) \gtrsim 2^j$, and the result follows as above. Hence, if $W_1 = 1$, the result follows easily by the requirement on α. Indeed, using Lemma A.1, one gets

$$\sum_{|j-\overline{j}|>1,k\in K_j} \frac{2^{2js}}{(1+|\omega|)^{2s}}|\Phi_{j,k}(\omega)|^2 \leq C_K\left(3 + \sum_j 2^{j(d-2\alpha)}\right), \tag{43}$$

which is clearly bounded. Now we can repeat this argument a finite number of time if $W_1 > 1$ and the result follows.

Lemma A.4 *Let $s \geq 0$ and φ such that $\Phi_{j,k}(\omega)$ satisfies hypothesis* (37)*. Define*

$$\widetilde{\varphi_{j,k,\lambda}}(t) = \frac{1}{2^{jd/2}} T_{2^{-j}\lambda}\, \mathrm{F}^{-1}_{\omega\mapsto t}\left(\frac{2^{js}}{(1+|\omega|)^s}\Phi_{j,k}(\omega)\right)(t); \tag{44}$$

then for $\nu \in (0,1]$ and all j,k the system of functions $\{\widetilde{\varphi_{j,k,\lambda}}(t)\}_{\lambda\in\nu\mathbb{Z}}$ is a Bessel sequence uniformly in j,k, that is

$$\sum_{\lambda\in\nu\mathbb{Z}} \left|\langle\widetilde{\varphi_{j,k,\lambda}}, f\rangle\right|^2 \leq C_\nu \|f\|^2_{L^2(\mathbb{R}^d)} \tag{45}$$

with C_ν independent on j,k.

Proof A well known result (see again e.g. [9][Thm 9.2.5 p.206]) states that the Bessel property (45) of $\left\{\widetilde{\varphi_{j,k,\lambda}(t)}\right\}_{\lambda\in\nu\mathbb{Z}}$, for fixed j, is equivalent to the following condition

$$\Xi_{j,k}(\gamma)=\sum_{m\in\mathbb{Z}^d}\left|\mathrm{F}\left(\widetilde{\varphi_{j,k,0}}\right)\left((\gamma-m)\frac{2^j}{\nu}\right)\right|^2\leq C_\nu\frac{\nu^d}{2^{jd}},\quad a.e.\ \gamma\in[0,1]^d.$$

By definition

$$\Xi_{j,k}(\gamma)=\frac{1}{2^{jd}}\sum_{m\in\mathbb{Z}^d}\left|\frac{2^{js}}{\left(1+\left|(\gamma-m)\frac{2^j}{\nu}\right|\right)^s}\Phi_{j,k}\left((\gamma-m)\frac{2^j}{\nu}\right)\right|^2.$$

Using the hypothesis (37) and relation (38), we can write

$$\begin{aligned}\Xi_{j,k}(\gamma)&\lesssim\frac{1}{2^{jd}}\left(1+\sum_{|m|>1}\frac{2^{jd}}{\left(1+d\left((\gamma-m)\frac{2^j}{\nu},I_{j,k}\right)\right)^{2\alpha}}\right)\\&\lesssim\frac{1}{2^{jd}}\left(1+\sum_{|m|>1}\frac{2^{jd}}{\left((|m|-1)\frac{2^j}{\nu}\right)^{2\alpha}}\right)\\&\lesssim\frac{1}{2^{jd}}\left(1+\frac{\nu^{2\alpha}}{2^{j(2\alpha-d)}}\sum_{|m|>1}\frac{1}{((|m|-1))^{2\alpha}}\right),\quad\text{a.e. }\gamma\in[0,1],\end{aligned}\tag{46}$$

where the second inequality follows from our assumption on the radius. Again, by our hypothesis on α, the sum in (46) is convergent, and uniformly bounded with respect to j.

Lemma A.5 *Let $\Phi_{j,k}$, $\widetilde{\varphi_{j,k,\lambda}}$ as in Lemma A.4 and $E_{j,k}$ as in* (13).
If $\nu\in(0,1]$ and $\operatorname{supp}\widehat{f}\cap E_{j,k}=\emptyset$, *then*

$$\sum_{\lambda\in\nu\mathbb{Z}}\left|\langle\widetilde{\varphi_{j,k,\lambda}},f\rangle\right|^2\leq\frac{C_\nu}{2^{j(2\alpha-d)}}\|f\|^2_{L^2(\mathbb{R}^d)},\tag{47}$$

with C_ν as in Lemma A.4, therefore independent on j,k.

Proof Since $\hat{f}(\omega)=0$ if $\omega\in E_{j,k}$

$$\sum_{\lambda\in\nu\mathbb{Z}}\left|\langle\widetilde{\varphi_{j,k,\lambda}},f\rangle\right|^2=\sum_{\lambda\in\nu\mathbb{Z}}\left|\langle\chi_{\mathbb{R}\setminus E_{j,k}}\mathrm{F}\left(\widetilde{\varphi_{j,k,\lambda}}\right),\mathrm{F}(f)\rangle\right|^2.$$

Therefore, using the same property of Lemma A.4, (47) is equivalent to prove that

$$\frac{1}{2^{jd}}\sum_{m\in\mathbb{Z}}\left|\chi_{\mathbb{R}\setminus E_{j,k}}(m_{\gamma,j,\nu})\frac{2^{js}}{(1+|m_{\gamma,j,\nu}|)^s}\Phi_{j,k}(m_{\gamma,j,\nu})\right|^2\leq\frac{1}{2^{j(2\alpha)}},\tag{48}$$

where $m_{\gamma,j,\nu}=(\gamma-m)\frac{2^j}{\nu}$. Since $\nu\leq1$, for each j,k, there exist a finite number of consecutive indices m such that $(\gamma-m)\frac{2^j}{\nu}\in E_{j,k}$ for some $\gamma\in[0,1]$. We set

$$M_{j,\nu}=\left\{m\in\mathbb{Z}:\ \exists\,\gamma\in[0,1]\text{ such that }(\gamma-m)\frac{2^j}{\nu}\in E_{j,k}\right\}.$$

We notice that $M_{j,\nu}$ is uniformly bounded with respect to j, by the properties of the partitioning and by the definition of $E_{j,k}$.

If $m \in M_{j,\nu}$ and $m_{\gamma,j,\nu} \in E_{j,k}$, then

$$\left|\chi_{\mathbb{R}\setminus E_{j,k}}\left(m_{\gamma,j,\nu}\right)\frac{2^{js}}{\left(1+|m_{\gamma,j,\nu}|\right)^{s}}\Phi_{j,k}\left(m_{\gamma,j,\nu}\right)\right|^{2}=0.$$

Otherwise $\chi_{\mathbb{R}\setminus E_{j,k}}\left(m_{\gamma,j,\nu}\right)=1$ and, using Lemma A.1,

$$\left|\frac{2^{js}}{\left(1+|m_{\gamma,j,\nu}|\right)^{s}}\Phi_{j,k}\left(m_{\gamma,j,\nu}\right)\right|^{2}\lesssim\left|\frac{2^{jd/2}}{\left(1+2^{j}\right)^{\alpha}}\right|^{2}\lesssim 2^{j(d-2\alpha)}.$$

Hence, (48) is bounded by

$$|M_{j,\nu}|\left(2^{j}\right)^{-2\alpha}+\frac{1}{2^{jd}}\sum_{m\notin M_{j,\nu}}\left|\chi_{\mathbb{R}\setminus E_{j,k}}\left(m_{\gamma,j,\nu}\right)\frac{2^{js}}{\left(1+\mathrm{d}\left(m_{\gamma,j,\nu},I_{j,k}\right)\right)^{s}}\Phi_{j,k}\left(m_{\gamma,j,\nu}\right)\right|^{2}. \tag{49}$$

The second term in the equation above may be bounded as follows

$$\begin{aligned}&\frac{1}{2^{jd}}\sum_{m\notin M_{j,\nu}}\left|\chi_{\mathbb{R}\setminus E_{j,k}}\left(m_{\gamma,j,\nu}\right)\frac{2^{js}}{\left(1+\mathrm{d}\left(m_{\gamma,j,\nu},I_{j,k}\right)\right)^{s}}\Phi_{j,k}\left(m_{\gamma,j,\nu}\right)\right|^{2}\\&\quad\lesssim\frac{1}{2^{jd}}\frac{\nu^{2\alpha}2^{jd}}{\left(2^{j}\right)^{2\alpha}}\sum_{|m|\geq 2}\frac{1}{\left(\left(|m|-1\right)\right)^{2\alpha}}\lesssim\frac{\nu^{2\alpha}}{\left(2^{j}\right)^{2\alpha}}\sum_{|m|\geq 2}\frac{1}{\left(\left(|m|-1\right)\right)^{2\alpha}}\lesssim\frac{\nu^{2\alpha}}{\left(2^{j}\right)^{2\alpha}}.\end{aligned}$$

Then the assertion follows as in Lemma A.4.

Lemma A.6 *Let $s\geq 0$ and $\varphi_{\bullet},\varphi$ be a system of functions such that there exists $a>0$ such that, for all $\omega\in\mathbb{R}^{d}$ then*

$$|\Phi_{\bullet}(\omega)|\geq a,\ \text{if }\omega\in I_{\bullet},\qquad\left|\Phi_{j,k}(\omega)\right|\geq a,\ \text{if }\omega\in I_{j,k}$$

and the constant a does not depend on j,k. Then

$$\frac{|\Phi_{\bullet}(\omega)|^{2}}{(1+|\omega|)^{2s}}+\sum_{j,k\in\Gamma}\frac{2^{2js}}{(1+|\omega|)^{2s}}|\Phi_{j,k}(\omega)|^{2}\geq C_{s}a^{2},\quad a.e.\ \omega\in\mathbb{R}^{d}\,, \tag{50}$$

with a constant C_s which depends on s only and is uniform with respect to ω.

Proof For $s=0$, the statement is trivial while for general s, notice that

$$(1+|\omega|)\asymp 2^{j},\qquad\omega\in I_{j,k},$$

while if $\omega\in I_{\bullet}$, then $(1+|\omega|)\asymp 1$.

Remark A.1 *Inequality* (50) *could be used as hypothesis on the window function weaker then ours. Since it is quite cumbersome to be checked, we prefer to work with a more transparent assumption.*

We state now the counterpart of Lemma A.4 and Lemma A.5 in the framework of seminorm discretisation.

Lemma A.7 *Let $s\geq 0$ and φ such that $\Phi_{j,k}(\omega)$ satisfies hypothesis* (39). *We define*

$$\begin{aligned}\widetilde{\varphi_{j,k,\lambda}}(t)&=\frac{1}{2^{jd/2}}T_{2^{-j}\lambda}\,\mathrm{F}^{-1}_{\omega\mapsto t}\left(\frac{2^{js}}{|\omega|^{s}}\Phi_{j,k}(\omega)\right)(t),\quad j\in\mathbb{N}\\\widetilde{\varphi_{-j,k,\lambda}}(t)&=\frac{1}{2^{jd/2}}T_{\lambda 2^{j}}\,\mathrm{D}_{2^{2j}}\,\mathrm{F}^{-1}_{\omega\mapsto t}\left(\frac{2^{js}}{|\omega|^{s}}\Phi_{j,k}(\omega)\right)(t),\quad j\in\mathbb{N}\setminus\{0\}\end{aligned} \tag{51}$$

then, for all $\nu \in (0,1]$ and j,k the system of functions $\left\{\widetilde{\varphi_{j,k,\lambda}}(t)\right\}_{\lambda\in\nu\mathbb{Z}}$ is a Bessel sequence uniformly in j,k, that is

$$\sum_{\lambda\in\nu\mathbb{Z}} \left|\langle\widetilde{\varphi_{j,k,\lambda}}, f\rangle\right|^2 \leq C_\nu \|f\|^2_{L^2(\mathbb{R}^d)} \tag{52}$$

with C_ν independent on j,k.

Lemma A.8 *Let $\Phi_{j,k}$, $\widetilde{\varphi_{j,k,\lambda}}$ as in Lemma A.7 and $E_{j,k}$ as in (13) for $j\in\mathbb{N}$ and as*

$$E_{-j,k} = \left\{x\in\mathbb{R} \mid 2^{2j}x \in E_{j,k}\right\}$$

for negative integers. If $\nu\in(0,1]$ and $\operatorname{supp}\widehat{f}\cap E_{j,k}=\emptyset$, then

$$\sum_{\lambda\in\nu\mathbb{Z}} \left|\langle\widetilde{\varphi_{j,k,\lambda}}, f\rangle\right|^2 \leq \frac{C_\nu}{|2^j|^{(2\alpha-d)}} \|f\|^2_{L^2(\mathbb{R}^d)} \tag{53}$$

with C_ν independent on j,k.

Proof (Lemma A.7, A.8) If $j\in\mathbb{N}$ the proofs for is the same of Lemma A.4 and Lemma A.5. In order to prove Lemma (A.7) for $-j$ with $j\in\mathbb{N}\setminus\{0\}$, we use the following relation

$$\sum_{\lambda\in\nu\mathbb{Z}} \left|\langle\widetilde{\varphi_{-j,k,\lambda}}, f\rangle\right|^2 = \sum_{\lambda\in\nu\mathbb{Z}} \left|\langle \mathrm{D}_{2^{2j}}\, T_{\frac{\lambda}{2^j}}\,\widetilde{\varphi_{j,k,0}}, f\rangle\right|^2 = \sum_{\lambda\in\nu\mathbb{Z}} \left|\langle T_{\frac{\lambda}{2^j}}\,\widetilde{\varphi_{j,k,0}}, \mathrm{D}_{2^{-2j}}\, f\rangle\right|^2$$

and the fact that $\|\mathrm{D}_{2^{-2j}}\, f\|_{L^2(\mathbb{R}^d)} = \|f\|_{L^2(\mathbb{R}^d)}$. For Lemma (A.8) notice also that $\operatorname{supp}\widehat{f}\cap E_{-j,k}=\emptyset$ implies $\operatorname{supp}\widehat{\mathrm{D}_{2^{-2j}}\, f}\cap E_{j,k}=\emptyset$.

B Interpolation techniques

We define a multi-resolution partition using the notation of [8]. Then, we adapt the interpolation result to our specific case.

Set $V = L^2(\mathbb{R}), Z = H^1(\mathbb{R})$ and let

$$V_{\overline{j}} = \operatorname{span}\left\{\varphi_{j,k,\lambda}\right\}_{j\leq\overline{j}} \tag{54}$$

where $\left\{\varphi_{j,k,\lambda} \mid j,k,\lambda\in\Delta\right\}$ is the frame defined as in (27). We also define the related projectors

$$\begin{aligned} P_{\overline{j}} : L^2 &\longrightarrow V_j \\ v &\longmapsto \sum_{j\leq\overline{j},k,\lambda} \langle\varphi_{j,k,\lambda}, v\rangle\varphi^D_{j,k,\lambda}, \end{aligned} \tag{55}$$

where $\varphi^D_{j,k,\lambda}$ is the dual window. By definition

$$\ldots V_{-j-1} \subseteq V_{-j} \subseteq \ldots \subseteq V_1 \subseteq \ldots \subseteq V_j \subseteq V_{j+1} \ldots \subseteq L^2(\mathbb{R}) \tag{56}$$

and also $V_j \subseteq H^s(\mathbb{R}^d), j\in\mathbb{Z}$. Finally, we notice that due to our definition

$$\lim_{j\to-\infty} P_j = 0. \tag{57}$$

Lemma B.9 *Let $r \in \mathbb{Z}$, then the following inequalities hold.*

$$|v|_r \lesssim 2^{\overline{j}r} \|v\|, \quad \forall v \in V_{\overline{j}}, \quad \overline{j} \in \mathbb{Z} \quad \textit{(Bernstein)} \tag{58}$$

and

$$\left\| v - P_{\overline{j}}(v) \right\| \lesssim 2^{-\overline{j}r} |v|_r \quad \forall v \in V_{\overline{j}}, \quad \overline{j} \in \mathbb{Z} \quad \textit{(Jackson)}. \tag{59}$$

Proof This follows immediately from Theorem 34 and a few observations. We notice that we can write the Sobolev seminorm as

$$|v|_r = \left\| v^{(r)} \right\|_2,$$

where $v^{(r)}$ is the $r-$th derivative of v. We also notice that if $\left\{\varphi_{j,k,\lambda}\right\}_{j,k,\lambda \in \Gamma}$ is a frame for H^r, then $\left\{2^{-jr}\varphi^{(r)}_{j,k,\lambda}\right\}_{j,k,\lambda \in \Gamma}$ is a frame for L^2. Indeed,

$$2^{-jr}\widehat{\varphi^{(r)}_{j,k,\lambda}}(\xi) = 2^{-jr} e^{-2\pi \mathrm{i}\, \omega \cdot 2^{-j}\lambda} |\xi|^r \Phi_{j,k}(\xi).$$

and the frequency window

$$\widetilde{\Phi_{j,k}}(\xi) = 2^{-jr}|\xi|^r \Phi_{j,k}(\xi),$$

clearly satisfies the requirements of Definition 33 and thus yields a frame, as claimed. Hence, we can represent the r-th derivative of the function v as

$$v^{(r)} = \sum_{j<\overline{j},k,\lambda} 2^{jr} \langle v, 2^{-jr}\varphi^{(r)}_{j,k,\lambda}\rangle \varphi^{D}_{j,k,\lambda}.$$

Finally, using the minimal property of the frame coefficients, see [16, Proposition 5.1.4.],

$$|v|_s^2 \lesssim \sum_{j<\overline{j},k,\lambda\in\Delta} 2^{2js} \left|\langle v, \varphi_{j,k,\lambda}\rangle\right|^2 \leq 2^{2\overline{j}s} \sum_{j<\overline{j},k,\lambda\in\Delta} \left|\langle v, \varphi_{j,k,\lambda}\rangle\right|^2 \leq 2^{2\overline{j}s} \|v\|,$$

as desired.

To prove Jackson's inequality, notice that, using again the minimal property of the frame coefficients,

$$\left\| v - P_{\overline{j}}(v) \right\|^2 \lesssim \sum_{j>\overline{j},k,\lambda\in\Delta} \left|\langle f, \varphi_{j,k,\lambda}\rangle\right|^2 \leq 2^{-2\overline{j}s} \sum_{j>\overline{j},k,\lambda\in\Delta} 2^{2\overline{j}s} \left|\langle f, \varphi_{j,k,\lambda}\rangle\right|^2 \leq 2^{-2\overline{j}s} |v|_s^2,$$

and this concludes the proof.

We now review Theorem 1 in [8] adapting it to our framework.

Theorem B.1 *Consider the spaces V_j and the associated partitions P_j that satisfies* (57) *and Jackson-Bernstein inequalities. Then, for any $0 < \alpha < 1, q > 1$ one has*

$$(L^2, H^1)_{\alpha,q} = \left\{ v \in L^2 \mid |v|_{\alpha,q} = \left[\sum_{j\in\mathbb{Z}} 2^{j\alpha q} \left(\sum_{k,\lambda\in\Delta_j} \left|\langle \varphi_{j,k,\lambda}, v\rangle\right|^2 \right)^{q/2} \right]^{1/q} < \infty \right\},$$

and a norm for this space is $\|\cdot\| = \|\cdot\|_2 + |\cdot|_{\alpha,q}$.

Proof The hypothesis of [8, Theorem 1] are fulfilled, hence the result follows.

Remark 9 *The same result holds if we interchange the role of the window function and its dual.*

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