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# Uniform Determinantal Representations*

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**Abstract.** The problem of expressing a specific polynomial as the determinant of a square matrix of affine-linear forms arises from algebraic geometry, optimization, complexity theory, and scientific computing. Motivated by recent developments in this last area, we introduce the notion of a *uniform determinantal representation*, not of a single polynomial but rather of all polynomials in a given number of variables and of a given maximal degree. We derive a lower bound on the size of the matrix, and present a construction achieving that lower bound up to a constant factor as the number of variables is fixed and the degree grows. This construction marks an improvement upon a recent construction due to Plestenjak and Hochstenbach, and we investigate the performance of new representations in their root-finding technique for bivariate systems. Furthermore, we relate uniform determinantal representations to vector spaces of singular matrices, and we conclude with a number of future research directions.

**Key words.** determinantal representation, uniform determinantal representation, system of polynomial equations, multiparameter matrix eigenvalue problem, space of singular matrices

**AMS subject classifications.** 13P15, 65H04, 65F15, 65F50

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## 1. Introduction and results.

Consider an $n$-variate polynomial of degree at most $d$:

$$p = \sum_{|\alpha| \leq d} c_\alpha x^\alpha,$$

where $x := (x_1, \ldots, x_n)$, $\alpha \in \mathbb{Z}^n_{\geq 0}$, $|\alpha| := \sum_i \alpha_i$, $x^\alpha := \prod_i x_i^{\alpha_i}$, and where each coefficient $c_\alpha$ is taken from a ground field $K$. A *determinantal representation* of $p$ is an $N \times N$-matrix $M$ of

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the form

$$M = A_0 + \sum_{i=1}^{n} x_i A_i,$$

where each $A_i \in K^{N\times N}$, with $\det(M) = p$. We call $N$ the *size* of the determinantal representation. Clearly, since the entries of $M$ are affine-linear forms in $x_1, \ldots, x_n$, the integer $N$ must be at least the degree of $p$. The minimal size of any determinantal representation of $p$ is called the *determinantal complexity* of $p$.

Determinantal representations of polynomials play a fundamental role in several mathematical areas: from *algebraic geometry* it is known that each plane curve ($n = 2$) of degree $d$ over an algebraically closed field $K$ admits a determinantal representation of size $d$ [9, 11]. Over nonalgebraically closed fields, and especially when restricting to symmetric determinantal representations, the situation is much more subtle [21]. For larger $n$, only certain hypersurfaces have a determinantal representation of size equal to their degree [3, 9]. In *optimization*, and notably in the theory of *hyperbolic polynomials* [41], one is particularly interested in the case where $K = \mathbb{R}$, $A_0$ is symmetric positive definite, and the $A_i$ are symmetric. In this case, the restriction of $p$ to any line through 0 has only real roots. For $n = 2$ the converse also holds, and indeed with a representation of size equal to the degree of $p$ [17] (via homogenization, this implies a conjecture of Lax [24]). For counterexamples to this converse holding for higher $n$, see [6]. In *complexity theory* a central role is played by Valiant's conjecture that the permanent of an $m \times m$-matrix does not admit a determinantal representation of size polynomial in $m$ [39]. Via the *geometric complexity theory* program [29] this leads to the study of polynomials in the boundary of the orbit of the $N \times N$-determinant under the action of the group $\mathrm{GL}_{N^2}(K)$ permuting matrix entries. Recent developments in this field include the study of this boundary for $N = 3$ [19] and the exciting negative result in [7] that Valiant's conjecture can *not* be proved using occurrence obstructions proposed earlier in [30].

Our motivation comes from *scientific computing*, where determinantal representations of polynomials have recently been proposed for efficiently solving systems of equations [34]. For this application, it is crucial to have determinantal representations not of a *single* polynomial $p$, but rather of all $n$-variate polynomials of degree at most $d$. Moreover, the representation should be easily computable from the coefficients of $p$. Specifically, in [34] determinantal representations are constructed for the bivariate case ($n = 2$) in which the entries of the matrices $A_0, \ldots, A_n$ themselves *depend affine-linearly* on the coefficients $c_\alpha$. This is what we call a *uniform determinantal representation* of the generic polynomial $p$ of degree $d$ in $n$ variables; see section 2 for a precise definition.

*Example* 1.1 (the bivariate quadric). The identity

$$c_{00} + c_{10}x + c_{01}y + c_{20}x^2 + c_{11}xy + c_{02}y^2 = \det \begin{bmatrix} -x & 1 & 0 \\ -y & 0 & 1 \\ c_{00} & c_{10} + c_{20}x + c_{11}y & c_{01} + c_{02}y \end{bmatrix}$$

exhibits the matrix on the right as a uniform determinantal representation of the generic bivariate quadric. ■

In applications, the matrix $M$ is used as input to algorithms in numerical linear algebra

that scale unfavorably with matrix size $N$, such as a complexity of $O(N^6)$ for $n = 2$ [34]. Consequently, we are led to consider the following fundamental question.

Question 1.2. *What is the minimal size $N^*(n,d)$ of any uniform determinantal representation of the generic polynomial of degree $d$ in $n$ variables?*

A construction from [34] shows that for fixed $n = 2$ and $d \to \infty$ we have $N^*(2,d) \leq \frac{1}{4}d^2 + O(d)$; this construction is reviewed in section 4. We improve the construction from [34] by giving a particularly elegant uniform determinantal representations of bivariate polynomials of size $2d+1$ in Example 4.1, and of size $2d-1$ in Example 4.6. In view of the obvious lower bound of $d$ this is clearly sharp up to a constant factor for $d \to \infty$, although we do not know where in the interval $[d, 2d-1]$ the true answer lies. We show in section 7 how to use these small determinantal representations of bivariate polynomials for solving systems of equations. Before that, we focus on the asymptotic behavior of $N^*(n,d)$ for fixed $n$ and $d \to \infty$. In this setting, we derive the following result.

Theorem 1.3. *For fixed $n \in \mathbb{Z}_{\geq 2}$ there exist positive constants $C_1, C_2$ (depending on $n$) such that for each $d \in \mathbb{Z}_{\geq 0}$ the smallest size $N^*(n,d)$ of a uniform determinantal representation of the generic polynomial of degree $d$ in $n$ variables satisfies $C_1 d^{n/2} \leq N^*(n,d) \leq C_2 d^{n/2}$. Moreover, $C_1$ can be chosen such that the determinantal complexity of any sufficiently general polynomial is at least $C_1 d^{n/2}$.*

In the last statement of Theorem 1.3, "sufficiently general" means that the coefficient vector of the polynomial lies in some (unspecified) Zariski-open and dense subset (when working over infinite fields), or should be interpreted in a suitable counting sense (when working over finite fields); see the proof of Theorem 1.3 in section 4. Note also that this statement does not require the determinantal representation to depend affine linearly on the coefficients of the polynomial.

We will compare our results with previous constructions, most notably with those by Quarez [35, Thm. 4.4], who proves the existence of a symmetric representation of size $\binom{n+\lfloor \frac{d}{2} \rfloor}{n}$. For fixed $n$ and $d \to \infty$, [35] therefore has the asymptotic rate $\sim d^n$, meaning that the results of this paper represent a clear improvement. For fixed $d$ and $n \to \infty$, [35] leads to the asymptotic behavior $\sim n^{\lfloor d/2 \rfloor}$, which is similar to our bounds; we will discuss more details in section 8.

In section 2 we formalize the notion of uniform determinantal representations, study their symmetries, and derive some simple properties. In particular, we relate uniform determinantal representations to spaces of singular $N \times N$-matrices. In section 3 we briefly review some of the existing literature on these singular spaces, and we prove that for $N > 4$ there are infinitely many equivalence classes of such objects; this poses an obstruction to a "brute-force" approach towards finding lower bounds on $N^*(n,d)$. In section 4 we present an efficient explicit construction and prove our main result, namely Theorem 1.3. We also construct alternative uniform determinantal representations that are worse in terms of size, but which are optimal under certain restrictions on the underlying singular matrix space. In section 5 we give upper bounds on $N^*(n,d)$ for small $n$ and $d$ and determine $N^*(2,2)$ and $N^*(3,2)$ exactly. We extend representations from scalar to matrix polynomials in section 6. In section 7 we give some numerical results that show that for $n = 2$ and small $d$ we get a competitive method

for computing zeros of polynomial systems. Finally, in section 8 we summarize our main conclusions and collect some questions that arise naturally from our work.

**2. Problem formulation and symmetries.** In this section we give a formal definition of uniform determinantal representations, and introduce a group that acts on such representations. We also show that a uniform determinantal representation gives rise to a vector space consisting entirely of singular matrices; such spaces are the topic of next section.

Let $K$ be a field and fix $d, n \in \mathbb{Z}_{\geq 0}$. Let $F_d$ denote the polynomials of degree at most $d$ in the polynomial ring $K[x_1, \ldots, x_n]$. Furthermore, let $p_{n,d}$ be the generic polynomial of that degree, i.e.,

$$p_{n,d} = \sum_{|\alpha| \leq d} c_\alpha x^\alpha, \tag{1}$$

where $x := (x_1, \ldots, x_n)$, $\alpha \in \mathbb{Z}^n_{\geq 0}$, $|\alpha| := \sum_i \alpha_i$, $x^\alpha := \prod_i x_i^{\alpha_i}$, and where we consider $c_\alpha$ as a variable for each $\alpha$.

Definition 2.1. *For $n, d \in \mathbb{Z}_{\geq 0}$, a* uniform determinantal representation *of $p_{n,d}$ is an $N \times N$-matrix $M$ with entries from $K[(x_1, \ldots, x_n), (c_\alpha)_{|\alpha| \leq d}]$, of degree at most 1 in each of these two sets of variables, such that $\det(M) = p_{n,d}$. The number $N$ is called the* size *of the determinantal representation.*

To be explicit, we require each entry of $M$ to be a $K$-linear combination of the monomials $1, x_i, c_\alpha, c_\alpha x_i$, $(i = 1, \ldots, n, |\alpha| \leq d)$. This means that we can decompose $M$ as $M_0 + M_1$, where $M_0$ contains all terms in $M$ that do not contain any $c_\alpha$, and where $M_1$ contains all terms in $M$ that do. We will use the notation $M = M_0 + M_1$ throughout this paper. When $n$ and $d$ are fixed in the context, we will also speak of a uniform determinantal representation without reference to $p_{n,d}$. Our ultimate aim is to determine the following quantity.

Definition 2.2. *For $n, d \in \mathbb{Z}_{\geq 0}$, $N^*(n, d) \in \mathbb{Z}_{>0}$ is the minimum among all sizes of uniform determinantal representations of $p_{n,d}$.*

This minimal size could potentially depend on the ground field $K$, but the bounds that we will prove do not. Note that in the definition of $N^*(n, d)$ we do not allow terms in $M$ of degree strictly larger than one in the $c_\alpha$. Relaxing this condition to polynomial dependence on the $c_\alpha$ might affect the exact value of $N^*$, but by Theorem 1.3 it can only affect the constant in front of $d^{n/2}$ for $n$ fixed and $d \to \infty$.

Given a uniform determinantal representation $M$ of size $N$, and given matrices $g, h$ in $\mathrm{SL}_N(K)$, the group of determinant-one matrices with entries in $K$, the matrix $gMh^{-1}$ is another uniform determinantal representation of $p_{n,d}$. In this manner, the group $\mathrm{SL}_N(K) \times \mathrm{SL}_N(K)$ acts on the set of uniform determinantal representations of $p_{n,d}$. Moreover, there exist further symmetries, arising from affine transformations of the $n$-space. Recall that these transformations form the group $\mathrm{AGL}_n(K) = \mathrm{GL}_n(K) \ltimes K^n$ generated by invertible linear transformations and translations.

Lemma 2.3. *The group $\mathrm{AGL}_n(K)$ acts on uniform determinantal representations of $p_{n,d}$.*

This statement is empty without making the action explicit, as we do in the proof.

*Proof.* Let $g \in \mathrm{AGL}_n(K)$ be an affine transformation of $K^n$, and expand

$$p_{n,d}(g^{-1}x, c) = \sum_{|\alpha| \le d} c'_\alpha x^\alpha,$$

where the $c'_\alpha$ are linear combinations of the $c_\alpha$. More precisely, the vector $c'$ can be written as $\rho(g)c$, where $\rho$ is the representation of $\mathrm{AGL}_n(K)$ on polynomials of degree at most $d$ regarded as a matrix representation relative to the monomial basis.

Now let $M = M(x, c)$ be a uniform determinantal representation of $p_{n,d}$. Then

$$\det(M(g^{-1}x, \rho(g)^{-1}c)) = p_{n,d}(g^{-1}x, \rho(g)^{-1}c) = p_{n,d}(x, c),$$

i.e., $M(g^{-1}x, \rho(g)^{-1}c)$ is another uniform determinantal representation of $p_{n,d}$. The action of $g$ is given by $M \mapsto M(g^{-1}x, \rho(g)^{-1}c)$. ∎

*Example* 2.4 (the bivariate quadric revisited). Take $n = d = 2$ and the affine transformation $g(x, y) := (y, x + 1)$ with inverse $g^{-1}(x, y) = (y - 1, x)$. We have

$$p_{2,2}(g^{-1}(x, y)) = (c_{00} - c_{10} + c_{20}) + (c_{01} - c_{11})x + (c_{10} - 2c_{20})y + c_{02}x^2 + c_{11}xy + c_{20}y^2.$$

We find

$$c' = \begin{bmatrix} c'_{00} \\ c'_{10} \\ c'_{01} \\ c'_{20} \\ c'_{11} \\ c'_{02} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_{00} \\ c_{10} \\ c_{01} \\ c_{20} \\ c_{11} \\ c_{02} \end{bmatrix} = \rho(g)c; \ \rho(g)^{-1} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

If we make the substitutions

$$\begin{array}{llll} c_{00} \mapsto c_{00} + c_{01} + c_{02}, & c_{10} \mapsto c_{01} + 2c_{02}, & c_{01} \mapsto c_{10} + c_{11}, & x \mapsto y - 1, \\ c_{20} \mapsto c_{02}, & c_{11} \mapsto c_{11}, & c_{02} \mapsto c_{20}, & y \mapsto x \end{array}$$

in the uniform determinantal representation of Example 1.1, then we arrive at the matrix

$$\begin{bmatrix} 1 - y & 1 & 0 \\ -x & 0 & 1 \\ c_{00} + c_{01} + c_{02} & c_{01} + c_{02} + c_{02}y + c_{11}x & c_{10} + c_{11} + c_{20}x \end{bmatrix}$$

whose determinant also equals $p_{2,2}$. ∎

The action of the affine group will be used in section 5 to determine the exact value of $N^*(n, 2)$ for $n = 2$ and 3. We now turn our attention to the component $M_0$ of a uniform determinantal representation $M$.

Lemma 2.5. *For any uniform determinantal representation $M = M_0 + M_1$ of size $N$, the determinant of $M_0$ is the zero polynomial in $K[x_1, \ldots, x_n]$. Moreover, at every point $\bar{x} \in K^n$, the rank of the specialization $M_0(\bar{x}) \in K^{N \times N}$ is exactly $N - 1$.*

*Proof.* The first statement follows from the fact that $\det(M_0)$ is the part of the polynomial $\det(M)$ which is homogeneous of degree zero in the $c_\alpha$; hence zero.

By specializing the vector $x$ of variables to a point $\bar{x} \in K^n$, the rank of $M_0$ cannot increase, so the rank of $M_0(\bar{x})$ is at most $N-1$. However, if it were at most $N-2$, then after column operations on $M$ by means of determinant-one matrices with entries in $K$ we may assume that $M_0(\bar{x})$ has its last two columns equal to 0. This means that all entries of $M(\bar{x}) = M_0(\bar{x}) + M_1(\bar{x})$ in these columns are linear in the $c_\alpha$. This in turn implies that any term in the polynomial $\det M(\bar{x})$ is at least quadratic in the $c_\alpha$. But on the other hand $\det M(\bar{x})$ equals $p_{n,d}(\bar{x})$, which is a nonzero *linear* polynomial in the $c_\alpha$ (nonzero since not every polynomial of degree at most $d$ vanishes at $\bar{x}$). This contradiction implies that the rank of $M_0(\bar{x})$ is $N-1$. ■

Lemma 2.6. *If $M = M_0 + M_1$ is a uniform determinantal representation of size $N$, then $V \subseteq F_{N-1}$ spanned by the $(N-1)\times(N-1)$-subdeterminants of $M_0$ satisfies $F_1 \cdot V \supseteq F_d$.*

Here, as in the rest of this paper, by the product of two spaces of polynomials we mean the $K$-linear span of all the products.

*Proof.* Let $D_{ij}$ be the determinant of the submatrix of $M_0$ obtained by deleting the $i$th row and the $j$th column. On the one hand, $\det(M) = p_{n,d}$ is linear in the $c_\alpha$ by assumption, and on the other hand, by expanding $\det(M)$ we see that the part of $\det(M)$ that is homogeneous of degree one in the $c_\alpha$ is

$$\sum_{i,j} (-1)^{i+j} (M_1)_{ij} D_{ij};$$

this therefore equals $p_{n,d}$. Hence any element $q$ of $F_d$ is obtained from the expression above by specializing the variables $c_\alpha$ to the coefficients of $q$. Since each $(M_1)_{ij}$ is then specialized to an element of $F_1$, we find $q \in F_1 \cdot V$. ■

## 3. Spaces of singular matrices.

Let $M = M_0 + M_1$ be a uniform determinantal representation of $p_{n,d}$. Writing $M_0 = B_0 + \sum_{i=1}^n x_i B_i$, Lemma 2.5 implies that the linear span $\langle B_0, \ldots, B_n \rangle_K \subseteq K^{N\times N}$ consists entirely of singular matrices (and indeed that this remains true when extending scalars from $K$ to an extension field). There is an extensive literature on such *singular matrix spaces*; see, e.g., [12, 14] and the references therein. The easiest examples are the following.

Definition 3.1. *A subspace $\mathcal{A} \subseteq K^{N\times N}$ is called a* compression space *if there exists a subspace $U \subseteq K^N$ with $\dim(\langle u^T A \mid A \in \mathcal{A}, u \in U\rangle_K) < \dim U$. We call the space $U$ a* witness *for the singularity of $\mathcal{A}$.*

Given any two subspaces $U, V \subseteq K^N$ with $\dim V = -1 + \dim U$, the space of all matrices which map $U$ into $V$ (acting on row vectors) is a compression space with witness $U$. It is easy to see that these spaces are inclusionwise maximal among all singular spaces.

If $\mathcal{A}$ is a singular matrix space, then so is $g\mathcal{A}h^{-1}$ for any pair $(g,h) \in \mathrm{GL}_N(K) \times \mathrm{GL}_N(K)$. We call the latter space *conjugate* to the former.

*Example* 3.2. For $N = 2$, every singular matrix space is a compression space, hence con-

jugate to a subspace of one of the two spaces

$$\left\{\begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix}\right\}, \quad \left\{\begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix}\right\},$$

where the $*$s indicate entries that can be filled arbitrarily. A witness for the first space is the span $\langle e_2 \rangle$ of the second standard basis vector, and a witness for the second space is $K^2$.

For $N = 3$, there are four conjugacy classes of inclusion-maximal singular matrix spaces, represented by the three maximal compression spaces

$$\left\{\begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 0 \end{bmatrix}\right\}, \quad \left\{\begin{bmatrix} * & * & * \\ 0 & 0 & * \\ 0 & 0 & * \end{bmatrix}\right\}, \quad \left\{\begin{bmatrix} 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}\right\},$$

and the space of skew-symmetric $3 \times 3$-matrices [13]; the latter is not a compression space. For $N = 4$, there are still finitely many (namely, 10) conjugacy classes of inclusion-maximal singular matrix spaces [13, 14], but this is not true for $N \geq 5$, as Theorem 3.4 below shows. This theorem is presumably folklore; we include a proof since we have not been able to find a literature reference for it. ■

Proposition 3.3. *For any $m, N \in \mathbb{Z}_{\geq 0}$ the locus $X_m$ in the Grassmannian $\mathrm{Gr}(m, K^{N\times N})$ of $m$-dimensional subspaces of $K^{N\times N}$ consisting of all* singular *subspaces is closed in the Zariski topology. Moreover, the locus $U_m$ in $X_m$ consisting of all* inclusionwise maximal *singular subspaces is open inside $X_m$.*

*Proof.* The first statement is standard. For the second statement, consider the incidence variety

$$Z := \{(\mathcal{A}, \mathcal{A}') \in X_m \times X_{m+1} \mid \mathcal{A} \subseteq \mathcal{A}'\} \subseteq X_m \times X_{m+1},$$

which is a closed subvariety of $X_m \times X_{m+1}$. The projection of $Z$ into $X_m$ is the complement of $U_m$, and it is closed because $X_{m+1}$ is a projective variety. ■

Theorem 3.4. *Assume that $K$ is infinite and of characteristic unequal to 2. For $N \geq 5$ there are infinitely many conjugacy classes of inclusionwise maximal singular $N \times N$-matrix spaces.*

*Proof.* Take $N \geq 5$. For sufficiently general skew-symmetric matrices $A_1, \ldots, A_N \in K^{N\times N}$ set $A := (A_1, \ldots, A_N)$ and define the space

$$\mathcal{B}_A := \{(A_1 x | \cdots | A_N x) \mid x \in K^N\} \subseteq K^{N\times N}.$$

Each matrix in this space is singular, since for $x \neq 0$ we have

$$x^T(A_1 x | \cdots | A_N x) = (x^T A_1 x, \ldots, x^T A_N x) = 0.$$

In [14] it is proved that, for a specific choice of the tuple $A$, the space $\mathcal{B}_A$ is maximal among the singular subspaces of $K^{N\times N}$. By Proposition 3.3, $\mathcal{B}_A$ is maximal for sufficiently general $A$ as well (note that we may first extend $K$ to its algebraic closure to apply the proposition). In the notation of that proposition, we have a rational map

$$\varphi : S^N \dashrightarrow U_N, \quad A \mapsto \mathcal{B}_A,$$

where $S \subseteq K^{N\times N}$ denotes the subspace of skew-symmetric matrices; the dashed arrow indicates that the map is defined only in an open dense subset of $S^N$. For any nonzero scalar $t$, $\varphi(tA) = \varphi(A)$. We claim that, in fact, the general fiber of $\varphi$ is indeed one-dimensional. As the fiber dimension is semicontinuous, it suffices to verify this at a particular point where $\varphi$ is defined. We take $A_i = E_{i,i+1} - E_{i+1,i}$ for $i = 1, \dots, N-1$ and $A_N$ general; here $E_{ij}$ is the matrix with zeros everywhere except for a 1 at position $(i, j)$. Let $B \in S^N$; if $\varphi(A) = \varphi(B)$, then there exists an invertible matrix $g \in \mathrm{GL}_N(K)$ such that

$$(A_1 gx|\cdots|A_N gx) = (B_1 x|\cdots|B_N x)$$

for all $x$, so that $A_i g = B_i$. Using skew-symmetry of $A_i$ and $B_i$, we find that $A_i g = g^T A_i$. Substituting our choice of $A_i$ for $i \in \{1, \dots, N-1\}$ yields $g_{i,j} = g_{j,i} = 0$ for all $j$ with $|i-j| > 1$, $g_{i,i+1} = -g_{i,i+1}$, and thus $g_{i,i+1} = 0$ since $\operatorname{char} K \neq 2$, and $g_{i,i} = g_{i+1,i+1}$. Hence, $g$ is a scalar multiple of the identity. It follows that the fiber of $\varphi$ through $A$ is one-dimensional as claimed.

Since $\dim S = \binom{N}{2}$, we have thus constructed an $(N\binom{N}{2} - 1)$-dimensional family inside $U_N$. Given any point $\mathcal{A}$ in $U_N$, its orbit under $\mathrm{GL}_N(K) \times \mathrm{GL}_N(K)$ has dimension at most $2\,(N^2 - 1)$ (scalars act trivially). Now for $N = 5$ we have

$$N\binom{N}{2} - 1 = 5 \cdot 10 - 1 = 49 \quad \text{and} \quad 2\,(N^2 - 1) = 48,$$

so that we have found (at least) a one-parameter family of conjugacy classes of singular spaces. For $N > 5$ the difference between $N\binom{N}{2} - 1$ and $2\,(N^2 - 1)$ is even larger. ■

For large $N$ it seems impossible to classify maximal singular matrix spaces. The construction above already gives an infinite number of conjugacy classes, but there are many other sources of examples. For instance, for infinitely many $N$ there exists a maximal singular matrix space in $K^{N\times N}$ of constant dimension 8, at least if we assume that $K$ has characteristic 0 [12]. On the other hand, if the singular matrix space $\mathcal{A}$ has dimension at least $N^2 - N$, then it is a compression space with either a one-dimensional witness or all of $K^N$ as witness [10] (and hence of dimension exactly $N^2 - N$). A sharpening of this result is proved in [14] (see also [8]).

It should be noted that in many cases not even the dimension of such singular matrix spaces is known, for fixed values of the size and rank of the matrices. There is a considerable body of work devoted to giving lower and upper bounds for such dimensions, both in the case of bounded and constant rank, but these bounds are rarely sharp; see, among many other references, [15, 20, 38, 42] and the more recent works on skew-symmetric matrices of constant rank [4, 25].

Hence, the fact that $M_0$ represents a singular matrix space of dimension (at most) $n + 1$ does not much narrow down our search for good uniform determinantal representations, except in small cases discussed in section 5. However, for our constructions in section 4 we will only use compression spaces where the witness has dimension 1 or about $\frac{1}{2}N$, and our lower bounds on $N^*(n, d)$ are independent of the literature on singular matrix spaces.

**4. Main result and explicit constructions.** In this section we look at determinantal representations $M = M_0 + M_1$, where $M_0$ represents a compression space, which will lead us to

the proof of our main Theorem 1.3. We then analyze the case where the compression space has a one-dimensional witness.

The basic example is the following.

*Example* 4.1. Let $p = \sum_{i+j\leq 4} c_{ij}x^iy^j$ be the generic polynomial of degree $d = 4$ in $n = 2$ variables. It has the following uniform determinantal representation:

$$p = \det \begin{bmatrix} -x & 1 & & & & & & & \\ & -x & 1 & & & & & & \\ & & -x & 1 & & & & & \\ & & & -x & 1 & & & & \\ c_{00} & c_{10} & c_{20} & c_{30} & c_{40} & -y & & & \\ c_{01} & c_{11} & c_{21} & c_{31} & & 1 & -y & & \\ c_{02} & c_{12} & c_{22} & & & & 1 & -y & \\ c_{03} & c_{13} & & & & & & 1 & -y \\ c_{04} & & & & & & & & 1 \end{bmatrix}, \tag{2}$$

where the empty positions denote zeros. Let $M = M_0 + M_1$ be the matrix on the right-hand side. In this case, $M_0$ represents a compression space with witness $U = \langle e_5, \ldots, e_9\rangle_K$, which is mapped into $\langle e_6, \ldots, e_9\rangle_K$.

To verify the identity above without too many calculations, note that the five maximal subdeterminants of the $4\times 5$-block with $x$'s are, consecutively, $1, -x, x^2, -x^3, x^4$, and similarly for $y$. The matrix obtained from $M$ by deleting the column corresponding to $x^i$ and the row corresponding to $y^j$ has determinant $x^iy^j$.

This example extends to a uniform determinantal representation of size $2d + 1$ for the generic bivariate polynomial $p$ of degree $d$. We get $p = \det(M)$, where

$$M = (-1)^d \begin{bmatrix} M_x & 0 \\ L & M_y^T \end{bmatrix},$$

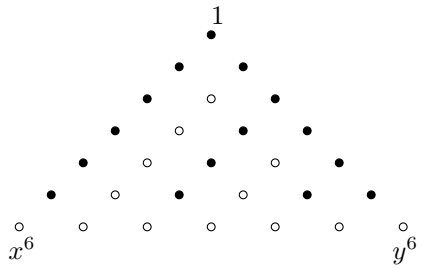
$M_x$ and $M_y$ are $d \times (d + 1)$ matrices with 1 on the first upper diagonal and $-x$ and $-y$, respectively, on the main diagonal, while $L$ is a $(d+1) \times (d+1)$ triangular matrix such that $\ell_{ij} = c_{j-1,i-1}$ for $i + j \leq d+2$ and 0 otherwise. We will slightly improve on the size $2d+1$ in Example 4.6. ■

To generalize Example 4.1, we will need the following fundamental notion.

Definition 4.2. *We say that a subspace $V \subseteq K[x_1, \ldots, x_n]$ is* connected to 1 *if it is nonzero and its intersections $V_e := V \cap F_e$ satisfy $F_1 \cdot V_e \supseteq V_{e+1}$ for each $e \geq 0$.*

Note that this implies that $V_0 = \langle 1\rangle$. The terminology is that of [27, Definition 2.5]. Moreover, the same notion already appears in [22], only defined for a set $S$ of monomials: such a set is called connected to 1 if $1 \in S$ and each nonconstant monomial in $S$ can be divided by some variable to obtain another monomial in $S$. The linear span of $S$ is then connected to 1 in our sense. Translating monomials to their exponent vectors, we will call a subset $S$ of $\mathbb{Z}^n_{\geq 0}$ *connected to* 0 if it contains 0 and for each $\alpha \in S \setminus \{0\}$ there exists an $i$ such that $\alpha - e_i \in S$, where $e_i$ is the $i$th standard basis vector.

*Example* 4.3. For $n = 2$ the following picture gives a space $V$, connected to 1 and spanned by the monomials marked with black vertices, such that $F_1 \cdot V = F_6$:




This is the construction of [34], which shows that there exists a uniform determinantal representation of the generic bivariate polynomial of degree $d$ of size $\frac{1}{4}\,d^2 + O(d)$ as $d \to \infty$. ■

Let $V$ be a finite-dimensional subspace of $K[x_1, \ldots, x_n]$ connected to 1. Choose a $K$-basis $f_1, \ldots, f_m$ of $V$ whose total degrees increase weakly. For each $i = 2, \ldots, m$ write

$$f_i = \sum_{j<i} \ell_{ij} f_j$$

for suitable elements $\ell_{ij} \in F_1$. Let $M_V$ be the $(m-1) \times m$-matrix whose $i$th row equals

$$(-\ell_{i1}, -\ell_{i2}, \ldots, -\ell_{i,i-1}, 1, 0, \ldots, 0).$$

Note that $M_V$ depends on the choice of basis and the $\ell_{ij}$, but we suppress this dependence in the notation, since the property of $M_V$ in the next lemma does not depend on the choice of basis.

Lemma 4.4. *The $K$-linear subspace of $K[x_1, \ldots, x_n]$ spanned by the $(m-1) \times (m-1)$-subdeterminants of $M_V$ equals $V$.*

*Proof.* By construction, $M_V$ has rank $m-1$ over the field $K(x_1, \ldots, x_n)$ and satisfies $M_V \cdot (f_1, \ldots, f_m)^T = 0$. By (a version of) Cramer's rule, the kernel of $M_V$ is also spanned by $(D_1, -D_2, \ldots, (-1)^{m-1} D_m)$, where $D_j$ is the determinant of the submatrix of $M_V$ obtained by removing the $j$th column. So these two vectors differ by a factor in $K(x_1, \ldots, x_n)$. Since $D_1 = 1 = f_1$ we find that they are, in fact, equal. Hence $\langle D_1, \ldots, D_m \rangle = V$ as claimed. ■

Proposition 4.5. *Let $V, W \subseteq K[x_1, \ldots, x_n]$ be subspaces connected to 1 such that $F_1 \cdot V \cdot W \supseteq F_d$. Then there exists a uniform determinantal representation of the generic $n$-variate polynomial of degree $d$ of size $-1 + \dim V + \dim W$.*

*Proof.* Set $m_1 := \dim V$ and $m_2 := \dim W$. Consider the matrix

$$M := \begin{bmatrix} M_V & 0 \\ L & M_W^T \end{bmatrix},$$

with $M_V$ and $M_W$ the matrices of sizes $(m_1 - 1) \times m_1$ and $(m_2 - 1) \times m_2$ from Lemma 4.4, and where $L = (\ell_{ij})_{ij}$ is an $m_2 \times m_1$-matrix to be determined. Note that the determinant of $M$ is linear in the entries of $L$. Indeed, setting $L = 0$ yields the singular matrix $M_0$, so $\det(M)$ contains no terms of degree 0 in the entries of $L$. Furthermore, deleting from $M$ two or more

of the first $m_1$ columns, we end up with a matrix that is singular since, when acting on rows, it maps the span of $\langle e_1, \ldots, e_{m_1-1}\rangle$ into a space of dimension at most $m_1 - 2$, so $\det(M)$ does not contain terms that are of degree $> 1$ in the entries of $L$.

Hence, the determinant equals $\sum_{ij} \pm \ell_{ij} D_j E_i$, where the $D_j$ are the maximal subdeterminants of $M_V$ and the $E_i$ are the maximal subdeterminants of $M_W$. By Lemma 4.4 we have $V = \langle D_1, \ldots, D_{m_1}\rangle_K$ and $W = \langle E_1, \ldots, E_{m_2}\rangle_K$. Hence, the assumption that $F_1 \cdot V \cdot W \supseteq F_d$ ensures that we can choose the $\ell_{ij} \in F_1$ in such a manner that the determinant of $M$ equals the generic polynomial $p$. ■

*Example* 4.6. Example 4.1 can be slightly improved to a representation of size $2d - 1$ by taking $V = \langle 1, x, \ldots, x^{d-1}\rangle$ and $W = \langle 1, y, \ldots, y^{d-1}\rangle$; note that, indeed, $F_1 \cdot V \cdot W \supseteq F_d$. A representation of size $2d - 1$ for the polynomial $p$ from (2) is

$$p = -\det \begin{bmatrix} -x & 1 & & & & & \\ & -x & 1 & & & & \\ & & -x & 1 & & & \\ c_{00} & c_{10} & c_{20} & c_{30} + c_{40}x & -y & & \\ c_{01} & c_{11} & c_{21} + c_{31}x & & 1 & -y & \\ c_{02} + c_{03}y & c_{12} + c_{22}x & & & & 1 & -y \\ c_{13}x + c_{04}y & & & & & & 1 \end{bmatrix}. \tag{3}$$

We expect that the factor 2 cannot be improved, but do not know how to prove this. ■

*Remark* 4.7. A representation of the form (3) can also be obtained from the linearizations based on dual basis from [36]. There, linearizations of a univariate polynomial are presented that use the basis of the form $\varphi_i(x)\psi_j(x)$, where $\varphi_i$ and $\psi_j$ are polynomials. If we use the same approach for a bivariate polynomial with the standard basis $\varphi_i = x^i$ and $\psi_j = y^j$, we get a representation of the form (3) up to permutations of rows and columns. ■

We will now prove our main theorem.

*Proof of Theorem* 1.3, *lower bound.* We will prove the statement that, for fixed $n$, the determinantal complexity of any sufficiently general polynomial $p \in F_d$ is bounded from below by a constant times $d^{n/2}$; this implies the same lower bound for $N^*(n, d)$. Consider the polynomial map $\varphi : F_1^{N\times N} \to F_N$ which takes an $N \times N$-matrix of affine-linear forms to its determinant.

First assume that $K$ is infinite. Then the Krull dimension of $\operatorname{im}\varphi$, with the topology induced from the Zariski topology on the finite-dimensional vector space $F_N$, is at most that of the domain $F_1^{N\times N}$ of $\varphi$, namely, $N^2(n + 1)$. On the other hand, if $\operatorname{im}\varphi$ contains an open, dense subset of $F_d$, then its dimension must be at least $\dim F_d$. We find the inequality

$$N^2(n + 1) \geq \dim F_d = \frac{d^n}{n!} + O(d^{n-1}), \tag{4}$$

from which the existence of $C_1$ follows.

On the other hand, if $K$ is finite with $q$ elements, then if $\operatorname{im}\varphi$ contains a positive fraction $c$ of $F_d$ for $d \to \infty$, we obtain the inequality $N^2(n + 1) \geq c\frac{d^n}{n!} + O(d^{n-1})$ by inspecting the exponents of $q$. ■

*Remark* 4.8. In the proof of the lower bound we can slightly improve the constant in front of $d^{n/2}$ as follows: by multiplying a representation of a nonzero polynomial $p$ from the left with a nonidentity matrix in $\mathrm{SL}_N(K)$, we obtain a distinct determinantal representation of $p$. Thus if $K$ is infinite, the fibers of $\varphi$ have dimension at least $N^2-1$, and we find the stronger inequality $N^2(n+1)-N^2+1 \geq \dim F_d$. A similar argument holds for finite $K$. One can perhaps repeat this argument with right multiplication, so as to peel off another term $N^2-1$ from the left-hand side, and use the group $\mathrm{AGL}_n(K)$ from section 2 to peel off another $n^2+n$—but for this one would need a more careful analysis of the stabilizer in $\mathrm{SL}_N(K)\times\mathrm{SL}_N(K)\times\mathrm{AGL}_n(K)$ of a determinantal representation. ■

*Proof of Theorem* 1.3*, upper bound.* We first give a simple construction for even $n$ that we upgrade later into a construction for odd $n$.

Assume that $n=2m$ with $m\in\mathbb{Z}_{\geq 1}$. Let $V$ be the space of polynomials in $x_1,\dots,x_m$ of degree at most $d-1$, and let $W$ be the space of polynomials in $x_{m+1},\dots,x_n$ of degree at most $d-1$. Then $V$ and $W$ are connected to 1 and we have $F_1\cdot V\cdot W\supseteq F_d$, so that by Proposition 4.5 we have $N^*(n,d)\leq -1+\dim V+\dim W$. Now compute

$$\dim V=\dim W=\binom{m+d-1}{m}=\frac{d^{n/2}}{(n/2)!}+O(d^{m-1}).$$

This implies the existence of $C_2$ for even $n$.

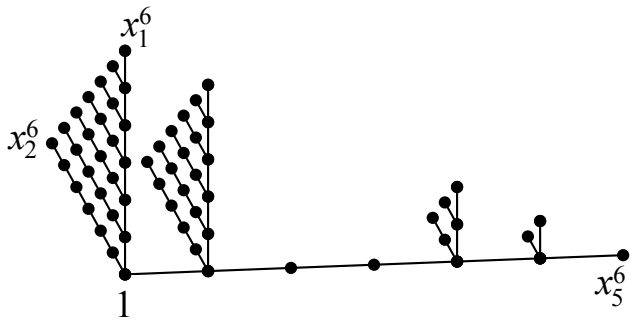
For $n=2m+1$ with $m\in\mathbb{Z}_{\geq 1}$ we upgrade the above construction using an idea for which we thank Aart Blokhuis. For $i=0,1$ let $B_i\subseteq\mathbb{Z}_{\geq 0}$ denote the set of nonnegative integers that can be expressed as $\sum_{j=0}^{e} b_j 2^{2j+i}$ with $b_j\in\{0,1\}$, i.e., whose binary expansions have ones only at even positions (for $i=0$, counting the least significant bit as zeroth position) or only at odd positions (for $i=1$). Observe that $B_0+B_1=\mathbb{Z}_{\geq 0}$ and that both $B_0$ and $B_1$ contain at most a constant times $\sqrt{d}$ of the first $d$ nonnegative integers for every $d$—they have "dimension 1/2". Now set $A_i=B_i\cap[0,d]$ for $i=0,1$ so that $A_0+A_1\supseteq\mathbb{Z}_{\geq 0}\cap[0,d]$. One can show that the number of elements of $A_i$ is at most $\sqrt{3d+1}$, and to see that the bound is sharp one may consider $d$ of the form $d=1+2^2+2^4+\cdots+2^{2s}$ and send $s$ to infinity.

Let $U_{0j}$ be the set of monomials in $x_1,\dots,x_m$ of degree at most $j$, and let $U_{1j}$ be the set of monomials in $x_{m+1},\dots,x_{n-1}$ of degree at most $j$. For $i=0,1$ now set $V_i$ as the space spanned by the monomials of the form $x_n^k\cdot\varphi$, where $k\in A_i$ and $\varphi\in U_{i,d-1-k}$, and the monomials $1,x_n,\dots,x_n^{d_i}$, where $d_i=\max(A_i)$. Then $V_0$ and $V_1$ are connected to 1 and $F_1\cdot V_0\cdot V_1\supseteq F_d$. For $i=0,1$ we get

$$\dim V_i\leq\sqrt{3d+1}\binom{m+d-1}{m}=\sqrt{3}\,\frac{d^{n/2}}{\lfloor n/2\rfloor!}+O(d^{m-1/2}),$$

which implies the existence of $C_2$ for odd $n$. ■

*Example* 4.9. For $n=5$ the following picture shows the monomials that span the space $V_0$ for $d=7$. Note that in this case $B_0=\{0,1,4,5\}$ and $B_1=\{0,2\}$ (the circles indicate the monomials and the edges show that $V_0$ is connected to 1):




The uniform determinantal representation constructed in the preceding proof involves a compression space with a witness of dimension roughly $N/2$. One can ask what happens to the bounds if we require the compression space to have a one-dimensional witness (or, dually by transposition, with a full-dimensional witness). The uniform representation of the bivariate quadric in Example 4.3 is of this form. It turns out that such representations have a bigger size than the general case, but in turn the constant factor that we find in the bound is sharp.

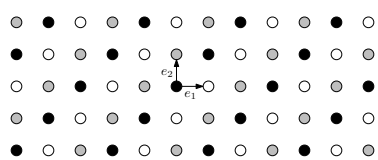
Theorem 4.10. *For fixed $n$, there exists a determinantal representation $M = M_0 + M_1$ of the generic $n$-variate polynomial of degree $d$ of size $\frac{1}{n \cdot n!}\, d^n + O(d^{n-1})$ such that the singular matrix space represented by $M_0$ is a compression space with a one-dimensional witness. Moreover, under this latter additional condition on $M_0$, the bound is sharp.*

*Proof.* By Proposition 4.5 (with $W = \langle 1 \rangle$) it suffices to show the existence of a subspace $V \subseteq F_d$ connected to 1 and such that $F_1 \cdot V = F_d$, where $\dim V = \frac{1}{n} \dim F_d + O(d^{n-1})$. We will, in fact, show that $V$ can be chosen to be spanned by monomials.

First, recall that there exists a lattice $\Lambda$ in $\mathbb{Z}^{n-1}$ such that $\mathbb{Z}^{n-1}$ is the disjoint union of $\Lambda$ and its cosets $e_i + \Lambda$ for $i = 1, \ldots, n-1$, namely, the root lattice of type $A_n$ generated by the rows of the $(n-1) \times (n-1)$-Cartan matrix

$$\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix},$$

where the empty positions represent zeros [5, Planche 1]. In particular, the index of $\Lambda$ in $\mathbb{Z}^{n-1}$ equals $n$. For example, if $n = 3$, here is the root lattice $\Lambda$ (in black) and its two cosets (in gray and white):




Now let $\Delta_d$ be the simplex in $\mathbb{R}^n$ with vertices $0, de_1, \ldots, de_n$, for $i = 1, \ldots, n$ let $S_i$ be the set of lattice points in $\Delta_d$ that have $i$th coordinate zero, and set $S_0 := (\mathbb{Z} \times \Lambda) \cap \Delta_d$. Define

$$S := S_1 \cup S_2 \cup \cdots \cup S_n \cup S_0,$$

a subset of the lattice points in $\Delta_d$. We claim that $S$ is connected to 0. Indeed, for each $i = 1, \ldots, n$ the set $S_i$ is connected to 0, and from each point $\alpha$ in $S_0$ one can walk within $S_0$ to $S_1$ by subtracting $\alpha_1$ times an $e_1$.

Next, we claim that for each $\alpha \in \Delta_d \cap \mathbb{Z}^n$ there exists a $\beta \in S$ with $\alpha - \beta \in \{0, e_2, \ldots, e_n\}$. Indeed, there is a (unique) $\beta'$ with this property in $\mathbb{Z} \times \Lambda$. If this $\beta'$ has nonnegative entries, then set $\beta := \beta' \in S_0$. Otherwise, $\alpha$ itself has a zero entry, say on the $i$th position, and we set $\beta := \alpha \in S_i$.

Furthermore, for $i = 1, \ldots, n$ the set $S_i$ contains $O(d^{n-1})$ vertices, and $S_0$ contains $\frac{1}{n} \cdot \frac{1}{n!} d^n + O(d^{n-1})$ vertices. This concludes the construction—note that in the construction of Proposition 4.5 the matrix $M_0$ has a zero row, so that it represents a compression space with a one-dimensional witness.

For sharpness, assume that $M = M_0 + M_1$ is a uniform determinantal representation of size $N$ such that the singular matrix space represented by $M_0$ is a compression space with a one-dimensional witness. After a choice of basis of $K^N$, we may assume that the first row of $M_0$ is identically zero; write $M_0 = [0|M_0']^T$ accordingly. Let $u^T$ be the first row of $M_1$ and write $M_1 = [u|M_1']^T$. Then we have

$$p = \sum_{|\alpha| \leq d} c_\alpha x^\alpha = \det[u|M_0' + M_1'].$$

Let $D_1, \ldots, D_N$ denote the $(N-1) \times (N-1)$-subdeterminants of $M_0'$. By Lemma 2.6, the space $V$ spanned by these satisfies $F_1 \cdot V \supseteq F_d$. This already gives a lower bound for $\dim(V)$ equal to $d^n/((n+1)n!) + O(d^{n-1})$. To improve the $n+1$ in the denominator into an $n$, we observe that by Cramer's rule the map

$$F_1^N \to K[x_1, \ldots, x_n], \ (\ell_1, \ldots, \ell_N) \mapsto \sum_i (-1)^i \ell_i D_i$$

has every column of $M_0'$ in its kernel. These columns are linearly independent over $K$ (indeed over $K(x_1, \ldots, x_n)$; see Lemma 2.5). We conclude that

$$N \cdot \dim F_1 - (N-1) \geq \dim F_d, \tag{5}$$

so that

$$N \geq ((\dim F_d) - 1)/n = d^n/(n \cdot n!) + O(d^{n-1}),$$

as desired. ■

**5. Small $n$ and $d$.** In this section we give several uniform representations of, to the best of our knowledge, the smallest possible size for cases where $n$ and $d$ are small. We start with the two cases where we can compute $N^*(n, d)$ exactly.

Proposition 5.1. $N^*(2, 2) = 3$.

*Proof.* Taking $V = \langle 1, x, y \rangle$ and $W = \langle 1 \rangle$ in Proposition 4.5 we see that $N^*(2, 2) \leq 3$; this is the representation of Example 1.1. Suppose that a uniform determinantal representation $M = M_0 + M_1$ of size $N = 2$ exists. Then, by Example 3.2, after acting with $\mathrm{SL}_2(K) \times \mathrm{SL}_2(K)$

and transposing if necessary, we may assume that the singular space represented by $M_0$ is a compression space with a one-dimensional witness. But then (5) reads

$$2 \cdot 3 - 1 = N \cdot \dim F_1 - (N-1) \geq \dim F_2 = 6,$$

a contradiction. Hence $N^*(2,2) = 3$. ■

Proposition 5.2. $N^*(3,2) = 4$.

*Proof.* Taking $V = \langle 1, x, y, z \rangle$ and $W = \langle 1 \rangle$ in Proposition 4.5 we see that $N^*(3,2) \leq 4$. Suppose that a uniform representation of size $N = 3$ exists. Up to transposition, there are three possibilities for the singular space $\mathcal{A}$ represented by $M_0$; see Example 3.2 (where the third compression space is conjugate to the transpose of the first):

1. Assume that $\mathcal{A}$ is a compression space with a one-dimensional witness, so that after acting with $\mathrm{SL}_3(K) \times \mathrm{SL}_3(K)$ we have

$$M_0 = \begin{bmatrix} 0 & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix}.$$

Let $D_j$ denote the determinant of the minor of $M_0$ obtained by deleting the first row and $j$th column. Then the linear map

$$\Omega : F_1^3 \mapsto K[x,y,z],\ (l_1, l_2, l_3) \mapsto l_1 D_1 - l_2 D_2 + l_3 D_3$$

has $F_2 \subseteq \operatorname{im} \Omega$. Now inequality (5) reads

$$3 \cdot 4 - 2 = N \cdot \dim F_1 - (N-1) \geq \dim F_2 = 10,$$

which holds with equality. This means that, in fact, $\operatorname{im} \Omega$ must *equal* $F_2$. In particular, $D_1, D_2, D_3$ must all be of degree one (or else $\operatorname{im} \Omega$ would contain cubic polynomials). The image of $\Omega$ depends only on the span $V := \langle D_1, D_2, D_3 \rangle \subseteq F_1$. If $1 \notin V$, then there exists an affine transformation in $\mathrm{AGL}_3(K)$ that maps $V$ into a subspace of $\langle x, y, z \rangle$. Then $1 \notin F_1 \cdot V = \operatorname{im} \Omega$, a contradiction. If $1 \in V$, then after an affine transformation we find $\langle 1 \rangle \subseteq V \subseteq \langle 1, x, y \rangle$. In that case, $z^2 \notin F_1 \cdot V$, another contradiction.

2. Assume that $\mathcal{A}$ is a compression space with a two-dimensional witness, so that after row and column operations we have

$$M_0 = \begin{bmatrix} 0 & 0 & q \\ 0 & 0 & r \\ s & t & * \end{bmatrix},$$

where $q, r, s, t \in F_1$. Write $M_1 = (m_{ij})_{ij}$. Using that $\det(M)$ is assumed to be linear in the $c_\alpha$s, we find that

$$\det(M) = -m_{11} rt + m_{12} rs + m_{21} qt - m_{22} qs.$$

Consequently, setting $V_1 := \langle q, r \rangle$ and $V_2 := \langle s, t \rangle$, we have $F_1 \cdot V_1 \cdot V_2 \supseteq F_2$. If $1 \notin V_1$, then by acting with a suitable element of $\mathrm{AGL}_3(K)$ we achieve that $V_1 \subseteq \langle x, y, z \rangle$. But then $F_1 \cdot V_1 \cdot V_2 \not\ni 1$. The same applies when $1 \notin V_2$. On the other hand, if $1 \in V_1 \cap V_2$, then by an element in $\mathrm{AGL}_3(K)$ we achieve that $\langle 1 \rangle \subseteq V_1, V_2 \subseteq \langle 1, x, y \rangle$. In that case, $z^2 \notin F_1 \cdot V_1 \cdot V_2$.

3. Finally, assume that $\mathcal{A}$ is conjugate to a space of skew-symmetric matrices, so that after conjugation
$$M_0 = \begin{bmatrix} 0 & q & r \\ -q & 0 & s \\ -r & -s & 0 \end{bmatrix},$$
where $q, r, s \in F_1$. Set $V := \langle q, r, s\rangle \subseteq F_1$. Then the space spanned by the $2 \times 2$-determinants of $M_0$ is $V \cdot V$ of dimension at most 6. Moreover, we have $F_1 \cdot V \cdot V \supseteq F_2$. If $1 \notin V$, then by acting with $\mathrm{AGL}_3(K)$ we achieve that $V \subseteq \langle x, y, z\rangle$, and hence $1 \notin F_1 \cdot V \cdot V$. If, on the other hand, $1 \in V$, then we achieve that $\langle 1\rangle \subseteq V \subseteq \langle 1, x, y\rangle$, and $z^2 \notin F_1 \cdot V \cdot V$.

In each of these cases we arrive at a contradiction. Consequently, $N^*(3, 2) = 4$ as claimed. ■

The proofs above use the classification of spaces of small singular matrices in an essential manner, as well as the action of $\mathrm{AGL}_n(K)$ on uniform determinantal representations. We conjecture that $N^*(4, 2) = 5$, and that this can still be proved in the same manner, using the classification of $4 \times 4$-singular matrix spaces from [14]. But as Theorem 3.4 shows, fundamentally new ideas will be needed to prove lower bounds in larger situations.

For some pairs of small $n$ and $d$ we now give the smallest uniform representations that we have been able to find. For the constructions we use Proposition 4.5 with subspaces $V, W \subseteq K[x_1, \ldots, x_n]$ spanned by the monomials and connected to 1. First, we give in Table 1 the minimal sizes known to us of uniform determinantal representations for some small values of $n$ and $d$.

**Table 1**

*Minimal known sizes of uniform determinantal representations we have been able to construct for $n$-variate polynomials of degree $d$; cf. Table 2.*

| $n$ | $d = 2$ | $d = 3$ | $d = 4$ | $d = 5$ | $d = 6$ | $d = 7$ | $d = 8$ | $d = 9$ |
|---|---|---|---|---|---|---|---|---|
| 2 | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 17 |
| 3 | 4 | 7 | 10 | 14 | 18 | 22 | 27 | 34 |
| 4 | 5 | 9 | 14 | 19 | 26 | 34 | 44 | |
| 5 | 6 | 11 | 18 | 26 | | | | |
| 6 | 7 | 13 | 22 | 33 | | | | |
| 7 | 8 | 15 | 27 | 39 | | | | |
| 8 | 9 | 17 | 32 | | | | | |

The corresponding representations for the entries in Table 1 for $n = 2$, which are of size $2d - 1$, are given in Example 4.6. For $d = 2$ we take $V = \langle 1, x_1, \ldots, x_n\rangle$ and $W = \langle 1\rangle$; therefore, $N^*(n, 2) \leq n + 1$, while for $d = 3$ we can take $V = W = \langle 1, x_1, \ldots, x_n\rangle$, and hence $N^*(n, 3) \leq 2n - 1$. In Table 2 we give sets $V$ and $W$ for the remaining nonzero entries in Table 1. The subspaces $V$ and $W$ have the form $V = V_0 \cup V_1$ and $W = W_0 \cup W_1$, where

$$\begin{aligned} V_0 &= \langle 1, x_1, \ldots, x_n, \ldots, x_1^e, \ldots, x_n^e\rangle, \\ W_0 &= \langle 1, x_1, \ldots, x_n, \ldots, x_1^f, \ldots, x_n^f\rangle \end{aligned} \tag{6}$$

for $e = \lceil (d-1)/2\rceil$ and $f = \lfloor (d-1)/2\rfloor$, which yields $d - 1 = e + f$. For clarity and brevity, the variables $x, y, z, w, u, v, q, s$ in Table 2 stand for $x_1, \ldots, x_8$, respectively.

**Table 2**

*List of monomials in $V_1$ and $W_1$ that, together with $V_0$ and $W_0$ from (6), lead to uniform representations of sizes as in Table 1.*

| $n$ | $d$ | $V_1$ | $W_1$ |
|---|---|---|---|
| 3 | 4 | – | – |
| 3 | 5 | – | $xy$ |
| 3 | 6 | – | $xy$, $x^2y$ |
| 3 | 7 | – | $x^2y$, $y^2z$, $z^2x$ |
| 3 | 8 | – | $x^2y$, $y^2z$, $z^2x$, $x^2y^2$, $z^2w^2$ |
| 3 | 9 | $x^3y$, $y^3z$, $z^3x$ | $x^2y$, $x^2z$, $y^2z$, $x^2y^2$, $x^2z^2$, $y^2z^2$ |
| 4 | 4 | – | $xy$ |
| 4 | 5 | – | $xy$, $zw$ |
| 4 | 6 | $x^2y$, $y^2z$, $z^2w$ | $xy$, $x^2y$, $zw$ |
| 4 | 7 | $x^2y$, $y^2x$, $z^2w$, $w^2x$, $xy$ | $x^2z$, $xz^2$, $y^2w$, $yw^2$ |
| 4 | 8 | $x^2y$, $x^2y^2$, $z^2x$, $x^3y$, $y^3z$, $z^3w$, $w^3x$ | $xy$, $xyz$, $xyw$, $y^2z$, $z^2w$, $w^2x$, $w^2y$, $x^2z$ |
| 5 | 4 | – | $xy$, $zw$ |
| 5 | 5 | $xy$, $yz$, $zw$ | $wu$, $xu$ |
| 6 | 4 | – | $xy$, $zw$, $uv$ |
| 6 | 5 | $xy$, $zw$, $uv$, $wy$ | $yz$, $wu$, $xv$, $xz$ |
| 7 | 4 | – | $xy$, $zw$, $uv$, $xq$, $yq$ |
| 7 | 5 | $xy$, $zw$, $uv$, $wy$, $qu$ | $yz$, $wu$, $vq$, $xz$, $wx$ |
| 8 | 4 | – | $xy$, $yz$, $xz$, $wu$, $wv$, $uv$, $qs$ |

*Example* 5.3. To show how things get complicated, let us consider the construction for $d = 4$. We take $V = \langle 1, x_1, \dots, x_n, x_1^2, \dots, x_n^2 \rangle$ and

$$W = \langle 1, x_1, \dots, x_n, x_{\alpha_1} x_{\beta_1}, \dots, x_{\alpha_m} x_{\beta_m} \rangle,$$

where $1 \le \alpha_i < \beta_i \le n$ and $m$ is as small as possible. If we take all possible pairs $x_\alpha x_\beta$, then clearly $F_1 \cdot V \cdot W \supseteq F_4$, while on the other hand, when $m = 0$, $F_1 \cdot V \cdot W$ does not contain any monomials of the form

$$x_i \, x_j \, x_k \, x_\ell \tag{7}$$

for $1 \le i < j < k < \ell \le n$. We need a minimal set of $x_\alpha x_\beta$ to cover all possible monomials (7), which is related to the following covering problem.

Given positive integers $r \le k \le n$, we say that a system $S$ of $r$-subsets (called blocks) of $\{1, \dots, n\}$ is called a Turán $(n, k, r)$-system if every $k$-subset of $\{1, \dots, n\}$ contains at least one block from $S$ [37]. The minimum size of $S$ is called the Turán number $T(n, k, r)$.

In our case, additional terms $x_{\alpha_1} x_{\beta_1}, \dots, x_{\alpha_m} x_{\beta_m}$ form a Turán $(n, 4, 2)$-system. While for most cases only upper and lower bounds for $T(n, k, r)$ are known, Turán proved that

$$T(n, 4, 2) = mn - 3\,\frac{m(m+1)}{2}, \tag{8}$$

where $m = \lfloor n/3 \rfloor$; see [37, Formula (25)]. To obtain the minimal set one has to divide $\{1, \dots, n\}$ into three nearly equal groups (their sizes do not differ for more than one) and then

take all pairs $x_\alpha x_\beta$ such that $\alpha$ and $\beta$ belong to the same group. As a result, such construction gives a uniform representation of size $N$, where $N = \frac{1}{6}\,n^2 + O(n)$, which therefore implies $N^*(n,4) \le \frac{1}{6}\,n^2 + O(n)$. ∎

**6. Matrix polynomials.** Suppose that we have a uniform representation $M$ of $p_{n,d}$ as in (1), and write

$$M = M_0 + M_1 = M_0 + \sum_{|\alpha|\le d} c_\alpha M_\alpha, \tag{9}$$

where each $M_\alpha \in F_1^{N\times N}$. Now consider the matrix polynomial (cf. (1))

$$P_{n,d} = \sum_{|\alpha|\le d} x^\alpha C_\alpha,$$

where $C_\alpha$ is a $k \times k$ matrix. We will show that under certain assumptions we can construct from $M$ a matrix $\widetilde{M}$ that represents $P_{n,d}$ in the sense that $\det(\widetilde{M}) = \det(P_{n,d})$. We obtain $\widetilde{M}$ from $M$ in the following way. Each element of the form $a_0 + a_1x_1 + \cdots + a_nx_n$ is replaced by the $k \times k$ matrix $(a_0 + a_1x_1 + \cdots + a_nx_n)I_k$, where $I_k$ is the $k \times k$ identity, and each $c_\alpha$ is replaced by the matrix $C_\alpha$.

Theorem 6.1. *Let (9) be a uniform representation of the generic polynomial (1) of degree $d$ in $n$ variables and assume that there exist matrices $Q$ and $Z$, whose elements are polynomials in $x_1, \ldots, x_n$ such that $\det(Q) = \det(Z) = 1$, and $QMZ$ is a triangular matrix with one diagonal element equal to $p_{n,d}$ and all other diagonal elements equal to 1. Then*

$$\widetilde{M} = M_0 \otimes I_k + \sum_{|\alpha|\le d} M_\alpha \otimes C_\alpha$$

*is a representation for the matrix polynomial $P_{n,d}$, i.e., $\det(\widetilde{M}) = \det(P_{n,d})$.*

*Proof.* It is easy to see that $(Q\otimes I_k)\widetilde{M}(Z\otimes I_k)$ is a block triangular matrix with one diagonal block $P_{n,d}$ while all other diagonal blocks are equal to $I_k$. Since $\det(Q\otimes I_k) = \det(Z\otimes I_k) = 1$, it follows that $\det(\widetilde{M}) = \det(P_{n,d})$. ∎

*Example* 6.2. Theorem 6.1 applies to the uniform representation (2). Indeed, take

$$Q = \begin{bmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 1 & y & y^2 & y^3 & y^4 \\ & & & & 1 & y & y^2 & y^3 \\ & & & & & 1 & y & y^2 \\ & & & & & & 1 & y \\ & & & & & & & 1 \end{bmatrix}, \qquad Z = \begin{bmatrix} 1 & & & & & & & \\ x & 1 & & & & & & \\ x^2 & x & 1 & & & & & \\ x^3 & x^2 & x & 1 & & & & \\ x^4 & x^3 & x^2 & x & 1 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{bmatrix},$$

then

$$QMZ = \begin{bmatrix} & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ p & \times & \times & \times & c_{40} & & & \\ \times & \times & \times & c_{31} & & 1 & & \\ \times & \times & c_{22} & & & & 1 & \\ \times & c_{13} & & & & & & 1 \\ c_{04} & & & & & & & 1 \end{bmatrix}.$$

It is easy to see that there exist permutation matrices $P_L$ and $P_R$ such that

$$P_L(QMZ)P_R = \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ c_{40} & \times & \times & \times & p & & & \\ & \times & \times & c_{31} & \times & 1 & & \\ & \times & c_{22} & & \times & & 1 & \\ & c_{13} & & & \times & & & 1 \\ & & & & c_{04} & & & & 1 \end{bmatrix}$$

is triangular and has the diagonal which satisfies Theorem 6.1. Therefore, we can apply (2) for matrix polynomials by using block matrices. This can be generalized to a uniform representation of size $2d+1$ of the form (2). In a similar way we can show that this also holds for representations of the form (3) of size $2d-1$. ■

Unfortunately, not all uniform determinantal representations induce a determinantal representation of a general matrix polynomial in this manner. As a counterexample, let $M$ be such a uniform determinantal representation of the polynomial $p_{n,d}$, $|\alpha|, |\beta| \leq d$, and construct a representation of larger size

$$M' = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}, \qquad \text{with} \quad N = \begin{bmatrix} 0 & c_\alpha & c_\beta & 1 \\ -c_\alpha & 0 & 1 & 0 \\ -c_\beta & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

Then $\det(M') = \det(M)\det(N) = p_{n,d}(1 + c_\alpha c_\beta - c_\beta c_\alpha) = p_{n,d}$, but $\widetilde{M'}$ is not a representation for the matrix polynomial $P_{n,d}$ as the coefficient matrices $C_\alpha$ and $C_\beta$ do not commute in general. This motivates the following definition.

Definition 6.3. *A uniform determinantal representation $M$ is* minimal *if there do not exist constant matrices $P$ and $Z$ such that* $\det(P) = \det(Z) = 1$ *and*

$$PMZ = \begin{bmatrix} * & * \\ 0 & M_2 \end{bmatrix}, \qquad \textit{where } M_2 \textit{ is square with} \quad \det(M_2) = 1.$$

We speculate that each minimal uniform representation gives rise to a representation for a matrix polynomial.

**7. Numerical experiments.** Recently, a new numerical approach for computing roots of systems of bivariate polynomials has been proposed in [34]. The main idea is to treat the system as a two-parameter eigenvalue problem using determinantal representations.

Suppose that we are looking for roots of a system of bivariate polynomials

$$\begin{aligned} p &= \textstyle\sum_{i+j\le d_1} \alpha_{ij} x^i y^j = 0, \\ q &= \textstyle\sum_{i+j\le d_2} \beta_{ij} x^i y^j = 0, \end{aligned} \tag{10}$$

where $p$ and $q$ are polynomials of degree $d_1$ and $d_2$ over $\mathbb{C}$. Let $P = A_0 + xA_1 + yA_2$ and $Q = B_0 + xB_1 + yB_2$, where $A_0, A_1, A_2 \in \mathbb{C}^{N_1\times N_1}$ and $B_0, B_1, B_2 \in \mathbb{C}^{N_2\times N_2}$, with $\det(P) = p$ and $\det(Q) = q$, be determinantal representations of $p$ and $q$, respectively. Then a root $(x, y)$ of (10) is an eigenvalue of the two-parameter eigenvalue problem

$$\begin{aligned} (A_0 + xA_1 + yA_2)\, u &= 0, \\ (B_0 + xB_1 + yB_2)\, v &= 0, \end{aligned} \tag{11}$$

where $u \in \mathbb{C}^{N_1}$ and $v \in \mathbb{C}^{N_2}$ are nonzero vectors. The standard way to solve (11) is to consider a joint pair of generalized eigenvalue problems [1]

$$\begin{aligned} (\Delta_1 - x\Delta_0)\, w &= 0, \\ (\Delta_2 - y\Delta_0)\, w &= 0, \end{aligned} \tag{12}$$

where

$$\Delta_0 = A_1 \otimes B_2 - A_2 \otimes B_1, \quad \Delta_1 = A_2 \otimes B_0 - A_0 \otimes B_2, \quad \Delta_2 = A_0 \otimes B_1 - A_1 \otimes B_0,$$

and $w = u \otimes v$.

In this particular application we can expect that the pencils in (12) are singular, i.e., $\det(\Delta_1 - x\Delta_0) \equiv 0$ and $\det(\Delta_1 - y\Delta_0) \equiv 0$. Namely, by Bézout's theorem a generic system (10) has $d_1 d_2$ solutions, while a generic problem (11) has $N_1 N_2$ eigenvalues. Unless $(d_1, d_2) = (N_1, N_2)$, both pencils in (12) are singular. In this case we first apply the staircase algorithm from [28] to extract the finite regular eigenvalues. The method returns smaller matrices $\widetilde{\Delta}_0$, $\widetilde{\Delta}_1$, and $\widetilde{\Delta}_2$ (of size $d_1 d_2 \times d_1 d_2$ for a generic (10)) such that $\widetilde{\Delta}_0$ is nonsingular and $\widetilde{\Delta}_0^{-1}\widetilde{\Delta}_1$ and $\widetilde{\Delta}_0^{-1}\widetilde{\Delta}_2$ commute. From

$$\begin{aligned} (\widetilde{\Delta}_1 - x\widetilde{\Delta}_0)\, \widetilde{w} &= 0, \\ (\widetilde{\Delta}_2 - y\widetilde{\Delta}_0)\, \widetilde{w} &= 0, \end{aligned}$$

we compute the eigenvalues $(x, y)$ using a variant of the QZ algorithm [18] and thus obtain the roots of (10).

The above approach is implemented in the MATLAB package BiRoots [33] together with the two determinantal representations from [34]. The first one, which we refer to as `Lin1`, is a uniform one from Example 4.3 of size $\frac{1}{4}\, d^2 + O(d)$ for a polynomial of degree $d$. The second one, which we refer to as `Lin2`, is not uniform and involves some computation to obtain a smaller size $\frac{1}{6}\, d^2 + O(d)$. Although the construction of `Lin2` is more time consuming, this pays off later, when the staircase algorithm is applied to (12).

Table 3 shows the sizes of determinantal representations for polynomials of small degree. As expected, the new uniform determinantal representation of size $2d-1$, which we refer to as `MinUnif`, returns smaller matrices, which reflects later in faster computational times. It is also important that `Lin1` and `MinUnif` return real matrices for polynomials with real coefficients, which is not true for `Lin2`.

**Table 3**

*Size of the matrices for* `Lin1` *and* `Lin2` *for bivariate polynomials ($n = 2$) and various degrees $d$.*

| Method | $d=3$ | $d=4$ | $d=5$ | $d=6$ | $d=7$ | $d=8$ | $d=9$ | $d=10$ | $d=11$ | $d=12$ |
|---|---|---|---|---|---|---|---|---|---|---|
| `Lin1` | 5 | 8 | 11 | 15 | 19 | 24 | 29 | 35 | 41 | 48 |
| `Lin2` | 3 | 5 | 8 | 10 | 13 | 17 | 20 | 24 | 29 | 34 |
| `MinUnif` | 5 | 7 | 9 | 11 | 13 | 15 | 17 | 19 | 21 | 23 |

It has been reported in [34] that the determinantal representation approach for solving systems of bivariate polynomials is competitive for polynomials of degree 9 or less. As we show below, the new uniform representation `MinUnif` extends this to degree 15 and, in addition, performs better than the existing representations for polynomials of degree 6 or more.

The numerical algorithm that uses `MinUnif` has complexity $\mathcal{O}(d^6)$, which is also the complexity of some numerical algorithms for systems of bivariate polynomials that are based on a resultant; see, e.g., [31]. Such an approach is thus not efficient for polynomials of large degree. We remark that there also exist probabilistic symbolic algorithms (see, e.g., [23, 26]) that aim for a smaller complexity, such as one not much higher than $\mathcal{O}(d^4)$.

In [34] the approach has been compared numerically to the following state-of-the-art numerical methods for polynomial systems: `NSolve` in Mathematica 9 [43], BertiniLab 1.4 [32] running Bertini 1.5 [2], NAClab 3.0 [44], and `PHCLab` 1.04 [16] running PHCpack 2.3.84 [40], which turned out to be the fastest of these methods. To show the improved performance of the new determinantal representation, we compare `MinUnif` to `Lin1`, `Lin2`, and `PHCLab` in Table 4. For each $d$ we run the methods on the same set of 50 real and 50 complex random polynomial systems of degree $d$ and measure the average time. For `Lin1` and `MinUnif`, where determinantal representations have real matrices for real polynomials, we report separate results for polynomials with real and complex coefficients. The timings for `Lin1` and `Lin2` are given only for $n \leq 10$ as for larger $n$ these two linearizations are no longer competitive.

Of course, the computational time is not the only important factor; we also have to consider the accuracy and reliability. In each step of the staircase algorithm a rank of a matrix has to be estimated numerically, which is a delicate task. After several steps it may happen that the gap between the important singular values and the meaningful ones that should be zero in exact computation virtually disappears. In such a case the algorithm fails and does not return any roots. As the number of steps in the staircase algorithm increases with degree of the polynomials, such problems occur more often for polynomials of large degree. A heuristic that usually helps in such cases is to apply the algorithm on a transformed system

$$\begin{aligned} \widetilde{p} &:= \quad cp + sq = 0, \\ \widetilde{q} &:= -sp + cq = 0 \end{aligned}$$

for random $c$ and $s$ such that $c^2 + s^2 = 1$. As this transformation does not change the

**Table 4**

*Average computational times in milliseconds for* `Lin1`, `Lin2`, `MinUnif`, *and* `PHCLab` *for random full bivariate polynomial systems of degree* 3 *to* 15. *For* `Lin1` *and* `MinUnif` *separate results are included for real* ($\mathbb{R}$) *and complex polynomials* ($\mathbb{C}$).

| $d$ | `Lin1` ($\mathbb{R}$) | `Lin1` ($\mathbb{C}$) | `Lin2` | `PHCLab` | `MinUnif` ($\mathbb{R}$) | `MinUnif` ($\mathbb{C}$) |
|---|---|---|---|---|---|---|
| 3 | 6 | 8 | 4 | 116 | 6 | 7 |
| 4 | 9 | 11 | 6 | 130 | 12 | 13 |
| 5 | 20 | 26 | 13 | 151 | 18 | 20 |
| 6 | 39 | 71 | 28 | 174 | 27 | 27 |
| 7 | 96 | 160 | 51 | 217 | 36 | 44 |
| 8 | 205 | 395 | 118 | 264 | 59 | 74 |
| 9 | 467 | 1124 | 279 | 329 | 95 | 125 |
| 10 | 1424 | 3412 | 600 | 414 | 147 | 221 |
| 11 | | | | 538 | 248 | 354 |
| 12 | | | | 650 | 361 | 530 |
| 13 | | | | 911 | 592 | 740 |
| 14 | | | | 1142 | 842 | 1148 |
| 15 | | | | 1531 | 1237 | 1835 |

conditioning of the roots, we can conclude that the difficulties with the staircase algorithm are not directly related to the conditioning. The trick does not work every time, and it seems that for some systems the only way to make the determinantal representation approach work is to increase the machine precision.

We can apply the same approach to systems of polynomials in more than two variables. However, since the size of the corresponding $\Delta$-matrices is the product of sizes of all representations, this is competitive only for $n = 3$ and $d \leq 3$. For a comparison, if we have a system of three polynomials in three variables of degree 3, then the size of the $\Delta$-matrices is $343 \times 343$. For degree 4 the size increases to $1000 \times 1000$ and PHCpack is faster. Finally, for $n = 4$ and the smallest nontrivial $d = 2$ we already get $\Delta$-matrices of size $625 \times 625$ and the method is not efficient.

**8. Outlook.** We have introduced *uniform determinantal representations*, which rather than representing a single polynomial as the determinant of a matrix of affine-linear forms, represent all polynomials of degree at most $d$ in $n$ variables as such a determinant. We have seen that in the bivariate case, these determinantal representations are useful for numerically solving bivariate systems of equations. In the general multivariate case we have determined, up to constants, the asymptotic behavior of $N^*(n, d)$, the minimal size of such a representation, for $n$ fixed and $d \to \infty$.

We now summarize several results that have been shown in this paper.

- For fixed $n$ and $d \to \infty$, $N^*(n, d) \sim d^{n/2}$, and indeed this is also the asymptotic behavior of the determinantal complexity of a sufficiently general polynomial; see Theorem 1.3. This is a noticeable improvement on [35], where an asymptotic rate of $N^*(n, d) \sim d^n$ is shown, with the remark that the representation in [35] is symmetric. However, symmetry currently cannot be exploited by methods that compute roots of multivariate polynomial systems.
- Tables 1 and 2 give constructions for the smallest representations that we have been

able to find for some small values of $n$ and $d$.

- $N^*(n,2) \leq n+1$; cf. Table 1.
- $N^*(n,3) \leq 2n+1$; cf. Table 1.
- $N^*(n,4) \leq \frac{1}{6}n^2 + O(n)$; see Example 5.3.
- $N^*(2,d) \leq 2d-1$; cf. Table 1. Note that this result satisfies Dixon [11] up to an asymptotic factor 2 whereby no computations are necessary for the determinantal representation. In particular, it is a major improvement on the $\sim \frac{1}{4}d^2$ of [34, 35].
- Due to the smaller sizes of the representations, the numerical approach for bivariate polynomials ($n=2$) is competitive to (say) Mathematica for degree $d$ up to $d \approx 15$ (see section 7); this in contrast to $d \approx 9$ as obtained in [34].
- Under some conditions, the results carry over to the case of matrix coefficients (see section 6).

There are still many interesting open questions, both of intrinsic mathematical interest and of relevance to polynomial system solving. First, in a situation where the degree $d$ is fixed and the number $n$ of variables grows, what is the asymptotic behavior of $N^*(n,d)$? Inequality (4) and $\dim F_d = \frac{n^d}{d!} + O(n^{d-1})$ for $n \to \infty$ yields a lower bound which is a constant (depending on $d$) times $n^{(d-1)/2}$. For odd $d$ we obtain a matching upper bound (with a different constant) by using Proposition 4.5 with $V = W = F_{(d-1)/2}$. However, for even $d$ we only know how to obtain $O(n^{d/2})$. We remark that in the regime of fixed $d$, $O(n^{\lfloor d/2 \rfloor})$ is the same bound as obtained in [35, Thm. 4.4] for *symmetric* uniform representations.

Second, in the case of fixed $n$ and varying $d$ studied in this paper, what are the best constants in Theorem 1.3? More specifically, for fixed $n$, does $\lim_{d\to\infty} \frac{N^*(n,d)}{d^{n/2}}$ exist, and if so, what is its value?

Third, how can our techniques for upper bounds and lower bounds be further sharpened? Can singular matrix spaces *other* than compression spaces be used to obtain tighter upper bounds (constructions) on $N^*(n,d)$? Can the action of the affine group be used more systematically to find lower bounds on $N^*(n,d)$?

Finally, is it true that each minimal uniform representation gives rise to a representation of the corresponding matrix polynomial (cf. section 6)?

**Appendix A. Algorithm.** We give an algorithm that constructs a determinantal representation for an $n$-variate polynomial (1) of degree at most $d$, where $n \geq 2$. It is based on the construction from the proof of Theorem 1.3 for even $n$, but can be applied to odd $n$ as well. For large $d$ the algorithm returns matrices of size $\mathcal{O}(d^{\lceil n/2 \rceil})$.

Let $m = \lfloor n/2 \rfloor$, and let $S_1$ be the list of all monomials in $x_1, \ldots, x_m$ of degree $d-1$ and $S_2$ the list of all monomials in $x_{m+1}, \ldots, x_n$ of degree $d-1$. For $\varphi \in S_i$ we denote by $\mathrm{pos}(\varphi, S_i)$ the position of $\varphi$ in $S_i$ for $i = 1, 2$.

We take for $V_i$ the span of all monomials in $S_i$ for $i = 1, 2$. The algorithm returns an $N \times N$ block matrix

$$M = \begin{bmatrix} M_{V_1} & 0 \\ L & M_{V_2}^T \end{bmatrix} \tag{13}$$

such that $\det(M) = \pm p$, where $M_{V_1}$ is of size $(N_1 - 1) \times N_1$, $M_{V_2}$ is of size $(N_2 - 1) \times N_2$, and $L$ is of size $N_2 \times N_1$, where $N_1 = \binom{m+d-1}{m}$, $N_2 = \binom{n-m+d-1}{n-m}$, and $N = N_1 + N_2 - 1$.

The outline of the algorithm is given in Algorithm 1. In Part 1 we construct matrices $M_{V_1}$ and $M_{V_2}$. Note that we do not have to construct $S_1$ and $S_2$ explicitly, we just need an efficient method to compute the position of a monomial from $S_1$ or $S_2$. We do not give the details, but by storing additional $\mathcal{O}(nd)$ parameters it is possible to implement the function $\mathrm{pos}(\varphi, S_i)$ so that its time complexity is $\mathcal{O}(d)$. To represent $p$ with (13) we take each nonzero term $c_\alpha x^\alpha$ of $p$ and write $x^\alpha = z \cdot \varphi_1 \cdot \varphi_2$, where $z \in \{1, x_1, \ldots, x_n\}$, $\varphi_1 \in S_1$, and $\varphi_2 \in S_2$. We put $c_\alpha z$ on the position $(j_2, j_1)$ in $L$, where $j_i = \mathrm{pos}(\varphi_i, S_i)$ for $i = 1, 2$.

If we write $M$ as a sparse matrix, then the complexity of Algorithm 1 is $\mathcal{O}(Nd)$ for Part 1 and $\mathcal{O}(kd)$ for Part 2, where $k$ is the number of nonzero coefficients in $p$. In the worst case, when all coefficients in $p$ are nonzero, the overall complexity is $\mathcal{O}\big(\binom{n+d}{n} d\big)$.

**Algorithm 1.** For a given $n$-variate polynomial $p$ of degree $d$, where $n \geq 2$, the algorithm returns matrix $M$ such that $\det(M) = \pm p$.

*Part* 1: *Construction of* $M_{V_1}$ *and* $M_{V_2}$

$m = \lfloor n/2 \rfloor, \quad N_1 = \binom{m+d-1}{m}, \quad N_2 = \binom{n-m+d-1}{n-m}$

$S_1$ is the list of all monomials in $x_1, \ldots, x_m$ of degree $d-1$

$S_2$ is the list of all monomials in $x_{m+1}, \ldots, x_n$ of degree $d-1$

for $i = 1, 2$

  set $M_{V_i}$ to a zero $(N_i - 1) \times N_i$ matrix

  $M_{V_i}(1 : N_i - 1, 2 : N_i) = I_{N_i - 1}$

  for all monomials $\varphi \in V_i$ of degree at least 1

    set $z = x_{i_1}$, where $\varphi = x_{i_1}^{\alpha_{i_1}} \cdots x_{i_r}^{\alpha_{i_r}}$ and $\alpha_{i_j} > 0$ for $j = 1, \ldots, r$

    set $M_{V_i}(j, k) = -z$, where $j = \mathrm{pos}(\varphi, V_i) - 1$ and $k = \mathrm{pos}(z^{-1} \cdot \varphi, V_i)$

*Part* 2: *Construction of* $L$

set $L$ to a zero $N_2 \times N_1$ matrix

for all nonzero terms $c_\alpha x^\alpha$ of polynomial $p$

  if $|\alpha| = d$

    set $z = x_{i_1}$, where $x^\alpha = x_{i_1}^{\alpha_{i_1}} \cdots x_{i_r}^{\alpha_{i_r}}$ and $\alpha_{i_j} > 0$ for $j = 1, \ldots, r$

  else

    set $z = 1$

  split $x^\alpha$ as $x^\alpha = z \cdot \varphi_1 \cdot \varphi_2$, where $\varphi_1 \in V_1$ and $\varphi_2 \in V_2$

  set $L(k, j) = c_\alpha z$, where $j = \mathrm{pos}(\varphi_1, V_1)$ and $k = \mathrm{pos}(\varphi_2, V_2)$

Return $M = \begin{bmatrix} M_{V_1} & 0 \\ L & M_{V_2}^T \end{bmatrix}$.

*Example* A.1. If we apply Algorithm 1 to

$$p = 2 + 3x_1^2 x_2 x_3 + 4x_1 x_2 x_3 + 5x_2^2 x_4 + 6x_2 x_3 x_4 + 7x_3 x_4 + 8x_5^4,$$

then $n = 5$, $d = 4$, and the algorithm uses monomial lists (ordered in the degree negative

lexicographic ordering)

$$S_1 = \{1, x_1, x_2, x_1^2, x_1x_2, x_2^2, x_1^3, x_1^2x_2, x_1x_2^2, x_2^3\},$$
$$S_2 = \{1, x_3, x_4, x_5, x_3^2, x_3x_4, x_3x_5, x_4^2, x_4x_5, x_5^2, x_3^3, x_3^2x_4, x_3^2x_5, \ldots, x_4^3, x_4^2x_5, x_4x_5^2, x_5^3\}.$$

The result of Part 1 is

$$M_{V_1} = \begin{bmatrix} -x_1 & 1 & & & & & & & & \\ -x_2 & & 1 & & & & & & & \\ & -x_1 & & 1 & & & & & & \\ & & -x_1 & & 1 & & & & & \\ & & -x_2 & & & 1 & & & & \\ & & & -x_1 & & & 1 & & & \\ & & & & -x_1 & & & 1 & & \\ & & & & & -x_1 & & & 1 & \\ & & & & & -x_2 & & & & 1 \end{bmatrix},$$

while $M_{V_2}$ is a $19 \times 20$ matrix with the following nonzero elements:

(a) 1 on the first upper diagonal,
(b) $-x_3$ on $(1,1), (4,2), (5,3), (6,4), (10,5), (11,6), (12,7), (13,8), (14,9)$, and $(15,10)$,
(c) $-x_4$ on $(2,1), (7,3), (8,4), (16,8), (17,9)$, and $(18,10)$,
(d) $-x_5$ on $(3,1), (9,4)$, and $(19,10)$.

In Part 2 the algorithm builds a $20 \times 10$ matrix $L$ with nonzero elements

$$\ell_{11} = 2, \quad \ell_{25} = 4, \quad \ell_{36} = 5, \quad \ell_{61} = 7, \quad \ell_{63} = 6 + 3x_1, \quad \text{and} \quad \ell_{20,1} = 8x_5.$$

The final result is a $29 \times 29$ matrix $M = \left[\begin{smallmatrix} M_{V_1} & 0 \\ L & M_{V_2}^T \end{smallmatrix}\right]$ that satisfies $\det(M) = -p$. ■

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