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LINEAR ELASTIC COMPOSITES WITH STATISTICALLY ORIENTED SPHEROIDAL INCLUSIONS

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Summary

The purpose of this chapter is to critically review some results that our groups obtained in previous works, which were devoted to the investigation of the elastic properties of composite materials with a statistical distribution of spheroidal inclusions. These studies were motivated by our interest in the description of mechanical properties of fibre-reinforced biological tissues (such as articular cartilage), starting from the internal structure of these tissues. After an introduction to tensor algebra, which defines the notation and clarifies the mathematical framework adopted in the chapter, we present, in a covariant setting inspired by Differential Geometry, Walpole's representation of isotropic and transversely isotropic second- and fourth-order tensors, along with its properties. Hence, starting from Eshelby's seminal work on the problem of an inclusion in an infinite matrix, we briefly review the theories developed by Hill, Walpole and Weng for the determination of the overall elasticity tensor of materials with one or more inclusion phases. Then, we discuss in detail the cases of composite materials with aligned spheroidal inclusions and with statistically oriented spheroidal inclusions. Emphasis is put on extending Walpole's formula to the case of inclusions aligned according to some probability density of orientation, both in the transversely isotropic and the isotropic case.

Keywords: covariant formalism, tensor algebra, material symmetries, inclusions, ellipsoidal inclusions, spheroidal inclusions, linear elasticity, elasticity tensor, elastic moduli, statistical orientation.

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Dedicated to the memory of Prof. Gaetano Giaquinta (1945-2016)

1 Introduction

From the 1950s to the 1970s, Eshelby published several papers (e.g., Eshelby, 1951, 1957, 1975) that turned out to be of fundamental importance in the development of the theory of materials with defects or inclusions. In this chapter, we are particularly interested in his work on the ellipsoidal inclusion (Eshelby, 1957), which is at the basis of the theory of materials reinforced by one or more phases of ellipsoidal inclusions, whose shape ranges from flat discs (which could represent cracks, if assigned a null elasticity tensor) to spherical inclusions to fibre-like inclusions.

The theory for the determination of the elasticity tensor of a composite material with inclusions has been developed by, among others, Hill (1963, 1965), Hashin (1963) and Walpole (1966a,b, 1969). The case of aligned inclusions has been thoroughly studied by Weng and his group (Weng, 1984, 1990; Qiu and Weng, 1990). A few cases of composites with non-aligned inclusions have been studied in the 1980s. The first work we are aware of is that by Chou and Nomura (1981), who studied a short-fibre composite in which the directions of alignment of the fibres lay on the surface of a cone. Tandon and Weng (1986), Weng (1990) and Qiu and Weng (1990) studied the case of randomly oriented spheroidal inclusions.

In this chapter, we report our method of solution for the general case of statistically oriented inclusions and, in particular, for the case of probability density being transversely isotropic with respect to a given direction. This is the core of one of our first works (Federico et al., 2004), which here we would like to present from a more mature point of view (twelve years are not so few...), and in a more general setting. Furthermore, we take this chance to correct a few imprecisions in our original work and in some subsequent ones.

Originally, we were motivated by our interest in modelling articular cartilage as a composite comprised of a proteoglycan matrix with spheroidal inclusions, representing the chondrocytes (i.e., cartilage cells) and collagen fibres (Federico et al., 2005). This method was able to predict the elastic behaviour of articular cartilage only for a given type of deformation (i.e., either in compression or in tension). Indeed, the method had been conceived to model “pure” linear elasticity, rather than to capture the tension-compression asymmetry caused by the fact that the collagen fibres bear load when extended, but almost no load when contracted. Such non-linear effect was highlighted by Soltz and Ateshian (2000), who modelled cartilage by means of the *conewise linear elastic model* developed by Curnier et al. (1995), which adopts different elasticity tensors in tension and compression. However, considering the sign of deformation explicitly in the original paper (Federico et al., 2004) and its application to articular cartilage (Federico et al., 2005) would have prevented direct averaging integration of the elasticity tensors over all possible directions. In fact, this difficulty emerged also in our subsequent non-linear works (see Federico, 2015, and references therein). Despite the limitations of our early paper (Federico et al., 2004), its methods have served as the basis for several other projects in our research groups, both in the linear and non-linear settings (see, again, Federico, 2015, for an account of all works in this “family”).

2 Theoretical Background

We shall exclusively deal with the theory of small deformations, and therefore we shall make no distinction between reference and current configuration of a deformable body, which we shall simply regard as an open subset \mathcal{B} of the physical space \mathcal{S} . Consequently, we shall not make distinction between uppercase and lowercase symbols as typically done in modern Continuum Mechanics (see, e.g., Marsden and Hughes, 1983), and shall exclusively use lowercase indices. However, we

44 decided to keep the distinction between vector and covector quantities, which is reflected in the
 45 distinction between contravariant and covariant indices, respectively. The rationale for this choice is
 46 twofold. First, we believe that, even when Cartesian coordinates are used and the difference between
 47 vectors and covectors fades under orthogonal transformations (and *exclusively* under orthogonal
 48 transformations), it is good practice to keep the distinction, from a didactical point of view. Second,
 49 the theory of composite materials has traditionally been developed under the tacit assumption of
 50 Cartesian coordinates, and our own past work is no exception; thus, we found appealing the idea
 51 of attempting to systematically employ a covariant formalism instead.

52 We start by presenting the description of the physical space \mathcal{S} as an affine space. Then, we
 53 introduce our general notation for tensors and tensor spaces, the metric tensor and contractions
 54 of tensors. We continue by introducing the symmetries of second- and fourth-order tensors,
 55 and the material symmetries of isotropy and transverse isotropy, along with the corresponding
 56 representations of second- and fourth-order tensors satisfying these symmetries. In particular, we
 57 present Walpole’s formalism for the representation of transversely isotropic fourth-order tensors.
 58 Finally, we briefly recall some key relations from the Theory of Linear Elasticity.

59 2.1 Affine Spaces, Open Subsets and Tangent Spaces

60 We cannot but agree with, e.g., Marsden and Hughes (1983) or Epstein (2010), when they say that
 61 *differentiable manifolds* are the most general and most appropriate setting for the description of
 62 Mechanics. However, in many cases, the much simpler structure of *affine space* is sufficient for a
 63 reasonably rigorous presentation. An affine space is in fact a *trivial* differentiable manifold (i.e., a
 64 differentiable manifold that can be covered by a single chart) and is the minimal structure that
 65 allows to develop Differential Calculus and to attach vectors and tensors at any point in space.

66 An affine space consists of a set \mathcal{S} , called the *point space*, a vector space \mathcal{V} , called the
 67 *modelling space*, and a map $\mathcal{F} : \mathcal{S} \times \mathcal{S} \rightarrow \mathcal{V}$ that, for every pair of points x, y in \mathcal{S} , yields a
 68 vector in \mathcal{V} denoted $\mathcal{F}(x, y) = y - x = \mathbf{w}$, called the *oriented segment* from x to y . The map
 69 \mathcal{F} must be anti-commutative, i.e., $[x - y] = -[y - x]$, and must satisfy the triangle rule, i.e.,
 70 $y - x = [y - z] + [z - x]$, and the axiom of arbitrary origin, i.e., for every $x \in \mathcal{S}$ and $\mathbf{w} \in \mathcal{V}$ there
 71 exists one, and only one, $y \in \mathcal{S}$, such that $y - x = \mathbf{w}$. At every point $x \in \mathcal{S}$, the set of all vectors
 72 emanating from x is defined by

$$T_x\mathcal{S} = \{\mathbf{w}_x = y - x : y \in \mathcal{S}\}, \quad (1)$$

73 where the notation \mathbf{w}_x means that “ \mathbf{w} is attached at x ”. The space $T_x\mathcal{S}$ is a vector space called
 74 *tangent space* of \mathcal{S} at point x . The *tangent bundle* of \mathcal{S} , denoted by $T\mathcal{S}$, is defined as the disjoint
 75 union of all tangent spaces $T_x\mathcal{S}$ for all $x \in \mathcal{S}$.

76 The definition of tangent space given in Equation (1), however, applies exclusively to affine
 77 spaces and *not* to subsets of an affine space. This is crucial, because deformable bodies are often
 78 seen as open subsets $\mathcal{B} \subset \mathcal{S}$. The problem for the case of a subset is that, if $x \in \mathcal{B}$, there exist
 79 tangent vectors $\mathbf{w}_x \in T_x\mathcal{B}$ such that $y = x + \mathbf{w}_x$ does *not* belong to \mathcal{B} , i.e., the “tip of the arrow”
 80 lies outside \mathcal{B} (see the example in Figure 1, right). Therefore, in order to properly define the tangent
 81 space $T_x\mathcal{B}$ at a point $x \in \mathcal{B}$, we need to use the definition inherited from Differential Geometry. In
 82 this definition, the tangent space $T_x\mathcal{S}$ is the set of all vectors that are tangent at x to all possible
 83 regular curves $\Gamma : [a, b] \rightarrow \mathcal{S} : s \mapsto \Gamma(s)$ such that $\Gamma(s_0) = x$, with $s_0 \in]a, b[$, i.e., the vectors

$$\mathbf{w}_x = \lim_{h \rightarrow 0} \frac{\Gamma(s_0 + h) - \Gamma(s_0)}{h} = \Gamma'(s_0) \in T_x\mathcal{S}, \quad (2)$$

84 in which the numerator of the limit is the difference between point $\Gamma(s_0 + h)$ and $\Gamma(s_0)$, which
 85 is a vector *secant* to Γ , and the limit is precisely the tangent at x (Figure 1, left). For the case
 86 of an affine space \mathcal{S} , this definition of tangent space $T_x\mathcal{S}$ coincides with that in Equation (1).
 87 However, although the definition in Equation (1) does not make sense for an open subset \mathcal{B} , that
 88 in Equation (2) can be automatically inherited by $T_x\mathcal{B}$ just by saying that $x \in \mathcal{B}$ and the curves Γ
 89 are such that $\Gamma : [a, b] \rightarrow \mathcal{B} : s \mapsto \Gamma(s)$, with $\Gamma(s_0) = x$, and $s_0 \in]a, b[$.

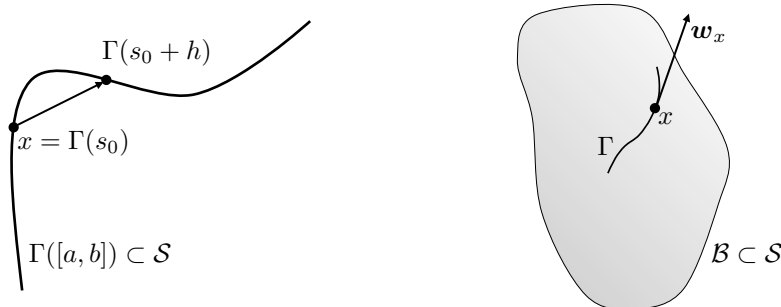


Figure 1: Left: The geometrical definition of tangent vector at a point x in the affine space \mathcal{S} as the tangent at x to a curve passing by x , obtained as the limit of the secant passing by x . Right: A body \mathcal{B} is an open subset of the physical space \mathcal{S} , which is considered as an affine space. Considering all regular curves Γ passing by $x \in \mathcal{B}$, the tangent space $T_x\mathcal{B}$ is the set of the tangent vectors \mathbf{w}_x that are each tangent at x to one of the curves Γ .

90 Usually, the affine space of Classical Mechanics is constructed by assuming that both the
 91 point space and the modelling space are \mathbb{R}^3 , and is often denoted by \mathbb{E}^3 . Here, we shall assume that
 92 $\mathcal{S} \equiv \mathbb{E}^3$.

93 2.2 Tensors

94 For our purposes, we shall refer to tensors on the tangent bundle $T\mathcal{B}$ of a material body \mathcal{B} , but
 95 these definitions are completely general and could be used in the tangent bundle $T\mathcal{S}$ of the physical
 96 space \mathcal{S} or even in a generic vector space \mathcal{V} of dimension n . Whenever we give examples, we use
 97 second- or fourth-order tensors, which are the types of tensors that are relevant in the subsequent
 98 sections of this chapter. This section is largely based on a previous work on non-linear elasticity
 99 (Federico, 2015), and is adapted to the setting of the small-displacement theory.

100 A *covector*, or *linear form*, or *one-form* is a linear map

$$\varphi : T\mathcal{B} \rightarrow \mathbb{R} : \mathbf{u} \mapsto \varphi \mathbf{u} \equiv \varphi(\mathbf{u}), \quad (3)$$

101 where we use simple juxtaposition to indicate the action of the covector φ on the vector \mathbf{u} . The
 102 space of all covectors on the tangent bundle $T\mathcal{B}$ is the *dual* of $T\mathcal{B}$, and is denoted $T^*\mathcal{B}$ and called
 103 *cotangent bundle*. If one looks at a point $x \in \mathcal{B}$, the dual of the tangent space $T_x\mathcal{B}$ is the *cotangent*
 104 *space* $T_x^*\mathcal{B}$. It is possible to prove that, given a basis $\{\mathbf{e}_i\}_{i=1}^3$ in $T\mathcal{B}$, the covectors $\{\mathbf{e}^i\}_{i=1}^3$, defined
 105 by

$$\mathbf{e}^i \mathbf{u} \equiv \mathbf{e}^i(\mathbf{u}) = u^i, \quad (4)$$

106 constitute a basis for the cotangent bundle $T^*\mathcal{B}$. The basis $\{\mathbf{e}^i\}_{i=1}^3$ is called the *dual basis* of $\{\mathbf{e}_i\}_{i=1}^3$.
 107 Each of the basis covectors \mathbf{e}^i has a very precise geometrical meaning, as it associates, with every

108 vector \mathbf{u} , the i -th component u^i of \mathbf{u} with respect to the vector basis $\{\mathbf{e}_i\}_{i=1}^3$ and, for this reason,
 109 the basis covectors \mathbf{e}^i are often called *projections*. The definition in Equation (4) implies

$$\mathbf{e}^i \mathbf{e}_j \equiv \mathbf{e}^i(\mathbf{e}_j) = \delta^i_j, \quad (5)$$

110 where δ^i_j is the Kronecker symbol. If, in the definition (3) of covector, we express vector \mathbf{u} in the
 111 basis $\{\mathbf{e}_i\}_{i=1}^3$, we have

$$\boldsymbol{\varphi} \mathbf{u} = \boldsymbol{\varphi}(u^i \mathbf{e}_i) = u^i \boldsymbol{\varphi} \mathbf{e}_i = u^i \varphi_i \quad (6)$$

112 where

$$\varphi_i = \boldsymbol{\varphi} \mathbf{e}_i \equiv \boldsymbol{\varphi}(\mathbf{e}_i). \quad (7)$$

113 Using the definition of basis covector, it follows that the covector $\boldsymbol{\varphi}$ can be decomposed as a linear
 114 combination of the basis covectors $\{\mathbf{e}^i\}_{i=1}^3$, i.e., as

$$\boldsymbol{\varphi} = \varphi_i \mathbf{e}^i, \quad (8)$$

115 where the coefficients φ_i take the meaning of components of $\boldsymbol{\varphi}$ with respect to $\{\mathbf{e}^i\}_{i=1}^3$.

116 In finite dimension, which is the case we are interested in, the relation between a vector
 117 space and its dual is symmetric in the sense that $T\mathcal{B}$ can be identified with the bi-dual space $T^{**}\mathcal{B}$
 118 (the set of all linear maps from $T^*\mathcal{B}$ into \mathbb{R}). Therefore, the vectors of $T\mathcal{B}$ can be made to act on
 119 the covectors of $T^*\mathcal{B}$ as linear forms, and the action of a vector \mathbf{u} on a covector $\boldsymbol{\varphi}$ is identical of
 120 that of $\boldsymbol{\varphi}$ on \mathbf{u}

$$\boldsymbol{\varphi} \mathbf{u} = \varphi_i u^i = u^i \varphi_i = \mathbf{u} \boldsymbol{\varphi}. \quad (9)$$

121 Consequently, the basis vectors $\{\mathbf{e}_i\}_{i=1}^3$ are the projections in $T^*\mathcal{B}$, i.e.,

$$\mathbf{e}_i \boldsymbol{\varphi} = \boldsymbol{\varphi} \mathbf{e}_i = \varphi_i. \quad (10)$$

122 We also say that the expression $\boldsymbol{\varphi} \mathbf{u} = \mathbf{u} \boldsymbol{\varphi} = u^i \varphi_i$ is the *contraction* of $\boldsymbol{\varphi}$ and \mathbf{u} .

123 A *tensor* of order $r + s = m$ on the tangent bundle $T\mathcal{B}$ is a *multilinear form*, i.e., a map of
 124 the type

$$\mathbb{T} : \underbrace{T^*\mathcal{B} \times \dots \times T^*\mathcal{B}}_{r \text{ times}} \times \underbrace{T\mathcal{B} \times \dots \times T\mathcal{B}}_{s \text{ times}} \rightarrow \mathbb{R}, \quad (11a)$$

$$\mathbb{T} : (\boldsymbol{\varphi}^1, \dots, \boldsymbol{\varphi}^r, \mathbf{u}_1, \dots, \mathbf{u}_s) \mapsto \mathbb{T}(\boldsymbol{\varphi}^1, \dots, \boldsymbol{\varphi}^r, \mathbf{u}_1, \dots, \mathbf{u}_s), \quad (11b)$$

125 that is linear in each of the $r + s$ arguments separately. The space of all tensors of the type in
 126 Equation (11) is denoted $[T\mathcal{B}]^r_s$, a notation that will be justified later on (Equation (17)).

127 The *tensor product* of the r vectors $\mathbf{v}_1, \dots, \mathbf{v}_r$ in $T\mathcal{B}$ and the s covectors $\boldsymbol{\psi}^1, \dots, \boldsymbol{\psi}^s$ in $T^*\mathcal{B}$
 128 is the tensor $\mathbf{v}_1 \otimes \dots \otimes \mathbf{v}_r \otimes \boldsymbol{\psi}^1 \otimes \dots \otimes \boldsymbol{\psi}^s$ in $[T\mathcal{B}]^r_s$ such that, for every $\boldsymbol{\varphi}^1, \dots, \boldsymbol{\varphi}^r$ in $T^*\mathcal{B}$ and for
 129 every $\mathbf{u}_1, \dots, \mathbf{u}_s$ in $T\mathcal{B}$,

$$[\mathbf{v}_1 \otimes \dots \otimes \mathbf{v}_r \otimes \boldsymbol{\psi}^1 \otimes \dots \otimes \boldsymbol{\psi}^s](\boldsymbol{\varphi}^1, \dots, \boldsymbol{\varphi}^r, \mathbf{u}_1, \dots, \mathbf{u}_s) = \mathbf{v}_1(\boldsymbol{\varphi}^1) \dots \mathbf{v}_r(\boldsymbol{\varphi}^r) \boldsymbol{\psi}^1(\mathbf{u}_1) \dots \boldsymbol{\psi}^s(\mathbf{u}_s). \quad (12)$$

130 The tensor $\mathbf{v}_1 \otimes \dots \otimes \mathbf{v}_r \otimes \boldsymbol{\psi}^1 \otimes \dots \otimes \boldsymbol{\psi}^s$ is said to have r *vector legs* $\mathbf{v}_1, \dots, \mathbf{v}_r$ and s *covector legs*
 131 $\boldsymbol{\psi}^1, \dots, \boldsymbol{\psi}^s$.

132 With the definition of tensor product of vectors and covectors, and using multilinearity, we
 133 can derive the component expression of any tensor \mathbb{T} in $[\mathcal{TB}]^r_s$ with respect to a given basis $\{\mathbf{e}_i\}_{i=1}^n$
 134 and dual basis $\{\mathbf{e}^i\}_{i=1}^n$. Indeed, we have

$$\begin{aligned}\mathbb{T}(\boldsymbol{\varphi}^1, \dots, \boldsymbol{\varphi}^r, \mathbf{u}_1, \dots, \mathbf{u}_s) &= \mathbb{T}(\varphi_{i_1}^1 \mathbf{e}^{i_1}, \dots, \varphi_{i_r}^r \mathbf{e}^{i_r}, u_1^{j_1} \mathbf{e}_{j_1}, \dots, u_s^{j_s} \mathbf{e}_{j_s}) \\ &= \varphi_{i_1}^1 \dots \varphi_{i_r}^r u_1^{j_1} \dots u_s^{j_s} \mathbb{T}(\mathbf{e}^{i_1}, \dots, \mathbf{e}^{i_r}, \mathbf{e}_{j_1}, \dots, \mathbf{e}_{j_s}) \\ &= \varphi_{i_1}^1 \dots \varphi_{i_r}^r u_1^{j_1} \dots u_s^{j_s} \mathbb{T}^{i_1 \dots i_r}_{j_1 \dots j_s},\end{aligned}\quad (13)$$

135 where

$$\mathbb{T}^{i_1 \dots i_r}_{j_1 \dots j_s} = \mathbb{T}(\mathbf{e}^{i_1}, \dots, \mathbf{e}^{i_r}, \mathbf{e}_{j_1}, \dots, \mathbf{e}_{j_s}) \quad (14)$$

136 are the components of \mathbb{T} . By analogy with the indices of vectors and covectors, the indices $i_1 \dots i_r$
 137 are called *contravariant* and the indices $j_1 \dots j_s$ are called *covariant*. Using Equations (4), (10) and
 138 (12), we obtain

$$\begin{aligned}\mathbb{T}(\boldsymbol{\varphi}^1, \dots, \boldsymbol{\varphi}^r, \mathbf{u}_1, \dots, \mathbf{u}_s) &= \varphi_{i_1}^1 \dots \varphi_{i_r}^r u_1^{j_1} \dots u_s^{j_s} \mathbb{T}^{i_1 \dots i_r}_{j_1 \dots j_s} \\ &= \mathbf{e}_{i_1}(\boldsymbol{\varphi}^1) \dots \mathbf{e}_{i_r}(\boldsymbol{\varphi}^r) \mathbf{e}^{j_1}(\mathbf{u}_1) \dots \mathbf{e}^{j_s}(\mathbf{u}_s) \mathbb{T}^{i_1 \dots i_r}_{j_1 \dots j_s} \\ &= \mathbb{T}^{i_1 \dots i_r}_{j_1 \dots j_s} [\mathbf{e}_{i_1} \otimes \dots \otimes \mathbf{e}_{i_r} \otimes \mathbf{e}^{j_1} \otimes \dots \otimes \mathbf{e}^{j_s}](\boldsymbol{\varphi}^1, \dots, \boldsymbol{\varphi}^r, \mathbf{u}_1, \dots, \mathbf{u}_s).\end{aligned}\quad (15)$$

139 By dropping the arguments $\boldsymbol{\varphi}^1, \dots, \boldsymbol{\varphi}^r, \mathbf{u}_1, \dots, \mathbf{u}_s$ on both sides of Equation (15), we obtain the
 140 component representation of \mathbb{T} in the *tensor basis* $\{\mathbf{e}_{i_1} \otimes \dots \otimes \mathbf{e}_{i_r} \otimes \mathbf{e}^{j_1} \otimes \dots \otimes \mathbf{e}^{j_s}\}_{i_1, \dots, i_r, j_1, \dots, j_s=1}^n$ of
 141 the tensor space $[\mathcal{TB}]^r_s$ as

$$\mathbb{T} = \mathbb{T}^{i_1 \dots i_r}_{j_1 \dots j_s} \mathbf{e}_{i_1} \otimes \dots \otimes \mathbf{e}_{i_r} \otimes \mathbf{e}^{j_1} \otimes \dots \otimes \mathbf{e}^{j_s}. \quad (16)$$

142 A tensor \mathbb{T} in $[\mathcal{TB}]^r_s$ is said to have r vector legs and s covector legs, and $[\mathcal{TB}]^r_s$ can be represented
 143 as the *tensor product of spaces* (Bishop and Goldberg, 1968)

$$[\mathcal{TB}]^r_s = \underbrace{\mathcal{TB} \otimes \dots \otimes \mathcal{TB}}_{r \text{ times}} \otimes \underbrace{\mathcal{T}^* \mathcal{B} \otimes \dots \otimes \mathcal{T}^* \mathcal{B}}_{s \text{ times}}. \quad (17)$$

144 With a widely accepted abuse of terminology, we shall often refer to a tensor in $[\mathcal{TB}]^r_s$ as
 145 being “ r times contravariant and s times covariant” although, rigorously speaking, the adjectives
 146 contravariant and covariant refer to tensor *indices* and tensor *components*.

147 Since a tensor leg can be a vector or a covector, there are 2^m possible spaces of tensors of
 148 order m . For instance, there is only one type of space of zero-order tensors (scalars in $[\mathcal{TB}]_0^0 \equiv \mathbb{R}$),
 149 two types of spaces of first-order tensors (vectors in $[\mathcal{TB}]_1^0 \equiv \mathcal{TB}$ and covectors in $[\mathcal{TB}]_0^1 \equiv \mathcal{T}^* \mathcal{B}$), 4
 150 types of spaces of second-order tensors, 16 types of spaces of fourth-order tensors. The table below
 151 summarises the situation and reports some examples of the 16 types of fourth-order tensors. Here,
 152 we shall exclusively deal with the first four types of fourth-order tensors.

Order	Types	Spaces	Components	Notes
0	2^0	$[T\mathcal{B}]_0^0 \equiv \mathbb{R}$	a	scalars
1	2^1	$[T\mathcal{B}]_0^1 \equiv T\mathcal{B}$	a^i	vectors
		$[T\mathcal{B}]_1^0 \equiv T^*\mathcal{B}$	a_i	covectors
2	2^2	$[T\mathcal{B}]_0^2$	a^{ij}	“contravariant”
		$[T\mathcal{B}]_2^0$	a_{ij}	“covariant”
		$[T\mathcal{B}]_1^1$	a^i_j	“mixed”
		$[T\mathcal{B}]_1^1$	a_i^j	“mixed”
4	2^4	$[T\mathcal{B}]_0^4$	A^{ijkl}	“contravariant”
		$[T\mathcal{B}]_4^0$	A_{ijkl}	“covariant”
		$[T\mathcal{B}]_2^2$	$A^i_j{}^k_l$	“mixed”
		$[T\mathcal{B}]_2^2$	$A_{ij}{}^{kl}$	“mixed”
		
		$[T\mathcal{B}]_1^1{}^1_1$	$A^i_j{}^k_l$	
		$[T\mathcal{B}]_1^1{}^1_1$	$A_{i^j}{}^k_l$	
...	...			

153

154 2.3 Tensor Contractions and Tensor as Linear Maps

155 So far, we have seen tensors as multilinear maps, whose legs are all contracted at the same time
156 with vectors or covector arguments, as appropriate. However, one could contract part of the legs of
157 a tensor with all or part of the legs of another tensor. In this work, we are going to see single and
158 double contractions.

159 Given a tensor whose last leg is a vector and another tensor whose first leg is a covector, or
160 vice versa, we call *single contraction* the contraction of the last leg of the first tensor with the first
161 leg of the second tensor, and denote it by simple juxtaposition. For instance, for a “contravariant”
162 second-order tensor \mathbf{a} in $[T\mathcal{B}]_0^2$ and a “covariant” second-order tensor \mathbf{c} in $[T\mathcal{B}]_2^0$, the contraction
163 $\mathbf{a}\mathbf{c}$ has components $a^{ij}c_{jk}$. The same type of contraction occurs between, e.g., a “mixed” tensor \mathbf{l}
164 in $[T\mathcal{B}]_1^1$ and a vector \mathbf{u} in $T\mathcal{B}$, and the single contraction is the usual $\mathbf{l}\mathbf{u}$ with components $l^i_j u^j$.

165 The double contraction of two tensors works similarly to the simple contraction, except that
166 one contracts the last two legs of the first tensor and the first two legs of the second tensor. As
167 with the single contraction, the contracting legs must be of opposite type. Double contraction is
168 denoted by a colon. For example, for a fourth-order tensor \mathbb{T} in $[T\mathcal{B}]_2^2$ and a second-order tensor
169 \mathbf{a} in $[T\mathcal{B}]_0^2$, the double contraction $\mathbb{T} : \mathbf{a}$ has components $\mathbb{T}^{ij}{}_{kl} a^{kl}$.

170 Tensors can also be regarded as linear maps between tensor spaces. For instance, a “mixed”
171 second-order tensor, i.e., a tensor \mathbf{l} in $[T\mathcal{B}]_1^1$ could be seen as the linear map $\mathbf{l} : T\mathcal{B} \rightarrow T\mathcal{B} : \mathbf{u} \mapsto \mathbf{l}\mathbf{u}$
172 (in components, $l^i_j u^j$). Similarly, a “contravariant” fourth-order tensor \mathbb{T} in $[T\mathcal{B}]_0^4$ could be regarded
173 as the linear map $\mathbb{T} : [T\mathcal{B}]_2^0 \rightarrow [T\mathcal{B}]_2^0 : \mathbf{c} \mapsto \mathbb{T} : \mathbf{c}$ (in components, $\mathbb{T}^{ijkl} c_{kl}$). Rigorously speaking, a
174 tensor seen as a linear map between two tensor spaces should be somehow notationally distinguished
175 from its multilinear form counterpart. However, since context and, above all, index notation prevent
176 any possible confusion, the customary practice is to use the same symbol \mathbb{T} for the tensor employed
177 in both manners.

178 2.4 Metric Tensor and Scalar Products

179 The physical space \mathcal{S} is assumed to be equipped with a metric tensor \mathbf{g} , which is inherited by the
180 body \mathcal{B} . A metric tensor is a symmetric and positive definite tensor in $[T\mathcal{B}]_2^0$, such that, for every
181 pair of vectors \mathbf{u} and \mathbf{v} in $T\mathcal{B}$,

$$\mathbf{g}(\mathbf{u}, \mathbf{v}) \equiv \mathbf{u} \mathbf{g} \mathbf{v} \equiv \langle \mathbf{u}, \mathbf{v} \rangle \equiv \mathbf{u} \cdot \mathbf{v} = u^i g_{ij} v^j. \quad (18)$$

182 Symmetry means that $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$, and positive definiteness means that, for every $\mathbf{u} \neq \mathbf{0}$,
183 $\langle \mathbf{u}, \mathbf{u} \rangle > 0$. The equivalent notations $\mathbf{g}(\mathbf{u}, \mathbf{v}) \equiv \mathbf{u} \mathbf{g} \mathbf{v} \equiv \langle \mathbf{u}, \mathbf{v} \rangle \equiv \mathbf{u} \cdot \mathbf{v}$ denote the *scalar product* of
184 the vectors \mathbf{u} and \mathbf{v} . The metric tensor induces the *Euclidean norm* $\|\mathbf{u}\| = \sqrt{\mathbf{g}(\mathbf{u}, \mathbf{u})} \equiv \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$.
185 A basis $\{\mathbf{e}_i\}_{i=1}^3$ is called *orthonormal* with respect to \mathbf{g} if $\mathbf{g}(\mathbf{e}_i, \mathbf{e}_j) \equiv \langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}$, i.e., if the
186 matrix representation of the metric tensor is the identity. Positive definiteness of \mathbf{g} also implies
187 invertibility, and the inverse is the tensor \mathbf{g}^{-1} valued in $[T\mathcal{B}]_0^2$ such that, for every pair of covectors
188 φ and ψ in $T^*\mathcal{B}$,

$$\mathbf{g}^{-1}(\varphi, \psi) \equiv \varphi \mathbf{g}^{-1} \psi \equiv \langle \varphi, \psi \rangle \equiv \varphi \cdot \psi = \varphi_i (\mathbf{g}^{-1})^{ij} \psi_j. \quad (19)$$

189 When considered as a linear map $\mathbf{g} : T\mathcal{B} \rightarrow T^*\mathcal{B}$, the metric tensor \mathbf{g} is said to be used
190 to “lower contravariant indices”, by mapping the vector \mathbf{u} into the associated covector $\mathbf{u}^\flat = \mathbf{g} \mathbf{u}$,
191 with components $u_i = g_{ij} u^j$. Analogously, the inverse metric tensor \mathbf{g}^{-1} , seen as the linear map
192 $\mathbf{g}^{-1} : T^*\mathcal{B} \rightarrow T\mathcal{B}$, is said to “raise covariant indices”, by mapping the covector φ into the associated
193 vector $\varphi^\sharp = \mathbf{g}^{-1} \varphi$, with components $\varphi^i = (\mathbf{g}^{-1})^{ij} \varphi_j$. The metric tensor and its inverse can be
194 used to lower and rise, respectively, the indices of tensors of any order. For instance, given the
195 “contravariant” fourth-order tensor \mathbb{T} in $[T\mathcal{B}]_0^4$, its “covariant” associated tensor is denoted \mathbb{T}^\flat and
196 has components $\mathbb{T}_{ijkl} = g_{ip} g_{jq} g_{kr} g_{ls} \mathbb{T}^{pqrs}$. In particular, if we raise the indices of the metric tensor
197 \mathbf{g} itself by means of the inverse metric tensor \mathbf{g}^{-1} , we have the important identity

$$\mathbf{g}^\sharp = \mathbf{g}^{-1} \mathbf{g} \mathbf{g}^{-1} = \mathbf{g}^{-1}, \quad \mathbf{g}^{il} = (\mathbf{g}^{-1})^{ij} g_{jk} (\mathbf{g}^{-1})^{kl} = (\mathbf{g}^{-1})^{il}. \quad (20)$$

198 The scalar product induced by the metric tensor \mathbf{g} can be extended to pairs of tensors of
199 the same type of any order, by contracting each pair of homologous indices by means of the metric
200 tensor or its inverse, as appropriate. For instance, given two tensors \mathbb{T} and \mathbb{Z} in $[T\mathcal{B}]_1^1$, their
201 scalar product is $\langle \mathbb{T}, \mathbb{Z} \rangle = \mathbb{T}^i \cdot \mathbb{Z}_i = g_{ip} g^{jq} g_{kr} g^{ls} \mathbb{T}^p q r s$.

202 Finally, we use a single low dot to indicate that the metric tensor (or its inverse) is involved
203 in the contraction of two tensors such that the last leg of the first tensor and the first leg of the
204 second tensor are of the same type. For instance, given two “contravariant” tensors \mathbf{a}, \mathbf{b} in $[T\mathcal{B}]_0^2$,
205 the expression $\mathbf{a} \cdot \mathbf{b}$ stands for $\mathbf{a} \mathbf{g} \mathbf{b}$, which has components $(\mathbf{a} \cdot \mathbf{b})^{il} = a^{ij} g_{jk} b^{kl}$.

206 2.5 Symmetries of Second- and Fourth-Order Tensors

207 For a “covariant” second-order tensor \mathbf{c} in $[T\mathcal{B}]_2^0$, the *transpose* is defined as the tensor \mathbf{c}^T in
208 $[T\mathcal{B}]_2^0$ such that, for every pair of vectors \mathbf{u}, \mathbf{v} in $T\mathcal{B}$, $\mathbf{u} \mathbf{c} \mathbf{v} = \mathbf{v} \mathbf{c}^T \mathbf{u}$, which in components reads
209 $u^i c_{ij} v^j = v^j (c^T)_{ji} u^i$, implying $(c^T)_{ji} = c_{ij}$. The transpose of a “contravariant” second-order tensor
210 is defined analogously. For the case of a “mixed” tensor \mathbf{a} in $[T\mathcal{B}]_1^1$, the transpose is the tensor \mathbf{a}^T
211 in $[T\mathcal{B}]_1^1$ such that, for every vector \mathbf{v} in $T\mathcal{B}$ and every covector φ in $T^*\mathcal{B}$, $\varphi \mathbf{a} \mathbf{v} = \mathbf{v} \mathbf{a}^T \varphi$, which in
212 components reads $\varphi_i a^i j v^j = v_j (a^T)_j^i u^i$, implying $(a^T)_j^i = a^i_j$. Note that, while a “covariant” and
213 its transpose, or a “contravariant” tensor and its transpose, belong to the *same* space, a “mixed”
214 tensor and its transpose belong to *different* spaces.

215 A “covariant” second-order tensor \mathbf{c} in $[\mathcal{TB}]_0^0$ is called symmetric if $\mathbf{c} = \mathbf{c}^T$, which, in
 216 components, means $c_{ij} = c_{ji}$. The symmetry of a “contravariant” tensor \mathbf{a} in $[\mathcal{TB}]_0^2$ is defined
 217 analogously. For the case of a “mixed” tensor, speaking about equality of the tensor and its transpose
 218 has no meaning, as they belong to *different* spaces. Thus, symmetry of a “mixed” tensor \mathbf{l} in $[\mathcal{TB}]_1^1$
 219 is defined in terms of the symmetry of its “covariant” counterpart $\mathbf{l}^\flat = \mathbf{g}\mathbf{l}$ or, equivalently, in terms
 220 of the symmetry of its “contravariant” counterpart $\mathbf{l}^\sharp = \mathbf{l}\mathbf{g}^{-1}$; indeed, we have that \mathbf{l}^\sharp is symmetric
 221 if, and only if \mathbf{l}^\flat is such.

222 When a fourth-order tensor is viewed as a linear map between spaces of second-order tensors,
 223 its transpose can be defined in a way similar to that of a second-order tensor. For the purposes
 224 of our presentation, let us restrict our attention to “covariant” tensors in $[\mathcal{TB}]_0^0$, “contravariant”
 225 tensors in $[\mathcal{TB}]_0^4$, and “mixed” tensors in $[\mathcal{TB}]_2^2$ and $[\mathcal{TB}]_2^2$ (actually, the transpose of fourth-order
 226 tensors of any other type is defined exactly in the same way, but we do not need these tensors
 227 here). For instance, the transpose of \mathbb{A} in $[\mathcal{TB}]_0^4$ is defined as the tensor \mathbb{A}^T in $[\mathcal{TB}]_0^4$ such that,
 228 for every \mathbf{c}, \mathbf{d} in $[\mathcal{TB}]_2^0$, the identity $\mathbf{c} : \mathbb{A} : \mathbf{d} = \mathbf{d} : \mathbb{A}^T : \mathbf{c}$ holds. In components, this reads
 229 $c_{ij} \mathbb{A}^{ijkl} d_{kl} = d_{kl} [\mathbb{A}^T]^{klij} c_{ij}$, i.e., $[\mathbb{A}^T]^{klij} = \mathbb{A}^{ijkl}$.

230 Fourth-order tensors admit a variety of symmetries. Here we are interested in those called
 231 *major* and *minor* symmetry. A “contravariant” fourth-order tensor \mathbb{A} in $[\mathcal{TB}]_0^4$ is said to have *major*
 232 *symmetry* (or *diagonal symmetry*) if $\mathbb{A}^T = \mathbb{A}$. The same definition holds for a “covariant” tensor
 233 \mathbb{B} in $[\mathcal{TB}]_0^4$. The case of “mixed” tensors is of course a little more complicated. We are interested
 234 in the case of a “mixed” tensor \mathbb{T} in $[\mathcal{TB}]_2^2$ or in $[\mathcal{TB}]_2^2$. The major symmetry of such tensor
 235 is checked by looking at the major symmetry of either its “contravariant” counterpart \mathbb{T}^\sharp or of
 236 its “covariant” counterpart \mathbb{T}^\flat , as \mathbb{T}^\sharp is major-symmetric if, and only if, \mathbb{T}^\flat is major-symmetric.
 237 Another important symmetry of fourth-order tensors is called *minor symmetry* (or *pair symmetry*),
 238 and it is straightforward to define for tensors in which the two legs (indices) within the first pair
 239 and within the second pair are of the same type, i.e., for tensors in $[\mathcal{TB}]_0^4$, $[\mathcal{TB}]_4^0$, $[\mathcal{TB}]_2^2$ and
 240 $[\mathcal{TB}]_2^2$. For instance, a tensor \mathbb{A} in $[\mathcal{TB}]_0^4$ is said to possess minor symmetry on the first pair of legs
 241 (indices) if, for every \mathbf{c} in $[\mathcal{TB}]_2^0$, one has $\mathbf{c} : \mathbb{A} = \mathbf{c}^T : \mathbb{A}$, and on the second pair of legs (indices) if
 242 $\mathbb{A} : \mathbf{c} = \mathbb{A} : \mathbf{c}^T$. In components, these symmetries read $\mathbb{A}^{ijkl} = \mathbb{A}^{jikl}$ and $\mathbb{A}^{ijkl} = \mathbb{A}^{ijlk}$, respectively.
 243 If a tensor enjoys minor symmetry on both the first and the second pair of legs, we simply say
 244 that it “enjoys minor symmetry”. When there is no danger of confusion, we say that a fourth-order
 245 tensor is *symmetric* if it enjoys both major and minor symmetry.¹

246 2.6 Isotropic Second- and Fourth-Order Tensors

247 Isotropy is the invariance of a material property under any arbitrary rotation. A “mixed”
 248 second-order tensor \mathbf{l} in $[\mathcal{TB}]_1^1$ is isotropic if, and only if, it is proportional to the identity tensor \mathbf{i} ,
 249 i.e., if $\mathbf{l} = l\mathbf{i}$ (in components, $l^i_j = l\delta^i_j$). A “contravariant” tensor \mathbf{a} in $[\mathcal{TB}]_0^2$ is said to be isotropic
 250 if the associated “mixed” tensor $\mathbf{a}\mathbf{g}$ is isotropic, which implies that $\mathbf{a} = \mathbf{a}\mathbf{g}^{-1}$ (i.e., $a^{ij} = a g^{ij}$).
 251 Similarly, a “covariant” tensor \mathbf{c} in $[\mathcal{TB}]_2^0$ is said to be isotropic if such is the associated “mixed”
 252 tensor $\mathbf{g}^{-1}\mathbf{c}$, from which $\mathbf{c} = \mathbf{c}\mathbf{g}$ (i.e., $c_{ij} = c g_{ij}$). We remark that, as a consequence of the definition
 253 of isotropic second-order tensor, it follows that any isotropic second-order tensor is symmetric.

254 Let us consider the subspace $([\mathcal{TB}]_2^2, \text{Sym})$ of $[\mathcal{TB}]_2^2$ of all tensors with major and minor
 255 symmetry. Since tensors in $[\mathcal{TB}]_2^2$ are “mixed”, major symmetry of a tensor \mathbb{T} is understood in the

¹In our past works, we have called a fourth-order tensor with both major and minor (diagonal- and pair-) symmetry “fully symmetric”, but we are not going to use this nomenclature here, as it can be confusing. Indeed, what is normally called fully or completely symmetric is a tensor that is invariant under *any* permutation of the indices.

256 sense of Section 2.5, i.e., in relation to the symmetry of the “contravariant” counterpart \mathbb{T}^\sharp or of the
 257 symmetry of the “covariant” counterpart \mathbb{T}^\flat . The *symmetric identity* in $([T\mathcal{B}]^2_2, \text{Sym})$ is defined
 258 with the help of the special tensor products $\underline{\otimes}$ and $\overline{\otimes}$ introduced by Curnier et al. (1995) as

$$\mathbb{I} = \frac{1}{2}(\mathbf{i} \underline{\otimes} \mathbf{i} + \mathbf{i} \overline{\otimes} \mathbf{i}), \quad \mathbb{I}^{ij}_{kl} = \frac{1}{2}(\delta^i_k \delta^j_l + \delta^i_l \delta^j_k), \quad (21)$$

259 where \mathbf{i} , with components δ^i_j , is the identity second-order tensor in $[T\mathcal{B}]^1_1$. Since \mathbb{I} is the identity, it
 260 is invariant under rotations and is therefore clearly isotropic. The symmetric identity is such that,
 261 for every *symmetric* second-order tensor \mathbf{a} in $[T\mathcal{B}]^2_0$, $\mathbb{I} : \mathbf{a} = \mathbf{a}$.

262 The tensor basis of the subspace $([T\mathcal{B}]^2_2, \text{Sym}, \text{Iso})$ of the symmetric *and* isotropic tensors
 263 is found by decomposing the symmetric identity into (Walpole, 1981, 1984; Federico, 2012)

$$\mathbb{I} = \mathbb{K} + \mathbb{M}, \quad (22)$$

264 where

$$\mathbb{K} = \frac{1}{3} \mathbf{g}^{-1} \underline{\otimes} \mathbf{g}, \quad \mathbb{K}^{ij}_{kl} = \frac{1}{3} g^{ij} g_{kl}, \quad (23a)$$

$$\mathbb{M} = \mathbb{I} - \mathbb{K}, \quad \mathbb{M}^{ij}_{kl} = \frac{1}{2}(\delta^i_k \delta^j_l + \delta^i_l \delta^j_k) - \frac{1}{3} g^{ij} g_{kl}, \quad (23b)$$

265 are the *spherical operator* and the *deviatoric operator*, such that, for every symmetric tensor \mathbf{a} in
 266 $[T\mathcal{B}]^2_0$, $\mathbb{K} : \mathbf{a} = \frac{1}{3} \text{tr}(\mathbf{a}) \mathbf{g}^{-1}$ is the spherical part of \mathbf{a} , and $\mathbb{M} : \mathbf{a} = \mathbf{a} - \frac{1}{3} \text{tr}(\mathbf{a}) \mathbf{g}^{-1}$ is the deviatoric
 267 part of \mathbf{a} , where $\text{tr}(\cdot)$ is the natural trace operator, such that $\text{tr}(\mathbf{a}) = \mathbf{g} : \mathbf{a} = g_{ij} a^{ij}$. The tensors
 268 $\{\mathbb{K}, \mathbb{M}\}$ constitute the basis of the space $([T\mathcal{B}]^2_2, \text{Sym}, \text{Iso})$ of the symmetric and isotropic tensors.
 269 We remark that *all* isotropic fourth-order tensors enjoy minor symmetry (Jog, 2006), and that
 270 there exist isotropic fourth-order tensors which do *not* enjoy major symmetry (the additional basis
 271 tensor is the *skew-symmetriser* $\mathbb{W} = \frac{1}{2}(\mathbf{i} \underline{\otimes} \mathbf{i} - \mathbf{i} \overline{\otimes} \mathbf{i})$; see Jog, 2006).

272 The bases of the spaces $([T\mathcal{B}]^4_0, \text{Sym}, \text{Iso})$ and $([T\mathcal{B}]^0_4, \text{Sym}, \text{Iso})$ are obtained by raising and
 273 lowering, respectively, the indices of $\{\mathbb{K}, \mathbb{M}\}$, or by decomposing the “contravariant” symmetric
 274 identity \mathbb{I}^\sharp and the “covariant” symmetric identity \mathbb{I}^\flat , respectively. The resulting tensors are
 275 (Federico, 2012)

$$\mathbb{I}^\sharp = \frac{1}{2}(\mathbf{g}^{-1} \underline{\otimes} \mathbf{g}^{-1} + \mathbf{g}^{-1} \overline{\otimes} \mathbf{g}^{-1}), \quad \mathbb{I}^{ijkl} = \frac{1}{2}(g^{ik} g^{jl} + g^{il} g^{jk}), \quad (24a)$$

$$\mathbb{K}^\sharp = \frac{1}{3} \mathbf{g}^{-1} \underline{\otimes} \mathbf{g}^{-1}, \quad \mathbb{K}^{ijkl} = \frac{1}{3} g^{ij} g^{kl}, \quad (24b)$$

$$\mathbb{M}^\sharp = \mathbb{I}^\sharp - \mathbb{K}^\sharp, \quad \mathbb{M}^{ijkl} = \frac{1}{2}(g^{ik} g^{jl} + g^{il} g^{jk}) - \frac{1}{3} g^{ij} g^{kl}, \quad (24c)$$

276 and

$$\mathbb{I}^\flat = \frac{1}{2}(\mathbf{g} \underline{\otimes} \mathbf{g} + \mathbf{g} \overline{\otimes} \mathbf{g}), \quad \mathbb{I}_{ijkl} = \frac{1}{2}(g_{ik} g_{jl} + g_{il} g_{jk}), \quad (25a)$$

$$\mathbb{K}^\flat = \frac{1}{3} \mathbf{g} \underline{\otimes} \mathbf{g}, \quad \mathbb{K}_{ijkl} = \frac{1}{3} g_{ij} g_{kl}, \quad (25b)$$

$$\mathbb{M}^\flat = \mathbb{I}^\flat - \mathbb{K}^\flat, \quad \mathbb{M}_{ijkl} = \frac{1}{2}(g_{ik} g_{jl} + g_{il} g_{jk}) - \frac{1}{3} g_{ij} g_{kl}. \quad (25c)$$

277 The tensors $\{\mathbb{K}^\sharp, \mathbb{M}^\sharp\}$ and $\{\mathbb{K}^\flat, \mathbb{M}^\flat\}$ constitute the bases of the spaces $([T\mathcal{B}]^4_0, \text{Sym}, \text{Iso})$ and
 278 $([T\mathcal{B}]^0_4, \text{Sym}, \text{Iso})$, respectively. It is important to recall how to obtain the representation of a
 279 symmetric isotropic tensor, and we show this in the case that is most important for our purposes,
 280 i.e., that of a “contravariant” tensor. A symmetric isotropic tensor \mathbb{T} in $([T\mathcal{B}]^4_0, \text{Sym}, \text{Iso})$ can be
 281 shown to admit the representation (Walpole, 1981, 1984)

$$\mathbb{T} = \langle \mathbb{T}, \mathbb{K}^\sharp \rangle \mathbb{K}^\sharp + \frac{1}{5} \langle \mathbb{T}, \mathbb{M}^\sharp \rangle \mathbb{M}^\sharp, \quad (26)$$

282 where

$$\langle \mathbb{T}, \mathbb{K}^\sharp \rangle = \mathbb{T}^{ijkl} g_{ip} g_{jq} g_{kr} g_{ls} \left(\frac{1}{3} g^{pq} g^{rs} \right) = \frac{1}{3} \mathbb{T}^i{}^k{}_i{}^k, \quad (27a)$$

$$\langle \mathbb{T}, \mathbb{M}^\sharp \rangle = \mathbb{T}^{ijkl} g_{ip} g_{jq} g_{kr} g_{ls} \left(\frac{1}{2} (g^{pr} g^{qs} + g^{ps} g^{qr}) - \frac{1}{3} g^{pq} g^{rs} \right) = \mathbb{T}^{ij}{}_{ij} - \frac{1}{3} \mathbb{T}^i{}^k{}_i{}^k. \quad (27b)$$

283 Note that, in the second of Equations (27), we obtain a term $\frac{1}{2}(\mathbb{T}^{ij}{}_{ij} + \mathbb{T}^{ij}{}_{ji})$, which reduces to $\mathbb{T}^{ij}{}_{ij}$
 284 because of the minor symmetry of \mathbb{T} . We remark that, if \mathbb{T} is a generic, not necessarily symmetric
 285 and isotropic tensor in $[\mathcal{TB}]_0^4$, the right-hand side of Equation (26) yields the *projection* of \mathbb{T} onto
 286 the isotropic subspace $([\mathcal{TB}]_0^4, \text{Sym}, \text{Iso})$ of $[\mathcal{TB}]_0^4$, i.e.,

$$\mathbb{T}_{\text{iso}} = \langle \mathbb{T}, \mathbb{K}^\sharp \rangle \mathbb{K}^\sharp + \frac{1}{5} \langle \mathbb{T}, \mathbb{M}^\sharp \rangle \mathbb{M}^\sharp \neq \mathbb{T}, \quad (28)$$

287 where the coefficients on the right-hand side are found in precisely the same way as in Equation (27).

288 A thorough analysis of the properties of *idempotence* and *orthogonality* (Walpole, 1981,
 289 1984) enjoyed by the tensors of each of the bases $\{\mathbb{K}, \mathbb{M}\}$, $\{\mathbb{K}^\sharp, \mathbb{M}^\sharp\}$ and $\{\mathbb{K}^\flat, \mathbb{M}^\flat\}$ is discussed in a
 290 previous work (Federico, 2012), in the same covariant formalism used here. The idempotence and
 291 orthogonality of isotropic basis tensors implies that multiplication and inversion of isotropic tensors
 292 are performed by multiplying and inverting the individual scalars of the decomposition (26).

293 2.7 Transversely Isotropic Second- and Fourth-Order Tensors

294 The set

$$\mathbb{S}^2\mathcal{B} = \{\mathbf{m} \in \mathcal{TB} : \|\mathbf{m}\| = 1\}, \quad (29)$$

295 where $\|\mathbf{m}\| = \sqrt{\mathbf{m} \cdot \mathbf{m}}$ is the Euclidean norm of vector \mathbf{m} , is the subset of all *unit* vectors in the
 296 tangent bundle \mathcal{TB} , and is called the (bundle) *unit sphere* in the body \mathcal{B} . When the point x is
 297 fixed, one speaks about the *unit sphere* $\mathbb{S}_x^2\mathcal{B} = \{\mathbf{m} \in T_x\mathcal{B} : \|\mathbf{m}\| = 1\}$ at x . Transverse isotropy
 298 with respect to \mathbf{m} is defined as the symmetry (i.e., the invariance) with respect to rotations about
 299 \mathbf{m} . The direction identified by \mathbf{m} is called *symmetry axis* and the class of equivalence of the planes
 300 orthogonal to \mathbf{m} is called *transverse plane*.

301 The subspace of $[\mathcal{TB}]_0^2$ of all second-order “contravariant” symmetric tensors with transverse
 302 isotropy with respect to a direction \mathbf{m} is denoted $([\mathcal{TB}]_0^2, \mathbf{m})$. The basis of $([\mathcal{TB}]_0^2, \mathbf{m})$ is given by
 303 (Walpole, 1981, 1984; Federico, 2012)

$$\mathbf{a} = \mathbf{m} \otimes \mathbf{m}, \quad (30a)$$

$$\mathbf{t} = \mathbf{g}^{-1} - \mathbf{a}, \quad (30b)$$

304 where \mathbf{t} is the complement of tensor \mathbf{a} to \mathbf{g}^{-1} , which serves as the “contravariant identity” in $[\mathcal{TB}]_0^2$.
 305 Evidently, both \mathbf{a} and \mathbf{t} are invariant under reflections $\mathbf{m} \mapsto -\mathbf{m}$, i.e., the sense of \mathbf{m} is irrelevant.
 306 Tensors \mathbf{a} and \mathbf{t} take the geometrical meaning of *axial projection operator* and *transverse projection*
 307 *operator*, respectively. Indeed, contraction of \mathbf{a} and \mathbf{t} , by means of the metric tensor \mathbf{g} , with a vector
 308 \mathbf{v} in \mathcal{TB} yields the axial and transverse *vectorial* components of \mathbf{v} , respectively, as²

$$\mathbf{v}_{\parallel} = \mathbf{a} \cdot \mathbf{v} = (\mathbf{m} \cdot \mathbf{v}) \mathbf{m}, \quad (31a)$$

$$\mathbf{v}_{\perp} = \mathbf{t} \cdot \mathbf{v} = \mathbf{v} - (\mathbf{m} \cdot \mathbf{v}) \mathbf{m}. \quad (31b)$$

²In two of our past works (Equations (2.8) in Federico and Grillo (2012), and Equations (96) in Federico (2015)), we regrettably forgot to set the font in bold for the parallel and transverse components of a vector \mathbf{v} with respect to a direction \mathbf{m} , and we may have therefore given the misleading impression that we were referring to scalar components when, in fact, we meant to speak about *vectorial* components.

309 In the jargon of composite and fibre-reinforced materials, tensor \mathbf{a} is often called the *structure tensor*
 310 or *fabric tensor* of direction \mathbf{m} . Tensor \mathbf{t} is often simply called *projector* (Bonet and Wood, 2008;
 311 Gurtin et al., 2010). It is sometimes convenient to explicitly indicate the dependence of \mathbf{a} and \mathbf{t} on
 312 the direction \mathbf{m} , in which case we say that $\{\mathbf{a}(\mathbf{m}), \mathbf{t}(\mathbf{m})\}$ is the basis of the space $([T\mathcal{B}]_0^2, \mathbf{m})$. As
 313 seen in the case of isotropy, transversely isotropic second-order tensors are necessarily symmetric.

314 The basis of the subspace of $[T\mathcal{B}]_0^4$ of all tensors with transverse isotropy with respect to
 315 direction \mathbf{m} , denoted $([T\mathcal{B}]_0^4, \mathbf{m})$, has been obtained in two different versions by Walpole (1981,
 316 1984). Initially, Walpole (1981) used a tensor basis allowing for a representation in the form of a
 317 6×1 array, which has been extensively used by other authors (Weng, 1990; Qiu and Weng, 1990;
 318 Bhattacharyya and Weng, 1994; Wu and Herzog, 2002; Federico et al., 2004, 2005). Later, Walpole
 319 (1984) perfected the representation, with new normalisation constants for the basis tensors, which
 320 allows for an extremely convenient representation in an array constituted by a 2×2 matrix and 2
 321 scalars (Walpole (1984) has also provided similar representations for all other symmetry groups).
 322 This later representation (Walpole, 1984) has been used in more recent works (Federico, 2015;
 323 Federico et al., 2015), developed within a covariant framework, and we do so in this chapter too.

324 The basis of $([T\mathcal{B}]_0^4, \mathbf{m})$ proposed by Walpole (1984) is obtained (similarly to that proposed
 325 in the older work; Walpole, 1981) by means of suitable tensor products, as

$$\mathbb{U}_{11} = \mathbf{a} \otimes \mathbf{a}, \quad (\mathbb{U}_{11})^{ijkl} = a^{ij} a^{kl}, \quad (32a)$$

$$\mathbb{U}_{12} = \frac{\sqrt{2}}{2} \mathbf{a} \otimes \mathbf{t}, \quad (\mathbb{U}_{12})^{ijkl} = \frac{\sqrt{2}}{2} a^{ij} t^{kl}, \quad (32b)$$

$$\mathbb{U}_{21} = \frac{\sqrt{2}}{2} \mathbf{t} \otimes \mathbf{a}, \quad (\mathbb{U}_{21})^{ijkl} = \frac{\sqrt{2}}{2} t^{ij} a^{kl}, \quad (32c)$$

$$\mathbb{U}_{22} = \frac{1}{2} \mathbf{t} \otimes \mathbf{t}, \quad (\mathbb{U}_{22})^{ijkl} = \frac{1}{2} t^{ij} t^{kl}, \quad (32d)$$

$$\mathbb{V}_1 = \frac{1}{2} (\mathbf{t} \otimes \mathbf{t} + \mathbf{t} \otimes \mathbf{t} - \mathbf{t} \otimes \mathbf{t}), \quad (\mathbb{V}_1)^{ijkl} = \frac{1}{2} (t^{ik} t^{jl} + t^{il} t^{jk} - t^{ij} t^{kl}), \quad (32e)$$

$$\mathbb{V}_2 = \frac{1}{2} (\mathbf{a} \otimes \mathbf{t} + \mathbf{a} \otimes \mathbf{t} + \mathbf{t} \otimes \mathbf{a} + \mathbf{t} \otimes \mathbf{a}), \quad (\mathbb{V}_2)^{ijkl} = \frac{1}{2} (a^{ik} t^{jl} + a^{il} t^{jk} + t^{ik} a^{jl} + t^{il} a^{jk}). \quad (32f)$$

326 The transversely isotropic basis in Equation (32) is denoted $\{\mathbb{U}_{\alpha\beta}, \mathbb{V}_\gamma\}_{\alpha,\beta,\gamma=1}^2$ and, when it is
 327 convenient to explicitly indicate the dependence of the basis tensors on the direction \mathbf{m} , one says
 328 that $\{\mathbb{U}_{\alpha\beta}(\mathbf{m}), \mathbb{V}_\gamma(\mathbf{m})\}_{\alpha,\beta,\gamma=1}^2$ is the basis of the space $([T\mathcal{B}]_0^4, \mathbf{m})$. A tensor \mathbb{T} in $([T\mathcal{B}]_0^4, \mathbf{m})$ is
 329 expressed as

$$\mathbb{T} = \bar{\mathbb{T}}^{\alpha\beta} \mathbb{U}_{\alpha\beta} + \bar{\mathbb{T}}^\gamma \mathbb{V}_\gamma, \quad (33)$$

330 where Einstein's summation convention is understood for $\alpha, \beta, \gamma \in \{1, 2\}$ and the components
 331 $\bar{\mathbb{T}}^{\alpha\beta}$ and $\bar{\mathbb{T}}^\gamma$ are obtained by the scalar product of \mathbb{T} with each of the basis tensors, with some
 332 normalisation constants:³

$$\bar{\mathbb{T}}^{\alpha\beta} = \langle \mathbb{T}, \mathbb{U}_{\alpha\beta} \rangle, \quad \bar{\mathbb{T}}^\gamma = \frac{1}{2} \langle \mathbb{T}, \mathbb{V}_\gamma \rangle. \quad (34)$$

333 In the basis of Equation (32) the tensors $\mathbb{U}_{\alpha\beta}$ constitute an algebra isomorphic to that of
 334 2×2 matrices (Walpole, 1984), which allows for grouping the Walpole components $\bar{\mathbb{T}}^{\alpha\beta}$ and $\bar{\mathbb{T}}^\gamma$ of
 335 Equation (34) into the array

$$\bar{\mathbb{T}} = \left\{ \begin{bmatrix} \bar{\mathbb{T}}^{11} & \bar{\mathbb{T}}^{12} \\ \bar{\mathbb{T}}^{21} & \bar{\mathbb{T}}^{22} \end{bmatrix}, \bar{\mathbb{T}}^1, \bar{\mathbb{T}}^2 \right\} = \{[\bar{\mathbb{T}}^{\alpha\beta}], \bar{\mathbb{T}}^\gamma\}, \quad (35)$$

³In some previous works (Federico, 2010a,b), we used the normalisation constants of the later work by Walpole (1984), but kept the formalism with the 6×1 array formalism of the earlier work by Walpole (1981). We candidly admit that this was an infelicitous choice on our part. Also, because of an incautious copy-and-paste operation from the definitions of the basis tensors, we reported (see Appendices in Federico, 2010a,b) the wrong coefficients for the scalar products in Equation (34).

336 which we call *Walpole array representation* of tensor \mathbb{T} . Note the compact notation $\overline{\mathbb{T}} = \{[\overline{\mathbb{T}}^{\alpha\beta}], \overline{\mathbb{T}}^\gamma\}$.

337 It is precisely for the form of the array in Equation (35) that we find Walpole’s formalism
 338 (Walpole, 1984) to be very convenient. Indeed, *all* operations on transversely isotropic tensors in
 339 $([\mathcal{TB}]_0^4, \mathbf{m})$ can be performed by working on the 2×2 matrix and the 2 scalars of the Walpole array of
 340 each tensor. Linear combination of tensors in $([\mathcal{TB}]_0^4, \mathbf{m})$ can be obtained via the linear combination
 341 of the matrices and the individual scalars. Given a tensor \mathbb{T} with Walpole array $\overline{\mathbb{T}} = \{[\overline{\mathbb{T}}^{\alpha\beta}], \overline{\mathbb{T}}^\gamma\}$,
 342 the Walpole array of the transpose \mathbb{T} is obtained by simply transposing the 2×2 matrix, i.e.,
 343 $\overline{\mathbb{T}^T} = \overline{\mathbb{T}}^T = \{[\overline{\mathbb{T}}^{\alpha\beta}]^T, \overline{\mathbb{T}}^\gamma\}$. Moreover, since $\mathbb{U}_{12}^T = \mathbb{U}_{21}$, major (diagonal) symmetry of a tensor \mathbb{T}
 344 is attained if $\overline{\mathbb{T}}^{12} = \overline{\mathbb{T}}^{21}$, in which case \mathbb{T} has only 5 independent components, rather than the
 345 6 independent components of the general case.⁴ We also remark that positive definiteness of a
 346 tensor \mathbb{T} in $([\mathcal{TB}]_0^4, \mathbf{m})$ can be checked extremely simply: \mathbb{T} is positive definite if, and only if, the
 347 2×2 matrix $[\overline{\mathbb{T}}^{\alpha\beta}]$ is positive definite and the 2 scalars $\overline{\mathbb{T}}^\gamma$ are strictly positive. We remark that
 348 *all* transversely isotropic fourth-order tensors (i.e., tensors of the space $([\mathcal{TB}]_0^4, \mathbf{m})$, spanned by the
 349 basis in Equation (32)) enjoy minor symmetry.

350 At this point, one may wonder how to treat transversely isotropic fourth-order tensors of
 351 type other than those in $([\mathcal{TB}]_0^4, \mathbf{m})$, for instance the tensors in $([\mathcal{TB}]_4^0, \mathbf{m})$, among which there are
 352 the inverses (when they exist) of those in $([\mathcal{TB}]_0^4, \mathbf{m})$. Fortunately, the representation with Walpole’s
 353 array of Equation (39) is *independent* of the type of fourth-order tensor at hand. For instance, if
 354 we transform a tensor \mathbb{T} in $([\mathcal{TB}]_0^4, \mathbf{m})$ into its “covariant” counterpart \mathbb{T}^b in $([\mathcal{TB}]_4^0, \mathbf{m})$, we have

$$\mathbb{T} = \overline{\mathbb{T}}^{\alpha\beta} \mathbb{U}_{\alpha\beta} + \overline{\mathbb{T}}^\gamma \mathbb{V}_\gamma \mapsto \mathbb{T}^b = \overline{\mathbb{T}}^{\alpha\beta} \mathbb{U}_{\alpha\beta}^b + \overline{\mathbb{T}}^\gamma \mathbb{V}_\gamma^b, \quad (36)$$

355 i.e., the transformation takes place on the basis tensors, leaving the Walpole components *untouched*.
 356 Thus, the double contraction of, e.g., a tensor \mathbb{T} in $([\mathcal{TB}]_2^2, \mathbf{m})$ and a tensor \mathbb{Z} in $([\mathcal{TB}]_4^0, \mathbf{m})$ is
 357 obtained by keeping in mind that the resulting tensor belongs to $([\mathcal{TB}]_0^4, \mathbf{m})$, and by performing
 358 the ordinary row-by-column product of the two matrices, and the multiplication of the homologous
 359 scalars, i.e.,

$$\overline{\mathbb{T}} : \overline{\mathbb{Z}} = \left\{ \begin{bmatrix} \overline{\mathbb{T}}^{11} & \overline{\mathbb{T}}^{12} \\ \overline{\mathbb{T}}^{21} & \overline{\mathbb{T}}^{22} \end{bmatrix} \begin{bmatrix} \overline{\mathbb{Z}}^{11} & \overline{\mathbb{Z}}^{12} \\ \overline{\mathbb{Z}}^{21} & \overline{\mathbb{Z}}^{22} \end{bmatrix}, \overline{\mathbb{T}}^1 \overline{\mathbb{Z}}^1, \overline{\mathbb{T}}^2 \overline{\mathbb{Z}}^2 \right\}. \quad (37)$$

360 Also, it is now clear how to represent the inverse (when it exists) of a tensor \mathbb{T} in $([\mathcal{TB}]_0^4, \mathbf{m})$.
 361 Indeed, the inverse of an invertible tensor \mathbb{T} in $([\mathcal{TB}]_0^4, \mathbf{m})$ is the tensor \mathbb{T}^{-1} in $([\mathcal{TB}]_4^0, \mathbf{m})$ such that
 362 $\mathbb{T} : \mathbb{T}^{-1} = \mathbb{I}$ and $\mathbb{T}^{-1} : \mathbb{T} = \mathbb{I}^T$, and has Walpole array representation

$$\overline{\mathbb{T}^{-1}} \equiv \overline{\mathbb{T}}^{-1} = \left\{ \begin{bmatrix} \overline{\mathbb{T}}^{11} & \overline{\mathbb{T}}^{12} \\ \overline{\mathbb{T}}^{21} & \overline{\mathbb{T}}^{22} \end{bmatrix}^{-1}, \frac{1}{\overline{\mathbb{T}}^1}, \frac{1}{\overline{\mathbb{T}}^2} \right\}. \quad (38)$$

363 In an orthonormal basis $\{\mathbf{e}_i\}_{i=1}^3$, such that $\mathbf{e}_1 = \mathbf{m}$, the components of the Walpole array $\overline{\mathbb{T}} =$
 364 $\{[\overline{\mathbb{T}}^{\alpha\beta}], \overline{\mathbb{T}}^\gamma\}$ of a tensor \mathbb{T} are related to the conventional components \mathbb{T}^{ijkl} by

$$\overline{\mathbb{T}} = \left\{ \begin{bmatrix} \mathbb{T}^{1111} & \sqrt{2} \mathbb{T}^{1122} \\ \sqrt{2} \mathbb{T}^{2211} & 2 \mathbb{T}^{2222} - 2 \mathbb{T}^{2323} \end{bmatrix}, 2 \mathbb{T}^{2323}, 2 \mathbb{T}^{1212} \right\}. \quad (39)$$

365 Since an isotropic tensor is transversely isotropic with respect to any direction \mathbf{m} , it is possible
 366 to express it in Walpole’s transversely isotropic representation. In particular, the “contravariant”

⁴In our original work (see text immediately following Equation (20) in Federico et al., 2004), we had stated that \mathbb{U}_{11} , \mathbb{U}_{22} , \mathbb{V}_1 and \mathbb{V}_2 (called \mathbb{B}_2 , \mathbb{B}_1 , \mathbb{B}_3 and \mathbb{B}_4 , respectively, in Federico et al., 2004) span the whole space of major- and minor-symmetric (transversely isotropic) tensors, which is of course incorrect, as we should have added also $\frac{1}{2}(\mathbb{U}_{12} + \mathbb{U}_{21})$ (corresponding to $\frac{1}{2}(\mathbb{B}_5 + \mathbb{B}_6)$ in Federico et al., 2004).

367 fourth-order identity, and the spherical and deviatoric operators in $[T\mathcal{B}]_0^4$, defined in Equation (24),
 368 have Walpole array representations

$$\overline{\mathbb{I}}^\sharp = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, 1, 1 \right\}, \quad (40a)$$

$$\overline{\mathbb{K}}^\sharp = \left\{ \begin{bmatrix} \frac{1}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{2}{3} \end{bmatrix}, 0, 0 \right\}, \quad (40b)$$

$$\overline{\mathbb{M}}^\sharp = \left\{ \begin{bmatrix} \frac{2}{3} & -\frac{\sqrt{2}}{3} \\ -\frac{\sqrt{2}}{3} & \frac{1}{3} \end{bmatrix}, 1, 1 \right\}. \quad (40c)$$

369 Thus, a generic symmetric isotropic tensor \mathbb{T} in $([T\mathcal{B}]_0^4, \text{Sym}, \text{Iso})$, which is written as

$$\mathbb{T} = 3k \mathbb{K}^\sharp + 2m \mathbb{M}^\sharp, \quad (41)$$

370 in the symmetric isotropic basis $\{\mathbb{K}^\sharp, \mathbb{M}^\sharp\}$ admits the representation

$$\overline{\mathbb{T}} = \left\{ \begin{bmatrix} k + \frac{4}{3}m & \sqrt{2}(k - \frac{2}{3}m) \\ \sqrt{2}(k - \frac{2}{3}m) & 2(k + \frac{4}{3}m) - 2m \end{bmatrix}, 2m, 2m \right\}, \quad (42)$$

371 where the coefficients $3k$ and $2m$ in Equation (41) echo those typical of isotropic linear elasticity
 372 (see Equation (54)), and are found as shown in Equation (26).

373 2.8 Basic Relations of the Theory of Linear Elasticity

374 Linear Elasticity can be developed as an independent branch of Mathematical Physics (see, e.g.,
 375 the text by Gurtin, 1972), or can be retrieved by linearising the general Theory of (Non-Linear)
 376 Elasticity (a covariant procedure is presented in the text by Marsden and Hughes, 1983). Linear
 377 Elasticity has a strong pedagogical character. Indeed, it often allows to find either analytical
 378 solutions or solutions in closed form to many problems of engineering relevance. Moreover, in
 379 many circumstances, it suffices to determine first-order approximations that, with relatively low
 380 computational costs, provide solutions to real-world problems even in the cases in which engineering
 381 materials undergo finite deformations. Perhaps because of these advantages, Linear Elasticity is
 382 what is usually taught to the vast majority of the students in structural/mechanical Engineering
 383 or Physics during their undergraduate studies. Linear Elasticity is so diffused that some call it
 384 ‘‘Classical Elasticity’’ and that, still today, quite many understand ‘‘Linear Elasticity’’, when they
 385 hear the word ‘‘Elasticity’’. One can choose to present the linear theory of elasticity either, and
 386 equivalently, by starting from stress or from energy. We choose the latter and we present this
 387 approach after having briefly introduced displacement, strain and stress.

388 In a body \mathcal{B} , the *displacement* is the vector field

$$\mathbf{u} : \mathcal{B} \rightarrow T\mathcal{B} : x \mapsto \mathbf{u}(x) \in T_x\mathcal{B}, \quad (43)$$

389 whose gradient (also called *covariant derivative*) is called *displacement gradient*,

$$\mathbf{h} = \text{grad } \mathbf{u}, \quad h^i{}_j = u^i|_j, \quad (44)$$

390 where the vertical bar denotes covariant differentiation.⁵ The *infinitesimal strain* is the “covariant”
 391 second-order tensor field

$$\boldsymbol{\epsilon} : \mathcal{B} \rightarrow [T\mathcal{B}]_2^0 : x \mapsto \boldsymbol{\epsilon}(x) \in [T_x\mathcal{B}]_2^0, \quad (45)$$

392 defined as the symmetric part of the “covariant” displacement gradient tensor $\mathbf{h}^b = \mathbf{g}\mathbf{h}$ (with
 393 components $h_{ij} = g_{ip}h^p_j$), i.e.,

$$\boldsymbol{\epsilon} = \frac{1}{2}(\mathbf{h}^b + \mathbf{h}^{bT}), \quad \epsilon_{ij} = \frac{1}{2}(h_{ij} + h_{ji}). \quad (46)$$

394 The *Cauchy stress* is defined as a “contravariant” second-order tensor field⁶

$$\boldsymbol{\sigma} : \mathcal{B} \rightarrow [T\mathcal{B}]_0^2 : x \mapsto \boldsymbol{\sigma}(x) \in [T_x\mathcal{B}]_0^2. \quad (47)$$

395 In the absence of external body forces and neglecting inertia, the balance of linear momentum
 396 reduces to the vanishing of the divergence of the Cauchy stress, i.e.,

$$\operatorname{div} \boldsymbol{\sigma} = \mathbf{0}, \quad \sigma^{ij}{}_{|j} = 0. \quad (48)$$

397 Often, the balance of angular momentum is invoked to obtain the condition of symmetry of the
 398 Cauchy stress:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T, \quad \sigma^{ij} = \sigma^{ji}. \quad (49)$$

399 Balance of linear and angular momentum constitute a system of 6 equations in 12 unknowns
 400 (the 3 components of the displacement and the 9 components of the Cauchy stress) or, equivalently,
 401 of 3 independent equations in 9 unknowns (the 3 components of the displacement and the 6
 402 independent components of the Cauchy stress). In order to be able to close the system, the need
 403 arises for 6 additional relations, called *constitutive laws*, expressing the stress tensor as a function
 404 of the strain tensor.

405 A material is said to obey a linear elastic constitutive law if there exists a quadratic function
 406 of the infinitesimal strain,

$$W(x) = \hat{W}(\boldsymbol{\epsilon}(x), x) = \frac{1}{2} \boldsymbol{\epsilon}(x) : \mathbb{L}(x) : \boldsymbol{\epsilon}(x) = \frac{1}{2} \epsilon_{ij}(x) L^{ijkl}(x) \epsilon_{kl}(x), \quad (50)$$

407 called (quadratic) *elastic potential* ($W : \mathcal{B} \rightarrow \mathbb{R} : x \mapsto W(x)$ denotes the *scalar field*, while
 408 \hat{W} denotes the corresponding *constitutive function*), such that the stress can be obtained as the
 409 derivative of \hat{W} with respect to the strain, i.e.,

$$\boldsymbol{\sigma}(x) = \frac{\partial \hat{W}}{\partial \boldsymbol{\epsilon}}(\boldsymbol{\epsilon}(x), x) = \mathbb{L}(x) : \boldsymbol{\epsilon}(x), \quad \sigma^{ij}(x) = \frac{\partial \hat{W}}{\partial \epsilon_{ij}}(\boldsymbol{\epsilon}(x), x) = L^{ijkl}(x) \epsilon_{kl}(x). \quad (51)$$

410 The “contravariant” fourth-order tensor field

$$\mathbb{L} : \mathcal{B} \rightarrow [T\mathcal{B}]_0^4 : x \mapsto \mathbb{L}(x) \in [T_x\mathcal{B}]_0^4 \quad (52)$$

⁵In Cartesian coordinates, covariant differentiation of a vector or tensor field reduces to the regular partial derivative and one writes, e.g., for a vector field, $u^i{}_{,j}$.

⁶In the general, large-deformation setting of Continuum Mechanics, the Cauchy stress is defined as a *spatial* tensor field, valued in $[T\mathcal{S}]_0^2$. In the small-deformation theory, however, the distinction between reference configuration (or body \mathcal{B}) and current configuration fades out, and it is legitimate to define also tensors, which by their nature would be spatial, in the body \mathcal{B} rather than in the space \mathcal{S} . In contrast, it is natural to define the infinitesimal strain $\boldsymbol{\epsilon}$ as a tensor field valued in $[T\mathcal{B}]_2^0$, since it can be thought of as the linearisation of the *material* Green-Lagrange strain \mathbf{E} .

411 is called the *linear elasticity tensor* and does *not* depend on x if the body is *homogeneous*.
 412 Equation (50) and the symmetry of ϵ imply that the elasticity tensor \mathbb{L} enjoys both major and
 413 minor symmetry. In order to guarantee the positiveness of the elastic potential, which implies the
 414 positivity of the internal work (or deformation work), one normally requires the positive definiteness
 415 of the elasticity tensor \mathbb{L} . The positive definiteness of \mathbb{L} in turn implies its invertibility. Note that
 416 the inverse of the “contravariant” elasticity tensor \mathbb{L} , which is also called *stiffness elasticity tensor*,
 417 is the “covariant” *compliance elasticity tensor* \mathbb{L}^{-1} , which is a tensor field valued in $[\mathcal{TB}]_4^0$.

418 For a transversely isotropic elasticity tensor \mathbb{L} , Walpole’s representation takes the form (see
 419 Equation (39))

$$\bar{\mathbb{L}} = \left\{ \left[\begin{array}{cc} n & \sqrt{2}\ell \\ \sqrt{2}\ell & 2c \end{array} \right], 2\mu_t, 2\mu_a \right\}, \quad (53)$$

420 where (see Hill, 1964) n is the *modulus in uniaxial strain* (also called *p-wave modulus* or also,
 421 in the literature on articular cartilage, *aggregate modulus*: see Holmes and Mow, 1990), c is the
 422 *plane-strain bulk modulus* (in the transverse plane of transverse isotropy), ℓ is the *cross modulus*
 423 (transversely isotropic analogue of the first Lamé’s constant $\lambda = \kappa - \frac{2}{3}\mu$ of isotropic linear elasticity:
 424 see Spencer, 1984), μ_t is the shear modulus in the transverse plane, and μ_a is the shear modulus
 425 in any plane containing the axis of symmetry \mathbf{m} of transverse isotropy.

426 An isotropic elasticity tensor \mathbb{L} can be represented in the basis $\{\mathbb{K}^\sharp, \mathbb{M}^\sharp\}$ of symmetric
 427 isotropic tensors in $[\mathcal{TB}]_0^4$ as (see Equation (41))

$$\mathbb{L} = 3\kappa\mathbb{K}^\sharp + 2\mu\mathbb{M}^\sharp, \quad (54)$$

428 where κ and μ are the bulk modulus and shear modulus, respectively. Using Equation (42) for the
 429 expression of an isotropic tensor in Walpole’s transversely isotropic array, we can represent the
 430 isotropic elasticity tensor of Equation (54) as

$$\bar{\mathbb{L}} = \left\{ \left[\begin{array}{cc} \kappa + \frac{4}{3}\mu & \sqrt{2}(\kappa - \frac{2}{3}\mu) \\ \sqrt{2}(\kappa - \frac{2}{3}\mu) & 2(\kappa + \frac{4}{3}\mu) - 2\mu \end{array} \right], 2\mu, 2\mu \right\}, \quad (55)$$

431 in which it is possible to recognise the first Lamé’s constant $\lambda = \kappa - \frac{2}{3}\mu$ and the modulus in uniaxial
 432 strain $n = \kappa + \frac{4}{3}\mu = \lambda + 2\mu$. In terms of the Lamé’s moduli λ and μ , the Walpole array reads

$$\bar{\mathbb{L}} = \left\{ \left[\begin{array}{cc} \lambda + 2\mu & \sqrt{2}\lambda \\ \sqrt{2}\lambda & 2(\lambda + 2\mu) - 2\mu \end{array} \right], 2\mu, 2\mu \right\}, \quad (56)$$

433 where $(\lambda + 2\mu) - \mu = \lambda + \mu$ is the isotropic equivalent of the plane-strain bulk modulus c of
 434 transverse isotropy (Equation (53)).

435 3 Composite Materials with Aligned Inclusions

436 We first recall the definitions of *Eshelby’s fourth-order tensor* \mathbb{S} introduced by Eshelby (1957) and
 437 of the closely related *strain concentration tensor* \mathbb{A} , which arises in the case of inclusions with
 438 material properties different from those of the matrix. Finally, we introduce composite materials
 439 with inclusions as described by the works of Hill (1963, 1965) and Walpole (1966a,b, 1969), and
 440 focus on the case of aligned inclusions.

441 3.1 Eshelby's Inclusion and Fourth-Order Tensor

442 Eshelby (1957) studied the problem of an inclusion in an infinite matrix, and in particular the case
 443 of an ellipsoidal inclusion. Eshelby constructed the inclusion problem in several steps (Eshelby,
 444 1957, last paragraph of page 376), which we report in our own words, following a previous work
 445 (Alhasadi and Federico, 2017).

- 446 1) A cavity is cut in a body \mathcal{B} and a *transformation strain* ϵ^* is applied to the geometry of the
 447 region \mathcal{D} occupied by the cavity, which is thus mapped into the new region \mathcal{D}^* ; the remaining
 448 region $\mathcal{M} = \mathcal{B} \setminus \mathcal{D}$ is called *matrix*;
- 449 2) The transformed region \mathcal{D}^* is now “filled” with a material, which could be the same as
 450 that of the matrix, with elasticity tensor \mathbb{L}_0 , or another one, with elasticity tensor \mathbb{L}_1 . The
 451 transformed region \mathcal{D}^* , now assigned with certain elastic properties, constitutes the *inclusion*,
 452 which *no longer fits* the original cavity \mathcal{D} ;
- 453 3) In order to make the inclusion occupying the transformed region \mathcal{D}^* fit again into the original
 454 cavity \mathcal{D} , tractions are applied on the boundary of the inclusion, so that it attains a strain
 455 $-\epsilon^*$, and then it is put back into the cavity;
- 456 4) Once the inclusion is back in place, the tractions on the boundary are removed, and so the
 457 inclusion and the surrounding matrix relax, causing a *cancelling strain* or *constrained strain*
 458 ϵ^c , which is *discontinuous* across the boundary of the inclusion.

459 At the end of this sequence of operations, and in the absence of external tractions applied on the
 460 boundary of the body \mathcal{B} , the *residual strain* due to the *geometrical misfit* is

$$\epsilon^b = \epsilon^c, \quad \text{in } \mathcal{M}, \quad (57a)$$

$$\epsilon^b = \epsilon^c - \epsilon^*, \quad \text{in } \mathcal{D}, \quad (57b)$$

461 where we emphasise again that the cancelling strain ϵ^c is *discontinuous* across the boundary of \mathcal{D}
 462 and thus must be studied and described *piecewise*.

463 In the absence of the inclusion, i.e., if the body \mathcal{B} were perfectly homogeneous (elasticity
 464 tensor equal to \mathbb{L}_0 everywhere) and without any region with geometrical misfit (identically vanishing
 465 transformation strain ϵ^*), the cancelling strain would vanish identically, and the application of
 466 traction forces on the boundary of \mathcal{B} would cause a stress state described by the continuous field
 467 σ^a everywhere in \mathcal{B} , which in turn would cause the continuous strain field

$$\epsilon^a = \mathbb{L}_0^{-1} : \sigma^a, \quad \text{everywhere in } \mathcal{B}. \quad (58)$$

468 In the presence of both inclusion and external tractions, the linearity of the problem allows to write
 469 the total strain as the superposition of that in Equation (57), which was obtained in the absence
 470 of external applied tractions, and of that in Equation (58), which was obtained in the absence of
 471 inclusion, as

$$\epsilon = \epsilon^a + \epsilon^b = \epsilon^a + \epsilon^c, \quad \text{in } \mathcal{M}, \quad (59a)$$

$$\epsilon = \epsilon^a + \epsilon^b = \epsilon^a + \epsilon^c - \epsilon^* \quad \text{in } \mathcal{D}. \quad (59b)$$

472 For the case of an ellipsoidal inclusion, it is clear that it remains an ellipsoid if, and only if,
 473 the transformation strain ϵ^* is uniform (Eshelby, 1957). In this case, also the cancelling strain ϵ^c

474 in the inclusion is uniform, and it is possible to relate it to the transformation strain ϵ^* by means
 475 of the relation (Eshelby, 1957)

$$\epsilon^c = \mathbb{S} : \epsilon^*, \quad \text{in } \mathcal{D}, \quad (60)$$

476 where the tensor \mathbb{S} in $[\mathcal{TB}]_2^2$ is the *fourth-order Eshelby tensor*, which depends on ratios of the
 477 elastic constants of the matrix and on the geometry of the inclusion. For an isotropic matrix, there
 478 is only one independent geometrical ratio of elastic constants, which is usually chosen to be the
 479 Poisson’s ratio ν_0 (e.g., Qiu and Weng, 1990). For the case of spheroidal inclusions, i.e., revolution
 480 ellipsoids, the only independent geometrical ratio is that of the major to the minor semi-axis (e.g.,
 481 Qiu and Weng, 1990). Note that the fourth-order Eshelby tensor can be seen as the linear operator
 482 $\mathbb{S} : [\mathcal{TB}]_2^0 \rightarrow [\mathcal{TB}]_2^0$ mapping the transformation strain into the cancelling strain. In components,
 483 Equation (60) reads

$$\epsilon_{ij}^c = S_{ij}{}^{kl} \epsilon_{kl}^*, \quad \text{in } \mathcal{D}. \quad (61)$$

484 We remark that the Eshelby tensor \mathbb{S} has minor symmetry on each pair of legs, but it lacks major
 485 symmetry (i.e., its “contravariant” counterpart \mathbb{S}^\sharp , with components $(\mathbb{S}^\sharp)^{ijkl} \equiv S^{ijkl} = g^{ip} g^{jq} S_{pq}{}^{kl}$,
 486 and its “covariant” counterpart \mathbb{S}^\flat , with components $(\mathbb{S}^\flat)_{ijkl} \equiv S_{ijkl} = S_{ij}{}^{rs} g_{rk} g_{sl}$, lack major
 487 symmetry). Therefore, assuming an isotropic matrix, for the case of an ellipsoidal inclusion with
 488 three distinct semi-axes, \mathbb{S} is a non-major-symmetric orthotropic tensor with 12 independent
 489 components and, for the case of a spheroidal inclusion with two equal semi-axes (i.e., a revolution
 490 ellipsoid), it is a non-major-symmetric transversely isotropic tensor with 6 independent components.

491 There are three possible inclusion problems:

- 492 • The “homogeneous inclusion”, with geometrical misfit caused by a transformation strain ϵ^* ,
 493 but material properties identical to those of the matrix, i.e., $\mathbb{L}_0 = \mathbb{L}_1$;
- 494 • The “inhomogeneous inclusion”, with no geometrical misfit, i.e., $\epsilon^* = \mathbf{0}$, but material
 495 properties different from those of matrix, i.e., $\mathbb{L}_0 \neq \mathbb{L}_1$;
- 496 • The “general inclusion”, with both geometrical misfit, i.e., $\epsilon^* \neq \mathbf{0}$, and material properties
 497 different from those of matrix, i.e., $\mathbb{L}_0 \neq \mathbb{L}_1$.

498 The “homogeneous” case is the fundamental one, and indeed the “inhomogeneous” and the
 499 “general” cases are solved by reducing the effect of the different material properties to an *equivalent*
 500 *transformation strain* (Eshelby, 1957; Mura, 1987; Alhasadi and Federico, 2017). In this work, we
 501 shall restrict our attention to the “inhomogeneous” case. Thus, we shall exclusively deal with
 502 inclusions with no geometrical misfit with the matrix, but with material properties different from
 503 those of the matrix.

504 3.2 Strain Concentration Tensor

505 The *strain concentration tensor* arises in the cases of the “inhomogeneous” inclusion and “general”
 506 inclusion, and is the object that captures the difference in material properties between matrix
 507 and inclusion within the method of the *equivalent transformation strain*, which is that fictitious
 508 transformation strain that has the same effect on the stress and strain fields that the mismatch in
 509 material properties has. This method is, again, described in detail by Eshelby (1957), and we also
 510 mention the classical book by Mura (1987).

511 The standard derivation of the strain concentration tensor is done in the case of the
 512 “inhomogeneous inclusion”, and we report its expression (for the details, see, e.g., Weng, 1984,
 513 1990; Alhasadi and Federico, 2017)

$$\mathbb{A} = [\mathbb{I}^T + \mathbb{S} : [\mathbb{L}_0^{-1} : \mathbb{L}_1 - \mathbb{I}^T]]^{-1}, \quad \text{in } \mathcal{D}, \quad (62)$$

514 which clearly depends on the Eshelby tensor, \mathbb{S} , and the elasticity tensors of matrix and inclusion,
 515 \mathbb{L}_0 and \mathbb{L}_1 . Like \mathbb{S} , tensor \mathbb{A} is in $[\mathcal{TB}]_2^2$ and is endowed with minor, but not major symmetry. Its
 516 component expression is

$$(\mathbb{A}^{-1})_{ij}{}^{kl} = (\mathbb{I}^T)_{ij}{}^{kl} + S_{ij}{}^{pq} [(\mathbb{L}_0^{-1})_{pqrs} (\mathbb{L}_1)^{rskl} - (\mathbb{I}^T)_{pq}{}^{kl}], \quad \text{in } \mathcal{D}. \quad (63)$$

517 Note the use of the *transpose* of the symmetric identity \mathbb{I} . Indeed, since the symmetric identity \mathbb{I}
 518 belongs to $[\mathcal{TB}]_2^2$, it is necessary here to use its transpose \mathbb{I}^T , which belongs to $[\mathcal{TB}]_2^2$, in order
 519 to be able to sum it to the other tensors. This distinction is unnecessary in Cartesian coordinates,
 520 and indeed in all papers and books we are aware of, including our own past works, one finds the
 521 expression in Equation (62) written with \mathbb{I} .

522 The strain concentration tensor gives the cancelling strain in the inclusion as

$$\boldsymbol{\epsilon}^c = (\mathbb{A} - \mathbb{I}^T) : \boldsymbol{\epsilon}^a, \quad \text{in } \mathcal{D}, \quad (64)$$

523 with components

$$\epsilon_{ij}^c = \left(A_{ij}{}^{kl} - (\mathbb{I}^T)_{ij}{}^{kl} \right) \epsilon_{kl}^a, \quad \text{in } \mathcal{D}. \quad (65)$$

524 It is important to note the structural similarity between Equation (60) and Equation (64). However,
 525 it is even more important to emphasise that, while in the case of the “homogeneous inclusion”
 526 (Equation (60)), the cancelling strain $\boldsymbol{\epsilon}^c$ is a constant once the transformation strain $\boldsymbol{\epsilon}^*$ is assigned,
 527 in the case of the “inhomogeneous inclusion”, the cancelling strain $\boldsymbol{\epsilon}^c$ is *linearly related* to the
 528 applied strain $\boldsymbol{\epsilon}^a$. Indeed, since there is no geometrical misfit ($\boldsymbol{\epsilon}^* = \mathbf{0}$), the cancelling strain $\boldsymbol{\epsilon}^c$ is
 529 identically zero when no tractions are applied, i.e., when the applied strain $\boldsymbol{\epsilon}^a$ is zero.

530 By adding the applied strain $\boldsymbol{\epsilon}^a$ to either side of Equation (64), we obtain the total strain
 531 in the “inhomogeneous” inclusion as

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^a + \boldsymbol{\epsilon}^c = \mathbb{A} : \boldsymbol{\epsilon}^a, \quad \text{in } \mathcal{D}, \quad (66)$$

532 with component expression

$$\epsilon_{ij} = \epsilon_{ij}^a + \epsilon_{ij}^c = A_{ij}{}^{kl} \epsilon_{kl}^a, \quad \text{in } \mathcal{D}. \quad (67)$$

533 Equation (66) gives \mathbb{A} in $[\mathcal{TB}]_2^2$ its physical meaning of *strain concentration tensor*: it can indeed
 534 be seen as the linear operator $\mathbb{A} : [\mathcal{TB}]_2^0 \rightarrow [\mathcal{TB}]_2^0$ that maps the applied strain $\boldsymbol{\epsilon}^a$ that would be
 535 attained in the absence of inclusion into the strain $\boldsymbol{\epsilon}$ *actually attained* by the inclusion.

536 3.3 Composites with Spheroidal Inclusions, and the Aligned Case

537 In the 1060s, Hill (1963, 1965), Hashin (1963) and Walpole (1966a,b, 1969) gave fundamental
 538 contributions to the development of techniques for the evaluation of the overall elasticity tensor
 539 (overall elastic moduli, in the terminology of the time) of a composite starting from the elasticity

540 tensors of the individual constituents of the composite. A composite differs from the system of
 541 Eshelby’s inclusion problem in that an inclusion is no longer a solitary *singularity* in an infinite
 542 matrix, but is one of many other inclusions, which could be of the same or of different type.

543 The problem of the evaluation of the overall elasticity tensor for a composite with one or
 544 more families of inclusions is therefore tackled by means of the concept of *representative element of*
 545 *volume* or, with the customary acronym, REV. The REV could be defined as the smallest region
 546 whose material properties are equivalent to those of the whole composite. We quote the definition
 547 of REV given by Hill (1963):

548 “*This phrase [representative element of volume] will be used when referring to a sample*
 549 *that (a) is structurally entirely typical of the whole mixture on average, and (b)*
 550 *contains a sufficient number of inclusions for the apparent overall moduli to be effectively*
 551 *independent of the surface values of traction and displacement, so long as these values*
 552 *are “macroscopically uniform”. That is, they fluctuate about a mean with a wavelength*
 553 *small compared with the dimensions of the sample, and the effects of such fluctuations*
 554 *become insignificant within a few wavelengths of the surface. The contribution of this*
 555 *surface layer to any average can be made negligible by taking the sample large enough.”*

556 Thus, the REV that we consider must contain a sufficient number of inclusions for the overall
 557 elasticity tensor (or the collection of the “apparent elastic moduli”, in Hill’s words) to be
 558 representative of that of the whole composite.

559 The composite is assumed to be comprised of $N + 1$ phases, with phase 0 referring to the
 560 matrix, and phases $r \in \{1, \dots, N\}$ referring to the r -th inclusion families. The inclusions are assumed
 561 to be *perfectly fitted* (Walpole, 1966a) in the matrix, which, in Eshelby’s terminology, means that
 562 they are “inhomogeneous inclusions”, i.e., inclusions with *no geometrical misfit* with the matrix,
 563 but with elastic properties *different* from those of the matrix. It is important to emphasise that, in
 564 order to be able to apply Eshelby’s theory as described in Sections 3.1 and 3.2, which is based on an
 565 *inclusion in an infinite matrix*, we must make sure that each inclusion is far enough from its prime
 566 neighbours and the interactions among inclusions can be neglected. This is achieved by imposing
 567 a reasonably low *volumetric fraction* for each of the inclusion phases. The volumetric fraction of
 568 each phase is defined as

$$\phi_r = \frac{\Omega_r}{\Omega}, \quad (68)$$

569 where Ω is the volume of the REV, and Ω_r is the volume of the portion of the REV occupied by
 570 phase r . The volumetric fractions obey the constraint

$$\sum_{r=0}^N \phi_r = 1. \quad (69)$$

571 The strain concentration tensor seen in Section 3.2 has been extensively used in the
 572 determination of the overall elastic properties of composite materials with inclusions (see, e.g.,
 573 Hill, 1963; Walpole, 1966a,b, 1969; Weng, 1984, 1990; Qiu and Weng, 1990). In the formalism
 574 introduced by Walpole (1966a,b, 1969) and Weng (1990), the overall elasticity tensor \mathbb{L} reads

$$\mathbb{L} = \left[\sum_{r=0}^N \phi_r \mathbb{L}_r : \mathbb{A}_r \right] : \left[\sum_{r=0}^N \phi_r \mathbb{A}_r \right]^{-1}, \quad (70)$$

575 where \mathbb{L}_r is the elasticity tensor of phase r and

$$\mathbb{A}_r = [\mathbb{I}^T + \mathbb{S}_r : [\mathbb{L}_0^{-1} : \mathbb{L}_r - \mathbb{I}^T]]^{-1} \quad (71)$$

576 is the strain concentration tensor of the r -th phase, in which \mathbb{S}_r is the Eshelby fourth-order tensor
 577 relative to the r -th phase, depending on the shape of the inclusions of phase r and the elastic
 578 constants of the matrix. Note that the strain concentration tensor \mathbb{A}_0 of the matrix is identically
 579 equal to the transpose \mathbb{I}^T of the symmetric identity. Indeed, *since the matrix is not an inclusion*
 580 *embedded in itself*, from Equation (71) we have

$$\mathbb{A}_0 = [\mathbb{I}^T + \mathbb{S}_0 : [\mathbb{L}_0^{-1} : \mathbb{L}_0 - \mathbb{I}^T]]^{-1} = [\mathbb{I}^T + \mathbb{S}_0 : \mathbb{O}]^{-1} = \mathbb{I}^T, \quad (72)$$

581 where \mathbb{O} is the zero tensor, regardless of the value of the tensor \mathbb{S}_0 .⁷

582 We remark that Equation (70) is analogical to that of the centre of mass of a system of
 583 particles, i.e.,

$$x_G = \frac{\sum_{r=0}^N m_r x_r}{\sum_{r=0}^N m_r}. \quad (73)$$

584 In this analogy, Equation (70) provides the “barycentric elasticity tensor” of a composite, in which
 585 the “masses” are the products $\phi_r \mathbb{A}_r$ and the “moment arms” are the elasticity tensors \mathbb{L}_r .

586 In principle, Equation (70) applies to any composite with ellipsoidal inclusions. However,
 587 for ellipsoidal inclusion families with different semi-axis ratios, different alignment of the semi-axes,
 588 and different alignment of the directions or planes of material symmetry, the overall elasticity
 589 tensor could be completely anisotropic. Equation (70) becomes immediately usable in the case of
 590 transverse isotropy with respect to a given direction \mathbf{m}_0 , which is obtained when:

- 591 (A1) The matrix is either isotropic or transversely isotropic with respect to direction \mathbf{m}_0 ;
 592 (A2) All inclusions in all families have their axis of symmetry oriented in direction \mathbf{m}_0 , are
 593 spheroidal (i.e., are revolution ellipsoids), and are either isotropic or transversely isotropic
 594 with respect to \mathbf{m}_0 .

595 When the two conditions (A1) and (A2) are satisfied, all tensors featuring in Equation (70) are
 596 transversely isotropic in direction \mathbf{m}_0 , and can be represented using Walpole’s formalism (Walpole,
 597 1981, 1984) presented in Section 2.7. This is the procedure followed by Weng (1990) and Qiu and
 598 Weng (1990), leading to the Walpole array representation

$$\overline{\mathbb{L}} = \left[\sum_{r=0}^N \phi_r \overline{\mathbb{L}_r : \mathbb{A}_r} \right] \left[\sum_{r=0}^N \phi_r \overline{\mathbb{A}_r} \right]^{-1}. \quad (74)$$

599 Qiu and Weng (1990) also noted that the lack of major symmetry of the strain concentration
 600 tensors \mathbb{A}_r causes in general the lack of major symmetry of the overall elasticity tensor obtained via
 601 Equation (70), except in the perfectly isotropic case mentioned above and when the aligned inclusion

⁷On one occasion, we had stated that \mathbb{S}_0 reduces to the identity \mathbb{I}^T (paragraph following Equation (35) in Federico et al., 2004) and, on another occasion, that it reduces to the zero tensor \mathbb{O} (paragraph following Equation (12) in Federico, 2010a). Both statements are incorrect, as this \mathbb{S}_0 is really arbitrary. One can think to obtain \mathbb{A}_0 by imagining to have an inclusion with an arbitrary ellipsoidal shape and an elasticity tensor \mathbb{L}'_0 , which defines a corresponding \mathbb{S}_0 , and then by performing the limit $\mathbb{L}'_0 \rightarrow \mathbb{L}_0$. This yields $\mathbb{A}_0 \rightarrow \mathbb{I}^T$ regardless of the value of \mathbb{S}_0 .

602 phases have all the same shape. Thus, the transversely isotropic overall elasticity tensor found from
 603 Equation (70) when the conditions (A1) and (A2) are satisfied has Walpole representation

$$\bar{\mathbb{L}} = \left\{ \begin{bmatrix} n & \sqrt{2}\ell \\ \sqrt{2}\ell' & 2c \end{bmatrix}, 2\mu_t, 2\mu_a \right\}, \quad (75)$$

604 where, in contrast with Equation (53), $\ell' \neq \ell$. A “brute force” solution to this problem was proposed
 605 by Wu and Herzog (2002), who took the (major) symmetric part of the overall elasticity tensor \mathbb{L}
 606 of Equation (75), i.e.,

$$\mathbb{L}_{\text{sym}} = \frac{1}{2}(\mathbb{L} + \mathbb{L}^T), \quad (\mathbb{L}_{\text{sym}})^{ijkl} = \frac{1}{2}(\mathbb{L}^{ijkl} + \mathbb{L}^{klij}), \quad (76)$$

607 which, in terms of the symmetrised cross modulus to be used in the Walpole representation $\overline{\mathbb{L}_{\text{sym}}}$
 608 of \mathbb{L}_{sym} , reads

$$\ell_{\text{sym}} = \frac{1}{2}(\ell + \ell'). \quad (77)$$

609 Finally, we note that, if the matrix is isotropic and all inclusions in all families are spherical
 610 and isotropic, isotropy is retrieved as a trivial particular case of transverse isotropy.

611 4 Composite Materials with Statistically Oriented Inclusions

612 In this section, we report, in the more recent notation presented in Section 2 (Federico, 2010a, 2015;
 613 Federico et al., 2015), our results for the general case of a composite with statistically oriented
 614 spheroidal inclusions, in which the orientation obeys a given probability density (Federico et al.,
 615 2004).

616 4.1 Generalised Walpole’s Formula

617 In order to univocally identify the orientation of an ellipsoid, we need three parameters, e.g.,
 618 the three Euler angles. In the case of a spheroid, by virtue of the rotational symmetry, only two
 619 parameters are required, and those could be, e.g., two Euler angles or, equivalently, the unit vector
 620 describing the direction of the axis of symmetry of the spheroid. We shall restrict our attention
 621 to the case of *spheroidal* inclusions. A phase of statistically oriented spheroidal inclusions (i.e.,
 622 inclusions all sharing the same geometry and elastic properties, but having different orientations)
 623 can be thought of as an infinity of phases, each oriented in a certain direction, so that the summation
 624 in Equation (70) becomes an integral on the unit sphere $\mathbb{S}^2\mathcal{B}$. In this integral, at every point x in
 625 the body \mathcal{B} , the weighing function is a probability density

$$\psi : \mathbb{S}^2\mathcal{B} \rightarrow \mathbb{R}_0^+ : \mathbf{m} \mapsto \psi(\mathbf{m}), \quad (78)$$

626 which describes the probability to find, at each point x in \mathcal{B} , an inclusion oriented in direction
 627 \mathbf{m} . In the case of inhomogeneous bodies, ψ depends explicitly on the point x in the body. In the
 628 present formulation, however, for the sake of a lighter notation, this dependence is omitted but
 629 understood. The probability density ψ must be normalised over the sphere and must be invariant
 630 for reflections $\mathbf{m} \mapsto -\mathbf{m}$, i.e.,

$$\int_{\mathbb{S}^2\mathcal{B}} \psi(\mathbf{m}) = 1, \quad (79a)$$

$$\psi(-\mathbf{m}) = \psi(\mathbf{m}). \quad (79b)$$

631 For any function f defined on the unit sphere $\mathbb{S}^2\mathcal{B}$ and valued in a tensor space of any order
 632 (including order zero, i.e., scalar functions), we denote by⁸

$$\langle\langle f \rangle\rangle = \int_{\mathbb{S}^2\mathcal{B}} \psi(\mathbf{m}) f(\mathbf{m}) \quad (80)$$

633 its *directional average*. Note that we do not explicitly indicate the area element (more precisely,
 634 the *area two-form*; see Epstein, 2010; Segev, 2013) in the integral. If the function f to be averaged
 635 enjoys the same symmetry as ψ , i.e., $f(-\mathbf{m}) = f(\mathbf{m})$, it is possible to restrict the integral to the
 636 *north hemisphere*, defined by

$$\mathbb{S}^{2+}\mathcal{B} = \{\mathbf{m} \in \mathbb{S}^2\mathcal{B} : \mathbf{m} \cdot \mathbf{m}_0 \geq 0\}, \quad (81)$$

637 where $\mathbf{m}_0 \in \mathbb{S}^2\mathcal{B}$ is the chosen *polar direction*. Naturally, since the integral is performed on *half*
 638 the domain (the north hemisphere $\mathbb{S}^{2+}\mathcal{B}$ is *half* of the sphere $\mathbb{S}^2\mathcal{B}$), one has to take *twice* the value
 639 of the integral (alternatively, one could re-normalise the probability density).

640 When the sum in Equation (70) becomes an integral, we need to transform the variables
 641 according to

$$\phi_r \mapsto \phi_1 \psi(\mathbf{m}), \quad (82a)$$

$$\mathbb{L}_r \mapsto \mathbb{L}_1(\mathbf{m}), \quad (82b)$$

$$\mathbb{A}_r \mapsto \mathbb{A}_1(\mathbf{m}), \quad (82c)$$

642 i.e., we can call the collection of all inclusions “phase 1”, and identify all orientations in $\mathbb{S}^2\mathcal{B}$ by
 643 means of the probability density ψ . Equation (70) then becomes

$$\mathbb{L} = [\phi_0 \mathbb{L}_0 + \phi_1 \langle\langle \mathbb{L}_1 : \mathbb{A}_1 \rangle\rangle] : [\phi_0 \mathbb{I}^T + \phi_1 \langle\langle \mathbb{A}_1 \rangle\rangle]^{-1}, \quad (83)$$

644 where the terms relative to the matrix account for the fact that the strain concentration tensor \mathbb{A}_0
 645 reduces to the transpose \mathbb{I}^T of the symmetric identity (Equation (72)), and we used the definition
 646 (80) of directional average in

$$\langle\langle \mathbb{L}_1 : \mathbb{A}_1 \rangle\rangle = \int_{\mathbb{S}^2\mathcal{B}} \psi(\mathbf{m}) \mathbb{L}_1(\mathbf{m}) : \mathbb{A}_1(\mathbf{m}), \quad (84)$$

$$\langle\langle \mathbb{A}_1 \rangle\rangle = \int_{\mathbb{S}^2\mathcal{B}} \psi(\mathbf{m}) \mathbb{A}_1(\mathbf{m}). \quad (85)$$

647 In the most general case, the composite is comprised of matrix (subscript 0), N_a inclusion
 648 phases, all aligned in a definite direction (subscript $r \in \{1, \dots, N_a\}$), and N_p inclusion phases with
 649 statistical orientation (subscript $r \in \{1, \dots, N_p\}$). Thus, the overall elasticity tensor reads

$$\begin{aligned} \mathbb{L} = & \left[\phi_0 \mathbb{L}_0 + \sum_{r=1}^{N_a} \phi_r \mathbb{L}_r : \mathbb{A}_r + \sum_{s=1}^{N_p} \phi_s \langle\langle \mathbb{L}_s : \mathbb{A}_s \rangle\rangle \right] \\ & : \left[\phi_0 \mathbb{I}^T + \sum_{r=1}^{N_a} \phi_r \mathbb{A}_r + \sum_{s=1}^{N_p} \phi_s \langle\langle \mathbb{A}_s \rangle\rangle \right]^{-1}, \end{aligned} \quad (86)$$

⁸Note that, in some previous works (Federico et al., 2004; Federico, 2010a), we used the symbol $\langle f \rangle$ for the integral in Equation (80) in the case of isotropic probability $\psi(\mathbf{m}) = 1/4\pi$, and called $\langle f \rangle$ the “average of f ”. We do not adopt this meaning of “average” here and, much more generally, we use “directional average” for the integral in Equation (80) with *any* probability density ψ .

650 where, for each phase s ,

$$\langle\langle \mathbb{L}_s : \mathbb{A}_s \rangle\rangle = \int_{\mathbb{S}^2 \mathcal{B}} \psi_s(\mathbf{m}) \mathbb{L}_s(\mathbf{m}) : \mathbb{A}_s(\mathbf{m}), \quad (87)$$

$$\langle\langle \mathbb{A}_s \rangle\rangle = \int_{\mathbb{S}^2 \mathcal{B}} \psi_s(\mathbf{m}) \mathbb{A}_s(\mathbf{m}) \quad (88)$$

651 are the directional averages of the product $\mathbb{L}_s : \mathbb{A}_s$ and of the strain concentration tensor \mathbb{A}_s ,
 652 respectively, and ψ_s is the probability density describing the orientation.

653 In both the case of Equation (83) with one phase of statistically oriented inclusions and
 654 the general case of Equation (86), it is important to remark that all inclusions in the same phase s
 655 have identical geometry and mechanical properties, which means that, for every pair of directions
 656 \mathbf{m}_α and \mathbf{m}_β , there are suitable rotation tensors \mathbf{Q} and \mathbf{R} (which coincide if the axes of geometrical
 657 symmetry coincide with the axes of material symmetry) such that

$$\mathbb{L}_s(\mathbf{m}_\alpha) = (\mathbf{Q} \otimes \mathbf{Q}) : \mathbb{L}_s(\mathbf{m}_\beta) : (\mathbf{Q}^T \otimes \mathbf{Q}^T), \quad (89a)$$

$$\mathbb{A}_s(\mathbf{m}_\alpha) = (\mathbf{R}^{-T} \otimes \mathbf{R}^{-T}) : \mathbb{A}_s(\mathbf{m}_\beta) : (\mathbf{R}^T \otimes \mathbf{R}^T). \quad (89b)$$

658 4.2 Transversely Isotropic Case: Preliminaries

659 In the general system with statistically oriented inclusions described by Equation (86), transverse
 660 isotropy in direction \mathbf{m}_0 is obtained with weaker conditions than those of the aligned case seen in
 661 Section 3.3. Specifically, while condition (S1) below is identical to condition (A1), condition (S2)
 662 below echoes (A2) but is valid only for the aligned phases $r \in \{1, \dots, N_a\}$, and a new condition (S3)
 663 must be stated for the statistically oriented phases $s \in \{1, \dots, N_p\}$

664 (S1) The matrix is either isotropic or transversely isotropic with respect to \mathbf{m}_0 ;

665 (S2) All inclusions in all aligned families $r \in \{1, \dots, N_a\}$ have their axis of symmetry oriented
 666 in direction \mathbf{m}_0 , are spheroidal (i.e., are revolution ellipsoids), and are either isotropic or
 667 transversely isotropic with respect to \mathbf{m}_0 .

668 (S3) All inclusions in all statistically oriented families $s \in \{1, \dots, N_p\}$ are spheroidal, are either
 669 isotropic or transversely isotropic with respect to their axis of geometrical symmetry, and
 670 the probability densities ψ_s are all transversely isotropic with respect to direction \mathbf{m}_0 , i.e.,
 671 $\psi_s(\mathbf{Q}\mathbf{m}) = \psi_s(\mathbf{m})$, where \mathbf{Q} is an orthogonal tensor such that $\mathbf{Q}\mathbf{m}_0 = \pm\mathbf{m}_0$.

672 Indeed, under the hypotheses (S1), (S2) and (S3), all elasticity tensors \mathbb{L}_0 and \mathbb{L}_r , the transpose
 673 $\overline{\mathbb{I}}^T$ of the symmetric identity, all strain concentration tensors \mathbb{A}_r , and all directional averages $\langle\langle \mathbb{L}_s : \mathbb{A}_s \rangle\rangle$
 674 $\langle\langle \mathbb{A}_s \rangle\rangle$ and $\langle\langle \mathbb{A}_s \rangle\rangle$ in Equation (86) are transversely isotropic with respect to \mathbf{m}_0 , which implies the
 675 transverse isotropy of the overall elasticity tensor \mathbb{L} . Thus, all tensors in Equation (86) can be
 676 decomposed in Walpole's transversely isotropic basis of Equation (32) relative to direction \mathbf{m}_0 ,
 677 i.e., $\{\mathbb{U}_{\alpha\beta}(\mathbf{m}_0), \mathbb{V}_\gamma(\mathbf{m}_0)\}_{\alpha,\beta,\gamma=1}^2$, so that all tensor contractions and inversions can be performed
 678 conveniently exploiting Walpole's array formalism, in which Equation (86) takes the form

$$\begin{aligned} \overline{\mathbb{L}} = & \left[\phi_0 \overline{\mathbb{L}}_0 + \sum_{r=1}^{N_a} \phi_r \overline{\mathbb{L}}_r : \overline{\mathbb{A}}_r + \sum_{s=1}^{N_p} \phi_s \overline{\langle\langle \mathbb{L}_s : \mathbb{A}_s \rangle\rangle} \right] \\ & \left[\phi_0 \overline{\mathbb{I}}^T + \sum_{r=1}^{N_a} \phi_r \overline{\mathbb{A}}_r + \sum_{s=1}^{N_p} \phi_s \overline{\langle\langle \mathbb{A}_s \rangle\rangle} \right]^{-1}, \end{aligned} \quad (90)$$

679 where the Walpole array $\overline{\mathbb{I}^T}$ is equal to $\overline{\mathbb{I}}$, identically.

680 The decomposition in the transversely isotropic basis $\{\mathbb{U}_{\alpha\beta}(\mathbf{m}_0), \mathbb{V}_\gamma(\mathbf{m}_0)\}_{\alpha,\beta,\gamma=1}^2$ is
681 straightforward for \mathbb{L}_0 , \mathbb{I}^T , \mathbb{L}_r and \mathbb{A}_r , which are all transversely isotropic in direction \mathbf{m}_0 by
682 hypothesis, but requires some additional work for the directional averages $\langle\langle \mathbb{L}_s : \mathbb{A}_s \rangle\rangle$ and $\langle\langle \mathbb{A}_s \rangle\rangle$.
683 Even though Equation (90) requires the determination of the Walpole components of $\langle\langle \mathbb{A}_s \rangle\rangle$ in
684 $[\mathcal{TB}]_2^2$, we shall decompose its “contravariant” counterpart $\langle\langle \mathbb{A}_s^\sharp \rangle\rangle$ in $[\mathcal{TB}]_0^4$, for which the formulae
685 (34) are applied. This can be done because, as seen in Section 4.3, $\langle\langle \mathbb{A}_s^\sharp \rangle\rangle$ and $\langle\langle \mathbb{A}_s \rangle\rangle$ have the same
686 Walpole array (see Section 2.7). We recall that $\langle\langle \mathbb{A}_s^\sharp \rangle\rangle$ is obtained from $\langle\langle \mathbb{A}_s \rangle\rangle$ by raising the first pair
687 of indices of \mathbb{A}_s through \mathbb{I}^\sharp (the “contravariant” symmetric identity defined in Equation (24a)), i.e.,
688 by computing $\mathbb{A}_s^\sharp = \mathbb{I}^\sharp : \mathbb{A}_s$, and averaging the resulting expression: $\langle\langle \mathbb{A}_s^\sharp \rangle\rangle = \langle\langle \mathbb{I}^\sharp : \mathbb{A}_s \rangle\rangle = \mathbb{I}^\sharp : \langle\langle \mathbb{A}_s \rangle\rangle =$
689 $\langle\langle \mathbb{A}_s \rangle\rangle^\sharp$. Note that the equality $\langle\langle \mathbb{A}_s^\sharp \rangle\rangle = \langle\langle \mathbb{A}_s \rangle\rangle^\sharp$ stems from the fact that directional averaging and
690 raising of indices commute with each other.

691 The whole problem of inclusions oriented according to a given transversely isotropic
692 probability density reduces to the evaluation of the Walpole array of the directional average of
693 the $2s$ tensors $\langle\langle \mathbb{L}_s : \mathbb{A}_s \rangle\rangle$ and $\langle\langle \mathbb{A}_s \rangle\rangle$. This latter problem is solved once we are able to evaluate the
694 Walpole array of the directional average of a generic tensor \mathbb{T} , which is the topic of Section 4.3. We
695 conclude this section by noting that isotropy is retrieved if the inclusions in the “aligned” phases
696 are spherical and isotropic, and if the probability density is imposed to be isotropic, which means
697 that the inclusions of the statistical phases are oriented randomly. Under this latter hypothesis,
698 the inclusions of the statistically oriented phases are allowed to be of spheroidal shape and either
699 isotropic or transversely isotropic with respect to the direction of their axis of geometrical symmetry.

700 4.3 Transversely Isotropic Case: Average of a Function of the Direction

701 Let

$$\mathbb{T} : \mathbb{S}^2\mathcal{B} \rightarrow [\mathcal{TB}]_0^4 : \mathbf{m} \mapsto \mathbb{T}_s(\mathbf{m}) \in ([\mathcal{TB}]_0^4, \mathbf{m}), \quad (91)$$

702 be a $([\mathcal{TB}]_0^4, \mathbf{m})$ -valued function, \mathbf{m}_0 a direction in $\mathbb{S}^2\mathcal{B}$ and ψ a probability density with transverse
703 isotropy with respect to \mathbf{m}_0 . Our purpose is to study the directional average

$$\langle\langle \mathbb{T} \rangle\rangle = \int_{\mathbb{S}^2\mathcal{B}} \psi(\mathbf{m}) \mathbb{T}(\mathbf{m}). \quad (92)$$

704 First, we note that, since the probability density ψ is transversely isotropic with respect to \mathbf{m}_0 ,
705 then $\langle\langle \mathbb{T} \rangle\rangle$ belongs to $([\mathcal{TB}]_0^4, \mathbf{m}_0)$ and we have the identity

$$\langle\langle \mathbb{T} \rangle\rangle = \overline{\langle\langle \mathbb{T} \rangle\rangle}^{\alpha\beta} \mathbb{U}_{\alpha\beta}(\mathbf{m}_0) + \overline{\langle\langle \mathbb{T} \rangle\rangle}^\gamma \mathbb{V}_\gamma(\mathbf{m}_0), \quad (93)$$

706 where $\{\mathbb{U}_{\alpha\beta}(\mathbf{m}_0), \mathbb{V}_\gamma(\mathbf{m}_0)\}_{\alpha,\beta,\gamma=1}^2$ is the basis of $([\mathcal{TB}]_0^4, \mathbf{m}_0)$, and

$$\overline{\langle\langle \mathbb{T} \rangle\rangle}^{\alpha\beta} = \langle\langle \mathbb{T} \rangle\rangle, \mathbb{U}_{\alpha\beta}(\mathbf{m}_0)\rangle, \quad \overline{\langle\langle \mathbb{T} \rangle\rangle}^\gamma = \frac{1}{2} \langle\langle \mathbb{T} \rangle\rangle, \mathbb{V}_\gamma(\mathbf{m}_0)\rangle, \quad (94)$$

707 according to Equation (34). Second, if we decompose $\mathbb{T}(\mathbf{m})$ in the basis $\{\mathbb{U}_{\mu\nu}(\mathbf{m}), \mathbb{V}_\pi(\mathbf{m})\}_{\mu,\nu,\pi=1}^2$
708 of the space $([\mathcal{TB}]_0^4, \mathbf{m})$ of transversely isotropic tensors with respect to \mathbf{m} , the directional average

709 in Equation (92) becomes

$$\begin{aligned}
\langle\langle \mathbb{T} \rangle\rangle &= \int_{\mathbb{S}^2 \mathcal{B}} \psi(\mathbf{m}) [\bar{\mathbb{T}}^{\mu\nu} \mathbb{U}_{\mu\nu}(\mathbf{m}) + \bar{\mathbb{T}}^\pi \mathbb{V}_\pi(\mathbf{m})] \\
&= \bar{\mathbb{T}}^{\mu\nu} \int_{\mathbb{S}^2 \mathcal{B}} \psi(\mathbf{m}) \mathbb{U}_{\mu\nu}(\mathbf{m}) + \bar{\mathbb{T}}^\pi \int_{\mathbb{S}^2 \mathcal{B}} \psi(\mathbf{m}) \mathbb{V}_\pi(\mathbf{m}) \\
&= \bar{\mathbb{T}}^{\mu\nu} \langle\langle \mathbb{U}_{\mu\nu} \rangle\rangle + \bar{\mathbb{T}}^\pi \langle\langle \mathbb{V}_\pi \rangle\rangle, \tag{95}
\end{aligned}$$

710 where the Walpole components $\bar{\mathbb{T}}^{\mu\nu}$ and $\bar{\mathbb{T}}^\pi$ *do not* depend on the direction \mathbf{m} and can be therefore
711 factorised out of the integral.

712 Now, we note that the directional averages $\langle\langle \mathbb{U}_{\mu\nu} \rangle\rangle$ and $\langle\langle \mathbb{V}_\pi \rangle\rangle$ are of the same type as the
713 average $\langle\langle \mathbb{T} \rangle\rangle$ in Equation (92), and thus are transversely isotropic in direction \mathbf{m}_0 . Therefore, we
714 can use Equation (93) to write $\langle\langle \mathbb{U}_{\mu\nu} \rangle\rangle$ and $\langle\langle \mathbb{V}_\pi \rangle\rangle$ as

$$\langle\langle \mathbb{U}_{\mu\nu} \rangle\rangle = \overline{\langle\langle \mathbb{U}_{\mu\nu} \rangle\rangle}^{\alpha\beta} \mathbb{U}_{\alpha\beta}(\mathbf{m}_0) + \overline{\langle\langle \mathbb{U}_{\mu\nu} \rangle\rangle}^\gamma \mathbb{V}_\gamma(\mathbf{m}_0), \tag{96a}$$

$$\langle\langle \mathbb{V}_\pi \rangle\rangle = \overline{\langle\langle \mathbb{V}_\pi \rangle\rangle}^{\alpha\beta} \mathbb{U}_{\alpha\beta}(\mathbf{m}_0) + \overline{\langle\langle \mathbb{V}_\pi \rangle\rangle}^\gamma \mathbb{V}_\gamma(\mathbf{m}_0), \tag{96b}$$

715 In this way, the problem of evaluating the directional average $\langle\langle \mathbb{T} \rangle\rangle$ of the tensor-valued function
716 $\mathbb{T}(\mathbf{m})$ is reduced to finding the averages $\langle\langle \mathbb{U}_{\mu\nu} \rangle\rangle$ and $\langle\langle \mathbb{V}_\pi \rangle\rangle$ of $\mathbb{U}_{\mu\nu}(\mathbf{m})$ and $\mathbb{V}_\pi(\mathbf{m})$. Since the scalar
717 product by a tensor that is independent of the direction \mathbf{m} and integration over all directions \mathbf{m}
718 commute, we have

$$\overline{\langle\langle \mathbb{U}_{\mu\nu} \rangle\rangle}^{\alpha\beta} = \left\langle \left(\int_{\mathbb{S}^2 \mathcal{B}} \psi(\mathbf{m}) \mathbb{U}_{\mu\nu}(\mathbf{m}) \right), \mathbb{U}_{\alpha\beta}(\mathbf{m}_0) \right\rangle = \int_{\mathbb{S}^2 \mathcal{B}} \psi(\mathbf{m}) \langle \mathbb{U}_{\mu\nu}(\mathbf{m}), \mathbb{U}_{\alpha\beta}(\mathbf{m}_0) \rangle, \tag{97a}$$

$$\overline{\langle\langle \mathbb{U}_{\mu\nu} \rangle\rangle}^\gamma = \frac{1}{2} \left\langle \left(\int_{\mathbb{S}^2 \mathcal{B}} \psi(\mathbf{m}) \mathbb{U}_{\mu\nu}(\mathbf{m}) \right), \mathbb{V}_\gamma(\mathbf{m}_0) \right\rangle = \frac{1}{2} \int_{\mathbb{S}^2 \mathcal{B}} \psi(\mathbf{m}) \langle \mathbb{U}_{\mu\nu}(\mathbf{m}), \mathbb{V}_\gamma(\mathbf{m}_0) \rangle, \tag{97b}$$

$$\overline{\langle\langle \mathbb{V}_\pi \rangle\rangle}^{\alpha\beta} = \left\langle \left(\int_{\mathbb{S}^2 \mathcal{B}} \psi(\mathbf{m}) \mathbb{V}_\pi(\mathbf{m}) \right), \mathbb{U}_{\alpha\beta}(\mathbf{m}_0) \right\rangle = \int_{\mathbb{S}^2 \mathcal{B}} \psi(\mathbf{m}) \langle \mathbb{V}_\pi(\mathbf{m}), \mathbb{U}_{\alpha\beta}(\mathbf{m}_0) \rangle, \tag{97c}$$

$$\overline{\langle\langle \mathbb{V}_\pi \rangle\rangle}^\gamma = \frac{1}{2} \left\langle \left(\int_{\mathbb{S}^2 \mathcal{B}} \psi(\mathbf{m}) \mathbb{V}_\pi(\mathbf{m}) \right), \mathbb{V}_\gamma(\mathbf{m}_0) \right\rangle = \frac{1}{2} \int_{\mathbb{S}^2 \mathcal{B}} \psi(\mathbf{m}) \langle \mathbb{V}_\pi(\mathbf{m}), \mathbb{V}_\gamma(\mathbf{m}_0) \rangle. \tag{97d}$$

719 This procedure reduces the number of integrals to be evaluated from $6 \times 81 = 486$ (indeed, each of the
720 6 tensors $\langle\langle \mathbb{U}_{\mu\nu} \rangle\rangle$, $\langle\langle \mathbb{V}_\pi \rangle\rangle$, with $\mu, \nu, \pi \in \{1, 2\}$, has 81 components) to $6 \times 6 = 36$ (or $16 + 8 + 8 + 4 = 36$,
721 if one looks at the four Equations (97)), i.e., 6 independent components for each of the 6 averages
722 $\{\langle\langle \mathbb{U}_{\mu\nu} \rangle\rangle, \langle\langle \mathbb{V}_\pi \rangle\rangle\}_{\mu\nu\pi=1}^2$. Moreover, this procedure eliminates many integrals, which vanish because of
723 the transverse isotropy of the system with respect to \mathbf{m}_0 , and could give numerical problems as they
724 could be highly oscillatory (Federico et al., 2004). As we shall show in Section 4.4, these integrals
725 can be expressed in spherical coordinates as a function of the co-latitude and longitude angles taken
726 from a reference frame in which the polar axis is the overall direction of symmetry \mathbf{m}_0 .

727 Using Equations (93), (95) and (96), we obtain the expression of the directional average
728 $\langle\langle \mathbb{T} \rangle\rangle$ in the basis $\{\mathbb{U}_{\alpha\beta}(\mathbf{m}_0), \mathbb{V}_\gamma(\mathbf{m}_0)\}_{\alpha,\beta,\gamma=1}^2$ as

$$\begin{aligned}
\langle\langle \mathbb{T} \rangle\rangle &= \bar{\mathbb{T}}^{\mu\nu} \left(\overline{\langle\langle \mathbb{U}_{\mu\nu} \rangle\rangle}^{\alpha\beta} \mathbb{U}_{\alpha\beta}(\mathbf{m}_0) + \overline{\langle\langle \mathbb{U}_{\mu\nu} \rangle\rangle}^\gamma \mathbb{V}_\gamma(\mathbf{m}_0) \right) \\
&\quad + \bar{\mathbb{T}}^\pi \left(\overline{\langle\langle \mathbb{V}_\pi \rangle\rangle}^{\alpha\beta} \mathbb{U}_{\alpha\beta}(\mathbf{m}_0) + \overline{\langle\langle \mathbb{V}_\pi \rangle\rangle}^\gamma \mathbb{V}_\gamma(\mathbf{m}_0) \right), \tag{98}
\end{aligned}$$

729 which can be rearranged into the final expression

$$\langle\langle \mathbb{T} \rangle\rangle = \left(\bar{\mathbb{T}}^{\mu\nu} \overline{\langle\langle \mathbb{U}_{\mu\nu} \rangle\rangle}^{\alpha\beta} + \bar{\mathbb{T}}^\pi \overline{\langle\langle \mathbb{V}_\pi \rangle\rangle}^{\alpha\beta} \right) \mathbb{U}_{\alpha\beta}(\mathbf{m}_0) + \left(\bar{\mathbb{T}}^{\mu\nu} \overline{\langle\langle \mathbb{U}_{\mu\nu} \rangle\rangle}^\gamma + \bar{\mathbb{T}}^\pi \overline{\langle\langle \mathbb{V}_\pi \rangle\rangle}^\gamma \right) \mathbb{V}_\gamma(\mathbf{m}_0). \tag{99}$$

730 The Walpole array form of Equation (99) is

$$\langle\langle \mathbb{T} \rangle\rangle = \left\{ \left[\overline{\mathbb{T}^{\mu\nu}} \langle\langle \mathbb{U}_{\mu\nu} \rangle\rangle^{\alpha\beta} + \overline{\mathbb{T}^\pi} \langle\langle \mathbb{V}_\pi \rangle\rangle^{\alpha\beta} \right], \overline{\mathbb{T}^{\mu\nu}} \langle\langle \mathbb{U}_{\mu\nu} \rangle\rangle^\gamma + \overline{\mathbb{T}^\pi} \langle\langle \mathbb{V}_\pi \rangle\rangle^\gamma \right\}. \quad (100)$$

731 4.4 Transversely Isotropic Case: Solution in the Polar Parametrisation

732 Assuming that the symmetry axis \mathbf{m}_0 of the transverse isotropy coincides with vector \mathbf{e}_1 of an
733 orthonormal basis $\{\mathbf{e}_i\}_{i=1}^3$, the generic direction \mathbf{m} can be expressed as a function

$$\mathbf{m}(\theta, \varphi) = \cos \theta \mathbf{e}_1 + \sin \theta \cos \varphi \mathbf{e}_2 + \sin \theta \sin \varphi \mathbf{e}_3, \quad (101)$$

734 where θ is the *co-latitude*, measured from the polar direction $\mathbf{m}_0 \equiv \mathbf{e}_1$, and φ is the *longitude*,
735 measured from the plane spanned by \mathbf{e}_1 and \mathbf{e}_2 (see Figure 2). Using the polar parametrisation of
736 the sphere, the probability density ψ can be written as

$$\varrho(\theta) = \psi(\mathbf{m}(\theta, \varphi)), \quad (102)$$

737 where the new function ϱ does *not* depend on the latitude angle φ because of the transverse isotropy
738 of ψ . In the polar parametrisation, the directional average (80) of a generic function f becomes

$$\langle\langle f \rangle\rangle = \int_0^{2\pi} \left[\int_0^\pi \varrho(\theta) f(\mathbf{m}(\theta, \varphi)) \sin \theta \, d\theta \right] d\varphi. \quad (103)$$

739 We recall the symmetry of ψ for reflections $\mathbf{m} \mapsto -\mathbf{m}$, and note that it is inherited by ϱ as a
740 symmetry in θ about the value $\pi/2$, i.e., $\varrho(\theta) = \varrho(\pi - \theta)$. Thus, for functions f invariant under
741 reflections $\mathbf{m} \mapsto -\mathbf{m}$ (such as all our fourth-order tensors, which depend on \mathbf{m} via the structure
742 tensor $\mathbf{a} = \mathbf{m} \otimes \mathbf{m}$), we can also write

$$\langle\langle f \rangle\rangle = 2 \int_0^{2\pi} \left[\int_0^{\pi/2} \varrho(\theta) f(\mathbf{m}(\theta, \varphi)) \sin \theta \, d\theta \right] d\varphi, \quad (104)$$

743 which is equivalent to integrating ψ over the *north hemisphere* $\mathbb{S}^{2+}\mathcal{B}$ as in Equation (81). For our
744 purposes, the generic function f has to be replaced by the integrands in Equation (97).

745 4.5 Some Relevant Particular Cases

746 The most “classical” particular cases of transversely isotropic probability density ψ are the case of
747 probability density converging to the Dirac-delta, describing orientation in one direction, the case
748 of in-plane random orientation, in which all directions within the same plane are equally probable,
749 and the case of random orientation, yielding an isotropic solution. In the solutions that we report
750 below, in order to minimise the possibility of making mistakes with the tedious integrals (97), we
751 employed *Wolfram Mathematica*.

752 The case of alignment in one definite direction can be tackled by means of a parametric
753 probability density peaked at $\theta = 0$, which can be made to converge, in the sense of distributions
754 (see, e.g., Kolmogorov and Fomin, 1999) to a Dirac delta, which is the approach we used in the
755 past (Federico et al., 2004). However, quite trivially, one can directly say that, if all inclusions are
756 oriented in the same direction, Equation (86) reduces to the Walpole solution (70) which Weng

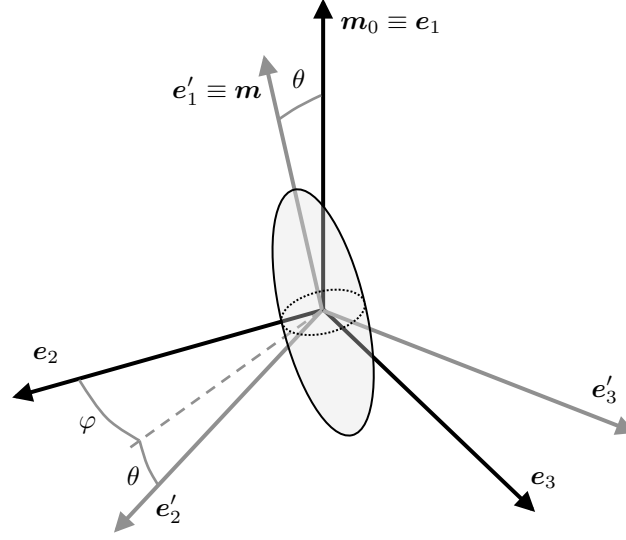


Figure 2: Representation of the generic direction $\mathbf{e}'_1 \equiv \mathbf{m}$ of the axis of symmetry of a spheroidal inclusion in terms of the colatitude angle θ and the longitude angle φ . The colatitude θ is calculated from the global direction of symmetry $\mathbf{e}_1 \equiv \mathbf{m}_0$, and the longitude φ is calculated from the plane spanned by \mathbf{e}_1 and \mathbf{e}_2 .

757 (1990) and Qiu and Weng (1990) used for aligned inclusions (and which we reported in the form
758 of a Walpole array in Equation (74)).

759 The case of random orientation in a plane can also be solved with a parametric probability
760 density peaked at $\theta = \pi/2$ that converges, in the sense of distributions, to a Dirac delta (Federico
761 et al., 2004). However, we can follow a simpler method. Rather than averaging the function
762 $f(\mathbf{m}(\cdot, \cdot))$ with values $f(\mathbf{m}(\theta, \varphi))$ of Equation (103) on the whole unit sphere $\mathbb{S}^2\mathcal{B}$, we average the
763 function $f(\mathbf{m}(\pi/2, \cdot))$ with values $f(\mathbf{m}(\pi/2, \varphi))$ on the equatorial unit circumference ($\mathbb{S}^1\mathcal{B}, \mathbf{m}_0$),
764 i.e., the circumference laying on the plane orthogonal to the direction of overall symmetry \mathbf{m}_0
765 (which is the plane spanned by \mathbf{e}_2 and \mathbf{e}_3 in Figure 2). This boils down to transforming the integral
766 in Equation (103) into

$$\langle\langle f \rangle\rangle = \int_0^{2\pi} \frac{1}{2\pi} f(\mathbf{m}(\pi/2, \varphi)) \sin(\pi/2) d\varphi = \frac{1}{2\pi} \int_0^{2\pi} f(\mathbf{m}(\pi/2, \varphi)) d\varphi, \quad (105)$$

767 where $1/2\pi$ is the constant value of the probability density on the equatorial unit circumference
768 ($\mathbb{S}^1\mathcal{B}, \mathbf{m}_0$), and equals the reciprocal of the amplitude of the interval $[0, 2\pi]$ within which the
769 longitude φ varies. Considering Equation (105), the components of the directional averages $\langle\langle \mathbb{U}_{\mu\nu} \rangle\rangle$
770 and $\langle\langle \mathbb{V}_\pi \rangle\rangle$ in Equation (97) can be obtained and represented via the Walpole arrays

$$\overline{\langle\langle \mathbb{U}_{11} \rangle\rangle} = \frac{1}{8} \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}, 2, 0 \right\}, \quad \overline{\langle\langle \mathbb{U}_{12} \rangle\rangle} = \frac{1}{8} \left\{ \begin{bmatrix} 0 & 0 \\ 4 & 2\sqrt{2} \end{bmatrix}, -\sqrt{2}, 0 \right\}, \quad (106a)$$

$$\overline{\langle\langle \mathbb{U}_{21} \rangle\rangle} = \frac{1}{8} \left\{ \begin{bmatrix} 0 & 4 \\ 0 & 2\sqrt{2} \end{bmatrix}, -\sqrt{2}, 0 \right\}, \quad \overline{\langle\langle \mathbb{U}_{22} \rangle\rangle} = \frac{1}{8} \left\{ \begin{bmatrix} 4 & 2\sqrt{2} \\ 2\sqrt{2} & 2 \end{bmatrix}, 1, 0 \right\}, \quad (106b)$$

$$\overline{\langle\langle \mathbb{V}_1 \rangle\rangle} = \frac{1}{8} \left\{ \begin{bmatrix} 4 & -2\sqrt{2} \\ -2\sqrt{2} & 2 \end{bmatrix}, 1, 4 \right\}, \quad \overline{\langle\langle \mathbb{V}_2 \rangle\rangle} = \frac{1}{8} \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, 4, 4 \right\}. \quad (106c)$$

771 For the case of random orientation, we follow the approach we used in the past (Federico
772 et al., 2004), with some minor notational and procedural differences. When the probability density

773 is given by $\psi(\mathbf{m}(\theta, \varphi)) = \varrho(\theta, \varphi) = 1/4\pi$ (random orientation), the integral in Equation (92) *must*
774 coincide with its isotropic projection (see Equation (28)), i.e., must necessarily be isotropic. Thus,
775 we have the identity

$$\langle\langle \mathbb{T} \rangle\rangle = \int_{\mathbb{S}^2 \mathcal{B}} \psi(\mathbf{m}) \mathbb{T}(\mathbf{m}) \equiv \left[\int_{\mathbb{S}^2 \mathcal{B}} \psi(\mathbf{m}) \mathbb{T}(\mathbf{m}) \right]_{\text{iso}} = \langle\langle \mathbb{T} \rangle\rangle_{\text{iso}}. \quad (107)$$

776 Since the operation of isotropic projection (28) and the averaging integral commute, we can also
777 write

$$\langle\langle \mathbb{T} \rangle\rangle = \int_{\mathbb{S}^2 \mathcal{B}} \psi(\mathbf{m}) \mathbb{T}(\mathbf{m}) \equiv \int_{\mathbb{S}^2 \mathcal{B}} \psi(\mathbf{m}) [\mathbb{T}(\mathbf{m})]_{\text{iso}} = \langle\langle \mathbb{T}_{\text{iso}} \rangle\rangle. \quad (108)$$

778 Moreover, the dependence on \mathbf{m} in $[\mathbb{T}(\mathbf{m})]_{\text{iso}}$ *must* disappear, as $[\mathbb{T}(\mathbf{m})]_{\text{iso}}$ is isotropic. Thus, we
779 can replace $[\mathbb{T}(\mathbf{m})]_{\text{iso}}$ by $[\mathbb{T}(\mathbf{m}_0)]_{\text{iso}}$, where \mathbf{m}_0 is an *arbitrary* direction, and factorise $[\mathbb{T}(\mathbf{m}_0)]_{\text{iso}}$
780 *outside* of the integral sign, to obtain the final expression

$$\langle\langle \mathbb{T} \rangle\rangle = [\mathbb{T}(\mathbf{m}_0)]_{\text{iso}}, \quad (109)$$

781 where we used the normalisation to one of the probability density. Since $\mathbb{T}(\mathbf{m}_0) = \bar{\mathbb{T}}^{\alpha\beta} \mathbb{U}_{\alpha\beta}(\mathbf{m}_0) +$
782 $\bar{\mathbb{T}}^\gamma \mathbb{V}_\gamma(\mathbf{m}_0)$, we can use linearity and write

$$\langle\langle \mathbb{T} \rangle\rangle = \bar{\mathbb{T}}^{\alpha\beta} [\mathbb{U}_{\alpha\beta}(\mathbf{m}_0)]_{\text{iso}} + \bar{\mathbb{T}}^\gamma [\mathbb{V}_\gamma(\mathbf{m}_0)]_{\text{iso}}. \quad (110)$$

783 The isotropic projections $[\mathbb{U}_{\alpha\beta}(\mathbf{m}_0)]_{\text{iso}}$ and $[\mathbb{V}_\gamma(\mathbf{m}_0)]_{\text{iso}}$ can be written either in the isotropic basis
784 $\{\mathbb{K}^\#, \mathbb{M}^\#\}$ or in Walpole's transversely isotropic representation as seen in Equation (42). In the
785 former case, we have

$$[\mathbb{U}_{11}]_{\text{iso}} = \frac{1}{3} \mathbb{K}^\# + \frac{2}{15} \mathbb{M}^\#, \quad [\mathbb{U}_{12}]_{\text{iso}} = \frac{\sqrt{2}}{3} \mathbb{K}^\# - \frac{\sqrt{2}}{15} \mathbb{M}^\#, \quad (111a)$$

$$[\mathbb{U}_{21}]_{\text{iso}} = \frac{\sqrt{2}}{3} \mathbb{K}^\# - \frac{\sqrt{2}}{15} \mathbb{M}^\#, \quad [\mathbb{U}_{22}]_{\text{iso}} = \frac{2}{3} \mathbb{K}^\# + \frac{1}{15} \mathbb{M}^\#, \quad (111b)$$

$$[\mathbb{V}_1]_{\text{iso}} = 0 \mathbb{K}^\# + \frac{2}{5} \mathbb{M}^\#, \quad [\mathbb{V}_2]_{\text{iso}} = 0 \mathbb{K}^\# + \frac{2}{5} \mathbb{M}^\#, \quad (111c)$$

786 and, in the latter, we have

$$\overline{[\mathbb{U}_{11}]_{\text{iso}}} = \frac{1}{15} \left\{ \begin{bmatrix} 3 & \sqrt{2} \\ \sqrt{2} & 4 \end{bmatrix}, 2, 2 \right\}, \quad \overline{[\mathbb{U}_{12}]_{\text{iso}}} = \frac{1}{15} \left\{ \begin{bmatrix} \sqrt{2} & 4 \\ 4 & 3\sqrt{2} \end{bmatrix}, -\sqrt{2}, -\sqrt{2} \right\}, \quad (112a)$$

$$\overline{[\mathbb{U}_{21}]_{\text{iso}}} = \frac{1}{15} \left\{ \begin{bmatrix} \sqrt{2} & 4 \\ 4 & 3\sqrt{2} \end{bmatrix}, -\sqrt{2}, -\sqrt{2} \right\}, \quad \overline{[\mathbb{U}_{22}]_{\text{iso}}} = \frac{1}{15} \left\{ \begin{bmatrix} 4 & 3\sqrt{2} \\ 3\sqrt{2} & 7 \end{bmatrix}, 1, 1 \right\}, \quad (112b)$$

$$\overline{[\mathbb{V}_1]_{\text{iso}}} = \frac{1}{15} \left\{ \begin{bmatrix} 4 & -2\sqrt{2} \\ -2\sqrt{2} & 2 \end{bmatrix}, 6, 6 \right\}, \quad \overline{[\mathbb{V}_2]_{\text{iso}}} = \frac{1}{15} \left\{ \begin{bmatrix} 4 & -2\sqrt{2} \\ -2\sqrt{2} & 2 \end{bmatrix}, 6, 6 \right\}. \quad (112c)$$

787 5 Discussion

788 In this chapter, we summarised and discussed in detail some selected results from previous studies
789 of ours, with the purpose of rephrasing in a more efficient, consequent, and formally correct way the
790 linear elastic formulation of our picture of composite materials with statistically oriented spheroidal
791 inclusions.

792 After presenting the covariant formulation of the linear algebra of isotropic and transversely
793 isotropic second- and fourth-order tensors, we addressed some fundamental aspects of composite
794 materials with aligned inclusions, which required to review Eshelby’s inclusion problem, Eshelby’s
795 fourth-order tensor \mathbb{S} , and the strain concentration tensor \mathbb{A} . Within this framework, we discussed
796 the conditions, pertaining the geometry of the inclusions and their orientation as well as the material
797 symmetries of the matrix, which lead to a globally transversely isotropic (or, in some special cases,
798 isotropic) composite. Then, we considered composite materials with statistically oriented inclusions.
799 To this end, we introduced the probability density describing the probability that the symmetry axis
800 of an inclusion is in a given direction, and generalised Walpole’s formula (70) (Walpole, 1966a,b,
801 1969; Weng, 1990) to the case of transversely isotropic materials with respect to a symmetry axis
802 \mathbf{m}_0 (cf. Equation (99)). In order to achieve this and to minimise the number of integrals to be
803 performed, we translated the directional averaging of tensor functions depending on the direction
804 into Walpole’s formalism, and obtained Equation (100), which determines the Walpole array of the
805 directional average of a given fourth-order tensor \mathbb{T} . Finally, we showed some explicit calculations
806 for the relevant cases of isotropy and transverse isotropy.

807 It is important to emphasise the difference in terms of conditions necessary to obtain
808 transverse isotropy between Walpole’s original formula and the generalised one. Walpole’s original
809 formula (70) necessitates only the two conditions (A1) and (A2) seen in Section 3.3 to be applicable
810 to transversely isotropic materials, in the Walpole array form of Equation (74). In contrast, the
811 generalised Walpole’s formula (86) considers N_a families of aligned inclusions and N_p families of
812 statistically oriented fibres, and the three conditions (S1), (S2) and (S3) of Section 4.2 are needed.
813 While condition (S1) is identical to the “old” condition (A1), and (S2) echoes condition (A2), but
814 only for the N_a aligned phases, a new condition (S3) needs to be stated for the N_p statistically
815 oriented families (i.e., the N_p probability densities ψ_s must be transversely isotropic with respect
816 to \mathbf{m}_0).

817 The theory of composite materials with statistical orientation of the inclusions is a rich
818 research field in which very diverse scientific interests converge. The trigger of our studies has
819 been the mechanical characterisation of soft biological tissues. These are highly organised media,
820 endowed with a complex internal structure, whose mechanical properties are vastly influenced by the
821 presence and orientation of collagen fibres. Tendons and ligaments are typical examples of tissues
822 in which the collagen fibres are aligned, and blood vessels and articular cartilage are examples
823 of tissues in which the collagen fibres have statistical orientation. Collagen fibres can indeed be
824 viewed as inclusions that provide structural reinforcement to the non-fibrous extracellular matrix,
825 and modulate several important bio-chemo-mechanical processes, which involve, for instance, the
826 flow of interstitial fluids as well as the diffusive-reactive dynamics of the chemical species populating
827 the tissues (nutrients and outputs of chemical reactions). These processes are associated with
828 both second- and fourth-order tensor quantities that, depending on the (either statistical or not)
829 arrangement of the fibres, can be represented by using the methods outlined in Sections 3 and
830 4. In the case of statistical orientation, the directional average defined in Equation (92) takes a
831 tensor describing how a given quantity is associated with the spatial direction \mathbf{m} of local fibre
832 alignment, and returns the overall tensor quantity defined in one point of the tissue. This allows
833 for obtaining microstructurally based constitutive laws and puts in evidence how the evolution of
834 the tissue’s internal structure yields an evolution of the averaged tensor quantity $\langle\langle \mathbb{T} \rangle\rangle$ associated
835 with the considered material property.

836 In two previous papers of ours (Grillo et al., 2012, 2015), we proposed a theory of remodelling
837 in fibre-reinforced materials, where by “remodelling” we mean here the structural reorganisation
838 of a body, be it a tissue or a non-biological material. In this theory, the evolution of the internal

839 structure of a given medium was described by the time change of the probability density ψ featuring
840 in the averaging integral (92). Under the hypothesis that the evolution of ψ does not modify the
841 transverse isotropy of the material with respect to the direction \mathbf{m}_0 (in fact, this requires ψ to
842 evolve by maintaining itself transversely isotropic, i.e., by maintaining itself independent of the
843 longitude angle, in the spherical coordinate setting of Section 4.4), the use of Walpole's notation in
844 Equation (100) makes it possible to isolate the effect of remodelling on the averaged tensor quantity
845 $\langle\langle \mathbb{T} \rangle\rangle$, expressed in terms of the array $\overline{\langle\langle \mathbb{T} \rangle\rangle}$. Indeed, while the averaged tensors of the Walpole's basis
846 for transverse isotropy with respect to the generic direction \mathbf{m} , i.e., $\langle\langle \mathbb{U}_{\mu\nu} \rangle\rangle$ and $\langle\langle \mathbb{V}_\pi \rangle\rangle$, evolve in time
847 as they are driven by the time change of ψ , the components $\overline{\langle\langle \mathbb{T} \rangle\rangle}^{\mu\nu}$ and $\overline{\langle\langle \mathbb{T} \rangle\rangle}^\pi$ do not. In turn, since the
848 direction \mathbf{m}_0 is assumed to be preserved by the considered remodelling process, only the arrays
849 $\{\overline{\langle\langle \mathbb{U}_{\mu\nu} \rangle\rangle}^{\alpha\beta}, \overline{\langle\langle \mathbb{U}_{\mu\nu} \rangle\rangle}^\gamma\}$, $\{\overline{\langle\langle \mathbb{V}_\pi \rangle\rangle}^{\alpha\beta}, \overline{\langle\langle \mathbb{V}_\pi \rangle\rangle}^\gamma\}$ vary in time. In conclusion, by adopting Walpole's arrays,
850 it is possible to study the influence of remodelling on a global property, expressed by the averaged
851 fourth-order tensor $\langle\langle \mathbb{T} \rangle\rangle$ (for example, the fourth-order elasticity tensor of the considered medium),
852 by looking at the evolution of the components of the averages of the Walpole's basis tensors $\mathbb{U}_{\mu\nu}$
853 and \mathbb{V}_π . This subject is among the topics of our current investigations.

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855 At the time in which we originally conceived the methods presented in this chapter, back in 2003 (see
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865 concentration tensor, developed as part of his PhD thesis work (see Alhasadi and Federico, 2017).

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873 In Memoriam

874 This work is dedicated to the memory of our Maestro and Mentor Prof. Gaetano Giaquinta (Catania,
875 Italy, 25 November 1945 – 13 August 2016). Prof. Giaquinta was professor of Structure of Matter
876 at the Università di Catania (Italy), and was our teacher of Physics and Alfio's PhD supervisor.
877 He had a profound influence on our scientific as well as humanistic formation, as he taught us how
878 to look at things in life. Among the many other things, his input for the original work that we are
879 revisiting in this chapter has been of fundamental importance. It is not possible to properly express
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